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Convergent sequences of perturbative approximations for the anharmonic oscillator II. Compact time approach*

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Abstract

We present an alternative pathway in the application of the variation improvement of ordinary perturbation theory exposed in [1] which can preserve the internal symmetries of a model by means of a time compactification.

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1 Introduction.

In the preceding paper [2], we presented a study of large orders of a variational method which provides convergent sequences of approximations for the quantum anharmonic oscillator. In this method, we introduced a mass term (with a mass ω) in the free Lagrangian and subtracted it in the interaction part. Then, we optimized at any order in terms of this variational parameter ω to get approximate values of different quantities.

We have in perspective the use of such a method in quantum field theory. For fermions like those of the Gross-Neveu model [3], or QCD quarks, introduction of a mass term causes *a priori* no problem, but for gluons it would break gauge invariance, and there is no obvious modification of the momentum dependent part of the gluon propagator and the structure of the ultraviolet infinities of the theory. However, a space compactification might provide some good approximation of quark confinement, like in the M.I.T. bag model [4].

In the M.I.T. bag model, the bag is a fundamental ingredient of the theory, which hence is not QCD, while we would advocate here an approach in which it would be added to the free gluon and quark action in the form of a modification of their propagators, the subtraction of this modification being done in perturbation theory. While formally when going to all orders one would have done nothing, and still be dealing with QCD, a finite given order would have to be optimized in the size of the bag as the closest one could get to the infinite order case. The advantage would be that different physics would be involved, hopefully closer to that of the full theory, with confinement, etc...

Compactification as an intermediate step in calculations in infrared divergent field theories has already been considered by several authors [5] , with reasonable success. These authors usually treat the theory in a spatial box of length l, and extrapolate the results to $l \to \infty$. We are here advocating a different, if related, strategy, in which the box is in some sense subtracted perturbatively, and the calculation at a given order in perturbation theory optimized with respect to the size of the box, this being supposed to mimick as well as it can the true all-orders theory, where formally there is no dependence

at all on the size of the box.

In this paper, we try out this idea on the case of the anharmonic oscillator, choosing an action with compactified time as our starting point. We find a configuration for the optima and the approximate values of the ground state energy very similar to those of the harmonic approach of this problem [2]. In the second section, we study the interaction part of the action to be used in order to formally leave the total action of a pure anharmonic oscillator unchanged. We make a choice which allows relatively simple perturbative calculations starting from a theory governed by a compactified action. The perturbation theory for such a compactified action is considered in the third section. In the fourth section, we establish the variational-perturbative expansion of the mean value of the Hamiltonian and optimize it in terms of our variational parameter (the size of the box) up to order 16. We interpret the configuration of the extrema using the large order behavior of the expansion, which is qualitatively much the same as in the harmonic approach of the previous paper, to which we compare our results. They turn out to be less accurate numerically for similar orders in the approximation, which reflects the fact that approximating an anharmonic potential by a compact time dimension is much worse than approximating it by a harmonic potential. Nevertheless, the convergence properties are qualitatively very similar.

2 Time compactification of a pure anharmonic oscillator.

The Euclidean action for a pure anharmonic oscillator writes:

$$S = \int_{-\infty}^{\infty} \left[\frac{1}{2} (\partial_t \phi)^2 + \frac{\lambda}{4} \phi^4 \right] dt \, . \tag{1}$$

Severe infrared divergences prevent any perturbation in powers of λ. A way to tame these divergences is of course to introduce an $\omega^2 \phi^2$ term as was done in the previous paper. Here, we would instead like to replace in some sense the free term of the action of equation (1) by the compactified free action $S^0(\tau)$:

$$S^0(\tau) \;=\; \int_{-\frac{\pi}{2}\tau}^{\frac{\pi}{2}\tau} \frac{1}{2}\,(\partial_t\phi)^2 \;dt\;, \tag{2}$$

and rewrite the total action S as:

$$S = S^0(\tau) + S^I(\tau). \tag{3}$$

Then, we will be able to perform a perturbative expansion of any quantity in powers of the interaction term $S^I(\tau)$ around the free term $S^0(\tau)$. To do so, let us consider the Euclidean action S_T of a time compactified pure anharmonic oscillator

$$S_T = \int_{-\frac{\pi}{2}T}^{\frac{\pi}{2}T} [\; \frac{1}{2}\,(\partial_t\phi)^2 \;+\; \frac{\lambda}{4}\phi^4 \;]\; dt. \tag{4}$$

With S_T, one can compute in perturbation theory in powers of λ at any given T (provided one uses appropriate boundary conditions, see below). To see how S and S_T may be related, we perform the following change of variable:

$$x = T\tan\frac{t}{T} \;\;;\;\; dx = (1+\frac{x^2}{T^2})\; dt, \tag{5}$$

under which the action S_T becomes:

$$S_T = \int_{-\infty}^{\infty} [\; \frac{1}{2}(1+\frac{x^2}{T^2})\,(\partial_x\phi)^2 \;+\; \frac{\lambda}{4}\frac{1}{(1+\frac{x^2}{T^2})}\phi^4 \;]\; dx. \tag{6}$$

On the other hand, we can rewrite the action S as

$$S \;=\; \int_{-\infty}^{\infty} \Big[\; \frac{1}{2}\,(1+\frac{x^2}{\tau^2}\,(1-\varepsilon))\,(\partial_x\phi)^2 + \frac{\lambda}{4}\frac{1}{1+\frac{x^2}{\tau^2}\,(1-\varepsilon)}\;\phi^4 \;\Big]\; dx, \tag{7}$$

where, strictly speaking, $\varepsilon = 1$. Comparing (7) with the expression of the compactified action (6), we see how to relate S and S_T: following the procedure of the previous paper, one computes the quantity of interest to some order in λ using S_T, sets $T = \tau(1-\varepsilon)^{-1/2}$, expands in powers of ε to the same total order as in λ, sets $\varepsilon = 1$ and optimizes the result with respect to τ. This should provide us with an approximate value of the same quantity computed with S.

In more detail, we can consider the action (7) as:

$$S \;=\; S^0(\tau) \;+\; S^I(\tau)$$

with $S^0(\tau)$ given by

$$S^0(\tau) \;=\; \int_{-\infty}^{\infty} \frac{1}{2}\left(1+\frac{x^2}{\tau^2}\right)\,(\partial_x\phi)^2\;dx \tag{8}$$

and

$$S^I(\tau) \;=\; \int_{-\infty}^{\infty}\Big[\,-\frac{1}{2}\,\varepsilon\,\frac{x^2}{\tau^2}\,(\partial_x\phi)^2 \;+\; \frac{\lambda}{4}\frac{1}{1+\frac{x^2}{\tau^2}\,(1-\varepsilon)}\,\phi^4\,\Big]\;dx. \tag{9}$$

The pth order in perturbation will contain all the terms proportional to $\lambda^n\varepsilon^m$ with $n+m\leq p$. Note that in this approach, we do not take into account the complete quartic term of (4) but that we reconstruct it more and more accurately as the order of perturbation increases. This feature does not seem to us to carry any pathology, but simply follows from the choice of S_T as our compactified action and the physics it contains.

This new choice for a free kinetic term parameterized by τ in S will give rise for every finite order p of perturbation to an explicit dependence on τ. Because the total action S remains unchanged at $\varepsilon = 1$, one expects this parameter τ to become irrelevant at infinite order of perturbation. As already explained, this suggests to fix this parameter by looking for the regions where the result to order p is stationary in τ.

Let us now discuss briefly the boundary conditions in the action of equation (4). Periodic conditions $\phi_{-T} = \phi_T$ are unsuitable for perturbation theory, as the Gaussian kernel has an isolated zero eigenvalue. Similarly for Neumann boundary conditions. Hence, we shall use Dirichlet boundary conditions in which

$$\phi_{\pm T} = 0. \tag{10}$$

which preserves the discrete $\phi \leftrightarrow -\phi$ symmetry. With this choice, all infrared problems disappear at finite T.

In order to compare the results obtained by this compact time method to those of the harmonic one [2], we would like to consider the ground state energy. However, the method we described above implies calculation using a compactified action which is therefore not invariant under translation in time. Then, the ground state energy is not a meaningful concept and we choose instead

to consider the mean value of the Hamiltonian operator at some time t_0:

$$< \hat{H}_{t_0} > \tag{11}$$

The variation over the size of the box T provides some new approach of the limit $T \to \infty$ (substituting $\tau \to T(1-\varepsilon)^{-1/2}$, expanding in powers of ε and taking $\varepsilon = 1$). In the limit of infinite order of perturbation, the box should disappear and we should recover the value of the ground state energy when calculating $< \hat{H}_{t_0} >$.

3 Perturbative expansion for the compactified action.

In the Euclidean path integral representation, the mean value of the Hamiltonian operator at time t_0 in the box writes:

$$< \hat{H}_{t_0} >= \frac{< 0; T \mid \hat{H}_{t_0} \mid 0; -T >}{< 0; T \mid 0; -T >}; \tag{12}$$

where

$$\begin{aligned}
< 0; T \mid \hat{H}_{t_0} \mid 0; -T > &= \int_{\phi_{\pm T}=0} D\phi \;\; H_{t_0} \; e^{-S_T[\phi]} \; , \\
< 0; T \mid 0; -T > &= \int_{\phi_{\pm T}=0} D\phi \; e^{-S_T[\phi]} \; , \\
-S_T[\phi] &= -\int_{-T}^{T} \Big[\frac{1}{2}(\partial_t \phi)_t^2 + \frac{\lambda}{4}\phi_t^4\Big] \, dt \; , \\
\text{and} \qquad H_{t_0} &= -\frac{1}{2}(\partial_t \phi)_{t_0}^2 + \frac{\lambda}{4}\phi_{t_0}^4 \; .
\end{aligned}$$

3.1 The partition function and the mean value of the Hamiltonian.

Dividing by the partition function $Z(T)$ in (12) cancels all the disconnected Feynman diagrams. We rewrite it:

$$Z(T) =< 0; T \mid 0; -T >=< 0 \mid \hat{U}_{2T} \mid 0 >$$

Here, in the Schrödinger representation, the evolution operator $\hat{U}_t$ of a quantum system described by $\hat{H}$ appears. It satisfies the following equations:

$$-\frac{d}{dt}\hat{U}_t = \hat{H}\,\hat{U}_t \quad \text{and} \quad \hat{U}_0 = 1. \tag{13}$$

Then, taking the derivative of the partition function with respect to the size of the box, we get:

$$\begin{aligned}\frac{d}{dT}Z(T) &= <0\mid \frac{d}{dT}\hat{U}_{2T}\mid 0> \\ &= -2\ <0\mid \hat{H}\hat{U}_{2T}\mid 0> \\ &= -2\ <0;T\mid \hat{H}_t\mid 0;-T>,\end{aligned}$$

so that

$$\frac{d}{dT}Z(T) = -2\ <-\frac{1}{2}(\hat{\partial_t\phi})_t^2 + \frac{\lambda}{4}\hat{\phi}_t^4>;\ \ \forall\, t\in[-T;T]. \tag{14}$$

Thus, the mean value of the Hamiltonian operator does not depend of the time when it is taken in the box. Moreover, its perturbative expansion in powers of T is obtained from the one of $Z(T)$. However, the variational procedure implies that at the order p in perturbation, the nth order contribution which is proportional to $\lambda^n\tau^\nu(1-\varepsilon)^{-\nu/2}$ has to be expanded in powers of ε up to order $p-n$. This n^{th} power of λ does not take into account the λ present in front of the quartic part of the Hamiltonian. But the two sides of equation (14) identify themselves in perturbation, mixing that λ with the perturbative one *. So, we have to know kinetic and potential contributions for each order in the expansion of $dZ(T)/dT$. Nevertheless, we will use this equality to compute at any order the value of $<(\hat{\partial_t\phi})_t^2>$ in term of $<\hat{\phi}_t^4>$ and $Z(T)$. We decided to compute the mean value of the Hamiltonian where its kinetic and potential contributions do not much vary and are the least dependent on the size of the box, *i.e.* in the middle of the box at $t=0$.

*For example, in the perturbative expansion of $dZ(T)/dT$, the zeroth order contribution of $<\hat{\phi}_t^4>$ is mixed with the first order of $<(\hat{\partial_t\phi})_t^2>$.

3.2 The perturbative calculation at any order.

In order to develop a feeling on how the variational method works, we have to reach high orders of perturbation, *i.e.* calculate a large number of coefficients. The Feynman diagrams approach implies the calculation of an increasing number of graphs. Instead, we prefer a perturbative expansion of the evolution operator closer to the usual Rayleigh-Ritz method. Expanding $\hat{U}_t$ in powers of the coupling constant λ, we get the following expansions for $Z(T)$, $< \hat{\phi}_0^4 >$ and $< (\hat{\partial_t \phi})_0^2 >$:

$$
\begin{aligned}
< 0; T|0; -T > &= \frac{1}{\sqrt{T}} \sum_{n=0}^{\infty} (-\lambda/4)^n T^{3n} \ \mathbf{Z}_n, \\
< 0; T|\hat{\phi}_0^4|0; -T > &= \frac{1}{\sqrt{T}} \sum_{n=0}^{\infty} (-\lambda/4)^n T^{3n+2} \ \mathbf{Q}_n, \\
< 0; T|(\hat{\partial_t \phi})_0^2|0; -T > &= \frac{1}{\sqrt{T}} \sum_{n=0}^{\infty} (-\lambda/4)^n T^{3n-1} \ \mathbf{P}_n,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbf{Z}_n &= \int_{1>t_n>\ldots t_1>-1} dt_1 \ldots dt_n < 0; 1|\hat{\phi}^4{}_{t_n} \ldots \hat{\phi}^4_{t_1}|0; -1 >_0 , \\
\mathbf{Q}_n &= \sum_{p=0}^{n} \int_{-1<t_0<\ldots<t_{p-1}<0} \int_{0<t_{p+1}<\ldots<t_n<1} dt_0 \ldots \check{dt}_p \ldots dt_n \\
&\quad \times < 0; 1|\hat{\phi}^4_{t_n} \ldots \hat{\phi}^4_{t_{p+1}} \ \hat{\phi}^4_0 \ \hat{\phi}^4_{t_{p-1}} \ldots \hat{\phi}^4_{t_0}|0; -1 >_0 , \\
\mathbf{P}_n &= \sum_{p=0}^{n} \int_{-1<t_0<\ldots<t_{p-1}<0} \int_{0<t_{p+1}<\ldots<t_n<1} dt_0 \ldots \check{dt}_p \ldots dt_n \\
&\quad \times < 0; 1|\hat{\phi}^4_{t_n} \ldots \hat{\phi}^4_{t_{p+1}} \ (\hat{\partial_t \phi})_0^2 \ \hat{\phi}^4_{t_{p-1}} \ldots \hat{\phi}^4_{t_0}|0; -1 >_0 \ .
\end{aligned}
$$

The zero subscripts appended to ket vectors mean that the corresponding expectation values are computed using free propagators with the boundary conditions discussed in section 2.

In the above expressions, we have extracted the T dependences of the coefficients $\mathbf{Z}_n$, $\mathbf{Q}_n$ and $\mathbf{P}_n$. In appendix, a systematic calculation of the $\mathbf{Z}_n$'s and $\mathbf{Q}_n$'s is performed with the help of a recursive formula. The values of the $\mathbf{P}_n$'s

are deduced from these using (14), which gives:

$$\mathbf{P}_n = (3n - 1/2)\mathbf{Z}_n - 2\mathbf{Q}_{n-1}.$$

After performing the division by $Z(T)$ order by order in perturbation, the expansion of the mean value of the Hamiltonian writes:

$$< \hat{H}_0 >_T^{(p)} = \sum_{n=0}^{p} [-\frac{1}{2}(\frac{-\lambda}{4})^n \, \mathbf{P}_n^c \, T^{3n-1} \; + \; \frac{\lambda}{4}(\frac{-\lambda}{4})^n \, \mathbf{Q}_n^c \, T^{3n+2}], \qquad (15)$$

where the c superscripts of $\mathbf{P}$ and $\mathbf{Q}$ refer to the contribution of the connected diagrams only. We have computed these coefficients up to order 16, and we give here the first few of them:

$$\begin{array}{lllllll}
\mathbf{Q}_0^c & = & \frac{3}{4} & & \mathbf{P}_0^c & = & \frac{-1}{2} \\
\mathbf{Q}_1^c & = & \frac{33}{10} & & \mathbf{P}_1^c & = & \frac{9}{10} \\
\mathbf{Q}_2^c & = & \frac{15661}{700} & & \mathbf{P}_2^c & = & \frac{349}{175} \\
\mathbf{Q}_3^c & = & \frac{3798833}{19250} & & \mathbf{P}_3^c & = & \frac{2363729}{250250} \\
\mathbf{Q}_4^c & = & \frac{634428694707}{297797500} & & \mathbf{P}_4^c & = & \frac{14511295339}{223348125} \\
\mathbf{Q}_5^c & = & \frac{10113181264708}{372246875} & & \mathbf{P}_5^c & = & \frac{8165883862419}{14145381250}.
\end{array} \qquad (16)$$

4 The results and their interpretation.

Starting from expansion (15) at order p, we perform the replacement $T^\nu \rightarrow \tau^\nu(1-\varepsilon)^{-\nu/2}$. Expanding in powers of ε up to order $p-n$, we get the required expansion, which then has to be optimized with respect to our variational parameter τ:

$$E_0^{(p)}(\tau) = \sum_{n=0}^{p} [-\frac{1}{2}(\frac{-\lambda}{4})^n \, \mathbf{P}_n^c \, \frac{\Gamma(p+n/2+1/2)}{\Gamma(3n/2+1/2)\,\Gamma(p-n+1)} \, \tau^{3n-1}$$
$$+\frac{\lambda}{4}(\frac{-\lambda}{4})^n \, \mathbf{Q}_n^c \, \frac{\Gamma(p+n/2+2)}{\Gamma(3n/2+2)\,\Gamma(p-n+1)} \, \tau^{3n+2}]. \qquad (17)$$

The values of this expression at its optima in τ will serve as variational estimates of the value found in [6]:

$$E_0^{exact} = \lambda^{1/3} 0.420805\ldots \qquad (18)$$

Such a polynomial (17) has a number of optima which increases with the order of perturbation p. Most of them are complex and give a small imaginary part to the estimated value of the energy as in the harmonic approach [2]. All of these estimated values exhibit the expected $\lambda^{1/3}$ factor. We take $\lambda = 1$ and compute all the optima up to order 16. Results and comments follow in the next section.

4.1 Solutions of our variational problem.

$E_0^{(p)}(\tau)$ is plotted in figures 1 and 2 versus the variational parameter for several even and odd orders.

Since τ is supposed to be irrelevant at infinite order of perturbation, as expected, as p increases, the curves $E_0^{(p)}(\tau)$ flatten around the exact value over an increasingly large range of values of τ as p increases.

We can follow on these figures the evolution of the optima with the orders of perturbation. The first minimum to appear provides a rather poor estimate ($E_0 \simeq 0.34\ldots$) but moves to the left of the figure as the order increases, and is replaced by a much more accurate maximum ($E_0 \simeq 0.43\ldots$). It is clear that a third extremum, a minimum, appears with a still more accurate value $E_0 \simeq 0.417\ldots$, and, from the trend of the odd orders, another, presumably even better maximum would appear if one would push the calculation to a couple more orders.

The polynomial $E_0^{(p)}$ to be optimized has an increasing number of stationary points as the order of perturbation increases. Plotting real optima versus the order of perturbation up to $p = 16$ (figure 3) reveals how they arrange themselves: Optimal values of τ belong to families characterized by the straight lines in figure 3 with slopes decreasing proportionally to the inverse of the square root of the order p. Such a family is created at first order, followed by another one at fourth order. One reasonably expects other families to appear at orders greater than 16, providing some even more accurate values. Figure 4 plots the location in $1/\tau^2$ of the all the optima up to order 16 in the first quadrant of the complex plane. One thus easily follows each family up to order 16.

Such a regular behavior of all these optima is explained in the next subsection by considering the asymptotic expression of $E_0^{(p)}(\tau)$ for large p.

4.2 The large order behavior.

For large p, using the Stirling formula, we find:

$$E_0^{(p)}(\tau) \simeq \sum_{n=0}[-\frac{1}{2}(\frac{-1}{4})^n \mathbf{P}_n^c \frac{1}{\Gamma(3n/2+1/2)} (\tau\, p^{1/2})^{3n-1} + \frac{1}{4}(\frac{-1}{4})^n \mathbf{Q}_n^c \frac{1}{\Gamma(3n/2+2)} (\tau\, p^{1/2})^{3n+2}], \tag{19}$$

where we have taken $\lambda = 1$. So, at large orders, the function to be optimized becomes a series in powers of $\tau\, p^{1/2}$. We voluntarily omitted its upper bound in n. Using our knowledge of the first coefficients, we establish empirically the following asymptotic behaviors:

$$\mathbf{P}_n^c \sim \frac{9}{20}\, 2^n \Gamma(n),$$

$$\mathbf{Q}_n^c \sim \frac{3}{4}\, 2^n \Gamma(n+2).$$

With these behaviors, the nth term of the series decreases like $1/\sqrt{n!}$ and the series has an infinite radius of convergence. For large orders p, the search for stationary values of $E_0^{(p)}(\tau)$ in τ is nothing but the search for those of the series (19) in $Y = 1/(\tau^2 p)$. Then, real extremal values of Y are the slopes of the families on figure 3. Using the optima already calculated, we can then extrapolate the τ optima and the corresponding values of the energy for large orders.

The optima of the real family appearing at first order behave asymptotically as:

$$1/(\tau^{opt})^2 \simeq 1.1612 + 2.31156\, p + \frac{0.025}{p}. \tag{20}$$

The corresponding value for the energy tends toward a rather bad approximation from above:

$$E_0^{opt} \simeq 0.335 + \frac{0.038}{p}. \tag{21}$$

For the second real family which appears at the fourth order

$$1/(\tau^{opt})^2 \simeq 0.4651 + 0.3578\, p + \frac{0.1245}{p}, \tag{22}$$

the corresponding energy tends towards a better value from below:

$$E_0^{opt} \simeq 0.434 - \frac{0.014}{p}. \tag{23}$$

The asymptotic value provided by a family is more accurate when the family appears later in perturbation. We also verify that the slopes tend to decrease. This corresponds to the successive extrema of (22) appearing at smaller and smaller values of Y, corresponding to those of the function $E_0^{as}(Y)$

$$E_0^{as}(Y) = \sum_{n=0}^{\infty}[-\frac{1}{2}(\frac{-1}{4})^n \mathbf{P}_n^c \frac{1}{\Gamma(3n/2+1/2)} Y^{-(3n-1)/2} + \frac{1}{4}(\frac{-1}{4})^n \mathbf{Q}_n^c \frac{1}{\Gamma(3n/2+2)} Y^{-(3n+2)/2}]. \tag{24}$$

Using the available coefficients up to order 16 is sufficient to see that this function has a minimum:

$$\bar{Y} \simeq 2.2629 \quad ; \quad E_0^{as}(\bar{Y}) \simeq 0.3315.$$

One recognizes the slope and the asymptotic energy of the first real family within a few percents. Using the empirical asymptotic behaviors of the coefficients $\mathbf{P}$ and $\mathbf{Q}$ would extend the range of Y where one could compute E_0^{as}, revealing the asymptotic properties of the other families.

5 Conclusion

The results presented in this paper are qualitatively very similar to those obtained in the harmonic approach [2]: In the same way as in the harmonic approach, the optima arrange themselves in families which are understood using a large order behavior. The set of these families provide a sequence of approximate values which converges to the exact one. However, for a given perturbative

order, our estimated values were more accurate in the harmonic approach. Indeed, for the first family, we obtain an estimated value of the ground state energy with a precision of 2.10^{-5} in the harmonic case and of 2.10^{-1} in the present approach.This is presumably due to several causes: in the compact time case, there is no time translation invariance, hence no Hamiltonian, and we only compute the expectation value of an operator which becomes the Hamiltonian in the large time box limit. Furthermore, as explained in section 2, our perturbative expansions do not take into account the complete ϕ^4 interaction term but only reconstruct it more and more accurately as the perturbative order increases. Finally, a compact Euclidean time is clearly quite far from the physics of an anharmonic oscillator, much farther than a harmonic approximation. One may even consider it quite remarkable that the procedure nevertheless seems to converge in the same manner, albeit much more slowly. One could thus contemplate using such a compactification, which could involve only the spatial coordinates or both space and time coordinates as a gauge invariant variational approach to gauge theories.

6 Appendix: Calculation of coefficients.

6.1 Calculation of the $\mathbf{Z}_n$'s

$$
\begin{aligned}
< 0; T|0; -T > &= \sum_{n=0}^{\infty} (\frac{-\lambda}{4})^n T^{3n} \mathbf{Z}_n \\
\mathbf{Z}_n &= \int_{-1<t_1<\ldots<t_n<1} dt_1 \ldots dt_n < 0; 1|\hat{\phi}^4_{t_n} \ldots \hat{\phi}^4_{t_1}|0; -1 >_0
\end{aligned}
$$

For each time $t_1, t_2, \ldots, t_n$, one uses the closure relation in position space. The $\hat{\phi}$ operator is diagonal in this basis. One thus rewrites the integrand in the following way:

$$
\begin{aligned}
< 0; 1|\hat{\phi}^4_{t_n} \ldots \hat{\phi}^4_{t_1}|0; -1 >_0 = \int_{-\infty}^{\infty} dx_n \ldots dx_1 \ < 0; 1 \mid x_n; t_n >_0 \ x_n^4 \\
\times < x_n; t_n \mid \ldots \mid x_1; t_1 >_0 \ x_1^4 \ < x_1; t_1 \mid 0; -1 >_0 .
\end{aligned}
$$

For $t' > t$, the free propagator writes:

$$< y; t'|x; t >= \frac{1}{\sqrt{2\pi(t'-t)}} \exp -\frac{(y-x)^2}{2(t'-t)}.$$

This product of expectation values of operators is a Gaussian function in terms of $x_1, x_2, \ldots, x_n$.One thus performs integrals over all positions simultaneously to obtain:

$$\begin{aligned} < 0; 1|\hat{\phi}^4_{t_n} \ldots \hat{\phi}^4_{t_1}|0; -1 >_0 &= \frac{1}{2\sqrt{\pi}} \; d^4_{j_1} \ldots d^4_{j_n} \;\; \exp(\frac{1}{4} J^T . A . J) \; |_{J\equiv 0} \\ &= \frac{1}{2\sqrt{\pi}} \; d^4_{j_1} \ldots d^4_{j_n} \;\; \frac{1}{(2n)!} \frac{1}{4^{2n}} \; (J^T . A . J)^{2n} \; , \end{aligned}$$

where

$$\begin{aligned} A_{ij} &= (t_i + 1)(1 - t_j) \;\; \text{for} \;\; j \geq i \;\;\; (t_j \geq t_i), \\ \det A &= 2^{n-1} \prod_{i=0}^{n} (t_{i+1} - t_i). \end{aligned}$$

$d^4_{j_n}$ is the fourth derivative with respect to j_n. Now, we build a recursive procedure which performs the four derivatives with respect to j_n ($d^4_{j_n}$) together with the integral over $t_n \in [t_{n-1}; 1]$. To do so, one will use the following identity:

$$\int_{t'}^{1} dt (1+t)^{\alpha} (1-t)^{\beta} = \frac{1}{1+\alpha+\beta} \frac{1}{C^{\beta}_{\alpha+\beta}} \sum_{j=0}^{\alpha} 2^j \, C^{\beta}_{\alpha+\beta-j} \, (1+t')^{\alpha-j} (1-t')^{\beta+1},$$

with

$$C^p_n = \frac{n!}{p!(n-p)!}.$$

Let us define:

$$\begin{aligned} \Omega(n, m, \alpha, \beta) &= \int_{-1}^{1} dt_1 \ldots \int_{t_{N-1}}^{1} dt_N \; d^4_{j_1} \ldots d^4_{j_N} \; (\underbrace{J.A.J}_{N})^n \; (\underbrace{J.A}_{N})^m \; |_{J\equiv 0} \\ &\quad \times (1 + t_N)^{\alpha} (1 - t_N)^{\beta} \; |_{J\equiv 0} \end{aligned}$$

$$\begin{aligned} = 4! \int_{-1}^{1} dt_1 \ldots \int_{t_{N-2}}^{1} dt_{N-1} \; d^4_{j_1} \ldots d^4_{j_{N-1}} \; C^i_n \, (\sum_{k=0}^{i} C^{4-i-k}_m \, C^k_i \, 2^{i-k}) \; (\underbrace{J.A.J}_{N-1})^{n-i} \\ \times (\underbrace{J.A}_{N-1})^{m+2i-4} \; (1 + t_{N-1})^{4-i+\alpha} (1 - t_{N-1})^{i+\beta} \; |_{J\equiv 0} \end{aligned}$$

where:

$$\underbrace{J.A.J}_{N} = \sum_{i,k=1}^{N} j_i A_{ik} j_k \quad \text{and} \quad \underbrace{J.A}_{N} = \sum_{i=1}^{N} j_i\,(t_i+1).$$

So that Ω satisfies:

$$\begin{aligned}\Omega(n,m,\ \ \alpha\ \ ,\beta) &= 4!\sum_{i=0}^{4}\Lambda_i(n,m)\sum_{j=0}^{4+\alpha-i}\Theta_{i,j}(\alpha,\beta)\\ &\times\ \Omega(n-i,m+2i-4,4+\alpha-i-j,i+\beta+1) \qquad (25)\end{aligned}$$

$$\Lambda_i(n,m) = C_n^i \sum_{k=0}^{i} C_m^{4-i-k}\, C_i^k\, 2^{i-k}$$

$$\Theta_{i,j}(\alpha,\beta) = \frac{C_{4+\alpha+\beta-j}^{i+\beta}}{C_{4+\alpha+\beta}^{i+\beta}}\,\frac{2^j}{5+\alpha+\beta}$$

with the boundary conditions:

$$\Omega(n,m,\alpha,\beta)=0 \quad \text{as soon as } n \text{ or } m \text{ is negative,}$$

$$\Omega(0,0,\alpha,\beta) = \lim_{t\to-1}(1+t)^\alpha(1-t)^\beta = \delta(\alpha)\,2^\beta,$$

for the last integration to be performed between -1 and 1. The coefficients $\mathbf{Z}_n$ of the perturbative expansion are:

$$\mathbf{Z}_n = \frac{1}{2\sqrt{\pi}}\frac{1}{(2n)!}\frac{1}{4^{2n}}\,\Omega(2n,0,0,0).$$

6.2 Calculation of the $\mathbf{Q}_n$'s.

$$\begin{aligned}<0;T|\hat\phi_0^4|0;-T> &= \frac{1}{\sqrt{T}}\sum_{n=0}^{\infty}(\frac{-\lambda}{4})^n T^{3n+2}\mathbf{Q}_n\\ \mathbf{Q}_{n-1} &= \sum_{p=1}^{n}\mathbf{Q}_{n-1}^p\\ \mathbf{Q}_{n-1}^p &= \int_{-1<t_1<\ldots<t_{p-1}<0<t_{p+1}<\ldots<t_n<1} dt_1\ldots\check{dt}_p\ldots dt_n\\ &\quad\times <0;1|\hat\phi_{t_n}^4\ldots\hat\phi_{t_{p+1}}^4\hat\phi_0^4\hat\phi_{t_{p-1}}^4\ldots\hat\phi_{t_1}^4|0;-1>_0\end{aligned}$$

The preceding method must now be adapted to a slightly different domain of integration over the times (now $t_p = 0$). We will consider four different regions in order to build the recursive definition of a new function $\Omega_p(n, m, \alpha, \beta)$;

$$\underbrace{\int_{-1}^{0} dt_1 \ldots \int_{t_{p-2}}^{0} dt_{p-1}}_{(4)} \underbrace{\text{“}(t_p = 0)\text{”}}_{(3)} \underbrace{\int_{0}^{1} dt_{p+1}}_{(2)} \underbrace{\ldots \int_{tn-1}^{1} dt_n}_{(1)}$$

These four different domains of integration can enter the preceding frame.The quantity $(2n+m)/4 = n, \ldots, 1$ counts the remaining iterations to be performed and decreases by one unit at each step. One then builds the Ω_p's for each of these steps, writing:

$$n' = n - i \quad , \qquad m' = m + 2i - 4,$$
$$\alpha' = \alpha - i - j \quad , \qquad \beta' = i + \beta + 1.$$

Sums over i and j are respectively performed between 0 and 4, and 0 and $4+\alpha-i$.

1. When $(2n+m)/4$ is greater than $p+1$: Ω_p is just as Ω:

$$\Omega_p(n, m, \alpha, \beta) \;=\; \sum_i \sum_j \Lambda_i(n, m)\, \Theta_{i,j}(\alpha, \beta)\, \Omega_p(n', m', \alpha', \beta').$$

2. When $(2n+m)/4$ equals $p+1$: Here t_p, the lower bound of the integral is driven to zero:

$$(1+t_p)^{\alpha'}\,(1-t_p)^{\beta'}\,|_{t_p=0} \;=\; 1 \;=\; (1+t_p)^0\,(1-t_p)^0$$

Then, choosing $\alpha', \beta' = 0$, one can incorporate this case in the preceding frame:

$$\Omega_p(n, m, \alpha, \beta) \;=\; \sum_i \sum_j \Lambda_i(n, m)\, \Theta_{i,j}(\alpha, \beta)\, \Omega_p(n', m', 0, 0)$$

3. When $(2n+m)/4$ equals p: At this step, no integral has to be taken over t_p so that there is no sum over j and the exponents α' and β' remain zero:

$$\Omega_p(n, m, 0, 0) \;=\; \sum_i \Lambda_i(n, m)\, \Omega_p(n', m', 0, 0)$$

4. When $(2n+m)/4$ is smaller than p: The domain of integration is obtained subtracting case (2) from case (1):

$$\int_{t_{i-1}}^{0} dt_i \;=\; \int_{t_{i-1}}^{1} dt_i \;-\; \int_{0}^{1} dt_i \;;$$

$$\begin{aligned}\Omega_p(n,m,\alpha,\beta) \;&=\; \sum_i \sum_j \Lambda_i(n,m)\,\Theta_{i,j}(\alpha,\beta) \\ &\quad \times(\Omega_p(n',m',\alpha',\beta') \;-\; \Omega_p(n',m',0,0)).\end{aligned}$$

This together with the boundary conditions

$$\Omega_p(n,m,\alpha,\beta) = 0 \quad \text{as soon as } n \text{ or } m \text{ is negative,}$$

$$\Omega_p(0,0,\alpha,\beta) = \delta(\alpha)\,2^{\beta}.$$

defines Ω_p for every p.

The coefficients $\mathbf{Q}_{n-1}$ are then:

$$\mathbf{Q}_{n-1} \;=\frac{1}{2\sqrt{\pi}}\frac{1}{(2n)!}\frac{1}{4^{2n}}\sum_{p=1}^{n}\Omega_p(2n,0,0,0).$$

The values of the $\mathbf{P}_n$'s are then deduced using formula (16) which writes in terms of the other coefficients:

$$\mathbf{P}_n \;=\; (3\,n \;-\; 1/2)\,\mathbf{Z}_n \;-\; 2\,\mathbf{Q}_{n-1} \quad \forall\, n. \tag{26}$$

7 Acknowlegments

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References

[1] A. Neveu, Nucl. Phys. B 18 (1990) 242.

[2] B. Bellet, P. Garcia and A. Neveu, "Convergent sequences of perturbative approximations for the anharmonic oscillator, I. Harmonic approach", Montpellier preprint, preceding paper.

[3] C. Arvanitis, F. Geniet and A. Neveu, "Variational solution of the Gross-Neveu model, I. The large-N limit", Montpellier preprint PM94-19 (hep-th/9506188).

[4] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn and V.F. Weisskopf, Phys. Rev. D 9 (1974) 3471;
A. Chodos, R.L. Jaffe, K. Johnson and C.B. Thorn, Phys. Rev. D 10 (1974) 2599;
K. Johnson, Lectures presented at the Scottish Universities Summer School, August 1976, St. Andrews, Scotland;
For a review: P. Hasenfratz and J. Kuti, Phys. Rep. 40 (1978) 75.

[5] M. Lüscher, Phys. Lett. 118 B (1982) 391;
M. Lüscher, DESY preprint 83-116 (1983);
M. Lüscher, Nucl. Phys. B 219 (1983) 233;
E.G. Floratos, Phys. Lett. 154 B (1985) 173;
E.G. Floratos and D. Petcher, Phys. Lett. 160 B (1985) 271;
E.G. Floratos and N.D. Vlachos, Phys. Lett. 189 B (1987) 137;
E.G. Floratos, N.D. Tracas and N.D. Vlachos, Phys. Lett. 253 B (1991) 399.

[6] C.M. Bender, K. Olaussen and P.S. Wang, Phys. Rev. D 16 (1977) 1780.

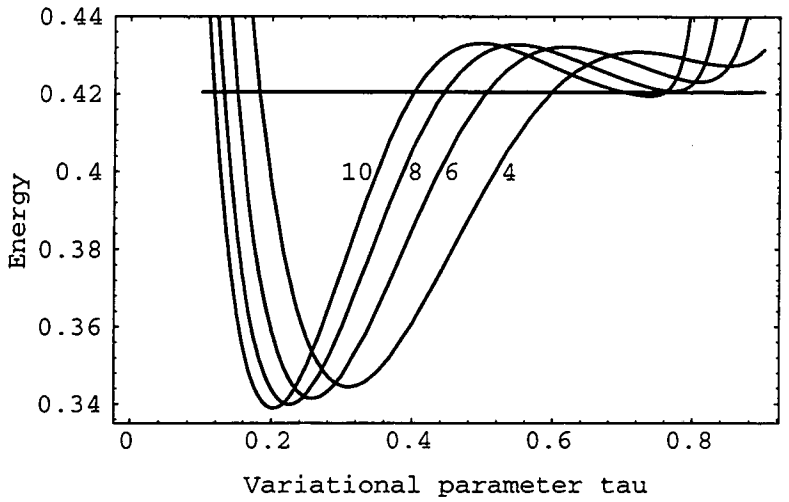

Figure 1: $E_0^{(p)}(\tau)$ versus τ for p =4, 6, 8 and 10.

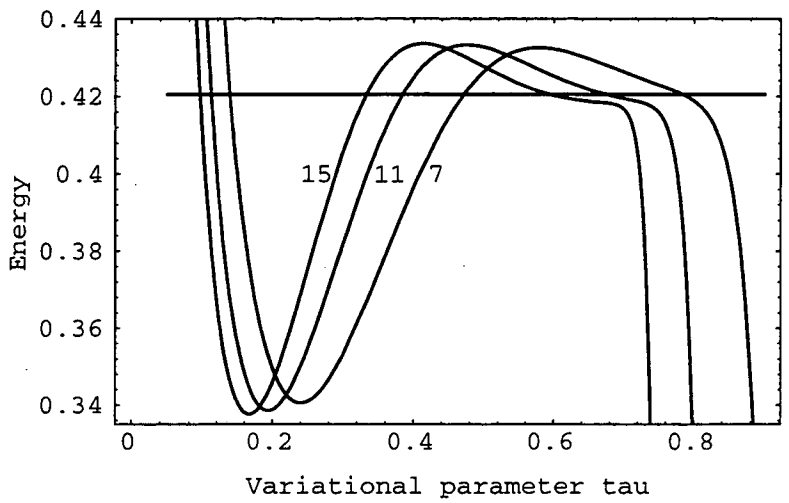

Figure 2: $E_0^{(p)}(\tau)$ versus τ for p =7, 11 and 15.

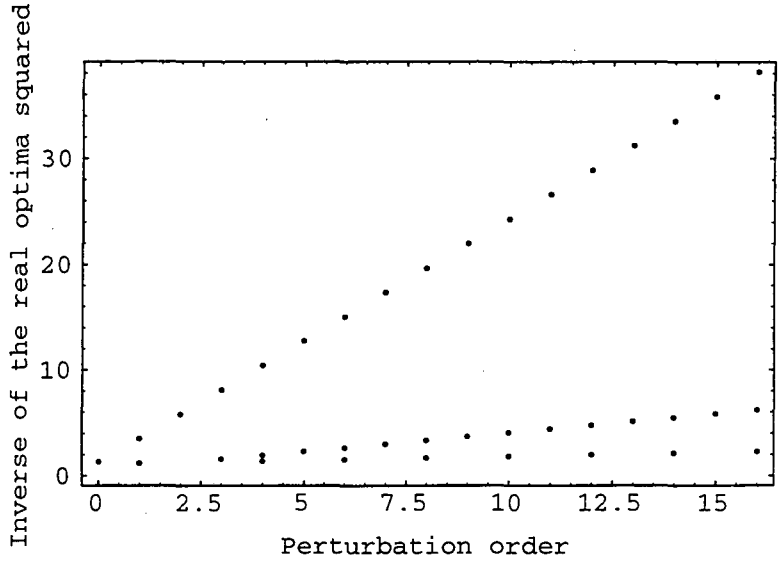

Figure 3: Real optima $(1/\tau^{opt})^2$ versus the order p.

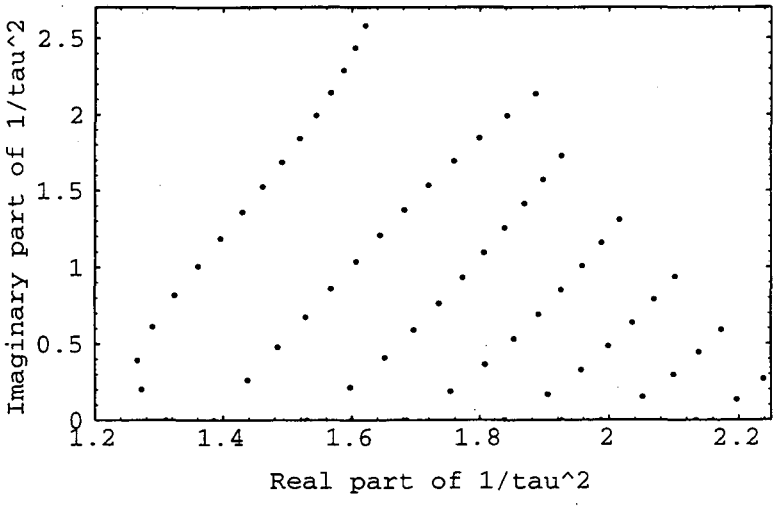

Figure 4: All the optima $(1/\tau^{opt})^2$ in the complex plane up to order 16.

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