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The "INVERSE PROBLEM" to the Evaluation of Magnetic Fields*.

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Abstract—**In the design of superconducting magnet elements, such as may be required to guide and focus ions in a particle accelerator, one frequently premises some particular current distribution and then proceeds to compute the consequent magnetic field through use of the laws of Biot and Savart or of Ampere. When working in this manner one of course may need to revise frequently the postulated current distribution before arriving at a resulting magnetic field of acceptable field quality. It therefore is of interest to consider an alternative ("inverse") procedure in which one specifies a desired character for the field required in the region interior to the winding and undertakes then to evaluate the current distribution on the specified winding surface that would provide this desired field.**

By evaluating the specified potential in the region interior to the winding along the interface, we have determined that a relaxation solution to the potential in the region outside the winding can be converged and used to calculate wire location. We have demonstrated this method by applying a slightly modified version of the program POISSON to a periodic alternating sinusoidal quadrupole field.

I. INTRODUCTION

In a very simple example it was desired to find a distribution of surface current density, on the surface of a *circular cylinder* of radius "a", that would provide in the interior a periodic alternating purely sinusoidal quadrupole field whose scalar magnetic potential would be proportional to $I_2\left(\frac{\pi\rho}{L}\right)\cos\left(\frac{\pi z}{L}\right)\sin 2\phi$ (cylindrical coordinates) – or (more generally) to an expression of the form $\left[\sum_m C_m I_2\left((2m-1)\frac{\pi\rho}{L}\right)\cos\left((2m-1)\frac{\pi z}{L}\right)\right]\sin 2\phi$. In this instance, with a *circular cylinder* selected as the form on which the current windings are to be placed, it may be evident that an analytic solution can readily be obtained and that indeed if additional azimuthal harmonics characterized by factors $\sin 6\phi$ or etc. were also present in the desired potential an analytical expression for the required current density could still be provided through superposition. We may note that in undertaking such an inverse procedure we would wish, on practical grounds, to avoid the use of any "double-layer" distributions of current on the winding surface but would not demand that no fields be generated in the exterior region, so that in this respect the goal would differ in detail from that discussed in [1].

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† Deceased

When a more general form of interface is considered desirable, $\rho = f(z)$ but still of circular cross-section, the coordinate system for a conventional analytic solution for the required current distribution may be lacking and we may wish to turn to some sort of relaxation process or processes in ρ,z space for computational solutions. We turn now to consideration of this option.

II. ANALYSIS

With a continuous interface ($\rho = f(z)$, and of circular cross-section, specified) for the surface on which current windings are to be placed the problem may then be specified as follows. [We may treat a single azimuthal component at a time in the course of the computational work, since the ϕ variation will be a separable variable and several harmonics, if present, may have their multipole fields superposed when required.]

The interface $\rho = f(z)$ will separate the ρ,z space into an interior region (Region I) and an exterior region (Region II). In each of these regions the magnetic field may be described by scalar potential functions $\Omega^I(\rho, z)\sin n\phi$ and $\Omega^{II}(\rho, z)\sin n\phi$ (where n represents the azimuthal harmonic number) [2],[3],[4] that should satisfy the differential equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Omega}{\partial\rho}\right) + \frac{\partial^2\Omega}{\partial z^2} - \frac{n^2}{\rho^2}\Omega = 0 \qquad (1)$$

The function Ω^I may be taken to be a *given function* that will vanish along the axis $\rho = 0$, while the function Ω^{II} (that remains to be found) should tend to zero at remote distances ($\rho \to \infty$). Each of the functions Ω^I & Ω^{II} should fulfill suitable boundary conditions [e.g., Dirichlet or Neumann $\left(\frac{\partial\Omega}{\partial n}\right) = 0$] at the side boundaries of a relaxation mesh.

The functions Ω^I & Ω^{II} should *not* be expected themselves to be continuous across the interface but we require instead that the *normal derivatives* should be continuous, e.g. at the interface $\frac{\partial\Omega^{II}}{\partial n}$ should become equal to the prescribed value of $\frac{\partial\Omega^I}{\partial n}$ at that same point (with n having the *same* direction in space on the two sides of the interface).

It appears that we have a mathematically well posed problem in ρ,z space for the function Ω^{II}. We thus may anticipate, in particular, that if the function Ω^I in some z region increases from zero to (say) some positive value as it approaches the interface, there then may be a jump to a negative value for Ω^{II} on the opposite side of the interface at that location and that Ω^{II} then will grow (to less negative values) as ρ increases further, thus maintaining continuity

at the interface of the normal derivatives of potential and permitting Ω^{II} to tend towards zero at large ρ.

If the problem thus posed becomes solved, as by a relaxation process applied to the function Ω^{II}, the values for the surface-current density on the interface then can be found. Thus, specifically, the value of the longitudinal component of current density (e.g., the component running along the interface in the ρ,z plane) is given by the discontinuity in the ϕ component of field [e.g by the difference between $-\frac{1}{\mu_0}\frac{\partial\Omega^{II}}{\rho\partial\phi}\sin n\phi$ and $+\frac{1}{\mu_0}\frac{\partial\Omega^{I}}{\rho\partial\phi}\sin n\phi$, or $\frac{n}{\mu_0}\frac{1}{\rho}\left(\Omega^I - \Omega^{II}\right)\cos n\phi$], while the discontinuity in the longitudinal derivative or longitudinal component of field similarly gives the ϕ component of current density as $\frac{1}{\mu_0}\left(\frac{\partial\Omega^{II}}{\partial s} - \frac{\partial\Omega^{I}}{\partial s}\right)\sin n\phi$ so as together will describe a current with zero surface divergence (as desired).

We might expect that the relaxation solution of the problem posed for the function Ω^{II} might be achieved by a slightly modified version of the program POISSON. We may now first mention, however, that the boundary condition $\Omega^{II} \rightarrow 0$ as $(\rho \rightarrow \infty)$ may not be easily realized on a necessarily finite mesh, so that one may need to have a recourse to some approximate treatment of this matter (such as imposing a boundary value $\Omega^{II} = 0$ at the outer edge of a quite extended mesh, or by some more sophisticated special treatment). A possible difficulty with regard to employing an available relaxation program, to solve the problem posed above for determining the function $\Omega^{II}(\rho, z)$, will arise if the program can accommodate a Neumann type of boundary condition *only if* in such cases the value specified for the normal derivative is *zero* (in contrast to the present requirement that the normal derivative of Ω^{II} shell be taken as equal to the known (prescribed) normal derivative of Ω^I).

In recognition of the possible occurrence of this difficulty, we now suggest a possible means of circumventing this difficulty, so that one could proceed by use of an available relaxation program for solving the relevant differential equation [subject to the provision the program "editor" will permit one to obtain correct values of *normal derivatives (right up to any boundary)* of solutions $\Omega^{II}(\rho, z)$ obtained by the relaxation process]. The suggested method may well be regarded as inefficient from the point of view of computer usage, but none the less its adoption may be regarded as appropriate method and we then may go on to illustrate the method by an extremely simple example that may serve to lend some confidence to the belief that the overall process will be convergent.

III. IMPLEMENTATION

The method to be outlined will omit the need to apply a Neumann boundary condition at the interface $\rho = f(z)$ in performing a relaxation sweep throughout the mesh wherein the function Ω^{II} is to be evaluated. The method instead will employ the assignment of "provisional" values for Ω^{II} at the node points along the interface, thus in effect introducing a Dirichlet type boundary condition at this boundary while relaxation processes are underway in Region II. Subsequent to the execution of several relaxation passes through Region II these provisional Dirichlet boundary values will be revised, in light of currently available provisional estimates of $\frac{\partial\Omega^{II}}{\partial n}$, in such a way that the desired continuity of $\frac{\partial\Omega}{\partial n}$ across the interface may become more closely attained and the relaxation process then will be resumed. Such readjustments can be performed repeatedly until a suitable close degree of convergence is attained and the then-available values of Ω^{II} and its derivative $\frac{\partial\Omega^{II}}{\partial s}$ employed (together with the corresponding known values of Ω^I *and* $\frac{\partial\Omega^{I}}{\partial s}$) to evaluate the implied values of current density on the interface surface.

Specifically we suggest that, in following this procedure, suitable *initial* provisional values for Ω^{II} at the interface may be taken simply as $-\Omega^I$ at the interface vertices. Moreover, when subsequently revising such provisional values of Ω^{II} at points on the interface, we propose that the values be scaled up simply by a provisional factor that is the average ratio of the known desired normal derivative of Ω^I to the normal derivative of the provisional present function Ω^{II} at that same point. ["over-relaxation factors" for these relaxation and revision process may well be acceptable, and even appropriate, but need not be regarded as necessary.]

Given an interface $\rho_w = f(z)$ for $0 \leq z \leq \frac{L}{4}$ $\left(or\ -\frac{L}{4} \leq z \leq \frac{L}{4}\right)$ with a local slope angle γ given by $\tan\gamma_i = T_i = \frac{d\rho_w}{dz}$ for which

$$\sin\gamma_i = \frac{T_i}{\sqrt{1+T_i^2}} \quad , \quad \cos\gamma_i = \frac{1}{\sqrt{1+T_i^2}}$$

The magnetic scalar potential in the inner region $\rho < \rho_w$ (region I) and the outer region $\rho > \rho_w$ (region II) is respectively of the form

$$V^I = \sum_n \Omega_n^I(\rho, z)\sin n\phi \ , \ with\ \ \Omega_n^I(0, z) = 0$$

$$V^{II} = \sum_n \Omega_n^{II}(\rho, z)\sin n\phi \ , \ with\ \ \Omega_n^{II}(\infty, z) \rightarrow 0$$

For each of the functions Ω_n^I & Ω_n^{II} we expect them to satisfy the differential equation (1) and to satisfy the boundary conditions

$$\begin{cases} \frac{\partial\Omega_n}{\partial z}\big|_{z=0} = 0 \\ \Omega_n\big|_{z=\frac{L}{4}} = 0 \end{cases} \quad or \quad \begin{cases} \Omega_n\big|_{z=-\frac{L}{4}} = 0 \\ \Omega_n\big|_{z=\frac{L}{4}} = 0 \end{cases}$$

The requisite connection between the functions Ω_n^I & Ω_n^{II} occurs at $\rho_w = f(z)$ and explicitly is $\frac{\partial\Omega^I}{\partial n} = \frac{\partial\Omega^{II}}{\partial n}$ where n denotes distance in the *normal* direction (in the same sense), or

$$\cos\gamma_i\frac{\partial\Omega_n^{II}}{\partial\rho} - \sin\gamma_i\frac{\partial\Omega_n^{II}}{\partial z} = \cos\gamma_i\frac{\partial\Omega_n^{I}}{\partial\rho} - \sin\gamma_i\frac{\partial\Omega_n^{I}}{\partial z}$$

when written in cylindrical coordinates, for each and every location z_i along the interface $\rho_{w,i} = f(z_i)$. We have used

the following relations to calculate derivatives of potentials normal and tangent to the interface :

$$\frac{\partial\Omega}{\partial n} = \frac{\partial\Omega}{\partial r}\cos\gamma - \frac{\partial\Omega}{\partial z}\sin\gamma$$
$$\frac{\partial\Omega}{\partial s} = \frac{\partial\Omega}{\partial r}\sin\gamma + \frac{\partial\Omega}{\partial z}\cos\gamma$$

A. Modifying "POISSON"

The program POISSON was modified to solve the revised differential equation, and to make the necessary provisional revisions on the interface. With the aide of the mesh generator program AUTOMESH we generate two regions (I & II), solve the inner region I first, and obtain the normal derivatives on each interface point. Second we turn off region I and turn on region II assuming the initial potentials on the interface are the negative of those in region I. Following several relaxation cycles (10 to 50), the potentials on the interface are multiplied by a common *Factor* (see below) and the updated potentials held constant during the next iteration cycle. The update *Factor* is the average ratio of the normal derivative on both sides of the interface.

$$Factor = \frac{1}{N}\sum_{i=1}^{N}\frac{\cos\gamma_i\frac{\partial\Omega_n^I}{\partial\rho} - \sin\gamma_i\frac{\partial\Omega_n^I}{\partial z}}{\cos\gamma_i\frac{\partial\Omega_n^{II}}{\partial\rho} - \sin\gamma_i\frac{\partial\Omega_n^{II}}{\partial z}}$$

N is the total number of points on the interface. Accordingly the interface potentials are revised :

$$\Omega_{i,new}^{II} = \Omega_{i,old}^{II}[1 + \lambda(Factor - 1)]$$

with λ being the relaxation factor. As the process converges the value of Factor tends towards 1. and $\Omega_{i,new}^{II} = \Omega_{i,old}^{II}$. The resulting current density on the interface can now be obtained using the potentials and derivatives on both sides of the interface.

$$J_s = \frac{n}{\mu_0}\frac{\Omega_n^I - \Omega_n^{II}}{\rho_w}\cos n\phi$$
$$J_\phi = -\frac{1}{\mu_0}\left(\frac{\partial\Omega_n^I}{\partial s} - \frac{\partial\Omega_n^{II}}{\partial s}\right)\sin n\phi \qquad (9)$$

(with $\mu_0 = \frac{4\pi}{10}$ in "Poisson units" of cm , amp, gauss), which should prove to be such that the surface-divergence of this surface-current density vanishes. Lines of current flow (or wire direction) are given by the differential equation

$$\frac{d\phi}{ds} = \frac{1}{\rho_w}\frac{J_\phi}{J_s} = -\frac{1}{\rho_w n}\left(\frac{\frac{\partial\Omega_n^I}{\partial s} - \frac{\partial\Omega_n^{II}}{\partial s}}{\Omega_n^I - \Omega_n^{II}}\right)\tan n\phi$$
$$= g(r,z)\tan n\phi$$

where s is a location distance along the interface curve $\rho_w = f(z)$. The above differential equation can be rewritten as a function of z and r (instead of s and r)

$$\frac{d\phi}{dz} - g(\rho,z)\sqrt{1 + \left(\frac{d\rho}{dz}\right)_w^2}\tan n\phi = 0 \qquad (11)$$

POISSON output provides tables of ρ, z, and g(ρ, z), which are used to calculate wire locations.

B. An alternating cylindrical quadrupole — EXAMPLE

To illustrate the procedure just outlined and possibly to give a sense of any issues concerning convergence, we apply this method to a simple problem in which a purely sinusoidal AG quadrupole field is to be formed by current windings placed on a circular cylinder. It will be recalled that for such a simple configuration it was known that for a current distribution at ρ=a

$$J = \cos\frac{\pi z}{L}\cos 2\phi\,\hat{e}_z + \frac{\pi a}{2L}\sin\frac{\pi z}{L}\sin 2\phi\hat{e}_\phi$$

it was determined by analytical means that the fields were derived from scalar potential functions such as for $\rho < a$ and for $\rho > a$ respectively

$$V^I = -\frac{2\pi}{10}\frac{\pi a^2}{L}K_2'\left(\frac{\pi a}{L}\right)I_2\left(\frac{\pi\rho}{L}\right)\cos\left(\frac{\pi z}{L}\right)\sin 2\phi$$
$$V^{II} = -\frac{2\pi}{10}\frac{\pi a^2}{L}I_2'\left(\frac{\pi a}{L}\right)K_2\left(\frac{\pi\rho}{L}\right)\cos\left(\frac{\pi z}{L}\right)\sin 2\phi$$

In the program we accordingly employ a mesh with the type of boundary condition indicated and from the formula shown for Ω^I the values of Ω^I are readily specified as fixed numerical values along the interface $\rho = a$. With this potential values, an input file into POISSON can be generated. The results of the given and converged potential values on the interface of regions I and II respectively are shown in the bottom of Fig. 1. Also shown are normal derivatives along the interface ($\frac{d\Omega}{dn} = \frac{d\Omega}{dr}$) with identical values for both regions I and II (as desired). Flux plots are illustrated in Fig. 2. The established potentials and tangential derivatives permit one to calculate the current density as prescribed by (2) plot their values in the ϕ and Z direction as shown in Fig. 3, generate the function g(ρ, z) as shown in Fig. 4 and solve the differential equation 3 obtaining the required wire distribution as shown in Fig. 5.

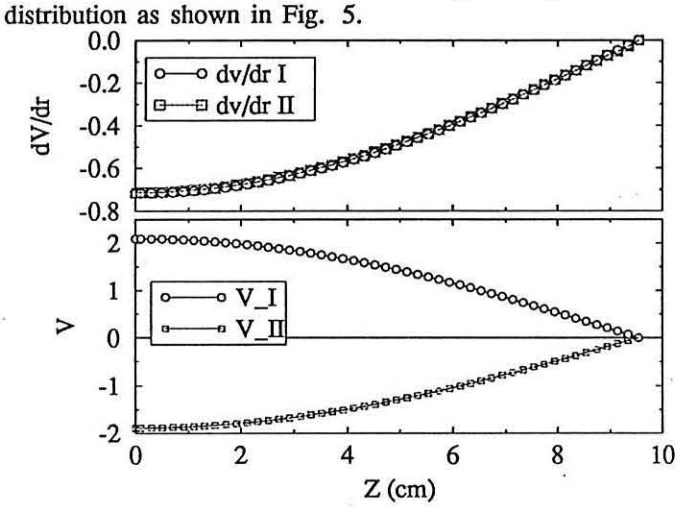

Fig. 1. Potentials and normal derivatives on the interface.

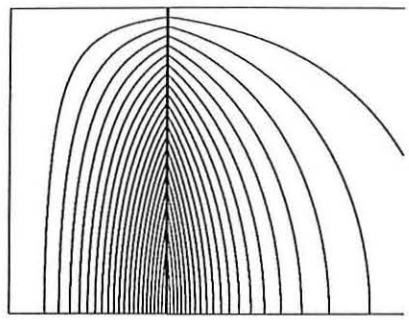

Fig. 2. Flux plot in inner region I (left) and outer Region II (right).

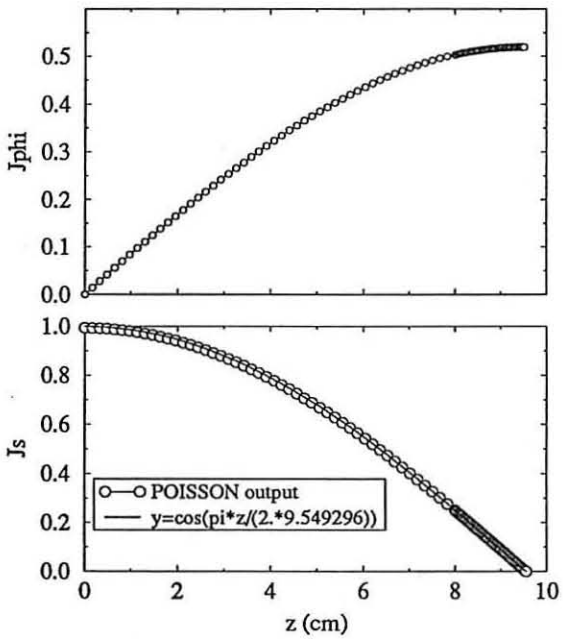

Fig. 3. Current density in the ϕ and Z directions.

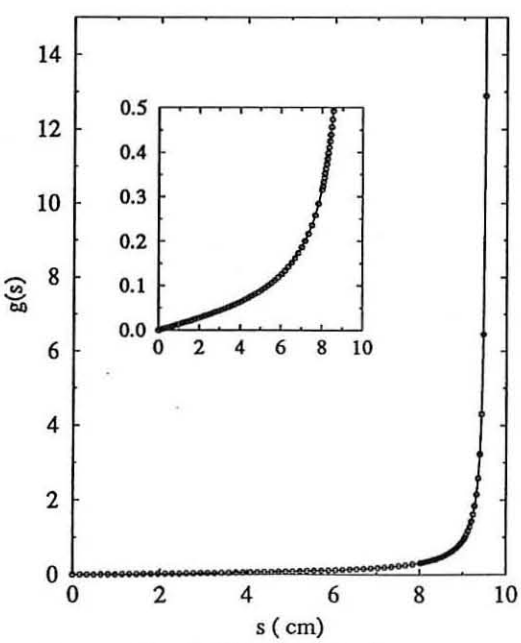

Fig. 4. The function g(s).

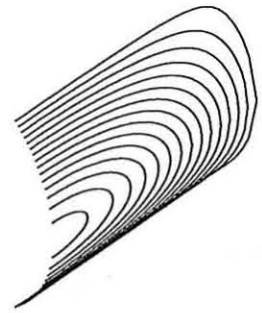

Fig. 5. A 3D CAD drawing of the quad wire distribution.

APPENDIX

A. *Interpolation & Differentiation in POISSON*

In the reprogramming of POISSON the interpolation and interpolation-differetiation are based on polynomials of the type shown in References [2],[5]. These functions are each of such a form that (for n>0) they *vanish* at r=0 and so (as intended) appear appropriate for use in an INTERIOR region that includes the *axis* of the cylindrical coordinate system. In application to the modified POISSON, such polynomials do *NOT* appear particularly suitable however for fitting values of the function Ω in *EXTERIOR* regions that extend essentially to ∞ where Ω may be expected to become zero.

In review of the circumstances noted above, the evaluation of interpolated Ω values and derivatives *in an EXTERNAL region* may require a basic replacement of functions so as to include expressions for n < 0 that should individually satisfy $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Omega}{\partial\rho}\right) + \frac{\partial^2\Omega}{\partial z^2} - \frac{n^2}{\rho^2}\Omega = 0$ and even with n a negative integer should not blow up. Suggested polynomials for interpolation and interpolation-differentiation have been incorporated in to the program POISSON and are tabulated in Reference [6].

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