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Basics of QCD

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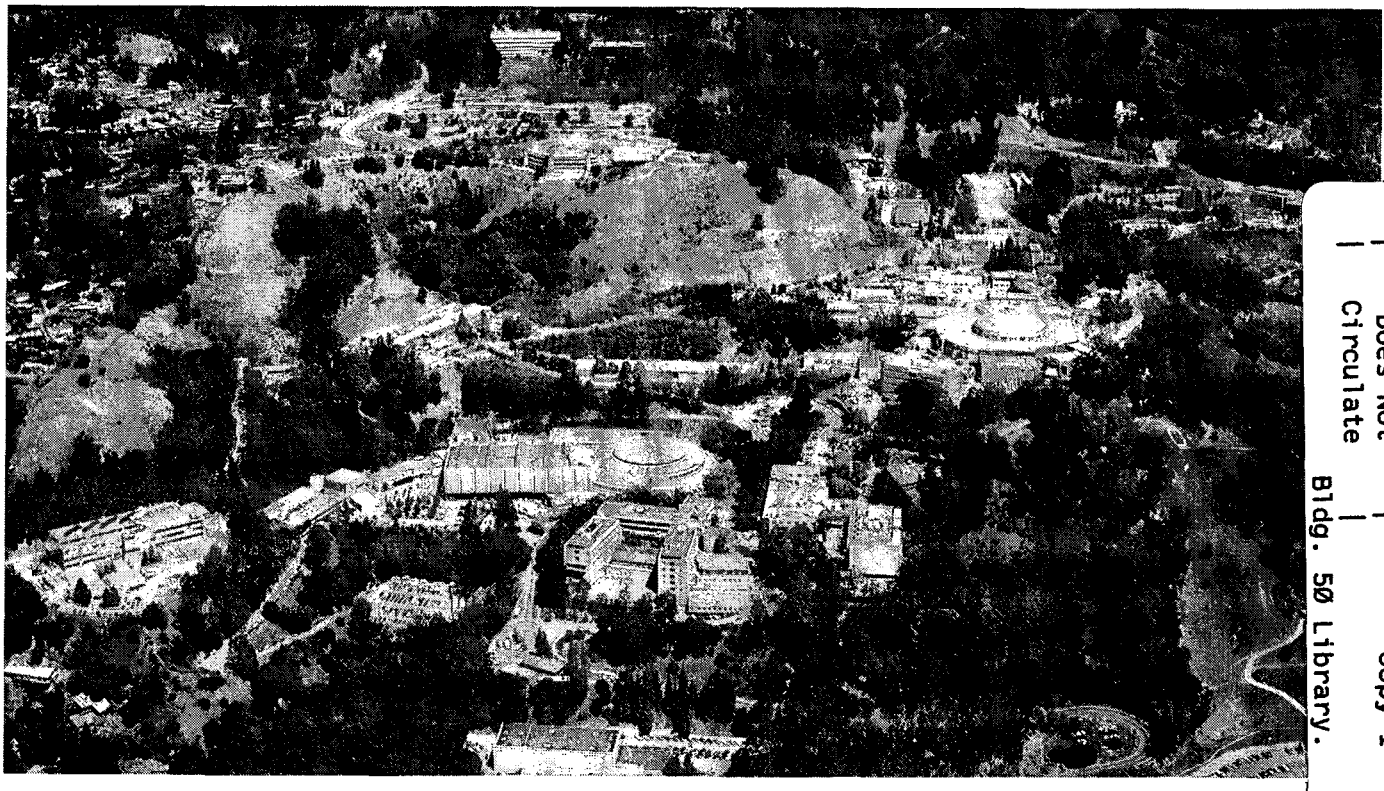

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Basics of QCD * †

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Abstract

These lectures provide an introduction to perturbative QCD and some of its current applications.

1 Introduction

In these lectures, I shall provide an introduction to perturbative QCD and to some of its applications. In the limited time available, I shall first concentrate on the basics of perturbative QCD and on the tools for calculations. After discussion the total hadronic cross-section and jet rates in e^+e^- annihilation, I shall discuss the QCD parton model. I will end with a discussion of how non-perturbative effects are parameterized by using heavy quark effective field theory as an example. There are many excellent references for the material in these lectures. Some recent sets of lecture notes and review articles should be consulted for more details and an alternative view.[1] [2].

2 The QCD Lagrangian.

The QCD Lagrangian describes the interactions of n_f flavors of quarks each of which has three colors (ψ_i) with an octet of gluon fields (G^a_μ) and may be written as follows:

$$-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \sum_{i,j}\bar{\psi}_i(i/D_{\mu ij} - \delta_{ij}m_j)\psi_j \tag{1}$$

The sum on j runs over quark flavors and gluonic filed strength tensor is written as,

$$F^a\mu\nu = \partial_\mu G^a_\mu - \partial_\nu G^a_\mu - igf_{abc}G^b_\mu G^c_\mu \tag{2}$$

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and the covariant derivative acting on fermion fields by

$$D_{\mu ij} = \delta_{ij}\partial_\mu - ig\, t^a_{ij}\, G^a_\mu \tag{3}$$

Here t^a are the 3×3 representation matrices and the structure constants f_{abc} of $SU(3)$ are given by $[t_a, t_b] = if_{abc}t_c$.

Apart from the quark masses (m_j), which have their origin in the Weinberg-Salam model of weak interactions, the theory has only one fundamental parameter, the coupling constant g. It is this coupling constant that provides us with an expansion parameter. The Lagrangian has a gauge invariance under which

$$\psi_i(x) \rightarrow u(x)_{ij}\psi_i(x) \tag{4}$$

and

$$t^a G^a(x)_\mu \rightarrow u t^a G^a(x)_\mu u^{-1} - \frac{1}{ig}(\partial_\mu u)u^{-1} \tag{5}$$

where $u(x)_{ij} = (exp(it^a\alpha_a(x))_{ij}$.

In order to quantize the theory, it is necessary to fix a gauge. In a covariant gauge, this is accomplished by adding a term

$$L_{gauge} = -\lambda/2(\partial^\mu G^a_\mu)(\partial^\mu G^a_\mu) \tag{6}$$

In this gauge unphysical degrees of freedom for the gluon fields are propagating, these are compensated by the ghost Lagrangian involving the propagation of an octet of unphysical fields η^a

$$L_{ghost} = (\partial_\mu \eta^{a\dagger} D^\mu_{ab}\eta_b) \tag{7}$$

Calculations of physical processes give results that are independent of λ. The most common choices are $\lambda = 1$ (Feynman gauge) and $\lambda = \infty$ (Landau gauge). Although covariant gauges are the most convenient for calculations, the presence of the ghost fields results in a loss of intuition. For this reason, some analyses are best carried out in axial gauge accomplished by adding a term

$$L_{gauge} = -\lambda/2(n^\mu G^a_\mu)(n^\mu G^a_\mu) \tag{8}$$

and taking the limit $\lambda \rightarrow \infty$ Here n is an arbitrary 4-vector. In this gauge, there are no ghosts and the gluon propagator has the form

$$G^{\mu\nu}(p) = \frac{i}{p^2 - i\epsilon}(-g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{(np)} - n^2\frac{p^\nu p^\mu}{(np)^2}) \tag{9}$$

In the limit $p^2 \rightarrow 0$, $p_\mu G^{\mu\nu}$ and $n_\mu G^{\mu\nu}$ both vanish demonstrating that only the propagator only propagates two physical degrees of freedom. Some aspects of QCD

particularly those associated with the parton model are most easily revealed in this gauge although higher order calculations can be technically awkward due to the unphysical singularities at $(np) = 0$.

If calculations are undertaken beyond the leading order in the coupling constant, ultra-violet divergences are encountered. These divergences must be regulated and reabsorbed into the fundamental parameters of the theory, *i.e.* the theory must be renormalized and a renormalized coupling constant defined. As an intermediate step a regulator must be introduced to render the divergences tractable; counter-terms are then added to remove the divergences and the regulator then removed. These counterterms correspond to renormalization of the scale of the gluon and quark fields and the coupling constant g. In order to be useful the regulator should preserve gauge invariance. The most convenient regulator is dimensional regularization.[4]

In order to understand the procedure, consider the calculation of the gluon self-energy at 1 loop in Feynman gauge. The dimension of space-time is continued to $n = 4 - 2\epsilon$ dimensions, *i.e.* in loop integrals $\frac{d^4k}{(2\pi)^4} \rightarrow \frac{d^nk}{(2\pi)^n}$. In n dimensions the coupling constant becomes dimensional $g \rightarrow g\mu^\epsilon$ and an arbitrary scale μ with the dimension of mass enters. There are 4 relevant Feynman diagrams, one involving a fermion loop, one involving a ghost loop and two with gluon loops, one of these being a tadpole that has no dependence on the external momentum. The resultant gluon self energy for a gluon of momentum p is

$$\Pi^{\mu\mu}_{ab}(p) = \delta_{ab}(g^{\mu\nu}p^2 - p^\mu p^\nu)\Pi(p^2) \tag{10}$$

where

$$\Pi(p^2) = g^2\mu^{2\epsilon}a(p^2)(-2n_f/3(1-\epsilon/3) + 5(1+\epsilon/15)) \tag{11}$$

with $a(p^2) = \frac{i}{16\pi^2}(D - \int_1^1 dx \ln(p^2x(1-x))$ and $D = 1/\epsilon + \ln(4\pi) - \gamma_E$, terms of order ϵ^2 or higher have been dropped and the quark mass has been set to zero (there are n_f quark flavors). The ultra-violet divergence is manifested in the $1/\epsilon$ poles. Suitable counter-terms involving $1/\epsilon$ can be added to the original Lagrangian such that these poles are eliminated from the calculation for a physical process and a finite result is obtained. The exact form of the counter-terms is prescribed once a renormalization *scheme* is specified. The resulting coupling constant parameter g is then dependent on that scheme.

In Quantum Electrodynamics an on-shell renormalization scheme is often used. Here the value of the $ee\gamma$ vertex is *defined* to be the renormalized charge e of the electron if the momentum of the photon is zero and both electrons have momenta satisfying $p^2 = m^2$. Such a scheme has a clear physical definition but it obscures an important fact; the renormalized charge e depends upon the choice the choice of momenta. This scheme does not work in QCD because at this low (zero) momentum

scale the theory is non-perturbative as discussed below. We are therefore forced to select an unphysical scheme. One choice (momentum space subtraction) involves defining the QCD coupling constant g in terms of the value of the three-gluon vertex at the point where the each of the gluon momenta satisfies $p^2 = -\mu^2$. Here again, the renormalized coupling is an implicit function of the arbitrary scale μ.

A far more convenient renormalization scheme in QCD is the $\overline{MS}$ scheme [3]. Here a process is computed using dimensional regularization then performing a Laurent expansion in $1/\epsilon$. This procedure is equivalent to adding a set of counter-terms to the original Lagrangian to remove the divergences. The terms involving D (see equation 11) are dropped and the limit $\epsilon \to 0$ then taken. The resulting expression for $\Pi(P^2)$ is

$$\Pi(P^2) = \frac{ig^2}{16\pi^2}(-2(-2n_f/3+5)\ln(q^2/\mu^2) - 2n_f/3 + 1) \tag{12}$$

The parameter g appearing in the resulting expression is then the renormalized coupling and depends implicitly on μ. Note the explicit dependence on μ in this result.

Now calculate a physical process $P(Q^2)$, which depends on some energy scale Q; P could, for example, represent a cross-section. It is convenient to choose the quantity P to be dimensionless; this can always be done by multiplying it by an appropriate power of Q. If we neglect quark masses, and follow the prescription given above then

$$P(Q^2) \sim F(\mu, Q^2, g) \tag{13}$$

Here F a function that is finite. Since P is dimensionless it has the form

$$P(Q^2) = F(Q^2/\mu^2, \alpha) \tag{14}$$

I have replaced g by α: $\alpha \equiv g^2/4\pi$ and the coupling constant is now in the $\overline{MS}$ scheme. The scale μ is arbitrary so that a physical quantity cannot depend upon its value

$$\frac{dP}{d\mu} = 0 \tag{15}$$

which implies

$$\left(\mu^2 \frac{\partial F}{\partial \mu^2} + \beta(\alpha)\frac{\partial F}{\partial \alpha}\right) = 0 \tag{16}$$

Here $\beta(\alpha)$ is defined by

$$\beta(\alpha) \equiv \mu^2 \frac{\partial \alpha}{\partial \mu^2} \tag{17}$$

We can introduce a momentum-dependent coupling $\alpha(t)$ via

$$t \equiv \int_\alpha^{\alpha(t)} \frac{dp}{\beta(p)} \tag{18}$$

where $t = \log(Q^2/\mu^2)$. Then Equation 16 has the solution

$$F(t, \alpha) = F(1, \alpha(t)) \tag{19}$$

Hence the only dependence on the scale Q or t is carried by $\alpha(t)$. We can expand β as a power series in α.

$$\beta = -b\frac{\alpha}{4\pi} - b'(\frac{\alpha}{4\pi})^2 + \ldots \tag{20}$$

Hence $\alpha(\mu^2)$ has the following form:

$$\alpha(\mu^2) = \frac{4\pi}{b\log(\mu^2/\Lambda^2)} + \ldots \tag{21}$$

Here $b = 11 - 2n_f/3$ where n_f is the number of quark flavors with mass less than μ. We can regard the fundamental parameter of QCD either as $\alpha(Q_0^2)$ or as the scale Λ. Notice that as μ becomes small, α becomes large. Therefore, perturbation theory cannot be used to discuss processes which involve momentum flows as small as a few times Λ. This is the reason that the on-shell subtraction scheme is impractical.

3 Processes in e^+e^- annihilation

As a specific example of QCD process, consider the total cross-section for $e^+e^- \rightarrow$ hadrons at center-of-mass energy $\sqrt{s}$. or the decay width of the Z boson into hadrons ($\Gamma \rightarrow hadrons$). If the coupling of the Z to fermions (ψ_i) is written as

$$Z^\mu \overline{\psi_i}(v_i\gamma_\mu + a_i\gamma_\mu\gamma_5)\psi_i \tag{22}$$

Then at lowest order in QCD (α_s^0), the cross-section for $e^+e^- \rightarrow hadrons$ at center of mass energy $\sqrt{s} = M_Z$ is calculated from the process $e^+e^- \rightarrow q\overline{q}$ and is given by

$$\sigma_h = \sigma_0 \tag{23}$$

$$= \frac{16G_F M_Z^4}{3\pi\Gamma_Z^2} \sum_{quarks} (v_i^2 + a_i^2) \tag{24}$$

where Γ_Z is the total decay width of the Z boson.

At next order in α_s two process are possible; $e^+e^- \rightarrow q\overline{q}$ and $e^+e^- \rightarrow q\overline{q} + gluon$. If we define x_1 and x_2 as the energy of the outgoing quark and anti-quark scaled by M_Z, *viz.* $x_1 = 2E_q/M_Z$ and $x_2 = 2E_{\overline{q}}/M_Z$., so that $0 \leq x_i \leq 1$, the cross section for the latter process can be written as

$$\sigma(q\overline{q}g) = \sigma_0\frac{2\alpha_s}{3\pi}\int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \tag{25}$$

The integral diverges when $x_1 \to 1$ or $x_2 \to 1$. If $x_1 = 1$, then either the energy of the gluon is zero ("soft divergence") , in which case $x_2 = 1$ also, or the gluon's momentum is parallel to the antiquark ("collinear divergence"). Neither of these divergences is physically observable since, in this limit, the final state of $q\overline{q} + gluon$ is indistinguishable from $q\overline{q}$. Therefore the 1-loop order α_s corrections to this process must also be considered. A regulator is needed to control these soft and collinear divergences. Again we can use dimensional regularization. Equation 25 is then replaced by

$$\sigma(q\overline{q}g) = \sigma_0 \frac{2\alpha_s}{3\pi} \frac{(1-\epsilon)^2}{(1-2\epsilon/3)\Gamma(1-2\epsilon)} \int dx_1 dx_2 \frac{x_1^2 + x_2^2 - \epsilon(2 - x_1 - x_2)}{(1-x_1)^{1+\epsilon}(1-x_2)^{1+\epsilon}} \quad (26)$$

$$= \sigma_0 \frac{2\alpha_s}{3\pi} \frac{(1-\epsilon)^2}{(1-2\epsilon/3)\Gamma(1-2\epsilon)} (\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2}) \quad (27)$$

The 1-loop corrections to the $q\overline{q}$ rate are also computed and added to this result. The singular terms as $\epsilon \to 0$ cancel and the limit $\epsilon \to 0$ is taken with the result

$$\sigma(q\overline{q}) + \sigma(q\overline{q} + gluon) = \sigma_0 \frac{\alpha_s}{\pi} \quad (28)$$

or equivalently

$$R_Z = \frac{\Gamma(Z \to hadrons)}{\Gamma(Z \to e^+e^-)} \quad (29)$$

$$= R_0(1 + \frac{\alpha_s}{\pi}) \quad (30)$$

with

$$R_0 = 3\frac{\sum_{quarks}(v_i^2 + a_i^2)}{v_e^2 + a_e^2} \quad (31)$$

This result is calculated to lowest non-trivial order in α_s and hence no QCD renormalization is needed. At higher orders, the renormalized coupling $\alpha_s(\mu)$ appears and we may write

$$R_Z = R_0(1 + \frac{\alpha_s(\mu)}{\pi} + \sum_{n \geq 2} c_n(M_Z^2/\mu^2)(\frac{\alpha_s(\mu)}{\pi})^n) \quad (32)$$

with $c_1 = 1.986 - 0.115 n_f - (\beta_0/4)\ln(M_Z^2/\mu^2)$, the result for c_2 can be found in reference [6]. The result looks simplest if we make the natural choice $\mu = M_Z$. However this is not necessary. Figure 3 shows the value of R_Z as a function of μ. It can be seen from this figure that the μ dependence of the result is reduced as successively more terms in the perturbation series are included. The μ dependence in $\alpha_s(\mu)$ is compensated by that in c_i. In general one should choose a value of μ that is comparable to the energy in the process. In this case therefore $\mu \sim \sqrt{s}$ in order to

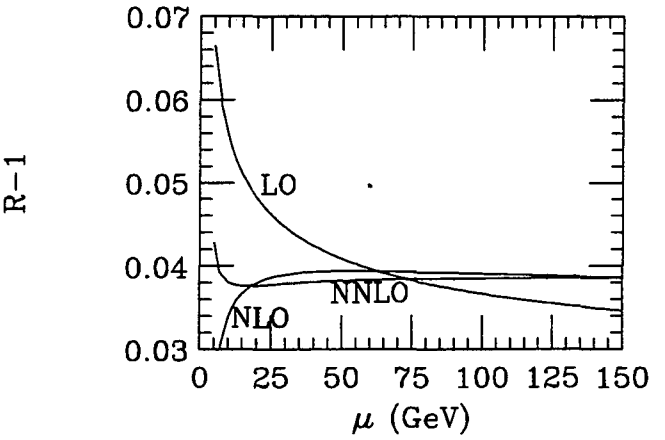

Figure 1: The value of R_Z (see text) as a function of μ. The dashed, dotted, and solid curves show the order α_s, α_s^2 and α_s^3 results respectively.

avoid large logarithms and therefore large values of c_i that indicate a poorly behaved perturbative expansion. Many schemes are available to select the appropriate scale [5]. However the variation of the result with μ is an indication of the size of the theoretical uncertainty.

Other tests of QCD in e^+e^- annihilation depend upon the study of the jets of particles produced in the final state from the hadronization of the produced quarks and gluons. At lowest order in α_s, the final state consists of a quark-antiquark pair; at next order we can get a state with an additional gluon. Since the quarks and gluons hadronize into jets of particles, this would seem to imply that the ratio #(3jets)/#(2jets) should be of order α_s. This is only partially true since it is necessary to define what is meant by a jet.

Consider the final state of two quarks and a gluon illustrated . The Feynman graphs contains an internal propagators which gives rise to the factor of $1/((1-x_1)(1-x_2))$ in equation 25. As discussed above, this factor becomes singular when either the gluon becomes very soft, or when it moves parallel to an outgoing quark or antiquark.

These soft and collinear divergences correspond precisely to those parts of phase space where a detector would only detect two jets. Consider an idealised detector consisting of a set of elements each of which covers an angular cone of opening angle δ and has an energy threshold ϵ. This detector will incapable of resolving two jets if one of them is very soft (energy ϵ or less), or if the two jets have an angular separation which is less than δ. We can define the f to be the fraction of total cross-section in which all but a fraction ϵ of the total energy is deposited into two cones of opening

angle δ. Then to order α_s,

$$(1-f) = \frac{\sigma_{3-jet}}{\sigma_{total}} \tag{33}$$

provides a definition of the three jet fraction.

We can calculate this fraction by restricting the range of the x_1 and x_2 integrals in Equation 25. Hence[7]

$$\begin{aligned}(1-f) &= \int_{\epsilon,\delta} \frac{1}{\sigma_{total}} \frac{d\sigma}{dx_1 dx_2} && (34)\\ &= \frac{4\alpha_s}{3\pi}(4log(1/\delta)log(1/2\epsilon) - 3log(1/\delta) + \pi^2/3 - 7/4) && (35)\end{aligned}$$

Notice that as ϵ and δ become very small the logarithms in this expression can become very large. Ultimately the perturbation expansion in α_s breaks down since there are terms in next order which are of order $\alpha_s^2 log^2(1/\delta)$. Since this is not small compared with $\alpha_s log(1/\delta)$, the expansion is not reliable. The situation can be improved by resumming these large logarithms to all orders.

This fixed angle cone scheme, although very intuitive, is difficult to extend to larger numbers of jets. Jet recombination algorithms are more general. In the JADE algorithm [8] particles of momenta p_i and p_j are combined into a pseudo-particle of momentum $p_i + p_j$ if the invariant mass of the pair is less than $y_0\sqrt{s}$. The process is then iterated until no more pairs of particles or pseudo-particles remain. The remaining number is then defined to be the number of jets in the event and can be compared to the QCD prediction. This algorithm can be applied directly to the perturbative QCD final state of quarks and gluons and to the experimental final state of hadrons. The Durham algorithm is slightly different, in computing the mass of a pair of partons it uses $M^2 = 2min(E_1^2, E_2^2)(1 - \cos\theta_{ij})$ for partons of energies E_i and E_j separated by angle θ_{ij} [9]. This is a more physical algorithm as it merges the slowest particle with the one closest to it in angle. Most tests of QCD using jet events in e^+e^- annihilation use one of these algorithms [17].

4 Parton Model

In order to discuss processes which involve hadrons in the initial state, we must discuss the parton model. Consider the case of electron-proton scattering ($ep \to eX$), where the cross-section can be written as

$$\frac{d\sigma}{dxdy} = \frac{4\pi\alpha_{em}^2 s}{Q^4}\left[\frac{1+(1-y)^2}{2}2xF_1(x,Q^2) + (1-y)(F_2(x,Q^2) - 2xF_1(x,Q^2))\right] \tag{36}$$

The variables are defined as follows (see Figure 4): q is the momentum of the

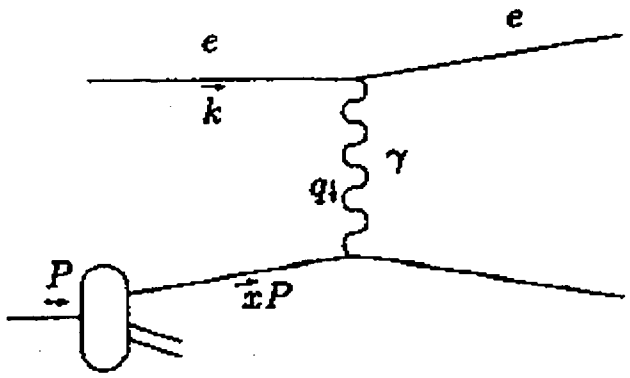

Figure 2: Diagram illustrating the variables in deep inelastic scattering electron + proton $\rightarrow$ electron + anything.

exchanged photon and P is the momentum of the target proton and k is that of the incoming electron

$$\begin{aligned} Q^2 &= -q^2 \\ \nu &= \frac{q \cdot P}{m_p} \\ x &= \frac{Q^2}{2m_p\nu} \\ y &= \frac{q \cdot p}{k \cdot p} \\ s &= 2p \cdot k + m_p^2 \end{aligned} \tag{37}$$

where m_p is the proton mass. I have neglected parity violating effects which arise from the exchange of a Z boson instead of a photon.

In the naive parton model the proton is viewed as being made up of a set of non-interacting partons. The structure functions F_1 and F_2 are related to the probability distribution $q_i(x)$ which represents the probability of finding a parton of type i (quark or gluon) inside the proton with fraction x of the proton's momentum, and the scattering cross-section for such a virtual photon from a parton.

$$F_1 = \frac{F_2}{2x} = \sum_i \int_x^1 \frac{dy}{y} q_i(y)[e_i^2 \delta(x/y - 1)] \tag{38}$$

where e_i is the charge of parton of type i. The δ-function appears from the cross-section for $q + \gamma \rightarrow q$ and corresponds to the constraint that the massless quark in the final state is on mass-shell.

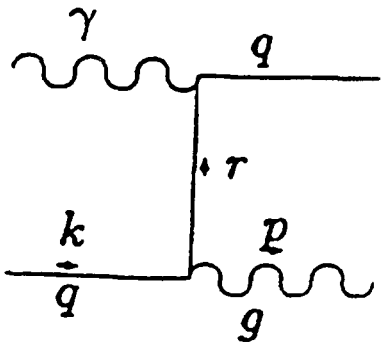

Figure 3: Diagram showing $g + \gamma \rightarrow q + \bar{q}$.

Let us consider QCD corrections to this scattering. At next order in α_s, there are contributions from gluon emission which lead to the final state $q + g$ and also from virtual gluons (see Figure 4).

The gluon momentum must be integrated over. In the cases discussed previously, the divergences arose from large momentum flows inside loop diagrams (ultra-violet divergences). In this case these divergences cancel. However here we encounter infrared and collinear divergences. The former cancel between the real and virtual diagrams, but, unlike the case of jet production in e^+e^- annihilation, the latter do not. Using dimensional regularization to regulate this divergence and keeping the terms that are singular or non-zero as $\epsilon \rightarrow 0$, to order α_s Equation 38 is replaced by

$$F_1 = \sum_i \int_x^1 \frac{dy}{y} q_i(y) \left[e_i^2 \delta\left(\frac{x}{y} - 1\right) + \sigma_i\left(\frac{x}{y}, Q^2\right)\right] \tag{39}$$

with

$$\sigma_i(z, Q^2) = \frac{\alpha_s}{2\pi} e_i^2 \left[(\ln(Q^2/\mu^2) + D) P_{qq}(z) + f(z) + 0\left(\frac{1}{Q^2}\right)\right] \tag{40}$$

and

$$P_{qq}(z) = \frac{4}{3} \frac{(1 + z^2)}{1 - z} \tag{41}$$

for $z \neq 1$. Here again the scale μ has appeared from dimensional regularization.

In order to see the origin of the singular terms, consider the graph of Figure 4 and work in a frame where $k_\mu = (k, k, 0, 0)$. If the transverse momentum of the gluon (p) relative to k is small then we can take $p = (\eta k + k_\perp^2/2\eta k,\ \eta k,\ k_\perp\ 0)$. (Terms of order $k_\perp^2$ are neglected.) The internal quark line now has invariant mass squared $r^2 = (k - p)^2 = k_\perp^2/\eta$, so that the squared amplitude from the graph will contain $1/k_\perp^4$. Now, at very small $k_\perp$ helicity conservation forbids the emission of a real gluon

from a quark line, so that one factor of $k_\perp^2$ appears in the numerator (this physical result is most easily revealed in axial gauge). We now have for the total cross-section $q + \gamma \to q +$ anything, a contribution

$$\sigma \sim \frac{\alpha_s}{2\pi} \int \frac{dk_\perp^2}{k_\perp^2} \tag{42}$$

which gives rise to a logarithmic singularity. Dimensional regularization causes this singularity to appear as the term D in equation 40. Note that for a massive quark the singularity becomes $\log(Q^2/m_q^2)$.

We have obtained a result that is not physically meaningful. But Equation 39 contains the unknown quantity $q_i(y)$. We can define

$$q_i(x, M^2) = q_i(x) + \frac{\alpha_s(D + \ln(M^2/\mu^2)}{2\pi} \int_x^1 \frac{dy}{y} q(y) P_{qq}\left(\frac{x}{y}\right) \tag{43}$$

Hence

$$F_1 = \sum_i \int_x^1 e_i^2 \frac{dy}{y} q_i(y, M^2) \left[\delta\left(\frac{x}{y} - 1\right) + \frac{\alpha_s}{2\pi}(f\left(\frac{x}{y}\right)\right] + \ln(Q^2/M^2) P_{qq}\left(\frac{x}{y}\right)) + 0(\alpha^2) \tag{44}$$

The singularity has been eliminated at the cost of introducing an M-dependent structure function. The scale M is called the factorization scale since it is the scale at which the parton distribution function $q_i(x, M^2)$ is evaluated. The natural choice in this case is $Q = M$ which simplifies result. However its choice is arbitrary, the value of the structure function $F_i(x, Q^2)$ does not depend in principal depend on it. Just as in the case of the μ dependence for R_Z, there is a compensation between the M dependence of $q_i(x, M^2)$ and the M dependence appearing explicitly in equation 44. As in the earlier case the M dependence of the full result is reduced as more terms in the perturbative expansion are included.

I have so far considered an oversimplification of the true problem. To order α_s there is an additional partonic process, namely $gluon + \gamma \to q + \bar{q}$. This process also contains a log (Q^2/μ^2) arising from the propagation of the internal quark close to its mass shell. This singularity results in the replacement of Equation 39 and 40 by

$$\begin{aligned} F_1(x, Q^2) \;\; &= \textstyle\int_x^1 \frac{dy}{y} \left[\sum_i e_i^2 q_i(y) \left[\delta(\frac{x}{y}) + \frac{\alpha_s}{2\pi}\left[(D + \ln(Q^2/\mu^2)) P_{qq}(\frac{x}{y}) + f_q(\frac{x}{y})\right]\right.\right. \\ &\quad \left.\left. + (\textstyle\sum_i e_i^2) g(y) \frac{\alpha_s}{2\pi}\left[(D + \ln(Q^2/\mu^2)) P_{qg}(\frac{x}{y}) + f_g(\frac{x}{y})\right]\right]\right. \end{aligned}$$

with $P_{qg}(x) = 1/2(x^2 + (1-x)^2)$. The singularities can be absorbed by defining

$$q_i(x, M^2) = q_i(x) + \frac{\alpha_s}{2\pi}(D + \ln(M^2/\mu^2)) \int_x^1 (q_i(y) P_{qq}(\frac{x}{y}) + g(y) P_{qg}(\frac{x}{y})) \frac{dy}{y} \tag{45}$$

so that the quark and gluon distributions ($q_i(x)$ and $g(x)$) are now coupled. This equation can be recast in the more familiar form (DGLAP equations) [10]

$$\frac{M^2 dq_i(x, M^2)}{d\ln M^2} = \frac{\alpha_s(M^2)}{2\pi}\int_x^1 (q_i(y)P_{qq}(\frac{x}{y}) + g(y)P_{qg}(\frac{x}{y}))\frac{dy}{y} \tag{46}$$

The equation for the evolution of the gluon distribution is

$$\frac{M^2 dg_i(x, M^2)}{d\ln M^2} = \frac{\alpha_s(M^2)}{2\pi}\int_x^1 (q_i(y)P_{gq}(\frac{x}{y}) + g(y)P_{gg}(\frac{x}{y}))\frac{dy}{y} \tag{47}$$

Given data from which $q_i(x, M_0)$ and $g(x, M_0)$ can be obtained as functions of x for a fixed M_0, these equations for the evolution of $q(x, M^2)$ and $g(x, M^2)$ with M can be solved to obtain them for all M. Note that structure functions at x_1 and M_1 depend only on those at $x > x_1$ provided $M_1 > M_0$. Since these equations are valid only to lowest order in α_s, M_0 must be sufficiently large for $\alpha_s(M_0)$ to be small enough so that the perturbation series can be trusted. If the equations are used to extrapolate to $M > M_0$ the series will become more trustworthy. The order α_s^2 terms in the DGLAP equations are known and are included in some parameterizations of $q_i(x, M^2)$ (see below). The structure functions fall to zero as x tends to 1. The quark and gluon distributions at $Q^2 = 20\ GeV^2$ as given by the MRS $D-'$ are shown for illustration in Figure 4 [12] Before leaving the DGLAP equations, I would like to discuss the behaviour of the structure functions at very small values of x. As the energy available in an ep collision increases it becomes possible to reach smaller and smaller values of x at fixed Q^2. Consider the behaviour of the gluon distribution at small x, We can neglect the generation of gluons from quarks since the gluon density is larger at small x (see figure 4). The DGLAP equation simplifies to

$$\frac{d}{dt}g(x,t) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y}g(y,t)P_{gg}\left(\frac{x}{y}\right). \tag{48}$$

where I have introduced $t = lnQ^2$. Furthermore $P_{gg}(x)$ may be approximated by

$$P_{gg}(x) = \frac{6}{x} \tag{49}$$

Equation 48 can be recast as

$$-x\frac{d^2(xg(x,t))}{dxdlogt} = \frac{12}{b}xg(x,t) \tag{50}$$

Here I have eliminated $\alpha_s(q^2)$ using Equation 21. Equation 50 can be solved to give

$$xg(x, Q^2) \propto exp\left(\sqrt{\frac{48}{b}log(1/x)loglog(Q^2)}\right) \tag{51}$$

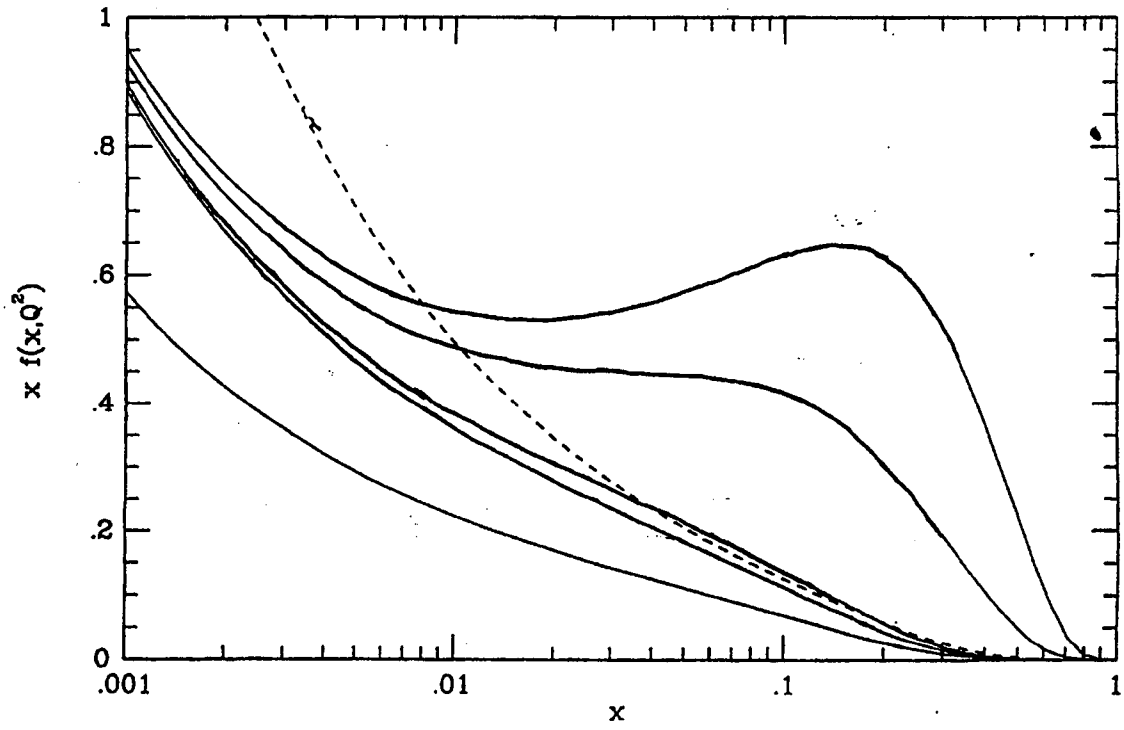

Figure 4: Diagram showing the behavior of the quark and gluon distributions as functions of x for various Q^2. Plotted is $xf(x)$ for gluons divided by 10 (dashed line) and as solid lines from top to bottom, u, d, $\overline{d}$, $\overline{u}$ and s $(= \overline{s})$ quarks at $Q^2 = 20\ GeV^2$.

This approximate result ("double log approximation") corresponds to summing terms of order $(\alpha_s \ln(1/x) \ln(Q^2))^n$ to all orders in perturbation theory. The full solution of the DGLAP equation is equivalent to summing all terms of the type $(\alpha_s \ln(Q^2))^n$ (and terms of order $(\alpha_s^n \ln(Q^2))^{n-1}$ if the next to leading order equation is used) with the full x dependence. The DGLAP result breaks down at very small values of x where $\alpha_s(Q) \ln(1/x) \sim 1$. An alternative formulation is available [11], that is complimentary to DGAP. This BFKL formulation sums terms of order $(\alpha_s \ln(1/x))^n$ and is applicable provided that $\alpha_s(Q) \ln(Q^2) << 1$. Data are available from experiments at HERA [13]; it is not clear whether accessible values of x are small enough for BFKL to be applicable.

The growth of $g(x)$ at small x is very rapid. It is eventually cut off when the equations break down [14]. We can estimate the position of this breakdown as follows. The DLAP/BFKL equations describe the growth of incoherent parton showers: the shower initiated by one parton is independent of that of the other partons. This assumption must eventually break down. Let us view the proton in a frame where is moving extremely fast, the appropriate frame for the parton picture. The proton looks like a pancake with area $1/m_\pi^2$. Viewed on a scale Q^2 it contains a set of partons each of size $1/Q$. The fractional area occupied by partons is

$$\frac{xg(x,Q^2)m_\pi^2}{Q^2}. \tag{52}$$

Provided this fraction is small the partons are not densely packed and the incoherent approximation is correct. If the fraction is of order one, the incoherent approximation breaks down and the growth of $g(x,Q^2)$ is cut off.

A vital property of QCD is that the distribution functions defined by equation 43 are universal. In order to illustrate this, consider the Drell-Yan process in proton-proton collisions. In the naive parton model, the cross-section for the production of a $\mu^+\mu^-$ pair of invariant mass M in a proton-proton collision (the Drell-Yan process) with total center-of-mass energy $\sqrt{s}$ is given by

$$\frac{d\sigma}{dM^2} = \frac{4\pi\alpha_{em}^2}{9M^2 s} \int dx_1 dx_2 [\sum_i q_i(x_1)\bar{q}_i(x_2)e_i^2 \delta(x_1 x_2 - M^2/s) + (1 \Leftrightarrow 2)] \tag{53}$$

Here $\bar{q}$ is an antiquark distribution. The fundamental process is quark-antiquark annihilation into $\mu^+\mu^-$. Consider the corrections to this at order α_s. As in the case of ep scattering these can involve either virtual or real gluons (see Figure 4). These corrections modify Equation 53, *viz.*,

$$\frac{d\sigma}{dM^2} = \frac{4\pi\alpha_{em}^2}{9M^2 s} \int \frac{dx_1}{x_1}\frac{dx_2}{x_2} [[e_i^2 q_i(x_1)\bar{q}_i(x_2) + (1 \Leftrightarrow 2)]$$

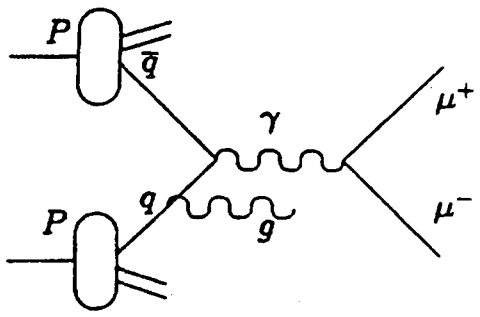

Figure 5: A parton diagram showing an order α_s correction to the Drell-Yan process, $p\overline{p} \rightarrow \mu^+\mu^- + X$

$$\begin{aligned}\Big[\delta(1-z) + \theta(1-z)\tfrac{\alpha_s}{2\pi}[2P_{qq}(z)(D + \ln(M^2/\mu^2)) + f'(z)]\Big] \\ +[\textstyle\sum_i e_i^2(q_i(x_1) + \bar{q}_i(x_1))G(x_2) + (1 \Leftrightarrow 2)] \\ [\theta(1-z)\tfrac{\alpha_s}{2\pi}[P_{qg}(z) + f''(z)]\end{aligned} \tag{54}$$

where $z = M^2/(sx_1x_2)$ [15]. The last part of the expression arises from the process $g + q \rightarrow \mu^+\mu^- + q$.

If we replace $q(x)$ by $q(x, M^2)$ defined by Equation 43 then the resulting expression will have no explicit divergences, *viz.*,

$$\frac{d\sigma}{dM^2} = \frac{4\pi\alpha_{em}^2}{9M^2s}\int dx_1dx_2[e_i^2q_i(x_1, M^2)\bar{q}_i(x_2, M^2)\delta(x_1x_2 - M^2/s) + (1 \Leftrightarrow 2) + \mathcal{O}(\alpha_s(Q^2))] \tag{55}$$

where the order $\alpha_s(M^2)$ terms are free of divergences. This absorption of the singular terms into $q(x, M^2)$ is known as factorization; it is a universal property which guarantees that hard processes can be reliably calculated in perturbative QCD and that the same set of structure functions should be used for all processes [16].

In summary, all cross-sections involving the transfer of large momentum (greater than $\sim$ 10 GeV) or the production of heavy particles in hadron hadron collisions can be calculated using the parton model. The cross-sections are given by

$$\sigma = \sum_{i,j}\int dx_1dx_2f_i(x_1, Q^2)f_j(x_2, Q^2)\hat{\sigma_{ij}} \tag{56}$$

Where the sum runs over the parton types (quarks and gluons) and $\hat{\sigma_{ij}}$ is the cross-section involving partons that is calculated using perturbative QCD.

5 Pre-Asymptotic Effects

In the processes discussed so far, I have concentrated on those characterised by some large energy scale Q which can then be calculated using perturbation theory. All processes are affected by contributions that cannot be calculated in perturbation theory. In general we can write the result for a QCD prediction in the form

$$P(Q) = f(\alpha_s(Q)) + g(\Lambda/Q) \tag{57}$$

where Λ is the scale where QCD perturbation theory breaks down ($\alpha_s(\Lambda) \sim 1$) and $g(0) = 0$. While a non-perturbative method is needed to compute $g(y)$, in many cases it is possible to perform an expansion $g(y) = \sum_i a_i y^i$, the first few terms will give a reasonable approximation at small y or large Q and hence the non-perturbative effects can be *parameterized* by a few constants. By using the underlying properties of QCD, it is often possible to relate the quantities a_i between processes. Hence a_i can be determined by experiment for some process and then used to predict others. One of the earliest applications of this technique is the "QCD sum rules" [20]. Students are encouraged to learn about this [21].

In these lectures, I shall give a discussion of another application: Heavy quark effective theory. There are a number of excellent review articles and I refer the reader to them for more details [2]. In the limited time available here, I shall provide an introduction to the subject. Consider a heavy quark Q (such as a b-quark) with mass $M_Q >> \Lambda$. Two types of meson states can exist using Q; the states of $Q\overline{Q}$ such as the family of Upsilon resonances and $Q\overline{q}$ states where q is a light quark (up, down, or strange). If M_Q is large enough so that the binding energy of the $Q\overline{Q}$ state is much bigger than Λ, then the full spectrum will be computable using perturbative QCD. However this is not the case for the Upsilon system. Some properties are computable such as the ratio of decay widths $\Gamma(\Upsilon \to hadrons)/\Gamma(\Upsilon \to \gamma + hadrons)$. Unfortunately the top quark lifetime is too short for a toponium system to form, although interesting effects are present in the process $e^+e^- \to t\overline{t}$ near threshold[19].

In order to study systems consisting of one heavy quark and one light antiquark, an effective Lagrangian approach can be used. If we are interested in momentum transfers less than the heavy quark mass, then pair production of $Q\overline{Q}$ will not be possible. Furthermore in the limit $M_Q \to \infty$, the dynamics of the binding become independent of the mass. The heavy quark acts like a static color charge as far as the light degrees of freedom are concerned. Therefore up to corrections of order $1/M_Q$, we can use the properties of one $Q\overline{q}$ system (such as B mesons) to predict the properties of another (such as D mesons). Start with the Lagrangian describing the interactions of a heavy quark (remember that the gluon field is hiding in the covariant derivative):-

$$L_Q = \overline{Q}(i\not{D} - M_Q) \tag{58}$$

If we are interested in momentum transfers k, that are small compared to M_Q, we can write the momentum p of the heavy quark as $p^\mu = M_Q v^\mu + k^\mu$, where $v^\mu v_\mu = 1$. Define $h_v(x) = exp(iM_Q v^\mu x_\mu)P_+ Q(x)$ and $H_v(x) = exp(iM_Q v^\mu x_\mu)P_- Q(x)$, where $P_\pm = (1 \pm /v)/2$. Hence $/v h_v = h_v$, $/v H_v = -H_v$ and $Q(x) = exp(-iM_Q v^\mu x_\mu)(h_v(x) + H_v(x))$ and

$$L_Q = \overline{h_v}(iv \cdot D)h_v + \overline{H_v}(iv \cdot D + 2M_Q)H_v + \overline{h_v} i/D_\perp H_v + \overline{H_v} i/D_\perp h_v \tag{59}$$

where $D^\mu \equiv D^\mu_\perp - v^\mu v \cdot D$ which implies $v \cdot D_\perp = 0$. Note that Lorentz invariance has been lost as we are working with a quark of fixed velocity. It can be recovered by summing over v [22]. The operator corresponding to H represents the excitation of a quantum of mass $2M_Q$, *i.e.*the creation of a $Q\overline{Q}$ pair. Solve the equation of motion for H

$$H_v = \frac{1}{2M_Q + iv \cdot D} i/D_\perp h_v \tag{60}$$

and insert the solution back into L_Q

$$L_Q = \overline{h_v}(iv \cdot D)h_v + \overline{h_v} i/D_\perp \frac{1}{2M_Q + iv \cdot D} i/D_\perp h_v \tag{61}$$

We can expand this as a power series in $1/M_Q$

$$L_Q = \overline{h_v}(iv \cdot D)h_v + \frac{1}{2M_Q}\overline{h_v}(i/D_\perp)^2 h_v + \frac{g}{4M_Q}\overline{h_v}\sigma_{\mu\nu}F^{\mu\nu}h + O(1/M_Q^2) \tag{62}$$

Note that the leading term does not contain the quark mass and that there is no spin structure; the gluon field couples to h_v as if it were a scalar field. The propagator for h_v is

$$\frac{i}{v \cdot k}\frac{1 + /v}{2} \tag{63}$$

and the $gluon - h_v$ vertex is

$$iT^a v^\mu \tag{64}$$

If we have N heavy quarks, the system will have and $SU(2N)$ heavy quark symmetry. Note that this symmetry is *not* a non-relativistic approximation. At order $1/M_Q$ there are two terms $(-1/2M_Q)\overline{h_v}D^2 h$ which can be thought of as a "residual kinetic energy" and $(g/M_q)\overline{h_v}(S \cdot B)h_v$ which represents the interaction of the heavy quark spin with the color magnetic field (B). The heavy quark symmetry immediately gives us a relation been the masses of the spin-0 and spin-1 D and B mesons:-

$$M_D^* - M_D = M_B^* - M_B + O(\Lambda/M_c) \tag{65}$$

Alternatively, since $\langle D| S \cdot B |D\rangle = \langle B| S \cdot B |B\rangle + O(1/M_Q)$

$$\frac{M_D^* - M_D}{M_B^* - M_B} = \frac{M_B}{M_C} \tag{66}$$

Experimentally the left side of this equation is $\frac{142MeV}{46MeV} = 3.1$ in very good agreement with the expectation.

One of the main applications of heavy quark effective field theory has been in the study of weak decays. Consider a heavy $(Q\overline{q})$ meson state of momentum p normalized so that $\langle p| \, |p'\rangle = \frac{2p^0}{M}(2\pi)^3\delta^3(p'-p)$ scattered into another state of momentum p' by the coupling of a heavy quark to an external current which changes the heavy quark momentum. The light (quark and gluon) degrees of freedom need to adjust in order to "catch up" with the recoiling heavy quark. This results in a form factor suppression for the transition amplitude that depends only on $v - v'$. As an example, suppose that the current is a vector current of the form $J_\mu = \overline{h_v}\gamma_\mu h_v$ then its matrix element between two pseudoscalar mesons has the form

$$\langle p(v')| \, J_\mu \, |p(v)\rangle = \xi(v \cdot v')(v + v')_\mu \tag{67}$$

ξ does not depend on M_Q and there is no form factor proportional to $(v - v'_\mu$ since $/vh_v = h_v$. The current J_μ is conserved and hence $\xi(1) = 1$. Transitions involving spin-1 mesons (V) are also controlled by ξ, since V can be obtained from P by flipping the spin of Q and in the effective theory this is a symmetry transformation. We can use this to show that the matrix element involving a vector meson of velocity v and polarization state ϵ is

$$\langle V(v', \epsilon| \, \overline{h'_v}\gamma_\mu(1-\gamma_5) \, |p(v)\rangle = i\epsilon^{\mu\nu\alpha\beta}\epsilon_\nu v'_\alpha v_\beta \xi(v\cdot v') - (\epsilon_\mu(1+v\cdot v' - v'_\mu(\epsilon\cdot v))\xi(v\cdot v') \tag{68}$$

Let us now use these results to derive some physical results. First consider the decay constant of a pseudoscalar meson X with momentum p^μ, defined by

$$\langle 0| \, A_\mu \, |X\rangle = f_X p_\mu \tag{69}$$

where the state $\langle X(p)|$ has conventional normalization. If X is a heavy meson state of velocity v, we have

$$\langle 0| \, A_\mu \, |X(v)\rangle = \tilde{f}_X v_\mu \tag{70}$$

and therefore taking into account the different normalizations for $\langle X(p)|$ and $\langle \dot{X}(v)|$ we have

$$\frac{f_B}{f_D} = \sqrt{\frac{M_D}{M_B}} + O(\Lambda/M_C) \tag{71}$$

This result does agree with recent results from lattice gauge theory where $\frac{f_B}{f_D} \sim 1$ [18] indicating that for this relation the Λ/M_Q corrections must be large.

The next application involves the semi-leptonic decay $B \to De\nu$. This process is mediated at the quark level by the interaction $b \to ce\nu$ which is given in terms of the Fermi constant and the Kobayashi Maskawa mixing matrix element V_{bc}. The

hadronic transition involves the matrix element of a vector current discussed above and hence

$$\langle D(p')|\, V_\mu \,|B(p)\rangle = \xi(v\cdot v')\sqrt{M_B M_D}(\frac{p_\mu}{M_B} + \frac{p'_\mu}{M_D}) \tag{72}$$

At the kinematic point where the leptons carry no momentum $v \cdot v' = 1$ and therefore an experimental determination of the decay width can be used to extract V_{bc}. Of course, the data must be extrapolated to this kinematic point. It can be shown that all of the $1/M_Q$ corrections to this result vanish at this special point [23], so great confidence can be given to the resulting value of V_{bc}. There are perturbative QCD corrections, but these can be computed.

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