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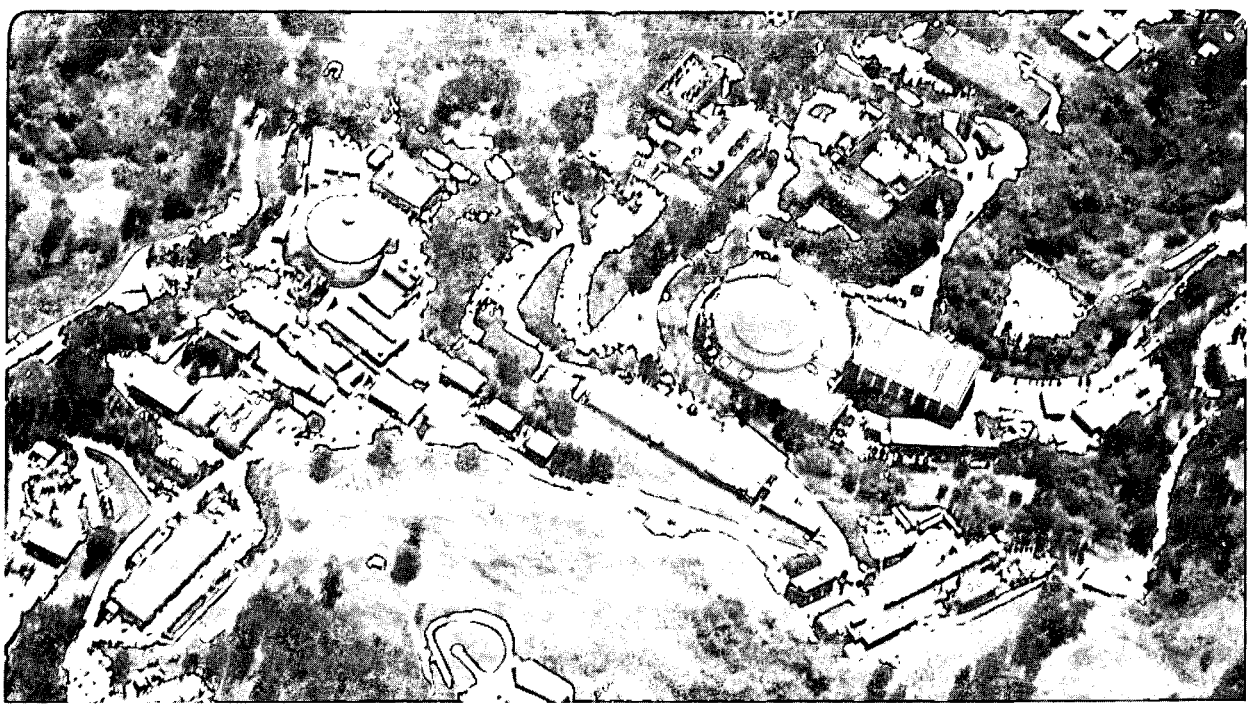

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# IMPOSING VISCOUS BOUNDARY CONDITIONS USING MAGNETIZATION VARIABLES[1]

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## Abstract

Buttke's [1] Lagrangian formalism expressing the equations of motion for incompressible fluid flow in terms of localized magnetization variables, is extended to the problem of bounded viscous flow. The approximate equations of motion for magnetization in the neighbourhood of a solid boundary are developed. A strategy for evolving magnetization from the boundary consistent with the no-slip condition is derived, which amounts to a boundary condition for magnetization. A hybrid method is also suggested which exploits the vortex sheet description of flow near a boundary. The methods are investigated in the context of a two-dimensional problem: viscous flow over a circular cylinder.

## 1. Introduction

Buttke [**1**,**2**], and others, have reformulated the equations of motion which govern incompressible inviscid flow (the Euler equations) in terms of the "velicity" (or the "magnetization variable"). In these variables, the Euler equations in three dimensions admit a discrete Lagrangian approximation which is Hamiltonian and in which the discrete invariants of the flow can be preserved. Specifically Buttke [**1**] has presented a numerical method based on this reformulation and its discretization.

The Lagrangian discretization of the Euler problem is the foundation upon which rests a family of solution procedures for the Navier-Stokes equations. These are collectively called Vortex Blob methods, and were introduced by Chorin [**3**] and have been widely analyzed and applied [**4**]. Vortex blob methods are based on the kinematics of the ensemble of vortex elements (i.e. 'blobs') which results from a particle discretization of the vorticity field of a flow. Typically the elements are ameliorated by high-order smoothing kernels [**5**] and the contribution of viscosity to the motion is introduced through stochastic modelling of diffusion (i.e. by applying random walk to the particles) or through some deterministic representation [**6**].

The particle system arising out of the vortex blob method – namely a system of interacting vortex elements, each element translated in the velocity field induced by the particle ensemble – applied to the *two-dimensional* Euler problem is, in itself, naturally Hamiltonian. It is this simplicity of the underlying dynamics that contributes to the robustness and efficiency of vortex blob methods applied to two-dimensional viscous problems [**2**].

The three-dimensional Euler problem is significantly more complicated than the two-dimensional case; specifically the phenomenon of "vortex stretching" obtains in three-dimensions whereas it does not in two-dimensions.

Nevertheless there have been a number of generalizations of the vortex method to three dimensions introducing various three-dimensional analogues to the two-dimensional blob element. The closest in concept to the two-dimensional case is the spherical blob [**5**]; also vortex tube-segments have been introduced [**7**] and closed vortex filaments or loops [**8**]. In the context of Euler flow such a collection of elements will not be necessarily Hamiltonian, nor will they preserve flow invariants. Furthermore, in the case of spherical blobs or segments, the solenoidal character of vorticity is not enforced, a situation which may lead to serious growth of error in the numerical

procedure.

It is for these reasons that Buttke's development of a Hamiltonian formulation of the discrete three-dimensional Euler problem (and one which respects the solenoidal character of vorticity) is important.

The question of applying the magnetization formulation to the case of viscous flow was addressed briefly by Buttke [**1**]. He presents the equations of motion in magnetization variables, and these can be shown to be equivalent to the projection form of the Navier-Stokes equations [**2**]. Buttke's discussion concentrates on problems of unbounded flow.

Closely allied to the question of viscous flow, is the representation of the no-slip condition and the generation of vorticity that this condition implies.

The work presented here is an attempt to express vortex creation in terms of magnetization variables and then to embed this strategy into a blob method applied to bounded viscous flow. An important motivation for this is to establish a creation algorithm consistent with a treatment of interior flow in which renormalization can be applied [**2**]. The approximations of the equations of motion appropriate to the neighbourhood of a solid boundary are determined. Hence the nature of a "magnet generation" analogue to "vortex generation" at a solid boundary is introduced.

To illustrate some of the issues which accompany such an implementation, the strategy is applied to the frequently studied two-dimensional problem of slightly viscous flow around a cylinder.

## 2. The equations of motion

The equations of motion for incompressible viscous flow, expressed in primitive variables, are the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p \tag{1}$$

where $\mathbf{u}$ is the velocity satisfying the condition $\nabla \cdot \mathbf{u} = 0$, and $p$ is the pressure (we assume uniform unit density). The paramter $\nu$ is the kinematic viscosity.

Vorticity is defined as the curl of velocity,

$$\xi = \nabla \times \mathbf{u} \tag{2}$$

The magnetization variable (or velicity) $\mathbf{m}$ is then defined as $\xi \equiv \nabla \times \mathbf{m}$. The equality $\nabla \times \mathbf{u} = \nabla \times \mathbf{m}$ implies the equivalence of $\mathbf{u}$ and $\mathbf{m}$ up to an additive gradient, i.e.

$$\mathbf{m} = \mathbf{u} + \nabla\phi \tag{3}$$

where $\phi$ is a scalar function. This amounts to a gauge condition: neither $\mathbf{m}$ nor $\phi$ is unique. Given a vector field $\mathbf{m}$, the velocity can be determined by taking the divergence of (3),

$$\Delta\phi = \nabla \cdot \mathbf{m} \tag{4}$$

Hence we can express the velocity in terms of the velicity as

$$\mathbf{u} = \mathbf{m} - \nabla\{\Delta^{-1}\{\nabla \cdot \mathbf{m}\}\} \tag{5}$$

where $\Delta^{-1}$ represents the inverse Laplacian operator.

Considering the curl of the Navier-Stokes equation (1), and invoking vector identities and the equivalence $\nabla \times \mathbf{u} \leftrightarrow \nabla \times \mathbf{m}$, we verify the following evolution equation for $\mathbf{m}$,

$$\frac{\partial \mathbf{m}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{m} + \mathbf{m} \cdot \nabla\mathbf{u} = \nu\Delta\mathbf{m}, \tag{6}$$

is equivalent to (1). It is in fact the gauge invariant form of (1). The evolution of $\mathbf{m}$ in an unbounded medium is determined by (6) upon prescribing an initial field $\mathbf{m}(\mathbf{r}, 0)$. Equation (6) can be written in terms of the material derivative, $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$, as

$$\frac{D\mathbf{m}}{Dt} = -\mathbf{m} \cdot \nabla\mathbf{u} + \nu\Delta\mathbf{m}.$$

Equation (6) bears some formal resemblance to the vorticity transport equation in three dimensions.

The vector $\mathbf{u}$ in a region $\Omega$ can be interpreted as the orthogonal projection of $\mathbf{m}$ in $\Omega$ on the space of solenoidal vectors parallel to the boundary $\partial\Omega$. This is to emphasize that (obviously) $\mathbf{m}$ is not generally solenoidal. Also there

exists a unique operator (upon specifying gauge) to project back down from $\mathbf{m}$ into the space of $\mathbf{u}$. The space of $\mathbf{m}$ is a more general (i.e. less constrained) space, and admits gauge freedom, which facilitates a localization principle.

A localization for the magnetization field can be conceived, analogous to that developed for the vorticity field in the context of vortex blob methods. Suppose vorticity vanishes outside a ball $S$ of radius $r$. Since velocity is irrotational exterior to $S$ there exists there a scalar potential such that $\mathbf{u} = -\nabla\psi$. Thus one can consider outside the ball that

$$\mathbf{m} = -\nabla\psi + \nabla\phi$$

The choice of gauge $\phi = -\psi$ outside the sphere makes $\mathbf{m}$ vanish there, and hence $\mathbf{m}$ has compact support to correspond with that of vorticity. This choice is not unique, and Buttke discusses situations where the support of magnetization and that of voriticity do not coincide.

In the case where $\mathbf{m}$ has compact support, the impulse

$$I = \int_{R^3} (\mathbf{r} \times \xi) dv$$

can be written

$$I = \int_{R^3} (\mathbf{r} \times \nabla \times \mathbf{m}) dv$$

which, after vector integration by parts, can be written

$$I = \int_{R^3} \mathbf{m} dv$$

This shows that velicity in the case of $\mathbf{m}$ of compact support can be identified as an impulse density.

Buttke offers a specific strategy for constructing a discretization of the $\mathbf{m}$ field in close analogy to that introduced to the vortex blob formulation. A smoothing function, $f_\delta$ is defined,

$$f_\delta(\mathbf{r}) = \frac{1}{\delta^p} f_0(\mathbf{r}/\delta)$$

where $f_0$ is a smooth function whose first $p-1$ moments vanish and has unit mass. A number of such functions of various order have been described in the literature [**1, 5**].

The discretization of $\mathbf{m}$ at time $t$ can be expressed in terms of an ensemble of $N$ particles as

$$\tilde{\mathbf{m}}(\mathbf{r},t) = \sum_{i=1}^{N} f_\delta(\mathbf{r}-\mathbf{r}_i)\mathbf{m}_i(t) \tag{7}$$

where each particle location translates according to

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{u}(\mathbf{r}_i) \tag{8}$$

and the magnetization strength itself evolves according to

$$\frac{d\mathbf{m}_i}{dt} = -\mathbf{m}_i \cdot \nabla\mathbf{u}(\mathbf{r}_i) \tag{9}$$

with the velocity approximated in terms of $\tilde{\mathbf{m}}$ as

$$\tilde{\mathbf{u}} = \tilde{\mathbf{m}} - \nabla\phi$$

and $\nabla \cdot \tilde{\mathbf{u}} = 0$. $\tilde{\mathbf{u}}$ is as spatially smooth as $f_\delta$, therefore we expect of a successively-refined discretization (characterized by discretization parameter, $h$) to show convergence of the form

$$\lim_{h\to 0} \tilde{\mathbf{m}} = \mathbf{m}$$

and

$$\lim_{h\to 0} \tilde{\mathbf{u}} = \mathbf{u}$$

as established in the case of vortex methods [**9**]. From (5) and (7) the velocity can be constructed explicitly as

$$\tilde{\mathbf{u}} = \sum_{i=1}^{N}\{\mathbf{m}_i f_\delta(\mathbf{r}-\mathbf{r}_i) - (\mathbf{m}_i \cdot \nabla)(\nabla\Delta^{-1} f_\delta(\mathbf{r}-\mathbf{r}_i))\} \tag{10}$$

We have now the ingredients of a Lagrangian particle representation of incompressible viscous flow. We have a strategy for representing magnetization of compact support, an evolution equation which can describe the subsequent motion of an ensemble of such particles from a prescribed initial distribution. It remains to show how these equations can be approximated in a boundary layer, and to describe a magnetization generation strategy consistent with the physical conditions which exist at a solid boundary.

## 3. Flow Near a Solid Boundary

In the neighbourhood of a solid wall, dimensional considerations allow us to determine approximate equations of motion analogous to the boundary layer equations discussed, for example, by Schlichting [**10**]. Following this example, consider a solid boundary at $z = 0$ (where $\mathbf{r} = (x, y, z)$ represents the Cartesian position vector). The region $z > 0$ is occupied by a fluid with viscosity $\nu$ and a velocity field $\mathbf{u} = (u_x, u_y, u_z)$. We conceive of a thin layer of fluid over the boundary of thickness $\delta$ such that the following dimensional considerations hold. Lateral length scales are taken to be of order 1, i.e. $(x, y) \sim O(1)$; the normal length scale is taken to be $O(\delta)$. This implies the following: $\partial u_x/\partial t \sim O(1)$; $u_x \sim O(1)$; $\partial u_x/\partial x \sim O(1)$; (similarly for $u_y$ and $\partial u_y/\partial y$). The continuity condition implies that $\partial u_z/\partial z \sim O(1)$, hence $u_z \sim O(\delta)$.

From these considerations we can infer the corresponding dimensional orders of the components of vorticity $\xi$. Specifically we determine: $\xi_x \sim O(1/\delta)$ and $\xi_y \sim O(1/\delta)$ and from the solenoidal property of vorticity we infer $\xi_z \sim O(1)$. From the relationship $\xi = \nabla \times \mathbf{m}$ we can infer the dimensional order of the components of $\mathbf{m}$, namely $m_x \sim O(1)$, $m_y \sim O(1)$, and $m_z \sim O(1/\delta)$.

We can now examine the three component equations of motion and determine the relative orders of these. Writing equation (6) in component form we have

$$\frac{\partial m_x}{\partial t} + u_x\frac{\partial m_x}{\partial x} + u_y\frac{\partial m_x}{\partial y} + u_z\frac{\partial m_x}{\partial z} = -m_x\frac{\partial u_x}{\partial x} - m_y\frac{\partial u_x}{\partial y} - m_z\frac{\partial u_x}{\partial z} + \nu\left(\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_x}{\partial y^2} + \frac{\partial^2 m_x}{\partial z^2}\right) \tag{11a}$$

$$\frac{\partial m_y}{\partial t} + u_x\frac{\partial m_y}{\partial x} + u_y\frac{\partial m_y}{\partial y} + u_z\frac{\partial m_y}{\partial z} = -m_x\frac{\partial u_y}{\partial x} - m_y\frac{\partial u_y}{\partial y} - m_z\frac{\partial u_y}{\partial z} + \nu\left(\frac{\partial^2 m_y}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} + \frac{\partial^2 m_y}{\partial z^2}\right) \tag{11b}$$

$$\frac{\partial m_z}{\partial t} + u_x \frac{\partial m_z}{\partial x} + u_y \frac{\partial m_z}{\partial y} + u_z \frac{\partial m_z}{\partial z} =$$

$$-m_x \frac{\partial u_z}{\partial x} - m_y \frac{\partial u_z}{\partial y} - m_z \frac{\partial u_z}{\partial z} + \nu \left( \frac{\partial^2 m_z}{\partial x^2} + \frac{\partial^2 m_z}{\partial y^2} + \frac{\partial^2 m_z}{\partial z^2} \right) \tag{11c}$$

Retaining the leading terms of $O(1/\delta)$ in equations (11) only equation (11c) survives, i.e. the scalar equation:

$$\frac{Dm_z}{Dt} = -(\mathbf{m} \cdot \nabla) u_z + \nu \frac{\partial^2 m_z}{\partial z^2}. \tag{12}$$

This equation resembles in form the three-dimensional Prandtl equation.

At $z = 0$, the velocity component normal to the wall, $u_z$, is zero (the impermeability condition). Also, in view of equation (12), we could, by way of boundary layer approximation, neglect the dimensionally lower-order components of $\mathbf{m}$, effectively letting $m_x = m_y = 0$. These approximations allow us to express a boundary condition for $m_z$ at the wall. Since we have

$$\xi = \nabla \times m_z \hat{\mathbf{z}}$$

(where $\hat{\mathbf{z}}$ is the unit vector normal to the boundary) we can invert this to determine a condition for $m_z$ on the boundary, namely

$$m_z = \int_o^x \xi_y dx - \int_0^y \xi_x dy \tag{13}$$

where the integration is taken along the boundary. This expression is precisely that developed in [**11**] — this is to say the normally-directed vector Hertz potential described in [**11**] can be identified with the normal component of a magnetization vector.

A particular discussion in reference [**12**] (pp 74-75) is relevant also to equation (12). We could choose a region of interest above the solid surface to be sufficiently thin, specifically we could choose it to be $O(1/\alpha)$ where $\alpha > \sqrt{Re}$, with $Re$ the Reynolds number. Within such a close proximity of the surface the transport is predominantly diffusive, and the convection terms in (12) could, as a first approximation, be neglected.

The no-slip condition at the wall can be approximated by introducing there a vortex sheet of vortex intensity $\kappa$, with component values given formally by the limit in the velocity discontinuity there, i.e.

$$\kappa_x = \lim_{d \to 0} \int_0^d \frac{\partial u_y}{\partial z} dz \simeq u_y$$

and

$$\kappa_y = \lim_{d \to 0} \int_0^d \frac{\partial u_x}{\partial z} dz \simeq u_x$$

If this sheet were partitioned into tiles of dimension $h \times h$ we could associate with each tile a circulation $\Gamma = h(-u_y, u_x)$ and a vorticity $\xi = h^{-1}(-u_y, u_x)$. If we consider equation (13) to be a discretized line integral we could express it as

$$m_z = h^{-1} \int_0^x u_x \, dx + h^{-1} \int_0^y u_y \, dy = h^{-1} \int_C \mathbf{u} \cdot d\mathbf{r} \tag{14}$$

with $C$ a path taken in the surface. For a *closed* path in the surface the line integral represents the circulation through the area spanned by the path; due to impermeability, and for sufficiently simple geometries, this should be zero (for example, if $C$ were in a simply-connected convex surface).

It will be noted that, to equations (13) or (14), attaches an arbitary additive constant of integration. This reflects the non-uniqueness of $\mathbf{m}$.

The level curves of $m_z$ should have a physical significance. The vorticity, $\xi$, in the boundary layer approximation, is tangential to the boundary. At any point of the boundary we have

$$\xi = \nabla \times m_z \hat{\mathbf{z}} = \nabla m_z \times \hat{\mathbf{z}}$$

Thus $\xi$ must be both normal to $\nabla m_z$ and to the normal vector $\hat{\mathbf{z}}$. If we define a unit vector aligned to the local vorticity, i.e. $\hat{\mathbf{S}} \equiv \xi / |\xi|$, then

$$\hat{\mathbf{S}} \cdot \nabla m_z = \frac{\partial m_z}{\partial S} = 0.$$

This is to say the level curves of $m_z$ are lines of equivorticity. They are lines along which one might wish to generate vortex filaments in a three-dimensional vortex filament method.

Equation (13) can be used to evolve magnetization from a no-slip surface in response to the developing dynamics of the flow. Let us say that at some

time $t$ the no-slip condition fails to be satisfied. The "slip field", i.e. the non-vanishing velocity field tangential to the surface, can be integrated in (13) to achieve a numerical value of $m_z$ at every point of the surface. Equation (12) represents an equation of motion for $m_z$ (and this equation may be further simplified to a diffusion equation within an appropriately small distance from the surface).

## 4. Matching Boundary Flow to the Interior

The principle of localization introduced for velicity in the context of interior flow (see Section 2) can be similarly applied in the boundary layer. The analogy is drawn with the vortex sheet construction [7]. We conceive of a magnetization $m_z$ determined at each point of the surface. The surface can be partitioned into tiles, each tile can bear the appropriate local value of magnetization, and this tile will interact with other tiles according to equation (12). This is predominantly a diffusion proceeding vertically from the solid surface. When the sheet leaves this neighbourhood of the wall, it can simply be re-designated as a magnetization blob element with equations of motion given by (8), (9) and (10).

This would constitute a "magnetization version" of the hybrid blob-sheet algorithms developed in the context of vortex methods. In fact, other strategies are also available, these stemming from the fact that there is a clear relationship between an element of magnetization and a closed vortex filament.

Consider a uniform partition of the solid boundary in terms of a mesh, each element having dimensions $h \times h$. Consider a $h \times h$ "tile" coincident with a particular mesh element. We integrate over this surface according to equation (13), using appropriate quadrature to establish the magnetization at each point. Over the particular $h \times h$ tile in question, (say the $j$ th tile) this value of magnetization is given by an integral of the form (taking integration

in $x$ to be generic),

$$m_j = \sum_{i=1}^{j} \int_{x_i}^{x_{i+1}} \xi_y \, dx$$

where $j \leq M - 1$ and $M$ indicates the closure node of the line-integral in the surface. The quantity $m_j$ is here to be understood as a magnitude of a vector normal to the surface.

The question of how one transforms this created surface field, $\{m_j\}$, of magnetization into interior elements can be addressed in a number of ways. We can, for example, make an equivalence [2] between the impulse of a vortex loop of circulation $\Gamma$ and that of a magnetization of compact support: this leads to

$$m \cdot v_{mag} = \Gamma_{loop} \cdot \mathrm{a}_{loop} \tag{15}$$

where $\Gamma_{loop}$ represents the circulation of the loop, $\mathrm{a}_{loop}$ represents the area spanned by the loop, and $v_{mag}$ represents the volume of the support of the magnetization element. Recalling the relationship between circulation and vorticity,

$$\Gamma_{loop} = \oint_{loop} \mathbf{u} \cdot \mathbf{d}\ell \; = \oiint_{\mathrm{a}_{loop}} \xi \cdot \hat{\mathbf{n}} \, ds \, ,$$

the line integration (13) can be expressed in terms of sheet circulation (since $\xi = \Gamma/h^2$ and $\Gamma = \kappa h$, where $\kappa$ is the sheet intensity). This circulation, in turn, can be expressed in terms of the local slip velocity over the tile (since $\kappa = u$). With these considerations we have

$$m_j = h^{-1} \sum_{i=1}^{j} \int \mathbf{u}_i \cdot d\ell \tag{16}$$

Consider again tile '$j$' on a partitioned "sheet of magnetization". This can be made to diffuse normally from the surface and at some stage become an interior magnetization element. We can reconstruct a loop of circulation $\Gamma_{loop}$ and area $\mathrm{a}_{loop}$ from this magnet by using (15) and (16). The magnetization of the tile can be related to the loop circulation, its spanned area, the support volume of the interior magnetization element, i.e.

$$\Gamma_{loop} = h^{-1} \sum_{i=1}^{j} \int \mathbf{u}_i \cdot d\ell \;\; \cdot \;\; \frac{v_{mag}}{a_{loop}}$$

In itself, this does not define a unique strategy. Among many possibilities we could, for example, choose an area of loop to correspond with the tile dimensions (in other words the discretization parameter of the solid surface partition, $h^2$) and choose the support of $\mathbf{m}$ to correspond to these tile dimensions. The $j$th loop circulation in this case would be

$$\Gamma_{\text{loop}} = \sum_{i=1}^{j} \int \mathbf{u}_i \cdot \mathbf{d}\ell = h\, m_j. \tag{17}$$

Buttke and Chorin [**2**] suggest an algorithm for transforming from vortex loops into magnetization variables, and vice versa.

We could similarly place the hybrid vortex sheet-blob method into the framework of the present surface partition. In this case the tiles have sheet intensity $\kappa = u_i$ and the blob will have a corresponding circulation,

$$\Gamma_{blob} = h\, u_i \tag{18}$$

The fact that the same dynamics is being represented by the two formalisms suggests that it should be possible to combine the two approaches. For example, one can easily create a layer of vortex tiles (diffusing and interacting according to the usual vortex methods discussed in [**4**]), but which *also* carry the magnetization expressed in (13). When such a tile leaves the sheet layer, it may become:

1. a vortex blob with circulation $\Gamma_{blob}$ from (18);
2. an element of localized magnetization with strength given by (16); or
3. an interior vortex loop with circulation $\Gamma_{loop}$ from (17).

In this same spirit we can construct, at the boundary, *sheets* in different ways. For example we can consider the surface partition to coincide with an assembly of polygonal "loops" in the boundary, these loops cancelling each other along mutual interfaces in such a way that they sum to the resultant required sheet-intensity. These polygonal loops can be represented as either vortex sheet clusters, or magnetization sheet clusters, and can be subject to the appropriate equations of motion, either the Prandtl equation in the case of vortex sheets, or equation (12) in the case of magnetization sheets. (This approach resembles the algorithm described in [**11**] in which a covering

of loops over the surface conforming to the level-curves of the vorticity is pursued.)

## 5. An Example in two Dimensions

To explore how such strategies of "magnetization creation" might function (and contriving to avoid the added complications of three-dimensional geometry) we consider a standard two-dimensional problem: flow around a cylinder. (See Cheer [**13**] for a hybrid sheet-blob modelling of this problem.)

In two dimensions equation (10) for magnetization consisting of a collection of variables of compact support $\mathbf{m}^{(j)} = (m_x^{(j)}, m_y^{(j)})$, has the explicit form:

$$\tilde{u}_x = \sum m_x^{(j)} f_\delta(\mathbf{r} - \mathbf{r}_j) - m_x^{(j)} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\Delta^{-1} f_\delta\right) - m_y^{(j)} \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\Delta^{-1} f_\delta\right)$$

$$\tilde{u}_y = \sum m_y^{(j)} f_\delta(\mathbf{r} - \mathbf{r}_j) - m_x^{(j)} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\Delta^{-1} f_\delta\right) - m_y^{(j)} \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\Delta^{-1} f_\delta\right)$$

These equations have been expressed in a very useful and convenient form by Cortez [**14**]. The inverse Laplacian of the radially symmetric function $f_\delta$ is denoted as $\psi(r)$. The functions are formed: $F'(r) = r f_\delta(r)$ and $\psi'(r) = r^{-1}F(r)$, where $F(r) = \int_0^r r' f_\delta(r') dr'$. With this definition, we can write

$$\tilde{\mathbf{u}} = \sum_j \left\{ \mathbf{m}^{(j)} \left[\frac{rF'(r) - F(r)}{r^2}\right] - \hat{\mathbf{x}}^{(j)}(\mathbf{m}^{(j)} \cdot \hat{\mathbf{x}}^{(j)}) \left[\frac{rF'(r) - 2F(r)}{r^2}\right] \right\} \quad (19)$$

Here $\mathbf{m}$ absorbs a constant pre-multiplier. This reveals explicitly the role of the smoothing function in the equations of motion. Since $f_\delta$ is smooth we expect the integral to exist; since the first moment of $f_\delta$ vanishes, we expect $\psi'(r)$ to tend to zero far from the support.

Buttke [**1**] has used, for his two dimensional example (the evolution of an initally radial velicity field) the support function:

$$f_0(r) = \frac{7}{2\pi} \begin{cases} 1 - 10r^3 + 15r^4 - 6r^5 & r \le 1 \\ 0 & r > 1 \end{cases}$$

It is not clear that the first moment of this vanishes since

$$\int_0^\infty r f_0(r) dr = \frac{7}{2\pi}\left\{\frac{r^2}{2} - 10\frac{r^5}{5} + 15\frac{r^6}{6} - 6\frac{r^7}{7}\right\}_0^1$$

$$= -\frac{6}{2\pi} \neq 0$$

This implies that the solutions for $\psi(r)$ inside and outside the support may have a discontinuity in gradient. This could present problems for the magnetization evolution equations (9) since they require gradients of $\mathbf{u}$ and hence of $F(r)$. It would seem the differential structure of the smoothing function, and in particular their moments and derivatives, enter into the equations of motion.

Another polynomial example of a smoothing kernel of order 4 is that due to Hald [9]:

$$f_0(r) = \frac{1}{2\pi}\begin{cases} -140(1-r)^3 + 420(1-r)^4 - 252(1-r)^5 & r \leq 1 \\ 0 & r > 1 \end{cases}$$

In practice the fourth-order Beale-Majda [5] kernel based on gaussians seems to be the most unproblematic for the present application. This is

$$f_0(r) = \frac{1}{2\pi}\left(4e^{-r^2} - e^{\frac{1}{2}r^2}\right) \tag{20}$$

To apply boundary conditions to the cylinder problem we simply perform at each time-step of the evolution the (discretized) integral over the circumference:

$$m_r(\theta_j) = \frac{1}{h}\int_0^{\theta_j} u_{\theta'} r d\theta'$$

where the discretization parameter, $h$, is the arc length $h = r\delta\theta$ and $\theta_j = j\delta\theta$. The resulting magnetization can be combined simply with the usual vortex sheet calculation. Fig. 1 illustrates the initial magnetization distribution on a cylinder of radius 1, specifically is plotted the quantity $h\, m_r(\theta) = -2\int_0^\theta \sin\theta' d\theta'$. For a 200-fold partition of the circle, $h = \pi/100 \simeq 0.0314$. Thus $m_r$ assumes a maximum at the windward stagnation where the initial velocity is a minimum. At $t = 0$ the closed-path integration over $(0, 2\pi)$ is zero as expected. As flow evolves, numerical error will compromise this

latter condition at any particular time, although hopefully it could remain established in time-average.

One can treat the sheets themselves like "magnetization sheets", i.e. each partition arc of the cylinder circumference is to consist of a radially-directed magnetization $m_r^{(j)}$. This is interpreted as two half-overlapping vortex sheets with opposing sense of vorticity, and each centred at opposite edges of the arc. Thus, when the effect of each sheet is added to that of the sheet coinciding in location but belonging to the adjacent arc, *in sum* the required net sheet intensity to establish no-slip at the edge is achieved (although the two sheets taken individually may have relatively large strength).

Limited success with this strategy was achieved: establishing no-slip was subject to greater numerical noise than that expected in vortex sheet methods. At any time step the no-slip could be established at the nodes, i.e. at the location of the center of the sheets, but this deviated considerably away from the nodes, for example at the mid-point of the arc. This was particularly a problem at the incident stagnation point of the cylinder, for reasons alluded to earlier. Fig. 2 shows the (static) no-slip achieved after 1 time-step with the strategy just described: at nodes no-slip is achieved, between nodes it is not. The static comparison is of marginal interest since the boundary conditions are to be achieved in time-average.

An alternative strategy to creating "magnetization sheets" is suggested by Chorin [**15**]. The suggestion is to create vortex sheets at the boundary, but also to evaluate the magnetization at each point of the boundary and to attach this value to the parameter list associated with each sheet. As the sheet enters the flow interior it can become an element of magnetization of compact support and be transported according to equation (8) and its strength evolve according to (9). The growth of magnetization implied by (9) makes the latter calculation difficult, and strategies must be applied in the interior to model this (see Cortez [**14**]).

For present purposes we neglect equation (9) since our aim is to illustrate a strategy for evolving vortices from the boundary into the interior, rather than a comprehensive modelling of interior flow using magnetization variables.

Fig. 3 illustrates the "slip velocity" at step 10, i.e. the tangential field to be extinguished at the next time step. The particle distribution consists of 1016 sheets and 971 magnet elements. This velocity was computed by using the "vortex blob" information in the elements' parameter list (i.e. using a Biot-Savart kernel), and alternatively using equation (8) with (10). A

smoothing parameter of 0.2 was used, and a time step of 0.1. The sheet elements are made to undergo random walk in the sheet layer in a radial direction; the interior elements undergo a two-dimensional random walk. The sheet layer was chosen to have thickness of three mean-free path lengths.

To use equation (10) in the context of bounded flow, the impermeability condition must be enforced at each time step. For particle models such as blob methods, the most efficient way to do this is to use images (if geometry permits). There are well known circle- and sphere- theorems (see [**16**, **17**] ) for irrotational flow problems. Matching impulse would suggest that an image magnetization element could be placed at the reciprocal point inside the circle, with opposite sign to the element itself, and of a strength enhanced by a factor $b$ where $b > 1$ is the position of the interior element from the origin. However with magnetization the problem seems to be complicated by the fact that part of the velocity formula (19) — or more obviously in (10) — is not harmonic in character (namely the support kernel). The question at this stage is not fully resolved, although a procedure was found which effected the impermeability condition by use of a magnetization 'image'. If a radially oriented magnet were at a position $b > 1$, then an image magnet could be placed at the reciprocal postion $1/b$. The image would be of opposite sign. The summation for the velocity calculation (19) was simply extended to include the images. However in the case of the images, the harmonic-type terms, i.e. the image terms deriving from $\psi(r)$, are multiplied by the factor $b^2$.

In a sense the problem of images is less awkward in the context of the three-dimensional problem. We can make the association (through impulse) between magnetization of compact support and a vortex loop. For a vortex loop outside a sphere there is a Sphere Theorem [**17**] to serve our needs.

Cortez [**14**] shows the relationship between magnetization of compact support in two dimensions and vortex pairs. The problems of imposing impermeability in the context of Biot-Savart interactions is well understood, so an interesting exercise is to transform vortex sheets into vortex pairs (via Cortez' analysis) upon entering the flow interior.

We perform this exercise choosing the discretization parameter $h$ as the pair separation (a choice which is arbitrary and unlikely to be optimal). A magnet of strength $m$ is replaced by a pair of two opposing vortices on a line parallel to the nearest tangent to the cylinder, and separated by a distance $h$, and each having vortex strengths $\pm m_j/2h$. These vortex pairs were kept

coupled, and continued to serve as interior elements as pairs.

Fig. 4 shows the slip velocity for this model over 8, 9, and 10 time steps ($dt = 0.1, \delta = 0.3, h = \pi/100$). Although this shows that no-slip is being imposed successfully, such a model does not represent an adequate calculation of interior flow: coupled vortex pairs are essentially an expression of magnetization and this must be subject to evolution equivalent to (9). When the centroid of the coupled vortex pairs re-enter the sheet layer, they simply revert to being a sheet and resume an radially outward diffusion into the interior.

Fig. 5 indicates how (for the coupled vortex pair model) the closed-path integration over the entire cylinder

$$m_r(2\pi) - m_r(0) = \frac{1}{h} \oint \mathbf{u} \cdot \mathbf{d}\ell$$

departs from zero as a function of time-step. Leaving aside the questions of the adequacy of the interior model, and the choice of numerical parameters, the evolution of integral deficit seems to oscillate in an encouraging way, i.e. in a way which tends to average out over time.

A more legitimate procedure results if, upon leaving the sheet layer, the element becomes a coupled vortex pair which is promptly de-coupled. We do not pursue an investigation of optimal (in respect of accuracy or efficiency) numerical parameters; rather, for purposes of illustration, we again choose $h$ to serve as initial separation. Fig. 6 shows the slip velocity that results in steps 8, 9, and 10; for the same parameters as Fig. 4, with the exception that the time-step has been halved (specifically $dt = 0.05$). The interior model no longer requires any evolution of magnetization (or its equivalent), however the matter of dealing with the re-entry of de-coupled elements into the sheet layer becomes more intricate: this operation was not pursued.

Figs. 7 and 8 are an indication of the location of interior elements after some 30 time-steps (albeit for different step sizes). The distribution of particles is quite different, reflecting perhaps the short-comings of the treatment of elements in the interior, and the quite different interaction we can expect of coupled pairs (essentially magnets) and vortex blobs. Nevertheless when complete interior solution strategies are available (see Cortez [**14**]) one should be able to expect the respective different Lagrangian kinematics to converge (in the appropriate statistical sense) to the common dynamics of the flow.

## 6. Conclusions

It has been demonstrated that a rationale exists for creating magnetisation at a solid boundary consistent with viscous boundary conditions. An (approximate) equation of motion appropriate for the neighbourhood of a solid boundary has been developed whereby created magnetization can be transported to the flow interior. The results allow the extension of Buttke's [**1**] Lagrangian formalism to problems of bounded incompressible viscous flow. Furthermore the present work places in a more general context the loop-generation algorithm of Summers [**11**].

A strategy can also be offered for evolving divergence-free vorticity from a solid surface which exploits vortex sheet methods in the neighbourhood of the solid wall.

The ability to generate divergence-free vorticity at a boundary opens up the prospect of applying renormalization strategies to challenging flow regimes, for example turbulent boundary layers or turbulent wakes.

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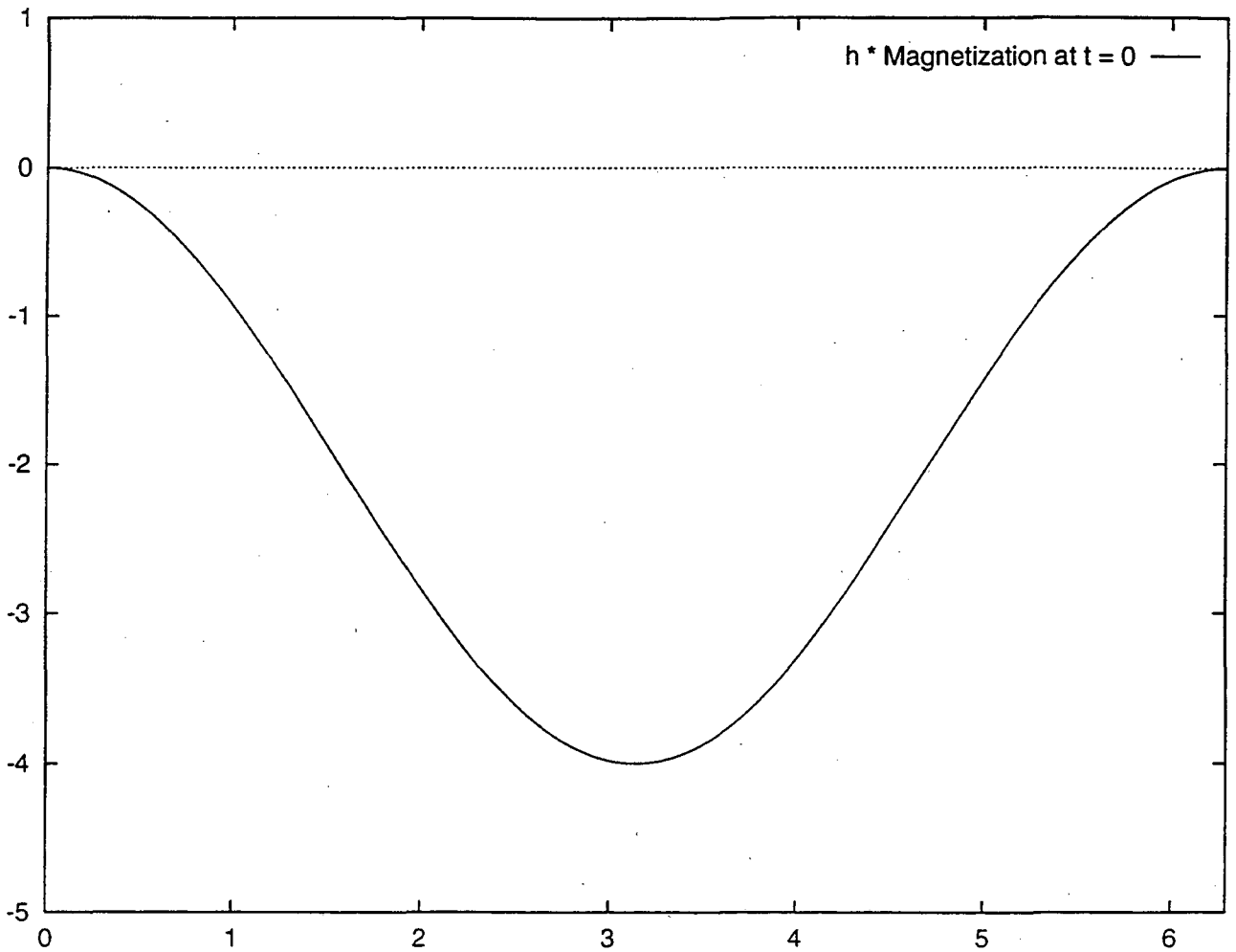


Fig. 1 Initial magnetization $h\, m_r(\theta)$ at $t = 0$.

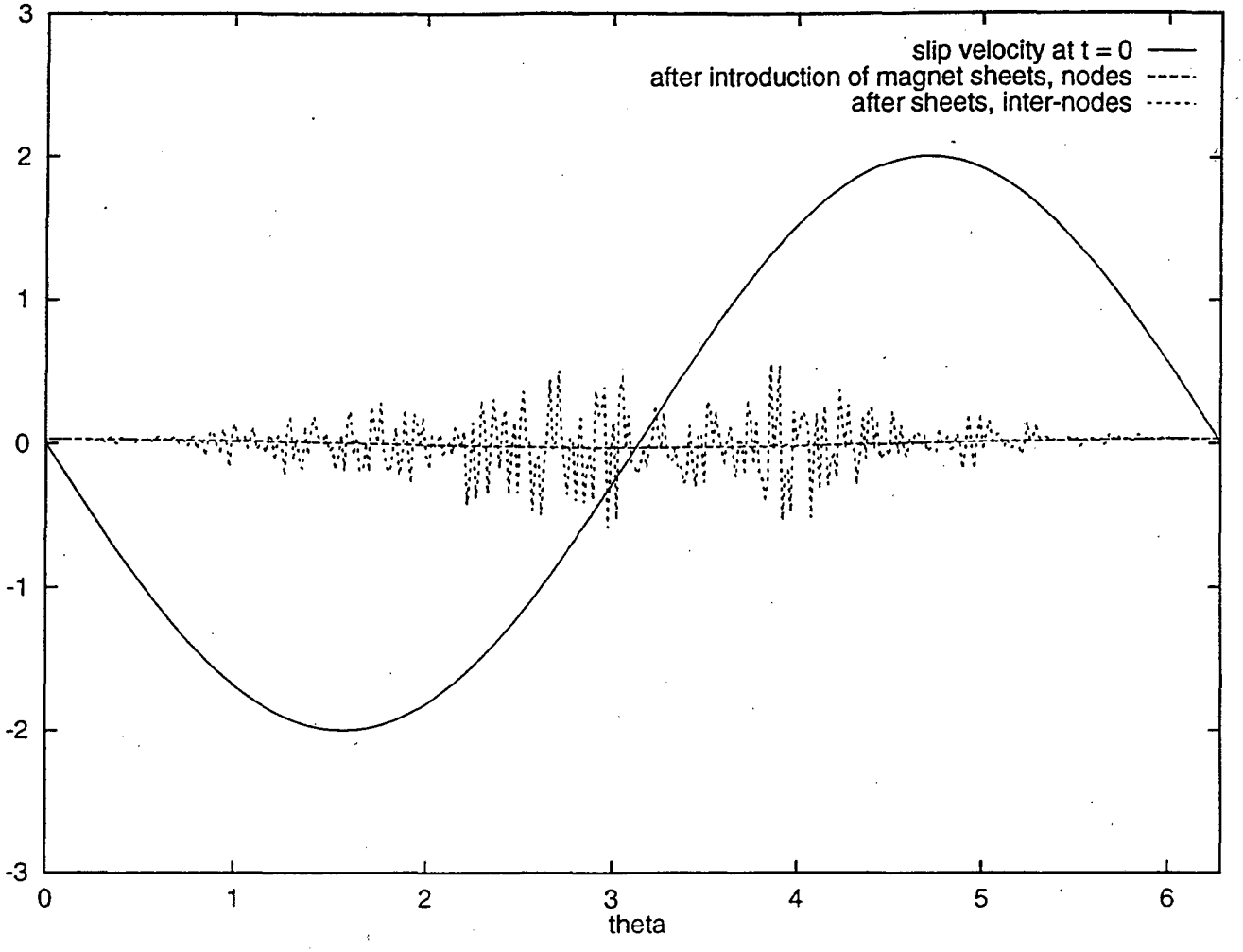


Fig.2 No slip established after introduction of "sheet magnetization" at $t = 0$, evaluated at the nodes of integration, and at mid-points of integration interval

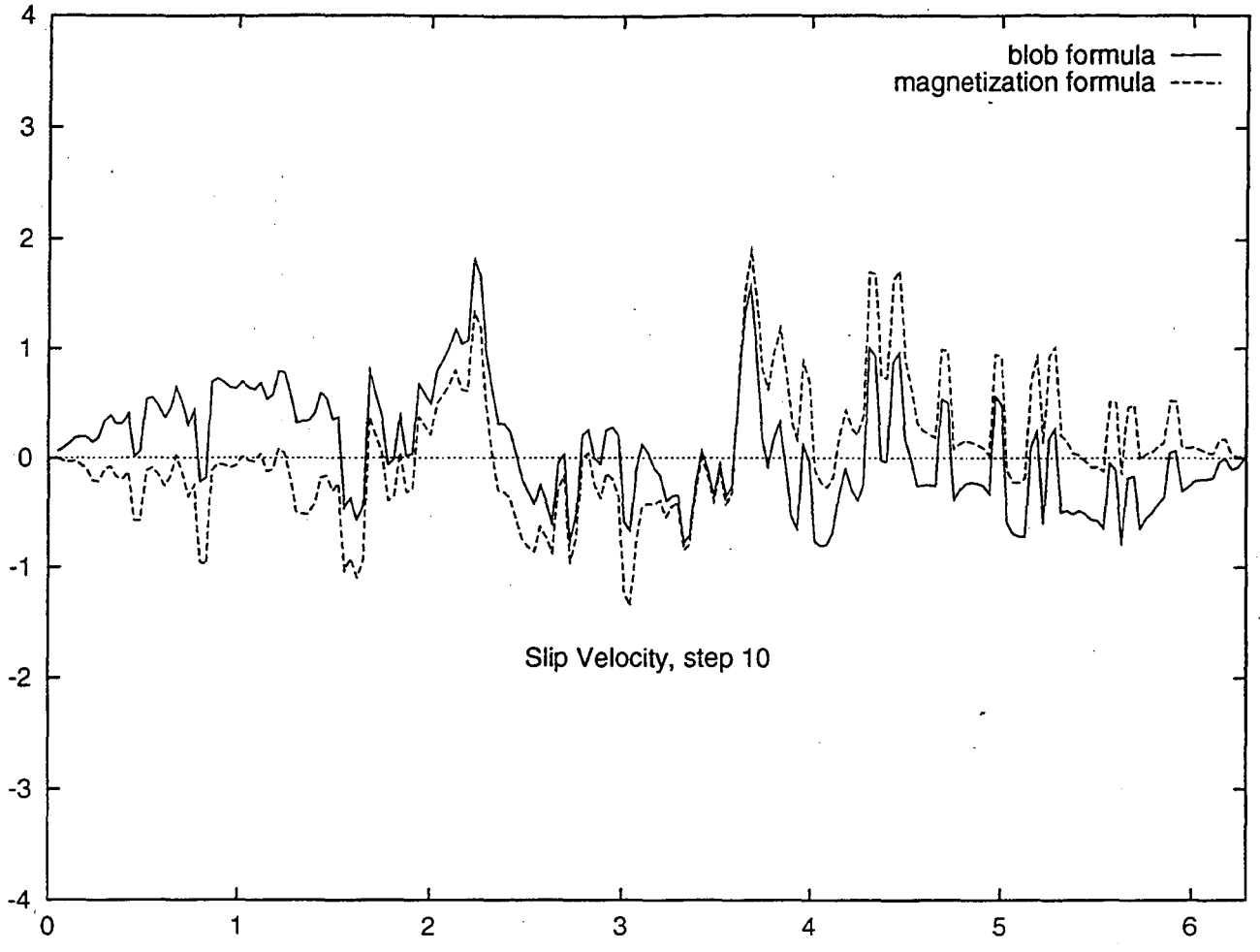


Fig. 3 Using magnetization variables (but with vortex blob information carried also) in the interior. Vortex sheets are used near the boundary. No magnet strength evolution. Tangential velocity on cylinder at step 10 — the 'slip' to be extinguished during next time step — evaluated using 1.) magnetization formula, or 2.) Biot-savart formula

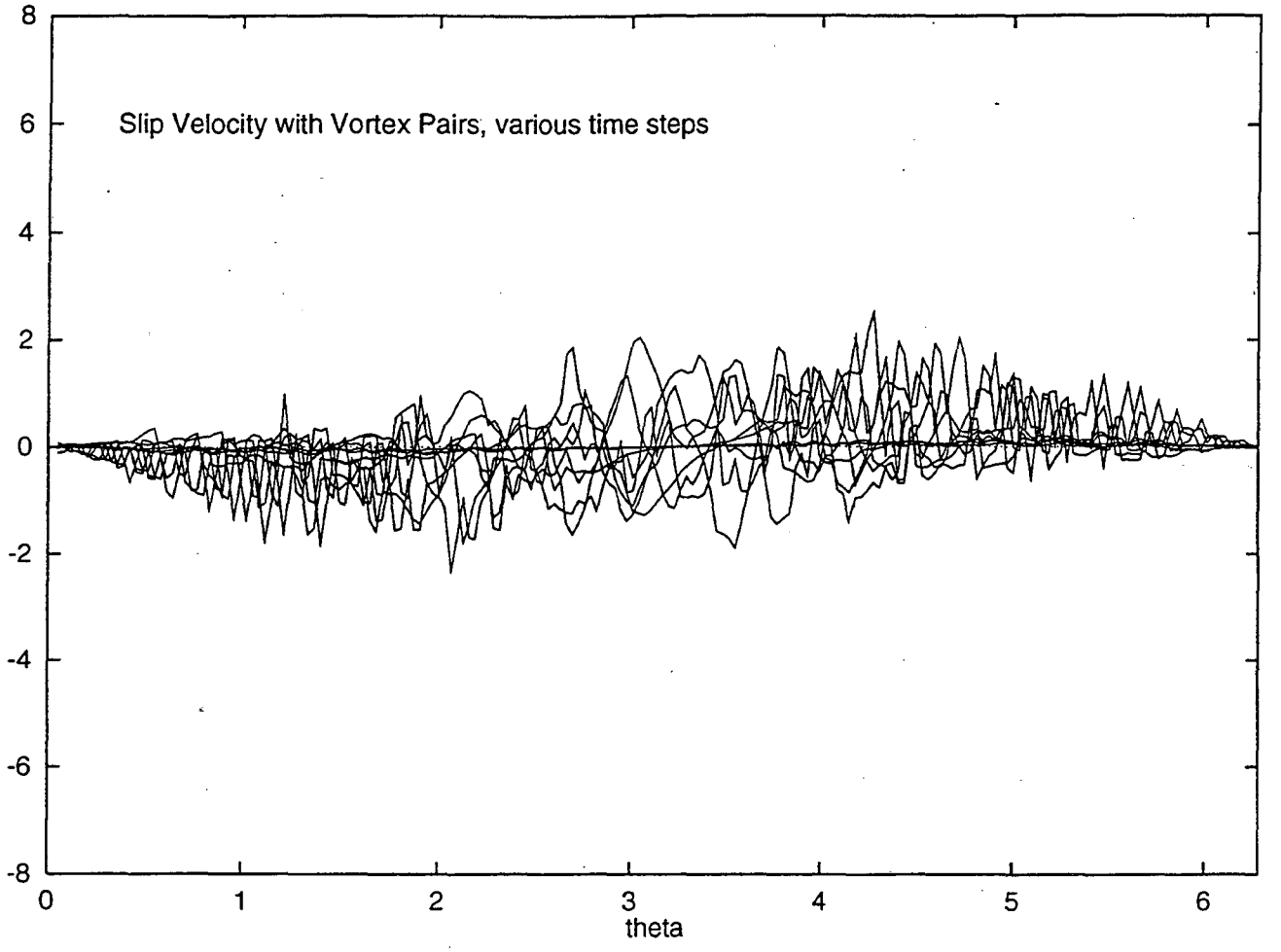


Fig. 4 Using Coupled Vortex pairs in the interior, vortex sheet near the boundary. The slip velocity over the cylinder for time steps 8, 9 and 10

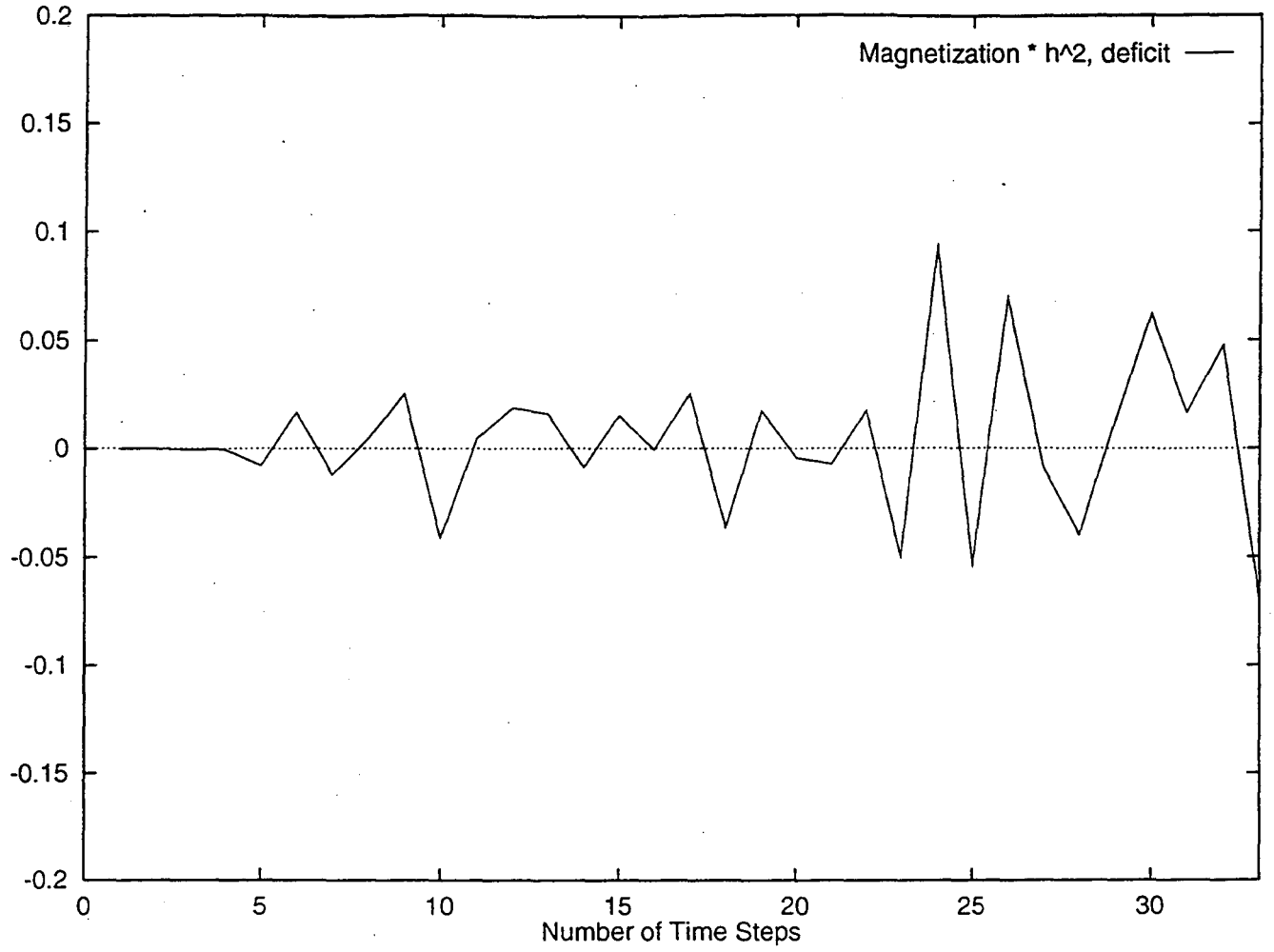


Fig. 5 Result of complete line-integration over the cylinder, the deficit evolving in time.

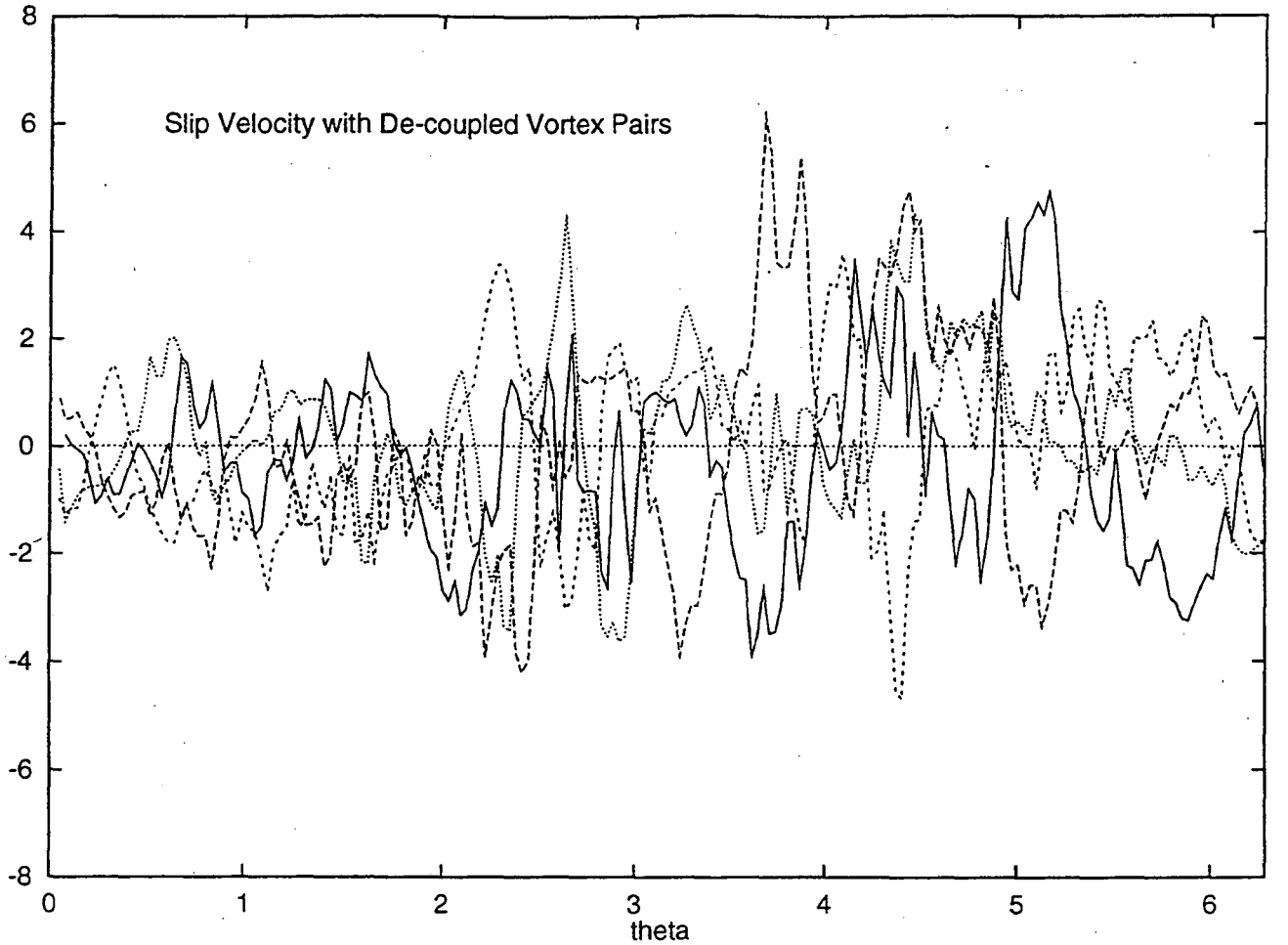


Fig. 6 Using De-coupled Vortex pairs in the interior, vortex sheets near the boundary. Steps 8, 9 and 10.

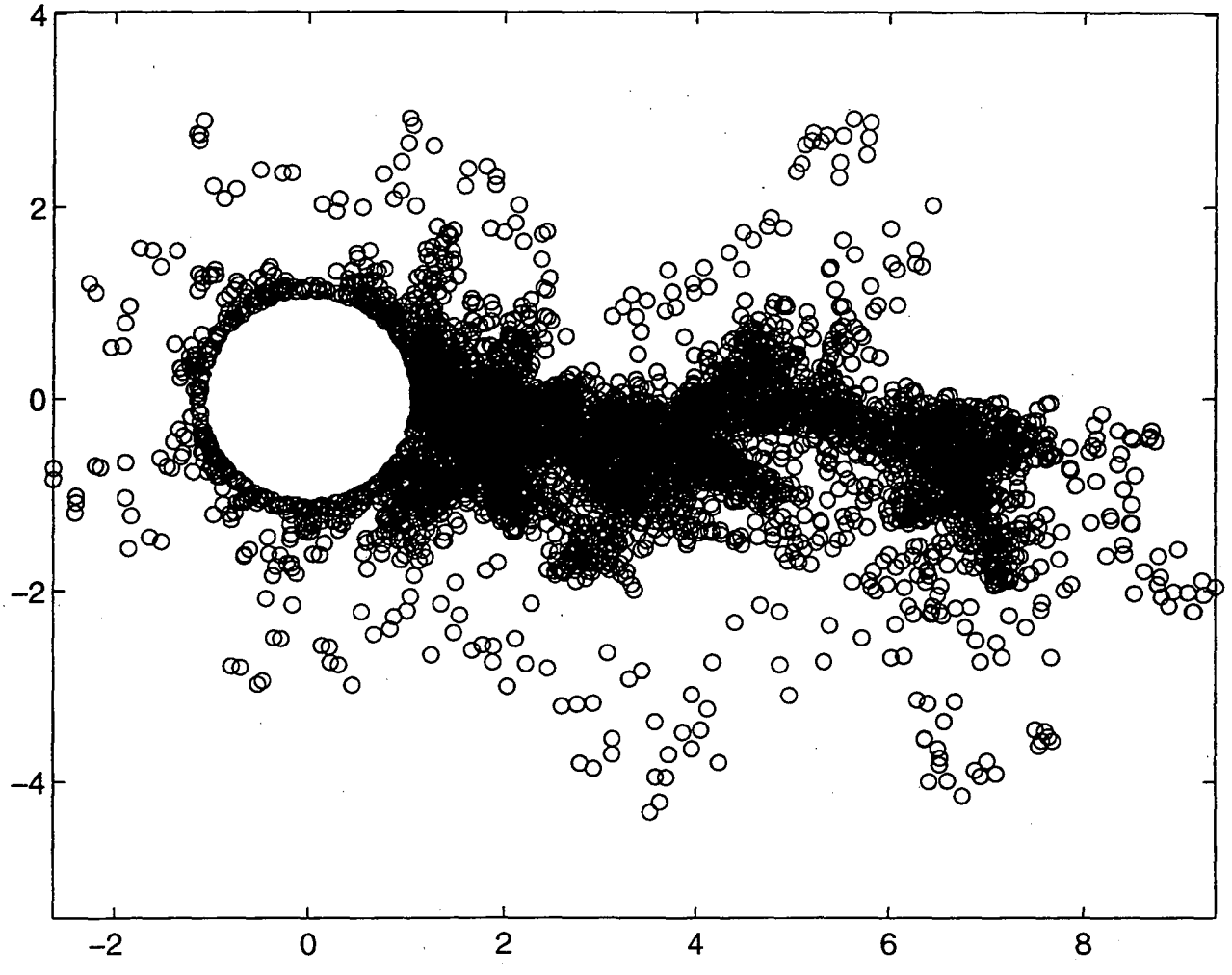


Fig. 7 After about 30 steps, the location of coupled vortex pairs.

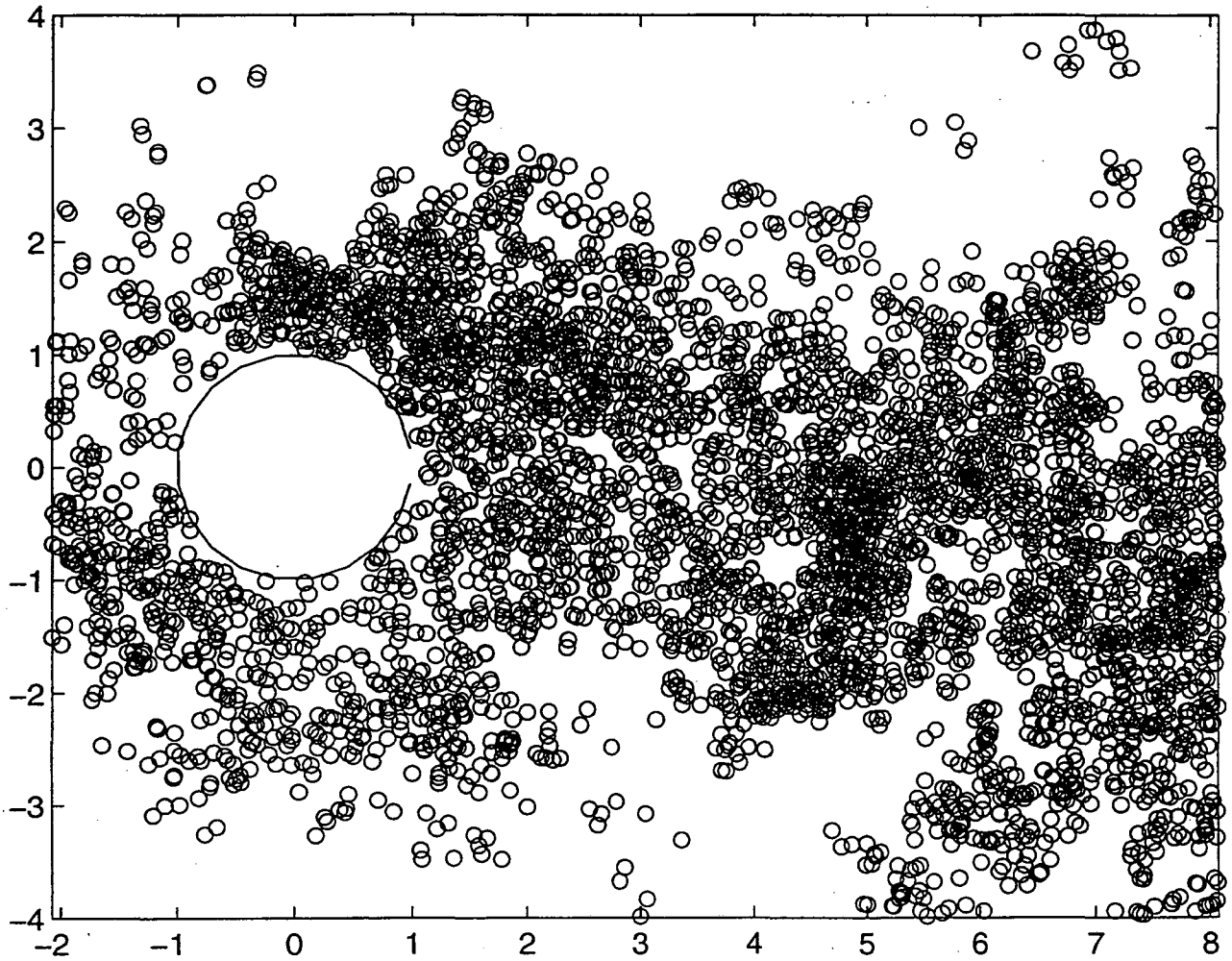


Fig. 8 After about 30 steps, the location of de-coupled vortex pairs

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