LBL-36926
UC-414
Preprint

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Submitted to Nuclear Physics B

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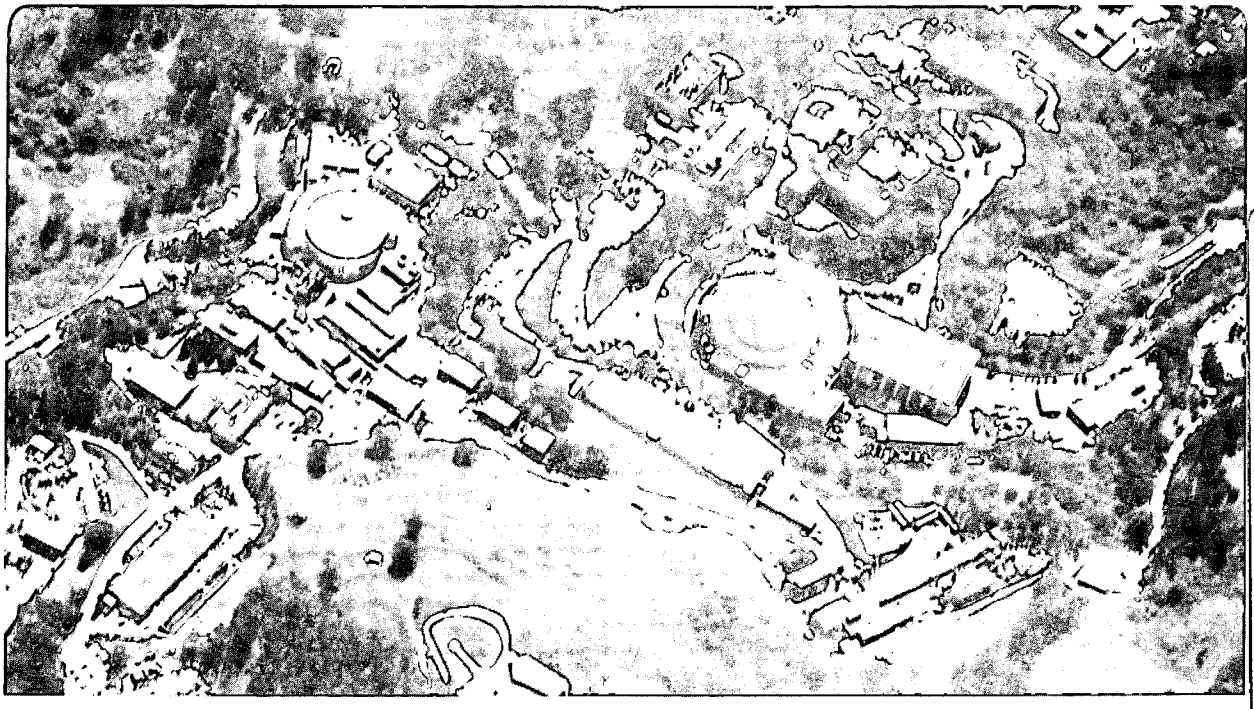
May 1995



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Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

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May 26, 1995

LBL-36926
UCB-PTH-95/07

The Conformal Points Of The Generalized Thirring Model II *

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Abstract

In the large N limit, conditions for the conformal invariance of the generalized Thirring model are derived, using two different approaches: the background field method and the Hamiltonian method based on an operator algebra, and the agreement between them is established. A free field representation of the relevant algebra is presented, and the structure of the stress tensor in terms of free fields (and free currents) is studied in detail.

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139.

†Supported by JNICT (Lisbon).

1. Introduction

In searching for new conformal field theories in two dimensions, a hitherto relatively less explored candidate is the generalized Thirring model. By generalized Thirring model, we mean a model of several massless fermions interacting through the most general Lorentz invariant four fermion couplings, including parity violating interactions. This is a further generalization of the parity invariant version considered in references [1] and [2]. In this paper, which is a sequel to [1], we will continue the investigation of this more general version of the Thirring model. The model is classically scale invariant , and although scale invariance is in general broken quantum mechanically, the hope is that there are isolated points in the coupling constant space where the invariance is restored. Since any local conformal field theory in two dimensions can serve as the basis for string compactification, the construction of new conformal theories of this type, apart from its own intrinsic interest, can lead to advances in string theory. Another possible area of application is the statistical mechanics of two dimensional systems.

A well known and somewhat trivial example of a conformal theory of this type is the original Thirring model [3], which is equivalent to a free field theory. A much less trivial example is the non-Abelian Thirring model, when the four fermion interaction is invariant under some Lie group. In a fundamental paper, Dashen and Frishman [4] showed that this model has conformal invariance at quantized values of the coupling constant, and that the stress tensor at the conformal points is given by the affine Sugawara construction [5]. Unfortunately, much less is known about the model when the coupling constants are not restricted by any symmetry. There is some evidence that [6] a model of this type may describe the world sheet of the string theory resulting from QCD, and if this indeed the case, it is important to learn more about possible conformal points in the coupling constant space. In the absence of exact solutions of the Dashen-Frishman type, recent investigations of this model treated the problem in the large N expansion, N being the number of fermions, and conditions on the coupling constants in the first non-trivial order in $1/N$ were derived [1,2,7]. If these results continue to hold in higher orders in $1/N$, the generalized Thirring model does indeed have conformal points in the coupling constant space.

In this paper, we will adress several questions related to conformal invariance of the generalized Thirring model in the large N limit. One of our aims is to compare both the methods and the conclusions of references [1] and [2]. We were lead to this reexamination because we found that the conditions on the coupling constants derived in these two references seemed to disagree. Since the methods used in these two papers are different, it seemed of interest to us to resolve this conflict. Although both papers start with bosonization, [2] uses the standard background field approach to examine conformal invariance, whereas [1] instead uses operator methods and the Hamiltonian picture in the light cone variables. We feel that it is worthwhile to supplement the Lagrangian approach with operator methods based on a Hamiltonian in order to learn more about the model, and so it is important to resolve possible conflicts between the two complementary methods. In section 2, we rederive the condition for conformal invariance, using the background field method. Our calculation is somewhat different from that of [2], and it serves as a good check on the results of this reference, since we use a different bosonization scheme which avoids the introduction of the dilaton. In the end, the conditions we derive turn out to be identical to the conditions derived in [2] for the case of parity conserving coupling constants. We also notice that these conditions are invariant under a set of transformations of the matrix representing the coupling constants. They consist of an inversion and multiplications by orthogonal matrices, and they remind us of a similar set of transformations encountered in toroidal compactification [8].

The confirmation of the results of reference [2] makes it clear that there must be something wrong with the conditions derived by operator methods in [1]. In section 3, we reexamine the operator approach and in particular the construction of the stress tensor. In a conformal theory, the stress tensor should be traceless and should satisfy the Virasoro algebra. Instead, we find an operator anomaly in the stress tensor which is equivalent to the well known trace anomaly. This anomaly, which was missed in [1], was the source of the disagreement between the operator and background field methods. By imposing the requirement that this anomaly should vanish, we derive conditions on the coupling constants in full agreement with the background field method. We also give the operator construction of the two chiral components of the stress tensor, show that they satisfy the Virasoro algebra without invoking any further

conditions on the coupling constants, and we compute the two (left,right) central charges. This then corrects and extends to parity non-invariant case the results of [1].

Another aim of this paper is to study further the operator algebra introduced in [1], which resulted from the quantization of the bosonized Thirring model. This is of some interest, since this algebra, which can be regarded as a generalization of the affine Lie algebra, to our best knowledge is new. In section 4, we present a (non-local) representation of this algebra in terms of free fields, incidentally establishing its consistency beyond any doubt. One reason for doing this is to see whether the model at hand can be mapped into a well-known conformal theory. In particular, we have in mind the free field theory, and less trivially, the affine Sugawara construction [5]. The mapping into free fields is non-local and complicated; however, there is always the hope that the stress tensor may turn out to be something simple and recognizable. Indeed,the expression for the stress tensor in terms of the free fields turns out to be quadratic, which is an unexpectedly simple result. However, in this expression there is an unusual term in which the second derivative of the free field appears. This term is responsible for the difference of the central charge from the free field value and it cannot be eliminated. In a similar fashion, one can reexpress the algebra in terms of currents that satisfy an affine Lie algebra, in the hope that the stress tensor may then admit an affine Sugawara construction [5]. The stress tensor is indeed quadratic in currents, however, there is again the term with two derivatives, which is not present in the affine Sugawara construction. As a result, the affine Sugawara construction, at least in its simplest form, does not work [9], unless additional conditions that go beyond requiring conformal invariance are imposed. This strongly suggests the emergence of a new conformal structure. Finally, the last section summarizes our conclusions and lists problems that await future investigation.

2. Background Field Method and Conformal Invariance

In this section, using bosonization and the background field method, we will investigate conformal invariance of the generalized parity non-invariant Thirring model. Our bosonization is based directly on the Polyakov-Wiegmann [10] method, whereas Tseytlin [2] used an approach based on the gauging of the

WZW model [11]. In this latter approach, one has to integrate over a gauge field, and the non-trivial integration measure requires the introduction of a dilaton field [12]. In contrast, we avoid this complication [13]. Our starting point is the parity violating generalized Thirring model given by

$$S_o = \int d^2x(\bar{\Psi} i\gamma^\mu \partial_\mu \Psi - \tilde{G}^{-1}_{ab}\ \bar{\Psi}_R \lambda_a \Psi_R\ \bar{\Psi}_L \lambda_b \Psi_L) \tag{1}$$

where R and L refer to the right and left chiral components of Ψ, and the coupling constant G_{ab} is not necessarily symmetric, resulting in parity violation. Upon bosonization [10,1,7], this gives *

$$S_o = W(g) + W(h^{-1}) - \frac{N}{2\pi}\int d^2x G_{ab}(ig^{-1}\partial_+ g)_a (ih^{-1}\partial_- h)_b \tag{2}$$

where X_a stands for $Tr(\lambda_a X)$ and

$$G_{ab} = \frac{1}{2}\delta_{ab} - \frac{\pi}{2N}\tilde{G}_{ab} \tag{3}$$

and W is the WZW action

$$W(g) = \frac{N}{8\pi}\left(\int d^2x Tr(\partial_\mu g^{-1}\partial^\mu g) + \frac{2}{3}\int Tr\left((g^{-1}dg)^3\right)\right). \tag{4}$$

Here, we have assumed that $(2G-1)$ is an invertible matrix. In the absence of sources, the equations of motion are equivalent to conservation of two currents:

$$\partial_+ J_- = \partial_- J_+ = 0, \tag{5}$$

where

$$\begin{aligned} J_+ &= i\frac{N}{4\pi}\left(-\partial_+ h h^{-1} + 2h\lambda_a h^{-1} G_{ba}(g^{-1}\partial_+ g)_b\right), \\ J_- &= i\frac{N}{4\pi}\left(-\partial_- g g^{-1} + 2g\lambda_a g^{-1} G_{ab}(h^{-1}\partial_- h)_b\right). \end{aligned} \tag{6}$$

Two implement the background field method, we add a term to the action which represents the coupling of two external sources $K_{+,-}$ to two suitable currents:

$$\Delta S = \frac{N}{2\pi}\int d^2x Tr\left(K_+(ih^{-1}\partial_- h)\right) + \frac{N}{2\pi}\int d^2x Tr\left(K_-(ig^{-1}\partial_+ g)\right). \tag{7}$$

*The metric in group space is just δ_{ab} and so there is no distinction between upper and lower indices.

The next step is to define classical fields by solving the equations of motion in the presence of sources. A special solution we are going to use is

$$\begin{aligned} K_{-,a} &= \left(-\frac{1}{2}(ig^{-1}\partial_- g)_a + G_{ab}(ih^{-1}\partial_- h)_b\right)_{classical} \\ K_{+,a} &= \left(-\frac{1}{2}(ih^{-1}\partial_+ h)_a + G_{ba}(ig^{-1}\partial_+ g)_b\right)_{classical} \end{aligned} \tag{8}$$

These sources $K_{+,-}$ can be substituted back in S to give the classical action $S^{(0)}$. This defines the classical (background) fields $g_{clas.}$ and $h_{clas.}$ around which we expand the full quantum fields g and h. In the appendix, we use background field perturbation theory to derive, to one loop order, the conditions that the coupling constants G_{ab} must satisfy to have conformal invariance. This is done by first expanding the action S around $S^{(0)}$, and by calculating the one loop divergent contribution to the action. This calculation is fairly standard [14], and for the sake of completeness, it is sketched in the appendix. The result is

$$S[\phi] = S^{(0)}[\phi_{clas.}] + S^{(2)}[\phi_{clas.}] \tag{9}$$

where $S^{(2)}$ is logarithmically divergent. Here ϕ (which stands for both ϕ and $\overline{\phi}$) is the field used to parametrize g (and h is parametrized by $\overline{\phi}$), and $\phi_{clas.}$ is defined by $g_{clas.} = g(\phi_{clas.})$. The divergent piece can be written as (from now on ϕ stands for $\phi_{clas.}$)

$$S^{(2)}[\phi] \cong \int \frac{d^2p}{p^2 - m^2} \int d^2x O(x) \tag{10}$$

where

$$\begin{aligned} O(x) &= Y^{(11)}_{ab} E^a_\alpha E^b_\beta \partial_+\phi^\alpha \partial_-\phi^\beta + Y^{(22)}_{ab} \overline{E}^a_\alpha \overline{E}^b_\beta \partial_+\overline{\phi}^\alpha \partial_-\overline{\phi}^\beta \\ &\quad + Y^{(21)}_{ab} \overline{E}^a_\alpha E^b_\beta \partial_+\overline{\phi}^\alpha \partial_-\phi^\beta + Y^{(12)}_{ab} E^a_\alpha \overline{E}^b_\beta \partial_+\phi^\alpha \partial_-\overline{\phi}^\beta \end{aligned} \tag{11}$$

and

$$\begin{aligned} Y^{(11)}_{ab} &= Tr[-4GH^{-1}G^T f_a \tilde{H}^{-1} f_b] \qquad & Y^{(22)}_{ab} &= Tr[-4H^{-1} f_a G^T \tilde{H}^{-1} G f_b] \\ Y^{(21)}_{ab} &= Tr[4GH^{-1} f_a G^T \tilde{H}^{-1} f_b] \qquad & Y^{(12)}_{ab} &= Tr[4H^{-1} G^T f_a \tilde{H}^{-1} G f_b] \end{aligned}$$

with

$$H = 1 - 4G^T G, \quad \tilde{H} = 1 - 4GG^T$$

and the matrices f_a are defined by $(f_a)_{bc} = f_{abc}$, where f_{abc} are the structure constants of the group. The E^a_α's are the vielbeins defined by

$$E^a_\alpha(\phi)\partial_+\phi^\alpha \equiv Tr(\lambda^a i g^{-1}\partial_\alpha g)\partial_+\phi^\alpha = Tr(\lambda^a i g^{-1}\partial_+ g)$$

with similar definitions for $\overline{E}^a_\alpha$'s in terms of h's. Now compare this divergent piece with the original Lagrangian, expressed in terms of classical fields

$$\begin{aligned} S^{(0)} &= W(g) + W(h^{-1}) + \frac{N}{2\pi}\int d^2x G_{ab} E^a_\alpha \overline{E}^b_\beta \partial_+\phi^\alpha \partial_-\overline{\phi}^\beta \\ &\quad -\frac{N}{4\pi}\int d^2x E^a_\alpha E_{a\beta}\partial_+\phi^\alpha\partial_-\phi^\beta - \frac{N}{4\pi}\int d^2x \overline{E}^a_\alpha \overline{E}_{a\beta}\partial_+\overline{\phi}^\alpha\partial_-\overline{\phi}^\beta \end{aligned} \quad (12)$$

Of the four distinct divergent terms defined by eq.(11), three correspond to wave function renomalizations and can be eliminated by field redefinitions. Conformal invariance is then imposed by requiring that the remaining divergence (the beta function) vanish. The field redefinitions that eliminate the spurious divergences are given by

$$\begin{aligned} (ig^{-1}\partial_+ g)_a &\longrightarrow (ig^{-1}\partial_+ g)_a + \lambda^{(11)}_{ab}(ig^{-1}\partial_+ g)_b + \lambda^{(12)}_{ab}(ih^{-1}\partial_+ h)_b, \\ (ih^{-1}\partial_- h)_a &\longrightarrow (ih^{-1}\partial_- h)_a + \lambda^{(21)}_{ab}(ih^{-1}\partial_- h)_b + \lambda^{(22)}_{ab}(ig^{-1}\partial_- g)_b \end{aligned} \quad (13)$$

where the λ's are first order in $1/N$. This corresponds, in the Polyakov-Wiegmann bosonization, to making the identification

$$\begin{aligned} A_{+,a} &= (\delta_{ab} + \lambda^{(11)}_{ab})(ig^{-1}\partial_+ g)_b + \lambda^{(12)}_{ab}(ih^{-1}\partial_+ h)_b, \\ A_{-,a} &= (\delta_{ab} + \lambda^{(22)}_{ab})(ih^{-1}\partial_- h)_b + \lambda^{(21)}_{ab}(ig^{-1}\partial_- g)_b, \end{aligned} \quad (14)$$

instead of

$$A_{+,a} = (ig^{-1}\partial_+ g)_a, \quad A_{-,a} = (ih^{-1}\partial_- h)_a$$

The same result can be obtained by introducing additional sources $L_{+,-}$,

$$\Delta S = \frac{N}{2\pi}\int d^2x Tr\left(L_-(ih^{-1}\partial_+ h)\right) + \frac{N}{2\pi}\int d^2x Tr\left(L_+(ig^{-1}\partial_- g)\right) \quad (15)$$

which are zero to lowest order, and by transforming $K_{+,-}$, $L_{+,-}$ linearly among themselves (source renormalization).

Under these field redefinitions the first order correction to $S^{(0)}[\phi]$ is

$$\begin{aligned}\Delta S^{(0)} &= -\frac{N}{4\pi}\int d^2x\,\Big((\lambda^{(21)}_{ab}+\lambda^{12}_{ba})\overline{E}^a_\alpha E^b_\beta\partial_+\overline{\phi}^\alpha\partial_-\phi^\beta \\ &\quad -2(\lambda^{(11)}_{ca}G_{cb}+\lambda^{(22)}_{cb}G_{ac})E^a_\alpha\overline{E}^b_\beta\partial_+\phi^\alpha\partial_-\overline{\phi}^\beta \\ &\quad +(\lambda^{(11)}_{ba}-2G_{ac}\lambda^{(21)}_{cb})E^a_\alpha E^b_\beta\partial_+\phi^\alpha\partial_-\phi^\beta \\ &\quad +(\lambda^{(22)}_{ab}-2G_{cb}\lambda^{(12)}_{ca})\overline{E}^a_\alpha\overline{E}^b_\beta\partial_+\overline{\phi}^\alpha\partial_-\overline{\phi}^\beta\Big)\,. \qquad (16)\end{aligned}$$

We now try to eliminate the divergent terms, and this leads to the matrix equations

$$\begin{aligned}-Y^{(11)}+\frac{N}{4\pi}((\lambda^{(11)})^T-2G\lambda^{(21)}) &= 0, \\ -Y^{(12)}-\frac{N}{4\pi}(2(\lambda^{(11)})^TG+2G\lambda^{(22)}) &= 0, \\ -Y^{(22)}+\frac{N}{4\pi}(\lambda^{(22)}-2(\lambda^{(12)})^TG) &= 0, \\ -Y^{(21)}+\frac{N}{4\pi}(\lambda^{(21)}+(\lambda^{(12)})^T) &= 0 \qquad (17)\end{aligned}$$

At first, one might think that these equations can be solved for the unknown λ's. If this were true, all the infinities would be absorbed by field redefinitions and conformal invariance would be automatic! In fact, the equations are linearly dependent, and for a solution to exist, the Y's must satisfy the following condition:

$$Y^{(12)}+2Y^{(11)}G+2GY^{(22)}+4GY^{(21)}G=0.$$

This condition is therefore equivalent to the vanishing of the beta function. Written out explicitly, this leads to the following equation between the coupling constants:

$$\begin{aligned}&Tr[H^{-1}G^Tf_a\tilde{H}^{-1}Gf_b]+4G_{aa'}G_{b'b}Tr[GH^{-1}f_{a'}G^T\tilde{H}^{-1}f_{b'}] \\ &-2G_{aa'}Tr[H^{-1}f_{a'}G^T\tilde{H}^{-1}Gf_b]-2G_{b'b}Tr[GH^{-1}G^Tf_a\tilde{H}^{-1}f_{b'}] = 0, \qquad (18)\end{aligned}$$

Eq.(18) is therefore the condition that determines the conformal points in the coupling constant space. For $G=G^T$, it agrees with the result obtained in [2], where the Q defined there is related to our G by

$$Q=2(I-G). \qquad (19)$$

We end this section by noticing that this equation is invariant under $(2G) \to (2G)^{-1}$ and under $G \to O_1^T G O_2$, where O_1 and O_2 are orthogonal transformations generated by rotations in group space. The first one is the standard duality transformation [8], already noticed in a classical context in [15]. The second set of transformations are generated by independent group rotations of left and right fermions:

$$\Psi_R \to U_R \Psi_R, \ \Psi_L \to U_L \Psi_L. \tag{20}$$

3. The OPE and Conformal Invariance

In this section, we shall reexamine the conformal invariance of the theory from the operator point of view, and show that again the same result as in the last section (eq.(18)) is obtained, reconciling the background field and operator methods. Our criterion for conformal invariance is the existence of a chirally conserved stress tensor: it is well known that this is equivalent to the absence of the trace anomaly in the stress tensor[16]. Our approach will be to solve the equations of motion for the quantized fields as a power series in $1/N$, and then use this result to construct the stress tensor explicitly. We will then see that there is an anomalous term which violates chiral conservation. Conformal invariance is restaured by demanding that this term vanish, and the resulting condition on the coupling constants agrees with the result derived in the last section using the background field method. Before discussing the quantum mechanical complications, we will first briefly review the classical situation. The two chiral components of the classical stress tensor, defined by

$$\begin{aligned} T(t,x) &= \frac{\pi}{\alpha^2} M_a(t,x) M_a(t,x) \\ \tilde{T}(t,x) &= \frac{\pi}{\alpha^2} N_a(t,x) N_a(t,x) \end{aligned} \tag{21}$$

where $\alpha = (4\pi/N)^{1/2}$, $t \equiv x_+, x \equiv x_-$, and

$$\begin{aligned} M_a &= (H^{\frac{1}{2}})_{ab}(ih^{-1}\partial_x h)_b, \quad H = 1 - 4G^T G, \\ N_a &= (\tilde{H}^{\frac{1}{2}}(2G^T)^{-1})_{ab}(ih^{-1}\partial_t h)_b, \quad \tilde{H} = 1 - 4GG^T, \end{aligned} \tag{22}$$

satisfy the conservation laws

$$\begin{aligned} \partial_t T(t,x) = 0, \quad &\longrightarrow \quad T_-(x) = T(t,x), \\ \partial_x \tilde{T}(t,x) = 0, \quad &\longrightarrow \quad T_+(t) = \tilde{T}(t,x), \end{aligned} \tag{23}$$

and also satisfy the classical (without central charge) Virasoro algebra [1]:

$$T(x)T(y) \cong \frac{1}{(x-y)^2}(T(x)+T(y)). \tag{24}$$

Now, in the quantum version of the stress tensor we replace the classical expression by (we will work with the $M_a(t,x)$'s, but the same applies to the $N_a(t,x)$'s),

$$T(t,x) = \frac{\pi}{\alpha^2} \lim_{y,u\to x,t} (C_{ab} M_a(t,x) M_b(u,y) - \text{sing.terms}) \tag{25}$$

where C_{ab} is a constant matrix which starts with classical value δ_{ab}, and has higher order corrections given by

$$C_{ab} = \delta_{ab} + \sum_{n=2}^{\infty} \alpha^n C_{ab}^{(n)} \tag{26}$$

due to renormalization. In reference [1], C_{ab} was incorrectly set equal to δ_{ab} to all orders in α; here, we will determine it by requiring that the stress tensor T satisfy the Virasoro algebra. To do this, and to find the singular terms to be subtracted, we need the OPE's (operator product expansion) between two M's. So we will expand

$$M_a(t,x) = \sum_{n=0}^{\infty} \alpha^n M_a^{(n)}(t,x) \tag{27}$$

and carry calculations up to second order. The strategy for computing OPE's is the following. We first define $M^{(n)}$'s at a fixed t, say $t=0$: $M_a^{(n)}(x) \equiv M_a^{(n)}(t=0,x)$. The OPE's depend only on x and they can be deduced from the Poisson brackets at fixed t [1]. The Poisson brackets between the M's and the OPE's that follow from them were computed in [1]; here we simply take over those results, generalizing them slightly to take into account of the fact that G is no longer self transpose:

$$\begin{aligned}
M_a^{(0)}(x) M_b^{(0)}(y) &\cong -\frac{1}{2\pi(x-y)^2}\delta_{ab}, \\
\sum_{n=0}^{1} M_a^{(n)}(x) M_b^{(1-n)}(y) &\cong -\frac{1}{4\pi(x-y)} F_{abc}\left(M_c^{(0)}(x) + M_c^{(0)}(y)\right), \\
\sum_{n=0}^{2} M_a^{(n)}(x) M_b^{(2-n)}(y) &\cong -\frac{1}{4\pi(x-y)} F_{abc}\left(M_c^{(1)}(x) + M_c^{(1)}(y)\right) \\
&\quad + \frac{1}{2\pi} E_{ab,a'b'} \log(x-y) M_{a'}^{(0)}(x) M_{b'}^{(0)}(y),
\end{aligned} \tag{28}$$

where the constants A_{abc} and F_{abc} are defined by

$$\begin{aligned} A_{abc} &= -2H_{aa'}^{-\frac{1}{2}}H_{bb'}^{-\frac{1}{2}}(\tilde{H}^{-\frac{1}{2}}G)_{cc'}f_{a'b'c'} + 4(H^{-\frac{1}{2}}G^T)_{aa'}(H^{-\frac{1}{2}}G^T)_{bb'}\tilde{H}_{cc'}^{-\frac{1}{2}}f_{a'b'c'}, \\ F_{abc} &= H_{aa'}^{-\frac{1}{2}}H_{bb'}^{-\frac{1}{2}}H_{cc'}^{-\frac{1}{2}}f_{a'b'c'} - 8(H^{-\frac{1}{2}}G^T)_{aa'}(H^{-\frac{1}{2}}G^T)_{bb'}(H^{-\frac{1}{2}}G^T)_{cc'}f_{a'b'c'}, \end{aligned}$$

and

$$E_{ab,cd} = A_{cae}A_{bde}. \tag{29}$$

The $N_a^{(n)}$'s obey similar OPE's, obtained from the above ones under $G \to G^T$, with the new constants $\tilde{A}_{abc}$ and $\tilde{F}_{abc}$,

$$\begin{aligned} \tilde{A}_{abc} &= -2\tilde{H}_{aa'}^{-\frac{1}{2}}\tilde{H}_{bb'}^{-\frac{1}{2}}(H^{-\frac{1}{2}}G^T)_{cc'}f_{a'b'c'} + 4(\tilde{H}^{-\frac{1}{2}}G)_{aa'}(\tilde{H}^{-\frac{1}{2}}G)_{bb'}H_{cc'}^{-\frac{1}{2}}f_{a'b'c'}, \\ \tilde{F}_{abc} &= \tilde{H}_{aa'}^{-\frac{1}{2}}\tilde{H}_{bb'}^{-\frac{1}{2}}\tilde{H}_{cc'}^{-\frac{1}{2}}f_{a'b'c'} - 8(\tilde{H}^{-\frac{1}{2}}G)_{aa'}(\tilde{H}^{-\frac{1}{2}}G)_{bb'}(\tilde{H}^{-\frac{1}{2}}G)_{cc'}f_{a'b'c'}, \end{aligned}$$

and

$$\tilde{E}_{ab,cd} = \tilde{A}_{cae}\tilde{A}_{bde}. \tag{30}$$

To extend these OPE's to $t \neq 0$, we solve the equations of motion up to second order, and express the M's and N's at arbitrary t in terms of the same variables at $t = 0$. Since the OPE's at $t = 0$ are already known, they are then easily extended to $t \neq 0$. From

$$g^{-1}(\partial_+ J_-)g = 0 \tag{31}$$

we have

$$-\partial_-(g^{-1}\partial_+ g) + 2[(g^{-1}\partial_+ g), \lambda_a]G_{ab}(h^{-1}\partial_- h)_b + 2\lambda_a G_{ab}\partial_+(h^{-1}\partial_- h)_b = 0. \tag{32}$$

Now solve for $(g^{-1}\partial_+ g)$ in terms of J_+,

$$(g^{-1}\partial_+ g)_a = (2G^T)_{ab}^{-1}(h^{-1}(\partial_+ - \frac{4i\pi}{N}J_+)h)_b. \tag{33}$$

The model is invariant under the gauge transformations $h \to u_+(x_+)h$; using this gauge invariance , we can set $J_+ = 0$ ($\partial_- J_+ = 0$, so J_+ depends only on $x_+ = t$). It is amusing to notice that the equations of motion can then be written as flatness conditions for two vector fields V and W:

$$\begin{aligned} \partial_+ V_- - \partial_- V_+ - i[V_+, V_-] &= 0, \\ \partial_+ W_- - \partial_- W_+ - i[W_+, W_-] &= 0, \end{aligned} \tag{34}$$

where,

$$V_{\pm,a} = (ih^{-1}\partial_{\pm}h)_a, \tag{35}$$

$$W_{+,a} = (2G^T)^{-1}_{ab}V_{+,b}, \quad W_{-,a} = (2G)_{ab}V_{-,b}. \tag{36}$$

These can be cast in a more useful form in terms of

$$M_a = (H^{\frac{1}{2}})_{ab}V_{-,b}, \quad N_a = (\tilde{H}^{\frac{1}{2}})_{ab}W_{+,b} \tag{37}$$

defined before. The equations of motion are then

$$\partial_t M_a = -\alpha A_{abc}M_b N_c, \tag{38}$$

$$\partial_x N_a = -\alpha \tilde{A}_{abc}N_b M_c. \tag{39}$$

The conservation laws of the (classical) stress tensors follow at once from these equations due to the antisymmetry of A_{abc} and $\tilde{A}_{abc}$ in the first two indices. The next step is to solve the equations of motion iteratively, using the expansion in α (eq.(27)), and a similar expansion for N.
The zeroth and first order solutions are

$$\begin{aligned} M_a^{(0)}(t,x) &= M_a^{(0)}(x), \\ N_a^{(0)}(t,x) &= N_a^{(0)}(t). \end{aligned} \tag{40}$$

$$\begin{aligned} M_a^{(1)}(t,x) &= M_a^{(1)}(x) - A_{abc}M_b^{(0)}(x)\int^t dt' N_c^{(0)}(t'), \\ N_a^{(1)}(t,x) &= N_a^{(1)}(t) - \tilde{A}_{abc}N_b^{(0)}(t)\int^x dx' M_c^{(0)}(x'), \end{aligned} \tag{41}$$

and to second order

$$\begin{aligned} M_a^{(2)}(t,x) &= M_a^{(2)}(x) - A_{abc}\int^t dt' M_b^{(0)}(x)N_c^{(1)}(t') \\ &+A_{abc}\tilde{A}_{cde}\int^x dx'\int^t dt' M_b^{(0)}(x)M_e^{(0)}(x')N_d^{(0)}(t') \\ &-A_{abc}\int^t dt' M_b^{(1)}(x)N_c^{(0)}(t') \\ &+A_{abc}A_{bde}\int^t dt'\int^{t'} dt'' M_d^{(0)}(x)N_c^{(0)}(t')N_e^{(0)}(t''). \end{aligned} \tag{42}$$

We will not need $N_a^{(2)}(t,x)$. Therefore, $M_a(t,x)$ can be expressed in terms of $M_a^{(n)}(x)$'s and $N_a^{(n)}(t)$'s, functions of only one variable. If we substitute the

above in the definition of T to second order, it is easy to see that classically all of the t dependent terms cancel, as they must, because we know from the equations of motion that this is true to all orders. However this does not happen in the quantum case, where $M_a^{(n)}(x), n = 0, 1, 2$, become operators that satisfy the OPE's given earlier (eq.(28)). First of all, as it stands, the above expression for $M_a(t, x)$ is not well defined, because we haven't defined yet the product of two or more M's at the same point. These products should be understood as nonsingular "normal ordered" products. For instance, $M_a^{(0)}(x)M_b^{(0)}(y)$ should be understood as

$$:M_a^{(0)}(x)M_b^{(0)}(y): \equiv M_a^{(0)}(x)M_b^{(0)}(y) + \frac{\delta_{ab}}{2\pi(x-y)^2} \tag{43}$$

and the same applies for the $N_a^{(n)}$'s. The product of a $M_a^{(n)}$ and a $N_a^{(n)}$ gives no problem since they are functions of different variables and commute with each other. This guarantees that $\lim_{y\to x} :M_a^{(0)}(x)M_b^{(0)}(y):$, and consequently the above expression for $M_a^{(2)}(t, x)$ is well defined.

Next, we examine eq.(25), to see what subtractions are needed to make the stress tensor well-defined, and whether it is t independent, as the conservation law (eq.(23)) demands. It turns out that to the order we are considering (second order in α), T can be made finite by making suitable subtractions, and that all of the terms in T, with the possible exception of one term, are t independent. The critical term in question, up to a multiplicative factor of π/α^2, is

$$T_{critical} = A_{abc}\tilde{A}_{cde}\int^x dx' \int^t dt' :M_b^{(0)}(x)M_e^{(0)}(x'): M_a^{(0)}(y)N_d^{(0)}(t') + (x \leftrightarrow y)$$

This term is finite as $y \to x$ and needs no subtraction. However, it is t dependent, and therefore, if it does not vanish, it violates the conservation law (eq.(23)). It does not automatically vanish because, while A_{abc} is antisymmetric in a and b, $:M_b^{(0)}(x)M_e^{(0)}(x'): M_a^{(0)}(y) + (x \leftrightarrow y)$ is not symmetric due to the normal ordering of the two M's. However, the completely normal ordered product

$$:M_b^{(0)}(x)M_e^{(0)}(x')M_a^{(0)}(y): + (x \leftrightarrow y)$$

is symmetric in a and b and vanishes when multiplied by A_{abc}. We now make use of the identity

$$:M_b^{(0)}(x)M_e^{(0)}(x'): M_a^{(0)}(y) =$$
$$-\frac{\delta_{ab}}{2\pi(x-y)^2}M_e^{(0)}(x') - \frac{\delta_{ae}}{2\pi(x'-y)^2}M_b^{(0)}(x) + :M_b^{(0)}(x)M_e^{(0)}(x')M_a^{(0)}(y):$$

to find

$$\begin{aligned}
\lim_{y\to x} T_{critical} &= -\lim_{y\to x} A_{abc}\tilde{A}_{cde}\int^t dt' N_d^{(0)}(t') \\
&\quad \left(\int^x dx' \frac{\delta_{ae}}{2\pi(x'-y)^2} M_b^{(0)}(x) + \int^y dx' \frac{\delta_{ae}}{2\pi(x'-x)^2} M_b^{(0)}(y)\right) \\
&= -\lim_{y\to x} A_{abc}\tilde{A}_{cde}\left(-\frac{1}{2\pi}\frac{M_b^{(0)}(x)-M_b^{(0)}(y)}{x-y}\right)\int^t dt' N_d^{(0)}(t') \\
&= +\frac{1}{2\pi} A_{bac}\tilde{A}_{dca} M_b'^{(0)}(x)\int^t dt' N_d^{(0)}(t')
\end{aligned}$$

To eliminate this anomaly and restaure conformal invariance, we have to set its coefficient equal to zero,

$$A_{acd}\tilde{A}_{bdc} = 0, \tag{44}$$

recovering the same condition as before (eq.(18)). We note that conformal invariance imposes no restrictions on $C_{ab}^{(2)}$. These constants can be determined by requiring that the stress tensor satisfy the Virasoro algebra to second order. We therefore need the OPE of the product of two stress tensors; this was given by eq.(5.5a) of reference [1]. This result has to be modified slightly to take into account that $C_{ab}^{(2)}$ no longer vanishes. With this modification, the OPE of two T's is

$$\begin{aligned}
T(x)T(y) \cong \frac{c}{2(x-y)^4} - \frac{\pi}{(x-y)^2}\Big(M_a^{(0)}(x)M_a^{(0)}(x) + M_a^{(0)}(y)M_a^{(0)}(y) \\
+\alpha^2 M_a^{(1)}(x)M_a^{(1)}(x) + \alpha^2 M_a^{(1)}(y)M_a^{(1)}(y)\Big) - \frac{\alpha^2}{4(x-y)^2} \\
(F_{aa'b}F_{aa'c} + 2E_{aa,bc} + 4\pi C_{bc}^{(2)})\left(M_b^{(0)}(x)M_c^{(0)}(x) + M_b^{(0)}(y)M_c^{(0)}(y)\right)
\end{aligned}$$

where c is the central charge.

Since the Virosoro algebra reads

$$T(x)T(y) \cong -\frac{1}{(x-y)^2}(T(x)+T(y)) + \frac{c}{2(x-y)^4}, \tag{45}$$

we must have

$$F_{aa'b}F_{aa'c} + 2E_{aa,bc} + 4\pi C_{bc}^{(2)} = 0, \tag{46}$$

which determines $C_{ab}^{(2)}$, and the central charge c is given by

$$c = D - \frac{\alpha^2}{4\pi}(3E_{aa,bb} + F_{abc}F_{abc}), \tag{47}$$

where D is the dimension of the flavor algebra. This is the central charge of the algebra generated by T. The central charge of the algebra generated by the other chiral component, $\tilde{T}$, (see eq.(21)) can be gotten from eq.(47), by replacing E by $\tilde{E}$ and F by $\tilde{F}$.

As a check on our formalism, we notice that, at $G_{ab} = 0$ in eq.(2), the action is a sum of two decoupled WZW models and therefore it is obviously conformal. $G = 0$ indeed satisfies the condition for conformal invariance given by eq.(18) and so the equation passes this test. There is a further check on the central charge. The stress tensor of the WZW model is given by the Sugawara construction in terms of the currents, with the standard formula for the central charge:

$$c = \frac{2kD}{2k + c_\psi^g}, \tag{48}$$

where k is the level number of the affine algebra, related to our N by $2k = N$ and

$$c_\psi^g \delta_{ab} = \sum_{c,d=1}^{D} f_{acd} f_{bcd}.$$

This formula is exact. We have to compare it with eq.(47) in the limit of large N (or k), with G set equal to zero. In this limit $E_{ab,cd} = 0$, $F_{abc} = f_{abc}$ and so from eq.(47)

$$c = D\left(1 - \frac{c_\psi^g}{N}\right)$$

which agrees with the standard formula eq.(48) to first order in $1/N$. This particular solution ($G = 0$) has some relation to the Dashen-Frishman conformal point [4]. It is natural to suspect such a relation, since both $G = 0$ and the Dashen-Frishman solution are $SU(n)$ symmetric. We do not know how to make a detailed comparison, except to note that the stress tensor of the Dashen-Frishman solution is given by the Sugawara construction and the central charge is therefore given by eq.(48). But, as we pointed out above, the $G = 0$ solution, being the sum of two WZW models, has also a Sugawara stress tensor and the standard formula for the central charge. Therefore, at the level of stress tensors, there is agreement.

4. Free Field Realization

In this section, we will express the fields $M_a^{(n)}(x)$ in terms of free fields $\phi_a(x)$'s so that the PB in the classical case [1], or the OPE in the quantum case (eq.(28)), between two $M_a(x)$'s is still satisfied. (These ϕ's are not to be confused with the ϕ's introduced in section 2). As in the rest of the paper, the calculations will be carried only to second order in α. Our motivation for doing this is twofold: first of all, one may ask whether the relatively complicated appearance of the OPE's given by eq.(28) is due to our choice of fields; with a different choice of fields, a simpler algebra might emerge. Indeed, we show that one can express everything in terms of free fields; however, the simplification achieved in this way is somewhat illusory, since the expressions connecting M's to free fields are non-local and complicated. Next, we reexpress the stress tensor in terms of free fields, hoping for a simple result. Indeed, the stress tensor turns out to be local and quadratic in free fields; on the other hand, an unusual term involving the second derivative of the fields makes its appearance. (The last term in eq.(54)) This term is responsible for the deviation of the central charge from the free field value and it cannot be eliminated. Although we will not present the details here, the M's can also be expressed in terms of currents that satisfy an affine Lie algebra; in fact, with minor modifications, eqs.(50) and (53) still hold if the $\phi'_a(x)$'s are replaced by currents. Again, the stress tensor is quadratic in the currents, as in eq.(54), which looks promising for an affine Sugawara construction. But again there appears the analogue of the last term in eq.(54), which, expressed in terms of the currents $J_a(x)$, looks like

$$J'_a(x)\int^x dy J_a(y)$$

and clearly does not belong in the affine Sugawara construction.

We start with the classical M fields and try to express them in terms of $\phi_a(x)$'s that satisfy the free field PB relations,

$$\{\phi_a(x),\phi_b(y)\} = -\log(x-y)\delta_{ab} \tag{49}$$

The solution to zeroth order ($M_a^{(0)}(x)$) is obvious, and the next two orders are easily constructed by guesswork. The result is,

$$\begin{aligned} M_a^{(0)}(x) &= \phi'_a(x), \\ M_a^{(1)}(x) &= \frac{1}{3}F_{abc}\phi'_b(x)\phi_c(x), \end{aligned}$$

$$
\begin{aligned}
M_a^{(2)}(x) &= -\frac{1}{36}(F_{ace}F_{bde} + F_{ade}F_{bce})\phi_b'(x)\phi_c(x)\phi_d(x) \\
&\quad + \left(\frac{1}{36}F_{abe}F_{cde} + \frac{1}{4}E_{ac,bd}\right)\phi_b'(x)\int^x dy\phi_c(y)\phi_d'(y). \qquad (50)
\end{aligned}
$$

This can easily be extended to operators. Define now quantum free fields by OPE's

$$
\phi_a(x)\phi_b(y) \cong -\frac{1}{2\pi}\log(x-y)\delta_{ab} \qquad (51)
$$

To avoid singular expressions we work with normal ordered fields, for example,

$$
\begin{aligned}
\phi_a(x)\phi_b(y)\phi_c(z) &= :\phi_a(x)\phi_b(y)\phi_c(z): -\frac{1}{2\pi}\phi_c(z)\log(x-y)\delta_{ab} \\
&\quad - \frac{1}{2\pi}\phi_b(y)\log(x-z)\delta_{ac} - \frac{1}{2\pi}\phi_a(x)\log(y-z)\delta_{bc} \qquad (52)
\end{aligned}
$$

In order to satisfy the OPE algebra given before (eq.(28)), we simply take over the classical expression, replacing products of fields by normal ordered products. It turns out that this almost works; however, additional terms are necessary to make it work. The final result is

$$
\begin{aligned}
M_a^{(0)}(x) &= \phi_a'(x), \\
M_a^{(1)}(x) &= -\frac{1}{3}F_{abc}:\phi_b'(x)\phi_c(x):, \\
M_a^{(2)}(x) &= -\left(\frac{1}{18\pi}F_{acd}F_{bcd} + \frac{1}{4\pi}E_{ab,cc}\right)\phi_b'(x) \\
&\quad + \left(\frac{1}{36\pi}F_{acd}F_{bcd} + \frac{1}{4\pi}E_{ab,cc}\right)\int^x \frac{dy}{y-x}(\phi_b'(y) - \phi_b'(x)) \\
&\quad -\frac{1}{36}(F_{ace}F_{bde} + F_{ade}F_{bce}):\phi_b'(x)\phi_c(x)\phi_d(x): \\
&\quad + \left(\frac{1}{36}F_{abe}F_{cde} + \frac{1}{4}E_{ac,bd}\right):\phi_b'(x)\int^x dy\phi_c(y)\phi_d'(y): . \qquad (53)
\end{aligned}
$$

Using these expressions we can construct the stress tensor, which to second order, is quadratic in the free fields and is given by

$$
\begin{aligned}
\frac{\alpha^2}{\pi}T(x) &= :\phi_a'(x)\phi_a'(x): \\
&\quad -\frac{\alpha^2}{24\pi}\left(F_{acd}F_{bcd} + 3E_{bc,ac}\right)\left(:\phi_a'(x)\phi_b'(x): + :\phi_a(x)\phi_b''(x):\right) \qquad (54)
\end{aligned}
$$

It can also be directly checked that, at least to second order, this construction in terms of free fields yields the Virasoro algebra with the correct central charge.

5. Conclusions

The main result of this paper is eq.(18), the condition on the coupling constants derived by imposing conformal invariance on the generalized Thirring model. This result, valid in the large N limit, is obtained by using two different approaches, the background field method and the operator method. It corrects and extends the results obtained in [1], bringing them in agreement with those of [2]. Among the problems that are still left open is the contribution of the higher order corrections in $1/N$ to both the condition for conformal invariance (eq.(18)), and to the operator algebra (eq.(28)).

We have also tried tried to shed some light on the structure of the operator algebra mentioned above by expressing it in terms of free fields and free currents. We have found some simplification in the expression for the stress tensor, but still the result could not be reproduced by any well-known construction. It appears very likely that we have a completely new conformal model

Acknowledgements

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139.

Appendix

In this appendix, we fill the gaps in the evaluation of $S^{(2)}[\phi_{clas.}]$ done in section 2. As explained there, we want to expand the action $S[\phi]$ around $S^{(0)}[\phi_{clas.}]$, the classical action. To do this, parametrize the fields g and h by:

$$g = g(\phi), \quad h = h(\overline{\phi}), \tag{55}$$

where $\phi(x)$ stands for $\phi^\alpha(x)$. The ϕ^α's are the coordinates in the group manifold where g takes values, and $x \equiv x^\mu$ are coordinates in Minkowski 2-space. The classical fields $\phi^\alpha_{clas.}$ are defined by $g_{clas.} = g(\phi_{clas.})$, and similarly for $\overline{\phi}^\alpha$. From

now on, unless otherwise stated, ϕ stands either for ϕ and $\overline{\phi}$.

Using the vielbeins $E^a_\alpha(\phi)$ and $\overline{E}^a_\alpha(\overline{\phi})$ defined in section 2, the source terms can be written as

$$\begin{aligned} Tr\,(K^-(ig^{-1}\partial_+ g)) &= Tr(K^-\lambda_a)E^a_\alpha\partial_+\phi^\alpha \equiv K^-_a E^a_\alpha\partial_+\phi^\alpha, \\ Tr\,(K^+(ih^{-1}\partial_- h)) &= Tr(K^+\lambda_a)\overline{E}^a_\alpha\partial_-\overline{\phi}^\alpha \equiv K^+_a\overline{E}^a_\alpha\partial_-\overline{\phi}^\alpha, \end{aligned} \tag{56}$$

and the action becomes

$$\begin{aligned} S &= W(g) + W(h^{-1}) - \frac{N}{2\pi}\int d^2x G_{ab}E^a_\alpha\overline{E}^b_\beta\partial_+\phi^\alpha\partial_-\overline{\phi}^\beta \\ &\quad + \frac{N}{2\pi}\int d^2x K^+_a\overline{E}^a_\alpha\partial_-\overline{\phi}^\alpha + \frac{N}{2\pi}\int d^2x K^-_a E^a_\alpha\partial_+\phi^\alpha. \end{aligned} \tag{57}$$

Now we expand this action $S[\phi]$ around the classical action $S^{(0)} = S[\phi_{clas.}]$ treating $K_{+,-}$ as classical sources, which can then be written in terms of $\phi_{clas.}$. To expand the action, let

$$\phi(x) \longrightarrow \phi(x,s)$$

so that $\phi(x,s=0) = \phi_{clas.}(x)$ and $\phi(x,s=1) = \phi(x)$ and define

$$\xi^\alpha = \frac{d}{ds}\phi^\alpha(x,s)\,|_{s=0} \tag{58}$$

The $\xi^\alpha(x)$'s span the tangent space at $\phi_{clas.}(x)$ and satisfy the geodesic equation

$$\frac{D}{Ds}\xi^\alpha = \frac{d}{ds}\xi^\alpha + \Gamma^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma = 0, \tag{59}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}E^\alpha_a\left(\frac{\partial}{\partial\phi^\beta}E^a_\gamma + \frac{\partial}{\partial\phi^\gamma}E^a_\beta\right) \tag{60}$$

is the Christoffel symbol. In general the $\xi^\alpha(x)$'s don't form an orthonormal basis but we can define new vectors

$$\zeta^a = E^a_\alpha\xi^\alpha \tag{61}$$

that span the tangent space at $\phi_{clas.}(x)$ and form an orthonormal basis. The inverse relation is given by

$$\xi^\alpha = E^\alpha_a\zeta^a \tag{62}$$

where $E_a^\alpha(\phi)$ is the inverse vielbein defined by

$$E_\alpha^a E_b^\alpha = \delta_{ab}. \tag{63}$$

Note that in the $\{\zeta^a\}$ basis the metric is δ_{ab} and so there is no diference between upper and lower indices, while in the $\{\xi^\alpha\}$ basis the metric is $g_{\alpha\beta} = E_\alpha^a E_{a\beta}$ and so an upper index is different from a lower index. The action can then be expanded as

$$S[\phi(x,s)]\,|_{s=1} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d}{ds}\right)^n S[\phi]\,|_{s=0} \equiv \sum_{n=0}^{\infty} S^{(n)}[\phi_{clas.},\zeta], \tag{64}$$

and keeping terms to second order in ζ's we have (from now on $\phi(x)$ stands for $\phi_{clas.}(x)$):

$$\begin{aligned}
S^{(0)}[\phi,\zeta] &= S[\phi], \\
S^{(1)}[\phi,\zeta] &= 0 \quad \text{if equations of motion hold} \\
S^{(2)}[\phi,\zeta] &= \frac{N}{8\pi}\int d^2x \left(\zeta^a(-\delta_{ab}\Box + A_{ab}^\mu\partial_\mu + D_{ab})\zeta^b \right. \\
&\quad +\overline{\zeta}^a(-\delta_{ab}\Box + \overline{A}_{ab}^\mu\partial_\mu + \overline{D}_{ab})\overline{\zeta}^b \\
&\quad +\zeta^a(2G_{ab}\Box + B_{ab}^\mu\partial_\mu + C_{ab})\overline{\zeta}^b \\
&\quad \left. +\overline{\zeta}^a(2G_{ba}\Box + \overline{B}_{ab}^\mu\partial_\mu + C_{ba})\zeta^b\right),
\end{aligned} \tag{65}$$

where

$$\begin{aligned}
A^\mu &= 2f_n E_\alpha^n \partial^\mu\phi^\alpha, \\
\overline{A}^\mu &= 2f_n \overline{E}_\alpha^n \partial^\mu\overline{\phi}^\alpha, \\
B^\mu &= -4f_n G E_\alpha^n(\eta^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\nu\phi^\alpha, \\
\overline{B}^\mu &= -4f_n G^T \overline{E}_\alpha^n(\eta^{\mu\nu} + \varepsilon^{\mu\nu})\partial_\nu\overline{\phi}^\alpha, \\
C &= 2f_n G f_m E_\alpha^n \overline{E}_\beta^m(\eta^{\mu\nu} + \varepsilon^{\mu\nu})\partial_\mu\phi^\alpha\partial_\nu\overline{\phi}^\beta, \\
D &= -f_m f_n E_\alpha^m E_\beta^n \partial_\mu\phi^\alpha\partial^\mu\phi^\beta, \\
\overline{D} &= -f_m f_n \overline{E}_\alpha^m \overline{E}_\beta^n \partial_\mu\overline{\phi}^\alpha\partial^\mu\overline{\phi}^\beta,
\end{aligned} \tag{66}$$

and the matrix f_n is defined by $(f_n)_{ab} = f_{nab}$. To compute the divergent counter term, we write $S^{(2)}$ in the form

$$S^{(2)} = \frac{N}{8\pi} \int d^2x Z^T (R\Box + P^\mu \partial_\mu + Q) Z \tag{67}$$

where

$$Z \equiv \begin{bmatrix} \zeta \\ \overline{\zeta} \end{bmatrix} \quad \text{and} \quad \zeta = \begin{bmatrix} \zeta^1 \\ \vdots \\ \zeta^n \end{bmatrix} \quad \overline{\zeta} = \begin{bmatrix} \overline{\zeta}^1 \\ \vdots \\ \overline{\zeta}^n \end{bmatrix} \tag{68}$$

and the matrices R, P^μ and Q are

$$R = \begin{bmatrix} -I & 2G \\ 2G^T & -I \end{bmatrix} \quad P^\mu = \begin{bmatrix} A^\mu & B^\mu \\ \overline{B}^\mu & \overline{A}^\mu \end{bmatrix} \quad Q = \begin{bmatrix} D & C \\ C^T & \overline{D} \end{bmatrix} \tag{69}$$

After integrating over Z, we get

$$\begin{aligned} S^{(2)} &\cong -\frac{1}{2} Tr \log(R\Box + P^\mu \partial_\mu + Q) \\ &\cong -\frac{1}{2} Tr \left(R^{-1} \frac{1}{\Box} Q - \frac{1}{2} R^{-1} \frac{1}{\Box} P^\mu \partial_\mu R^{-1} \frac{1}{\Box} P^\nu \partial_\nu \right) \\ &\cong \int \frac{d^2p}{p^2 - m^2} \int d^2x O(x), \end{aligned} \tag{70}$$

where $O(x)$ was defined in section 2.

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