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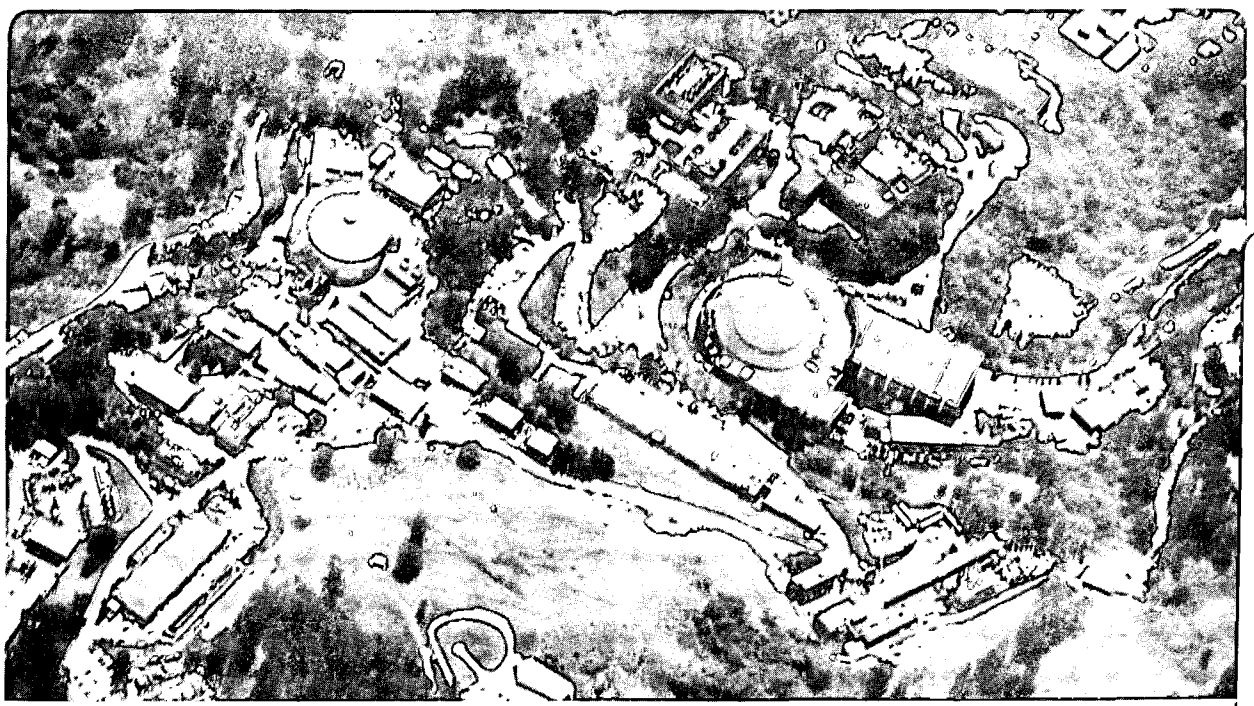

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THE STRUCTURE OF SPIRAL DOMAIN PATTERNS*

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The structure of spiral domain patterns

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Abstract

We analyse cellular patterns which appear spontaneously in a number of nonequilibrium systems governed by the dynamics of a complex field and show that their structure can be understood in very simple terms. In the case of the complex Ginzburg-Landau equation disordered cells of effectively frozen spirals appear, separated by thin walls on a scale much larger than the basic wavelength of the spirals. We show that the walls are segments of hyperbolae and that their transverse structure depends on the angle with the phase contours and that it can be oscillatory.

Pattern formation in nonequilibrium systems is an extremely rich subject which relates many different branches of the natural sciences. It is fascinating that even the simplest nonlinear partial differential equations (PDEs) can exhibit a bewildering variety of patterned, disordered or turbulent states. In this paper we are interested in systems describable in terms of a complex "order parameter" [1,2] whose phase field exhibits phase singularities and shocks. Examples include oscillatory chemical reactions [3–6], surface catalysis [7], multimode lasers [8], intracellular waves [9], colonies of social amoebae [10], and cardiac arrhythmias [5,11]. In large-scale simulations, stable (or metastable) disordered cellular (glass-like) structures have recently been observed in such systems [12]. An example is shown in figure 1. The dark dots in figure 1a represent phase singularities and the network

of walls are shocks (of finite width), where the phase gradients become almost discontinuous. Superficially these structures resemble soap films [13] or Voronoi cells [14], but, as we shall show, there are fundamental differences from the Voronoi construction due to the coexistence of spiraling phases of a complex scalar field. The presence of shocks has been noted in numerous experiments. See *e.g.* [15].

The so-called complex Ginzburg-Landau (CGL) equation is perhaps the simplest PDE modelling this behaviour. It describes an extended medium for which the parameter values are in the vicinity of a Hopf bifurcation, *i.e.*, a system where a stable equilibrium has just become a global limit cycle. [6,1,2]. It exhibits two kinds of simple travelling waveforms: plane waves and spiral waves. The latter can be seen as defects in the phase pattern of the plane waves since the iso-phase contours (figure 1b) terminate on the spiral cores. The CGL equation for the scalar complex field A is

$$\dot{A} = A - (1 + i\alpha)|A|^2 A + (1 + i\beta)\nabla^2 A \tag{1}$$

It has long been known [16] that the uniformly rotating solution $A(\vec{x}, t) = e^{-i\alpha t}$ becomes unstable to long-wavelength phase disturbances, when $1+\alpha\beta$ becomes negative, the so-called Benjamin-Feir (BF) transition (from the analogous instability of ocean waves). It was only recently realized, however, that (i) the transition to turbulence [6,17] is *not* governed by the BF instability - it can indeed occur even in the BF-stable regime - and (ii) the excitations that drive this transition, the *spiral waves*, can form bound states which play an important role [18–22,12].

In [18] such bound states were termed *entangled.* But, in fact, looking at the *modulus* of the field (figure 1a), one sees that the spiral waves (or *vortices*) simply divide the space between them. Over most of space the modulus is uniform, but at the vortex centers, where the phase becomes ill-defined, the modulus goes to zero. Furthermore the territory of each vortex is clearly marked by an increase of the modulus on a thin *shock line* or *wall.* Each of these domains (or *cells*) is occupied by an almost perfect spiral.

The existence of such well-defined vortex domains was a mystery for a while, but seems

now, at least for the case of two vortices to be understood. Early work on the interaction of vortices was based on linear superposition of spiral waves [23,24] and predicted a power-law interaction, disallowing bound vortex states. Subsequently the importance of shock lines was noted [25,21] and the strong screening of the interaction produced by the shocks was calculated [22,26].

Patterns like figure 1 appear to consist very accurately of a collection of unperturbed spirals separated by thin shocks. Each vortex is thus described by a phase ϕ such that the i'th vortex has the phase of a simple Archimedian spiral

$$\phi_i = \sigma_i \theta_i - q r_i + C_i \tag{2}$$

where the direction of spiraling is specified by the charge σ_i ($= \pm 1$) of the i'th spiral (which seems, on figure 1, to be rather randomly distributed). θ_i and r_i are the polar coordinates measured from the center of that spiral (with respect to a fixed direction), and C_i is a phase constant for the spiral. The wavenumber q is the same for all spirals; it is the *selected* wavenumber for a given set of parameters α and β. On the shock lines, the phases of two vortices are equal, and the domain of each vortex is simply the region where its phase is larger than the phase of any others.

This phase matching strongly constrains the geometry of the pattern. If, say, three shocks meet at a corner, the location of the corner as well as the locations and charges of the three surrounding vortices uniquely determine the three shock curves emanating from that corner. In particular, since the three phases ϕ_i match at the corner, the three constants C_i are determined only up to an additive constant. We have checked this using figure 1, by examining groups of four vortices meeting at two corners and we find that the above construction reproduces the wall structure precisely to within the wall thickness. Also it is possible to see why corners predominantly have three, and not four or more shocks emanating from them: two phases become equal on a line, three phases are typically equal only at single points and four phases only become equal under special conditions.

To a good approximation, the shocks are *segments of hyperbolae with the two nearest*

vortices as foci. This has been noted from numerical computation [27] and can be easily understood. The shocks follow lines where the phases of two vortices, ϕ_1 and ϕ_2 are equal. If we assume that the distance from each vortex to the shock is much larger than the characteristic wavelength $\lambda = 2\pi/q$, we find simply

$$r_1 - r_2 = (C_1 - C_2)/q \tag{3}$$

and the locus of points having constant difference in distance from two centers is a hyperbola with focal points at those centers.

Within this approximation there is a simple rule giving the local direction of a shock line: For any point P on a shock line, the direction of the line is such that it bisects the angle from P to the two neighbouring vortices. This follows directly from the fact that the difference between the distances has to remain constant along the shock. The distances to the two vortex centers changes proportionally to $\cos\theta_i$, where θ_i is the angle between the tangent of the shock line at P and the ray from the i'th vortex center to P ($i = 1$ or 2). Thus $\cos\theta_1 = \cos\theta_2$ and $\theta_1 = \theta_2$.

In figure 2a we display a close-up of the wall-pattern. In figure 2b we have reproduced almost the same pattern (aside from the small edge-vortices) by feeding the position of the vortex centers and the constants C_i, estimated from the distances to the walls, into (3). In contrast, the so-called Voronoi construction [14] finds the region nearest to any vortex by constructing the perpendicular bisectors of all connecting lines. Obviously the vortex domains are *non-Voronoi*, but at the same time our construction represents a generalization of the Voronoi construction, to which it corresponds when all $r_i \gg \lambda$ and all the C_i are identical.

The sharpness of the walls between the spirals is not uniform. As can be seen in figure 2a, parts of the walls are broad and shallow to the point of disappearing, whereas others are sharp and oscillatory in the sense that the transverse profile of $|A|$ is non-monotonic. This can be simply understood by approximating the two impinging states as plane waves, and looking for kink-like solutions of (1) connecting these two states (a *sink* in the terminology

of [28]). From the above analysis, we know that the wall must bisect the angle between the two plane waves. Thus, to analyse a wall through a point P, we choose a system of coordinates with the y-axis along the tangent of the wall through P, and x orthogonal to it. The wavevector to the left of the wall is then, say $\vec{k_1} = (-k_x, k_y)$ with $\vec{k_2} = (k_x, k_y)$ to the right of the wall. Thus the angle θ is the angle between the wall and the iso-phase contours for the plane waves, so that $k_x = \cos\theta$ and $k_x = \sin\theta$. $\theta = 0$ means that the waves move head-on into the wall, whereas $\theta \to \pi/2$ is the limit where the plane waves move parallel to the wall and the walls become very weak. Thus the structure of the walls depend crucially upon θ.

For a given set of parameters, α and β in (1) the spiral waves have a selected wavelength λ. To find the transverse structure of the walls by the approach outlined above, we assume that the field can be written as $A = R_k e^{ik_y y - i\omega t} r(x) e^{i\phi(x)}$, where the frequency ω, the modulus R_k and the wavenumber k, through (1), must satisfy $\omega = \alpha {R_k}^2 + \beta k^2$ and $R_k = \mu - k^2$. Introducing a dimensionless distance as $\xi = x k_x = x k \cos\theta$ and a local dimensionless wavenumber as $y = \phi'(\xi)$ we find the following equations for $r(x)$ and $y(x)$:

$$r'' - r(y^2 - 1) = Ar(r^2 - 1) \tag{4}$$

$$ry' + 2r'y = Br(r^2 - 1) \tag{5}$$

where $A = \frac{1+\alpha\beta}{1+\beta^2}\frac{R_k^2}{k_x^2}$ and $B = \frac{\alpha-\beta}{1+\beta^2}\frac{R_k^2}{k_x^2}$. To find the wall structure, we should solve (4) and (5) on $-\infty < \xi < \infty$ such that $r \to 1$ and $y \to \pm 1$ as $\xi \to \pm\infty$. But then the position of the wall would not be given. Therefore it is easier to fix the wall center at 0 by solving (4) and (5) on $0 < \xi < \infty$ with the boundary condition $y(0) = r'(0) = 0$ and $r, y \to 1$ as $\xi \to \infty$. That leaves as unknown the wall height $r(0)$, which has to be found by solving the equations.

Figure 3 shows the wall structure at angles $\theta = 0$ and $\theta = \pi/3$. Clearly the structure is not monotonic - the modulus oscillates and creates "ripples" around the walls. For $\theta = \pi/3$ the wall has become weaker and very close to $\theta = \pi/2$ they become monotonic as can be

seen from linear stability analysis around the two plane asymptotic plane wave states. In figure 2 the dashed line is the solution of (4) and (5) and the fully drawn line is a result of a lattice simulation of 1. Obviously the plane wave approximation leading to (4) and (5) works very well. The structure of the shocks will be explored in more detail in a forthcoming publication [29].

We have emphasized that the pattern depends strongly on the phase constants C_i. One would expect that a simplified particle-like theory for these systems should be based not only on dynamical equations for the spiral centers $\vec{R}_i$, but on coupled equations for both $\vec{R}_i$ and C_i. The formulation of such a theory is a challenging task to which we hope to contribute in the future.

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FIGURES

FIG. 1. The pattern obtained by simulating a discrete form [18] of (1) with $\alpha = 0.792$ and $\beta = -1$, with periodic boundary conditions and starting from random initial conditions. Figure 1a shows the amplitude of the complex field A and figure 1b is a contour plot of the phase of A. In the regime of *transient turbulence* [19,22] a turbulent state appears, but after a period of nucleation [12], the system settles into a frozen disordered state as shown. Here some of the vortices have developed several spiral windings and become large, while others, termed *edge vortices*, get pushed into the corners of the networks formed by the shocks between the big ones. Note that the group velocity is directed away from the vortex centers into the walls, thus sweeping the edge-vortices into the corners. The state is *glassy* or *metastable* in the sense that small rearrangements can continue, especially where several edge vortices form a *braid* [27], whereas the big vortices and the shocks between them are fixed in time.

FIG. 2. a) Close-up of the wall structure. b) Reconstruction of the wall structure using the hyperbolic approximation (3).

FIG. 3. Transverse structure of the walls or shocks separating the spiral waves. The dashed line is the solution of the static equations (4) and (5) for a shock between two plane waves and the full lines are the result of a lattice simulation. The large figure shows the $head - on$ case where the angle between the phase contours and the shock is 0. The inset shows the case, where that angle is $\pi/3$. The parameters were $\alpha = 0.792$ and $\beta = -1$. Note that the structure of the wall is non-monotonic for the cases shown.

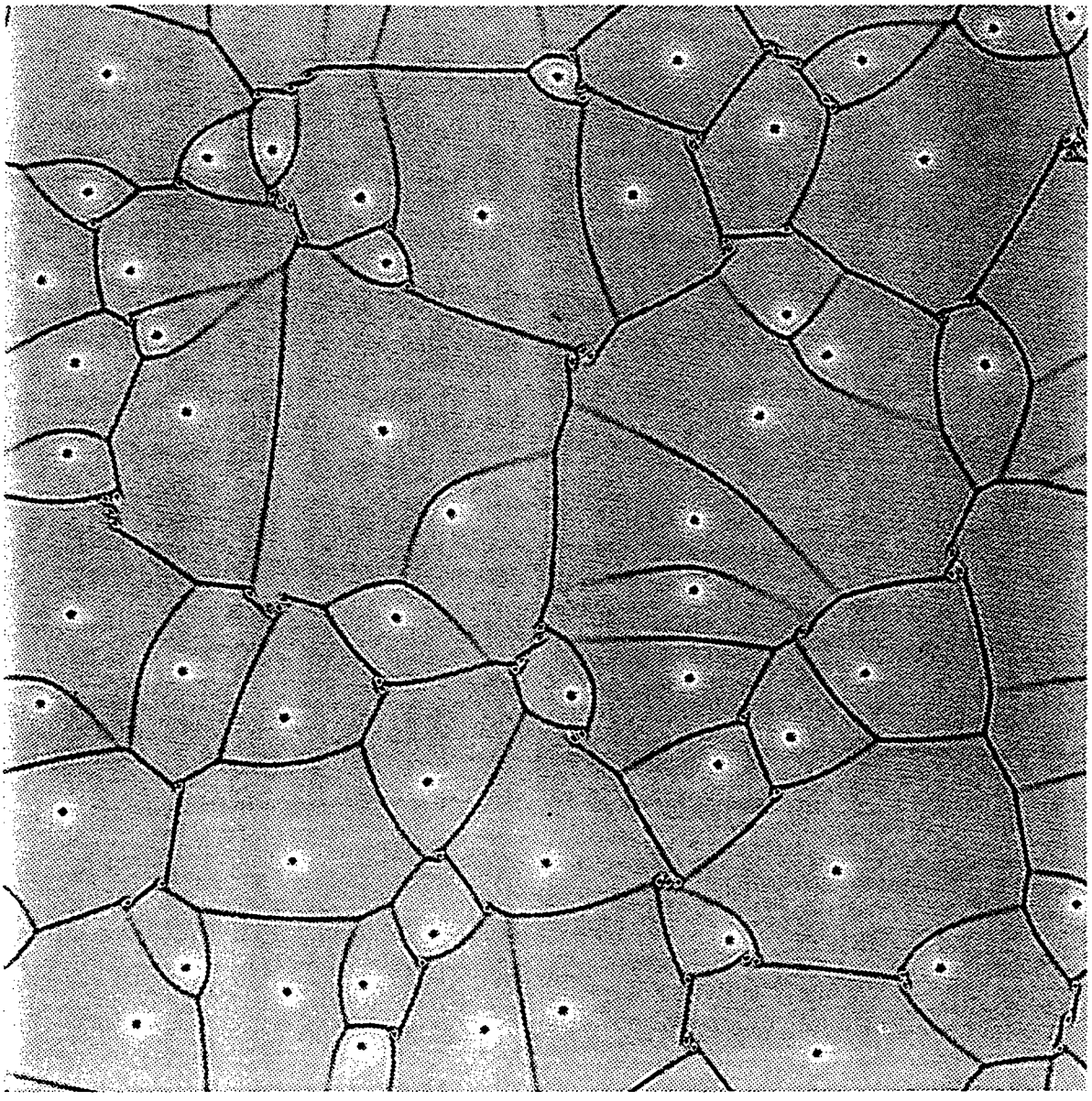


Figure 1(a)

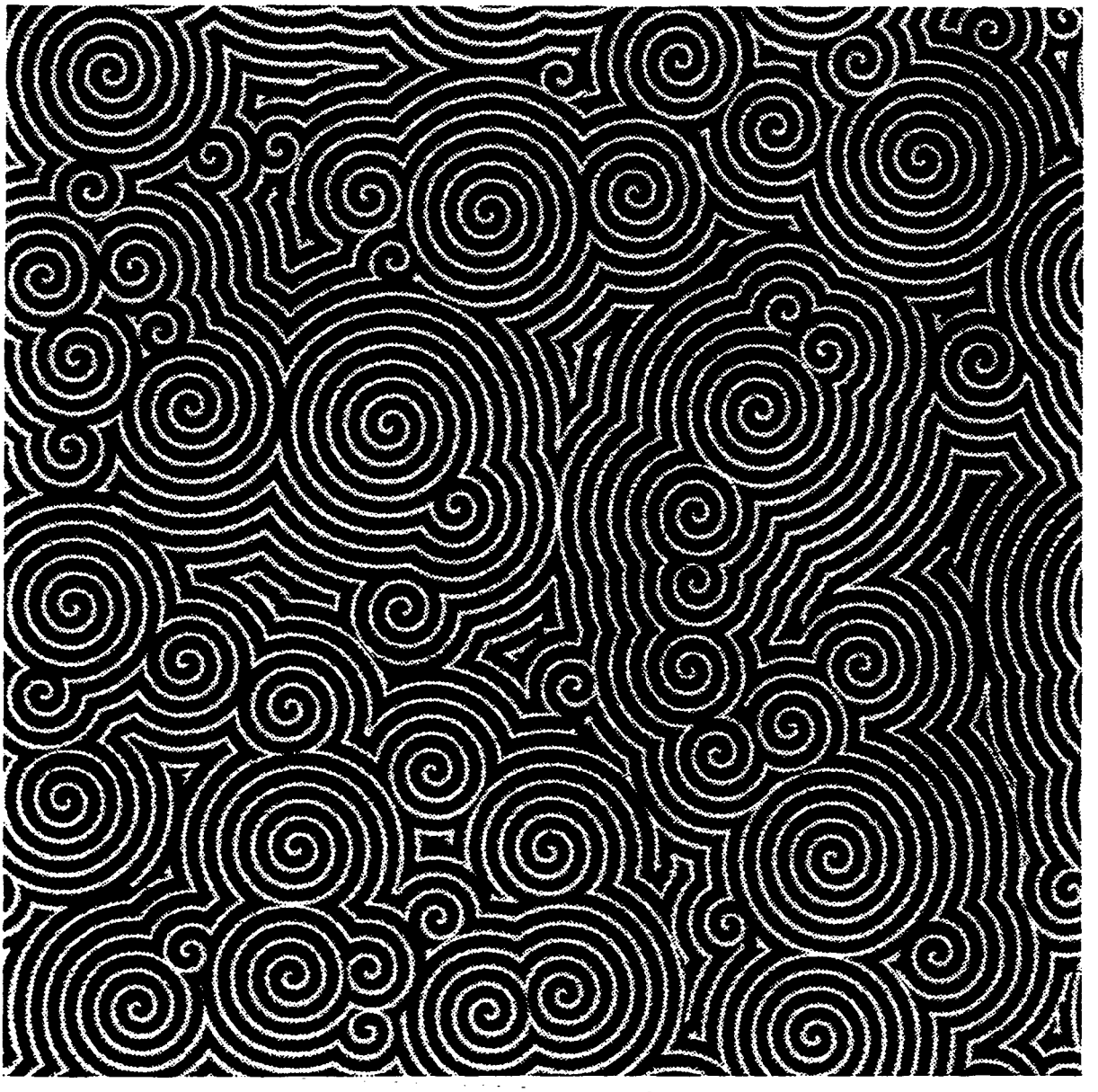


Figure 1(b)

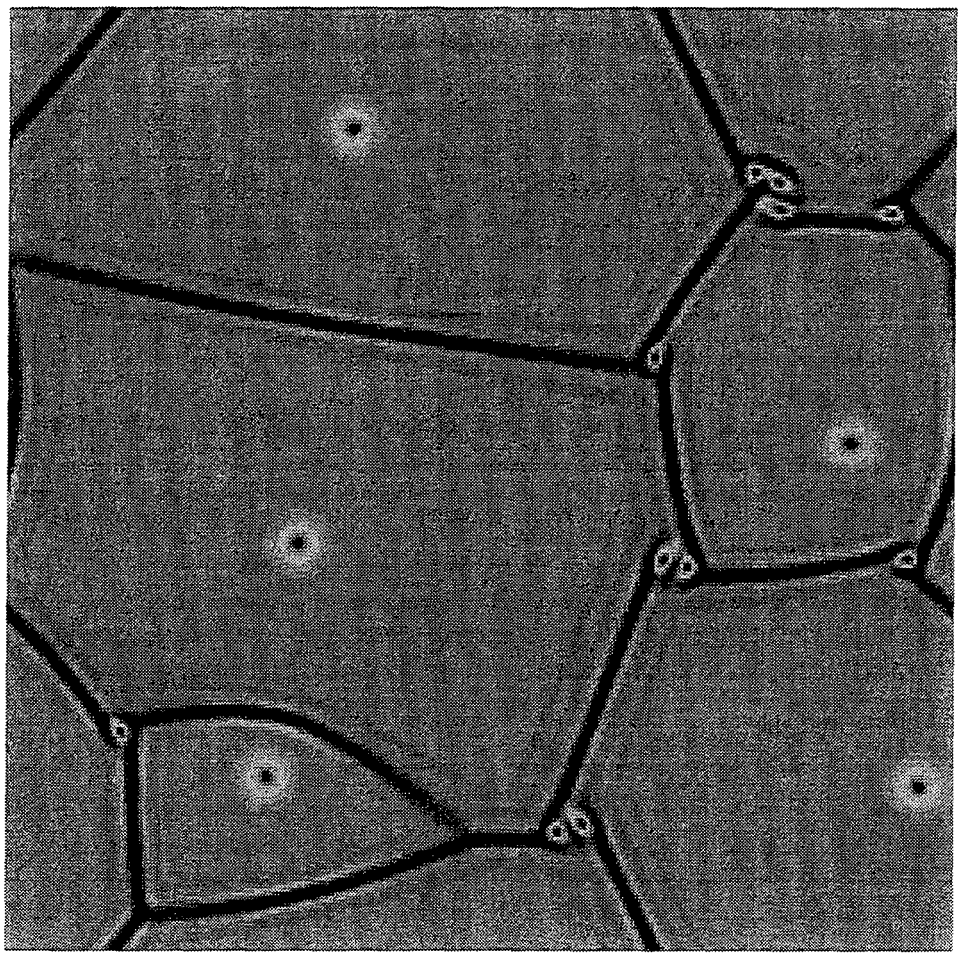


Figure 2(a)

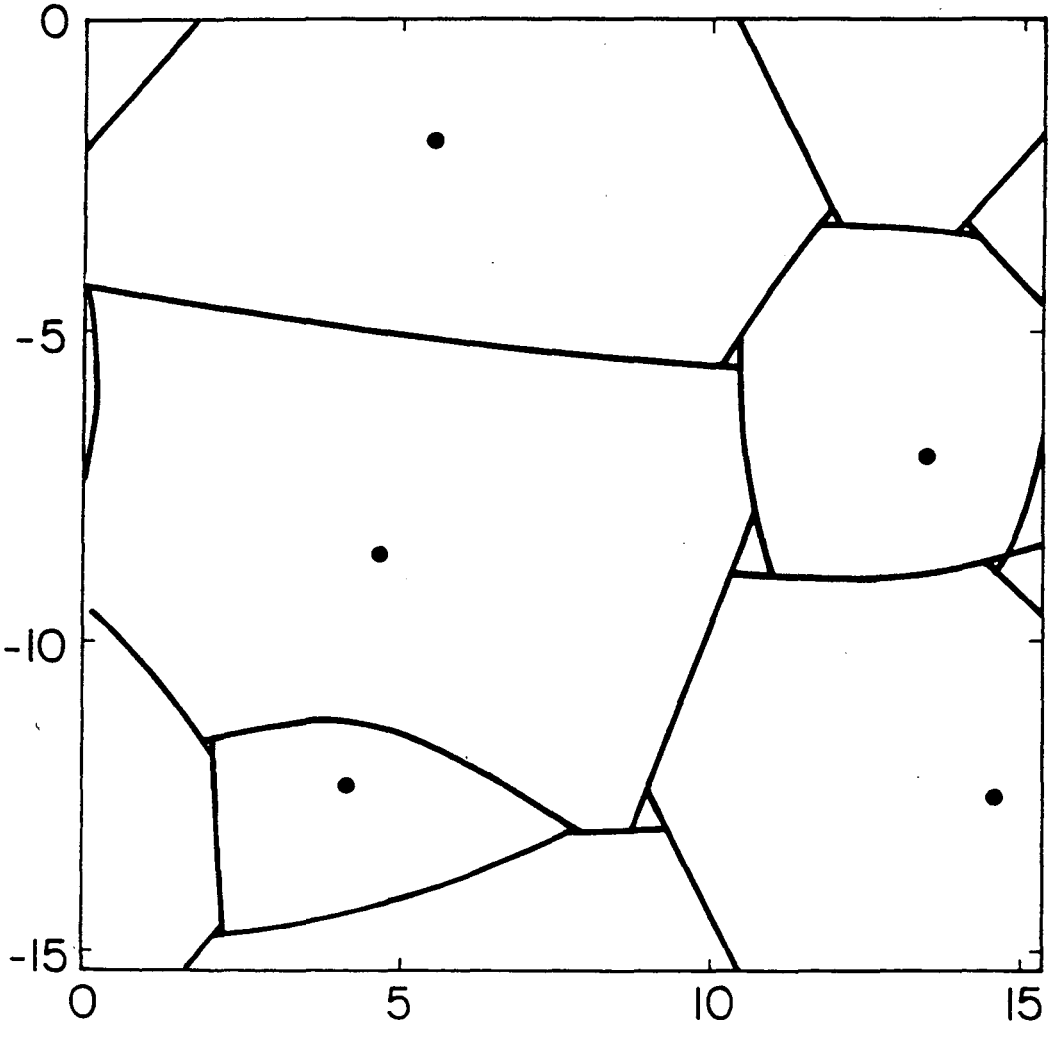
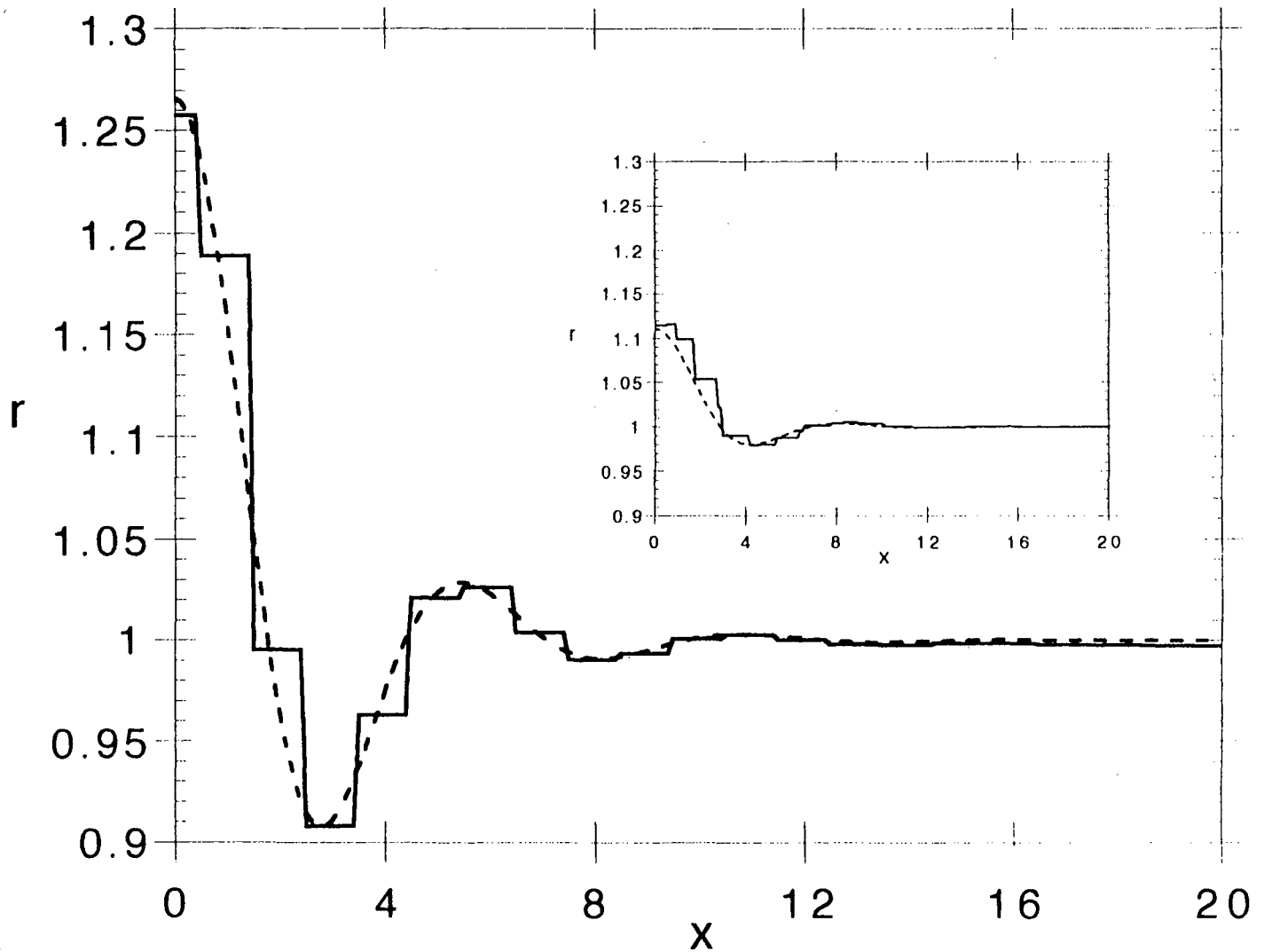


Figure 2(b)

Figure 3



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