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Semi-Classical Blocks and Correlators in Rational and Irrational Conformal Field Theory

M.B. Halpern and N.A. Obers
Physics Division

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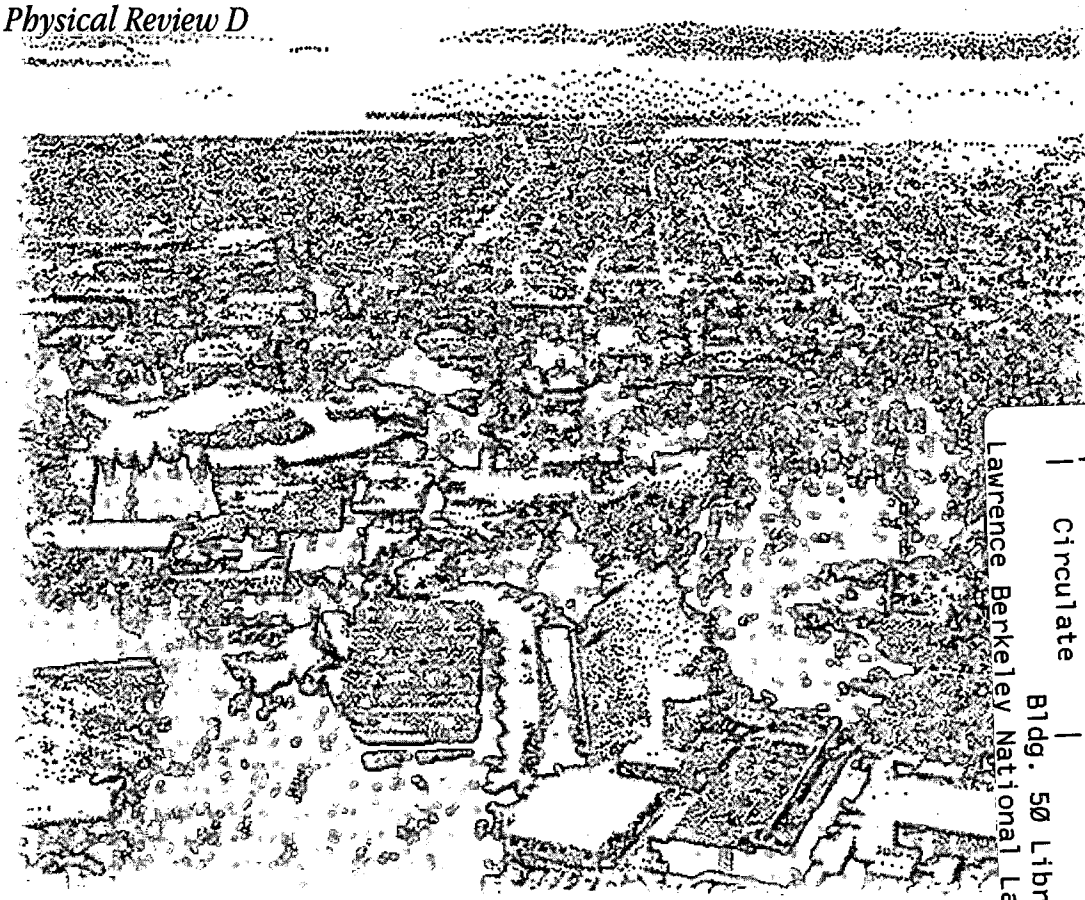

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Semi-Classical Blocks and Correlators in Rational and Irrational Conformal Field Theory

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ABSTRACT

The generalized Knizhnik-Zamolodchikov equations of irrational conformal field theory provide a uniform description of rational and irrational conformal field theory. Starting from the known high-level solution of these equations, we first construct the high-level conformal blocks and correlators of all the affine-Sugawara and coset constructions on simple g. Using intuition gained from these cases, we then identify a simple class of irrational processes whose high-level blocks and correlators we are also able to construct.

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1 Introduction

In recent years we have learned that the generic conformal field theory has irrational central charge, even when the theory is unitary. The study of this subject is called irrational conformal field theory (ICFT), which properly includes rational conformal field theory (RCFT) as a small subspace,

$$\text{ICFT} \supset\supset \text{RCFT} \tag{1.1}$$

where RCFT is understood here as the affine-Sugawara [1-6] and coset constructions [1,2,7,8]. A comprehensive review of ICFT is found in Ref.[9].

The foundation of ICFT is affine Lie algebra [10,1] and the general affine-Virasoro construction [11,12],

$$T = L^{ab} \,{:}\, J_a J_b \,{:} \tag{1.2}$$

on the currents J_a, $a = 1 \ldots \dim g$ of the general affine algebra. The construction (1.2) is summarized by the Virasoro master equation [11,12] for the inverse inertia tensor L^{ab}, and the system may be understood as a conformal spinning top.

The solutions of the master equation show a symmetry hierarchy [13] in ICFT,

$$\text{ICFT} \supset\supset H\text{-invariant CFTs} \supset\supset \text{Lie } h\text{-invariant CFTs} \supset\supset \text{RCFT} \tag{1.3}$$

where the H-invariant CFTs, which are also generically irrational, include all theories with a symmetry H, where H may be a finite group or a Lie group. In this hierarchy, the RCFTs are understood as special cases of exceptionally high symmetry, with ever-increasing symmetry breakdown to the left. The generic ICFT is completely asymmetric.

The central computational tools of the subject are the generalized Knizhnik-Zamolodchikov (KZ) equations of ICFT [14], which provide a unified description of rational and irrational conformal field theory, including powerful new tools for RCFT. In particular, the recent solution of these equations for the general coset correlators [15,16,14] appears to be inaccessible by other methods.

Moreover, the semi-classical or high-level solution of the generalized KZ equations has been known for some time, providing a uniform and apparently simple description of all ICFT $\supset\supset$ RCFT on simple g. The high-level solution is deceptively simple, however, because it is expressed in a Lie algebra basis, which is not the block basis in which conformal blocks are conventionally expressed, and it is only in solving the general problem.

- Lie algebra basis $\rightarrow$ block basis

that one confronts the full complexity of the ever-increasing symmetry breakdown of ICFT.

In this paper we begin the study of the known high-level solutions, obtaining the high-level conformal blocks and non-chiral correlators of the simplest and most symmetric cases.

In particular, we will first find closed-form expressions for the high-level conformal blocks and correlators of all the affine-Sugawara and coset constructions. Both results are new.

Using intuition gained in this analysis, we then identify what we believe to be the simplest and most symmetric class of correlators in ICFT, which we call

- the $L(g;H)$-degenerate processes in the H-invariant CFTs.

This is the set of correlators each of whose external states has completely degenerate conformal weights. The set includes all the affine-Sugawara correlators, a highly-symmetric set of coset correlators and a presumably large set of irrational correlators, examples of which are known. For this class of processes, we are also able to find general expressions for the high-level conformal blocks and non-chiral correlators, and we discuss an irrational example with octahedral symmetry in some detail.

2 The High-Level Chiral Correlators of ICFT

Our starting point is the set of high-level four-point chiral correlators of ICFT,

$$Y_L^\sigma(y) = v_g^\beta[\mathbb{1} + 2L_\infty^{ab}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y))]_\beta{}^\sigma + \mathcal{O}(k^{-2}) \tag{2.1a}$$

$$L_\infty^{ab} = \frac{P^{ab}}{2k} \quad , \quad a,b = 1 \ldots \dim g \tag{2.1b}$$

on simple[a] compact G, where G is the Lie group whose algebra is g, and k is the level of affine g. These correlators were conjectured in Ref.[15], derived in Ref.[16], and were also obtained as solutions of the generalized Knizhnik-Zamolodchikov (KZ) equations of ICFT in Ref.[14].

In what follows, we discuss the conventions, notation and concepts involved in the result (2.1).

A. Logarithms. The variable y in (2.1a) is complex, and the logarithms in (2.1a) are defined with natural cuts: $\ln y$ is defined for $|\arg(y)| < \pi$, with its cut to the left, and $\ln(1-y)$ is defined for $|\arg(1-y)| < \pi$ (or equivalently $|\arg(-y)| < \pi$), with its cut to the right for $y > 1$.

B. Inverse inertia tensors. The symmetric matrix L_∞^{ab} in (2.1a,b) is the high-level form $L \rightarrow L_\infty$ of the inverse inertia tensor of any high-level smooth solution of the Virasoro master equation. The matrix P^{ab}, which must solve the relation [17,18]

$$P^{ac}\eta_{cd}P^{db} = P^{ab} \tag{2.2}$$

is the high-level projector of the L theory and η_{ab} is the Killing metric of g. The high-level central charge of the theory is $c(L_\infty) = \text{rank } P$.

[a] The chiral correlators (2.1) and the results of this paper also apply to ICFT on semisimple compact $g = \oplus_I g_I$ with $k_I = k, \forall I$.

The chiral correlators (2.1) provide a uniform high-level description of the rational and irrational conformal field theories on g, including

$$P_g^{ab} = \eta^{ab} \quad , \quad P_{g/h}^{ab} = P_g^{ab} - P_h^{ab} \tag{2.3}$$

for the affine-Sugawara and coset constructions respectively, where η^{ab} is the inverse Killing metric of g. More generally, the projectors P are closely related to the adjacency matrices of graph theory [19] and generalized graph theory [20] in the partial classification of ICFT. For example, one has [19]

$$P_{ij,kl} = \theta_{ik}(\mathcal{G}_n)\delta_{ij,kl} \quad , \quad 1 \le i < j \le n \quad , \quad 1 \le k < l \le n \tag{2.4}$$

in the graph theory ansatz on $SO(n)$, where $a = (ij)$ is the adjoint index and $\theta(\mathcal{G}_n)$ is the adjacency matrix of any graph $\mathcal{G}_n$ of order n. The level-families classified by the graphs and generalized graphs are generically unitary and irrational on non-negative integer levels of the affine algebras.

C. Matrix irreps. The matrices

$$(\mathcal{T}_a^i)_{\alpha_i}{}^{\beta_i} \quad , \quad \alpha_i, \beta_i = 1 \ldots \dim \mathcal{T}^i \quad , \quad i = 1 \ldots 4 \tag{2.5}$$

are irreducible matrix representations (irreps) of g, which satisfy

$$[\mathcal{T}_a, \mathcal{T}_b] = i f_{ab}{}^c \mathcal{T}_c \quad , \quad a, b, c = 1 \ldots \dim g \tag{2.6}$$

where $f_{ab}{}^c$ are the structure constants of g. The labels $\alpha, \beta, \ldots$ are composite indices, e.g. $\alpha = (\alpha_1\alpha_2\alpha_3\alpha_4)$, and multiplication of irreps is by tensor product, so that

$$(\mathbb{1})_\alpha^\beta = \delta_\alpha^\beta \equiv \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}\delta_{\alpha_3}^{\beta_3}\delta_{\alpha_4}^{\beta_4} \tag{2.7a}$$

$$(\mathcal{T}_a^1)_\alpha{}^\beta \equiv (\mathcal{T}_a^1)_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}^{\beta_2}\delta_{\alpha_3}^{\beta_3}\delta_{\alpha_4}^{\beta_4} \tag{2.7b}$$

$$(\mathcal{T}_a^1\mathcal{T}_b^2)_\alpha{}^\beta \equiv (\mathcal{T}_a^1)_{\alpha_1}{}^{\beta_1}(\mathcal{T}_b^2)_{\alpha_2}{}^{\beta_2}\delta_{\alpha_3}^{\beta_3}\delta_{\alpha_4}^{\beta_4} \quad . \tag{2.7c}$$

D. Broken affine primary fields. The chiral correlators (2.1) may be understood schematically as the high-level form of the averages

$$Y_L^\alpha \sim \langle R_L^{\alpha_1}(\mathcal{T}^1)R_L^{\alpha_2}(\mathcal{T}^2)R_L^{\alpha_3}(\mathcal{T}^3)R_L^{\alpha_4}(\mathcal{T}^4)\rangle \tag{2.8}$$

where $R_L^\alpha(\mathcal{T})$, $\alpha = 1 \ldots \dim \mathcal{T}$ is the broken affine primary field of the L theory corresponding to irrep $\mathcal{T}$ of g. The correlators are written assuming an L-basis [15] for each $\mathcal{T}^i$, where the conformal weight matrix of the broken affine primary field $R_L^{\alpha_i}(\mathcal{T}^i)$ is diagonal.

$$(L^{ab}\mathcal{T}_a^i\mathcal{T}_b^i)_{\alpha_i}{}^{\beta_i} = \Delta_{\alpha_i}(\mathcal{T}^i)\delta_{\alpha_i}^{\beta_i} \quad , \quad \Delta_{\alpha_i}(\mathcal{T}^i) = \mathcal{O}(k^{-1}) \quad . \tag{2.9}$$

Fig.1 shows our conventions for the s and t-channels of the correlators, and the 13 channel is the u-channel.

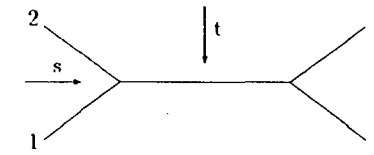

Fig. 1. The correlators.

In ICFT, broken affine primary states are the only states whose conformal weights are $\mathcal{O}(k^{-1})$. In the affine-Sugawara constructions, conformal weights of the form integer plus $\mathcal{O}(k^{-1})$ are integer descendants of affine primary states, but this is not necessarily true for the coset constructions[b] and beyond, where we know only that the corresponding states are broken affine secondary.

E. Global Ward identity. The objects v_g^α are arbitrary linear combinations of g-invariant tensors of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$, which satisfy the g-global Ward identity,

$$v_g^\beta(\sum_{i=1}^4 \mathcal{T}_a^i)_\beta{}^\alpha = 0 \quad , \quad a = 1 \ldots \dim g \quad . \tag{2.10}$$

F. Hermiticity. The matrix irreps $\mathcal{T}_a$ satisfy the hermiticity condition,

$$\mathcal{T}_a^\dagger = \rho_a{}^b \mathcal{T}_b \tag{2.11a}$$

$$(\mathcal{T}_a^\dagger)_\alpha{}^\beta \equiv \eta_{\alpha\rho}\eta^{\beta\sigma}(\mathcal{T}_a)_\sigma{}^{\rho *} \tag{2.11b}$$

where star is complex conjugation and $\eta_{\alpha\beta} = \eta_{\alpha\beta}^*$ is the carrier space metric of irrep $\mathcal{T}$. Moreover, we will consider only unitary theories (non-negative integer level of the affine algebra and $L^\dagger(m) = L(-m)$), for which the inverse inertia tensor satisfies

$$L^{ab*} = L^{cd}(\rho^{-1})_c{}^a(\rho^{-1})_d{}^b \tag{2.12}$$

and similarly for L_∞. It follows that all the matrices in (2.1) are hermitean, e.g.

$$(2L_\infty^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)^\dagger = 2L_\infty^{ab}\mathcal{T}_a^1\mathcal{T}_b^2 \tag{2.13}$$

with orthonormal, complete sets of eigenvectors and real eigenvalues.

G. $SL(2,\mathbb{R})$ gauge. The chiral correlators (2.1) are given in the 2-3 symmetric KZ gauge [4],

$$Y^\alpha(y) = (\prod_{i<j}^4 z_{ij}^{\gamma_{ij}})A^\alpha(z_1, z_2, z_3, z_4) \quad ; \quad y = \frac{z_{12}z_{34}}{z_{14}z_{32}} \tag{2.14a}$$

$$\gamma_{12} = \gamma_{13} = 0 \quad . \quad \gamma_{14} = 2\Delta_{\alpha_1} \quad . \quad \gamma_{23} = \Delta_{\alpha_1} + \Delta_{\alpha_2} + \Delta_{\alpha_3} - \Delta_{\alpha_4} \tag{2.14b}$$

[b] See for example the conformal weights of Ref.[15] under the coset construction $(SU(n)_{k_1} \times SU(n)_{k_2})/SU(n)_{k_1+k_2}$ when $k_1 = k_2 = k$.

$$\gamma_{24} = -\Delta_{\alpha_1} + \Delta_{\alpha_2} - \Delta_{\alpha_3} + \Delta_{\alpha_4} \quad , \quad \gamma_{34} = -\Delta_{\alpha_1} - \Delta_{\alpha_2} + \Delta_{\alpha_3} + \Delta_{\alpha_4} \tag{2.14c}$$

where $A^\alpha(z)$ are the non-invariant chiral four-point correlators.

H. Limiting behavior. For any conformal field theory in the KZ gauge, the conformal weights $\Delta_{(s)}$, $\Delta_{(u)}$ and $\Delta_{(t)}$ of the s, u and t-channel intermediate states appear in the limiting behavior,

$$Y^\alpha(y) \sim \begin{cases} y^{\Delta_{(s)}-\Delta_{\alpha_1}(\mathcal{T}^1)-\Delta_{\alpha_2}(\mathcal{T}^2)} & , \ y \to 0 \\ (1-y)^{\Delta_{(u)}-\Delta_{\alpha_1}(\mathcal{T}^1)-\Delta_{\alpha_3}(\mathcal{T}^3)} & , \ y \to 1 \\ \left(\frac{1}{y}\right)^{\Delta_{(t)}+\Delta_{\alpha_1}(\mathcal{T}^1)-\Delta_{\alpha_4}(\mathcal{T}^4)} & , \ y \to \infty \end{cases} . \tag{2.15}$$

Here, we will use these facts in the high-level form

$$y^{-\Delta_{\alpha_1}(\mathcal{T}^1)-\Delta_{\alpha_2}(\mathcal{T}^2)} = 1 - [\Delta_{\alpha_1}(\mathcal{T}^1) + \Delta_{\alpha_2}(\mathcal{T}^2)] \ln y + \mathcal{O}(k^{-2}) \tag{2.16}$$

where we have recalled that the conformal weights of the broken affine primary fields are $\mathcal{O}(k^{-1})$.

I. High-level OPEs. In Ref.[16,14], it was shown that the high-level chiral correlators (2.1) have physical singularities in all channels, and that the high-level fusion rules among the broken affine primaries follow the Clebsch-Gordan coefficients of their corresponding matrix irreps. In further detail, the high-level OPEs of the broken affine primaries can be written schematically as

$$\begin{aligned} R(\mathcal{T}^1, z)^{\alpha_1} R(\mathcal{T}^2, w)^{\alpha_2} &= \sum_{i,\alpha_i} \frac{[C(\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^i) + \mathcal{O}(k^{-1})]^{\alpha_1\alpha_2}{}_{\alpha_i} R(\mathcal{T}^i, w)^{\alpha_i}}{(z-w)^{\Delta_{\alpha_1}(\mathcal{T}^1)+\Delta_{\alpha_2}(\mathcal{T}^2)-\Delta_{\alpha_i}(\mathcal{T}^i)}} \\ &\quad + \mathcal{O}(k^{-1}) \cdot (\text{broken affine secondaries}) \end{aligned} \tag{2.17}$$

where the level-independent tensor $C(\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^i)$ is proportional to the Clebsch-Gordan coefficients and the broken affine secondaries enter only at the next order of the high-level expansion.

Symmetry hierarchy in ICFT

The high-level correlators (2.1) provide a uniform description of all ICFT on simple g, which is a bewildering variety [9] of theories and correlators. In this paper we make the first attempt to identify simpler, more symmetric correlators among these varieties.

Towards this end, we remind the reader of the symmetry hierarchy [13] in ICFT,

$$\text{ICFT} \supset\supset H\text{-invariant CFTs} \supset\supset \text{Lie } h\text{-invariant CFTs} \supset\supset \text{RCFT} \tag{2.18}$$

which organizes the space of ICFTs on G according to the residual symmetry group $H \subset G$ of the theory. As seen in this hierarchy, the generic ICFT has no residual symmetry group[c], and these generic theories are expected to be the most complex. Consequently, we focus here on the theories with a symmetry, which are also generically irrational.

[c] In the graph theory ansatz [19] on $SO(n)$, whose high-level projectors are given in (2.4), this corresponds to the fact that the generic graph has no symmetry.

The set of all ICFTs with a non-trivial symmetry group H (which may be a discrete subgroup of G or a Lie subgroup) is called the set of H-invariant CFTs. Among the H-invariant CFTs, the subspace of theories with a Lie symmetry is called the set of Lie h-invariant CFTs, where $h \subset g$. This subspace includes the affine-Sugawara and coset constructions as a much smaller subspace.

When a theory L is an H-invariant CFT, the correlators (2.1) also satisfy the global H-invariance condition,

$$Y_H \Omega(H) = Y_H \quad , \quad \Omega(H) \in G \quad , \quad \Omega(H)_\alpha{}^\beta = \prod_{i=1}^{4} \Omega(H, \mathcal{T}^i)_{\alpha_i}{}^{\beta_i} \tag{2.19}$$

where $\Omega(H, \mathcal{T}^i)_{\alpha_i}{}^{\beta_i}$ is the subgroup H in matrix irrep $\mathcal{T}^i$. When the theory is a Lie h-invariant CFT, the condition (2.19) reduces to the h-global Ward identity

$$Y_{\text{Lie}\,h} \sum_{i=1}^{4} \mathcal{T}_a^i = 0 \quad , \quad a = 1 \ldots \dim h \tag{2.20}$$

which applies for example in the cases of the affine-Sugawara construction (with $h = g$) and the g/h coset constructions.

For the affine-Sugawara and g/h coset constructions, it is known [4,15] that the resolution of chiral correlators into conformal blocks is a basis change from the Lie algebra basis to the block basis, using the h-invariant tensors defined by (2.20). More generally, one expects that the H-invariant tensors defined by (2.19) will play an analogous role in finding the block bases of the H-invariant CFTs.

3 The Affine-Sugawara Constructions

3.1 The affine-Sugawara blocks

The simplest and most symmetric conformal field theories are the affine-Sugawara constructions [1-6] on G, whose high-level correlators are described by (2.1) with

$$L_{g,\infty}^{ab} = \frac{\eta^{ab}}{2k} \tag{3.1a}$$

$$Y_g(y) \sum_{i=1}^{4} \mathcal{T}_a^i = 0 \quad , \quad a = 1 \ldots \dim g \tag{3.1b}$$

where $L_g^{ab} = \eta^{ab}$ is the inverse Killing metric of g. In this case, the correlators (2.1) are the high-level solutions of the KZ equations [3,4] for any correlator on simple g.

We begin by defining the s-channel block basis of g-invariants $v(s, g)^m$ as the solutions of the simultaneous eigenvalue problem and g-global condition

$$(2L_{g,\infty}^{ab} \mathcal{T}_a^1 \mathcal{T}_b^2)_\alpha{}^\beta v(s, g)_\beta^m = (\Delta_{(s)}^g(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)) v(s, g)_\alpha^m \tag{3.2a}$$

$$\sum_{i=1}^{4}(\mathcal{T}_a^i)_\alpha{}^\beta v(\mathrm{s},g)_\beta^m = 0 \quad , \quad a = 1\ldots\dim g \quad . \tag{3.2b}$$

The g-global condition (3.2b) is compatible with the eigenvalue problem because the generators $\sum_{i=1}^4 \mathcal{T}_a^i$ commute with $L_g^{ab}\mathcal{T}_a^1\mathcal{T}_b^2$. Here $\Delta^g(\mathcal{T}^i)$, $i = 1,2$ are the high-level forms of the conformal weights of external affine primary states,

$$\Delta_{\alpha_i}(\mathcal{T}^i)|_{L=L_g} = \Delta^g(\mathcal{T}^i) = \frac{I(\mathcal{T}^i)}{x+\tilde{h}} = \frac{I(\mathcal{T}^i)}{x} + \mathcal{O}(x^{-2}) \tag{3.3a}$$

$$x = \frac{2k}{\psi_g^2} \tag{3.3b}$$

where ψ_g, $\tilde{h}$, $I(\mathcal{T})$ and x are respectively the highest root and dual Coxeter number of g, the invariant Casimir of irrep $\mathcal{T}$ and the invariant level of the affine algebra. The high-level form of the relation

$$2L_g^{ab}\mathcal{T}_a^1\mathcal{T}_b^2 = L_g^{ab}(\mathcal{T}_a^1+\mathcal{T}_a^2)(\mathcal{T}_b^1+\mathcal{T}_b^2) - (\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^2))\mathbb{1} \tag{3.4}$$

tells us that the quantities in (3.2a)

$$\Delta^g_{(\mathrm{s})}(m) \equiv \Delta^g(\mathcal{T}^m) \tag{3.5}$$

are the high-level conformal weights of an irrep $\mathcal{T}^m$ in $\mathcal{T}^1 \otimes \mathcal{T}^2$, hence the conformal weights of affine primary states exchanged in the s-channel. The dual eigenvalue problem is

$$\bar{v}(\mathrm{s},g)_m^\beta(2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2)_\beta{}^\alpha = \bar{v}(\mathrm{s},g)_m^\alpha(\Delta^g_{(\mathrm{s})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)) \tag{3.6a}$$

$$\bar{v}(\mathrm{s},g)_m^\beta \sum_{i=1}^{4}(\mathcal{T}_a^i)_\beta{}^\alpha = 0 \quad , \quad a = 1\ldots\dim g \tag{3.6b}$$

where $\bar{v}(\mathrm{s},g)_m^\alpha = v(\mathrm{s},g)_\beta^{m*}\eta^{\beta\alpha}$ and $\eta_{\alpha\beta} = \prod_{i=1}^4 \eta_{\alpha_i\beta_i}$ is the product of the carrier space metrics.

Because $2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2$ is hermitean we know that the eigenvectors are orthonormal and complete,

$$\bar{v}(\mathrm{s},g)_m v(\mathrm{s},g)^n = \delta_m^n \quad , \quad v(\mathrm{s},g)_\alpha^m \bar{v}(\mathrm{s},g)_m^\beta = (I_g)_\alpha^\beta \tag{3.7}$$

where I_g is the projector onto the G-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$. The relation

$$[L_{g,\infty}^{ab}\mathcal{T}_a^i\mathcal{T}_b^j, I_g] = 0 \quad , \quad 1 \le i,j \le 4 \tag{3.8}$$

also holds on the G-invariant subspace defined by (3.2). An explicit solution to the eigenvalue problem and global condition in (3.2) is known [16]

$$v(\mathrm{s},g)_m^\alpha = \sum_{\alpha_r\alpha_r} w_s(r,\xi)^{\alpha_1\alpha_2\alpha_r} w_s(r,\xi')^{\alpha_3\alpha_4\alpha_r}\eta_{\alpha_r\alpha_r} \quad , \quad m = (r,\xi,\xi') \tag{3.9a}$$

$$w_s(r,\xi)^{\beta_1\beta_2\beta_r}(\mathcal{T}_a^1+\mathcal{T}_a^2+\mathcal{T}_a^r)_{\beta_1\beta_2\beta_r}{}^{\alpha_1\alpha_2\alpha_r} = 0 \quad , \quad a = 1\ldots\dim g \tag{3.9b}$$

$$w_s(r,\xi)^{\beta_1\beta_2\alpha_r}[2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2]_{\beta_1\beta_2}{}^{\alpha_1\alpha_2} = w_s(r,\xi)^{\alpha_1\alpha_2\alpha_r}[\Delta^g(\mathcal{T}^r) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)] \tag{3.9c}$$

where $w_s(r,\xi)^{\alpha_1\alpha_2\alpha_r}$ are the Clebsch-Gordan coefficients of $\mathcal{T}^i \otimes \mathcal{T}^j$ into irrep $\mathcal{T}^r$, ξ labels copies of the same irrep $\mathcal{T}^r$ and $\bar{r}$ is the conjugate representation of r. Using (3.9), it is easy to check directly that $\Delta^g_{(\mathrm{s})}(m) = \Delta^g(\mathcal{T}^m)$ in (3.2a) is the conformal weight of irrep m under the affine-Sugawara construction.

As an explicit example, one finds for $n\bar{n}\bar{n}n$ correlators on $SU(n)$ that the invariant tensors (3.9) are

$$v(\mathrm{s},SU(n))_V^\alpha = v(\mathrm{s},SU(n))_\alpha^V = \frac{1}{n}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \tag{3.10a}$$

$$v(\mathrm{s},SU(n))_A^\alpha = v(\mathrm{s},SU(n))_\alpha^A = \frac{1}{\sqrt{n^2-1}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} - \frac{1}{n}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}] \tag{3.10b}$$

where V and A are vacuum and adjoint. This is the original example [4] considered by Knizhnik and Zamolodchikov, although our Clebsch basis (3.9), (3.10) is slightly different than theirs (see Appendix B).

From (2.1),(3.1) and the completeness relation (3.7), we use eigenvector expansions to define the s-channel conformal blocks $\mathcal{F}_g^{(\mathrm{s})}(y)$ of the affine-Sugawara construction

$$\bar{v}_g^\alpha = \sum_m d(\mathrm{s})^m \bar{v}(\mathrm{s},g)_m^\alpha \tag{3.11a}$$

$$Y_g^\alpha(y) = \sum_{m,n} d(\mathrm{s})^m \mathcal{F}_g^{(\mathrm{s})}(y)_m{}^n \bar{v}(\mathrm{s},g)_n^\alpha \tag{3.11b}$$

$$\mathcal{F}_g^{(\mathrm{s})}(y)_m{}^n = \bar{v}(\mathrm{s},g)_m[\mathbb{1} + 2L_{g,\infty}^{ab}(\mathcal{T}_a^1\mathcal{T}_b^2\ln y + \mathcal{T}_a^1\mathcal{T}_b^3\ln(1-y))]v(\mathrm{s},g)^n + \mathcal{O}(k^{-2}) \tag{3.11c}$$

as the coefficients of the chiral correlators expanded in the block basis. Here, $d(\mathrm{s})^m$ are a set of undetermined constants.

To study the small y behavior of the s-channel blocks, we rearrange (3.11c) as follows,

$$\mathcal{F}_g^{(\mathrm{s})}(y)_m{}^n = \bar{v}(\mathrm{s},g)_m[\mathbb{1} + 2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2\ln y][\mathbb{1} + 2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^3\ln(1-y)]v(\mathrm{s},g)^n + \mathcal{O}(k^{-2}) \tag{3.12a}$$

$$= [1 + (\Delta^g_{(\mathrm{s})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2))\ln y] \times \bar{v}(\mathrm{s},g)_m[\mathbb{1} + 2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^3\ln(1-y)]v(\mathrm{s},g)^n + \mathcal{O}(k^{-2}) \tag{3.12b}$$

$$= y^{\Delta^g_{(\mathrm{s})}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)}\left[\delta_m^n - c(\mathrm{s},g)_m{}^n\sum_{p=1}^{\infty}\frac{y^p}{p}\right] + \mathcal{O}(k^{-2}) \tag{3.12c}$$

$$c(\mathrm{s},g)_m{}^n = \bar{v}(\mathrm{s},g)_m 2L_{g,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^3 v(\mathrm{s},g)^n \tag{3.12d}$$

where we have used the dual eigenvalue problem (3.6a) to obtain (3.12b) and the high-level relation (2.16) to obtain (3.12c). We note in particular that the eigenvector resolution correctly guarantees that each block has a unique leading singularity.

$$\mathcal{F}_g^{(\mathrm{s})}(y)_m{}^n \underset{y\to 0}{\sim} \Gamma_g^{(\mathrm{s})}(m,n)y^{\Delta^g_{(\mathrm{s})}(m,n) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2)} + \mathcal{O}(k^{-2}) \tag{3.13a}$$

$$\Delta^g_{(\mathrm{s})}(m,n) = \begin{cases} \Delta^g_{(\mathrm{s})}(m) + \mathcal{O}(k^{-2}) & , \ n = m \\ 1 + \mathcal{O}(k^{-1}) & , \ n \neq m \end{cases} \tag{3.13b}$$

$$\Gamma_g^{(s)}(m,n) = \begin{cases} 1 + \mathcal{O}(k^{-2}) & , \ n = m \\ -c(s,g)_m{}^n + \mathcal{O}(k^{-2}) & , \ n \neq m \end{cases} \tag{3.13c}$$

labelled by m and n, which is followed by integer-spaced secondaries from $\ln(1-y)$. According to eqs.(2.15) and (3.13), the leading singularities of the $n = m$ blocks correspond to the s-channel exchange of affine primary states, with residue $\Gamma_g(m,m) = \mathcal{O}(k^0)$, while the leading singularities of the $n \neq m$ blocks are affine secondaries, with $\Gamma_g(m, n \neq m) = \mathcal{O}(k^{-1})$. This pattern is in agreement with the general OPE (2.17). Beyond the leading residues, diagonal blocks begin at $\mathcal{O}(k^0)$ and off-diagonal blocks begin at $\mathcal{O}(k^{-1})$.

If $c(s,g)_m{}^n = 0$ for some $n \neq m$, then this block begins at $\mathcal{O}(k^{-2})$, and we obtain no information beyond this fact in our approximation. Although we are not aware of any examples of this phenomenon among the affine-Sugawara blocks, examples do occur in the coset constructions and irrational processes (see Appendix D and Section 6).

We also note that, although we have solved the generalized KZ equations through $\mathcal{O}(k^{-1})$, we are not able to determine the $\mathcal{O}(k^{-1})$ part of the $n \neq m$ conformal weights in this approximation. Of course, under the affine-Sugawara constructions all conformal weights have the form $\Delta^g(\mathcal{T})$+integer, so we can guess the exact result

$$\Delta^g_{(s)}(m,n) = \Delta^g_{(s)}(m) + 1 - \delta_{m,n} \quad , \quad \forall\, m,n \tag{3.14}$$

for the conformal weights of the blocks, which we believe to be correct (see Appendix B).

To define block bases for the other channels, we also introduce the u and t-channel g-invariants as solutions to their corresponding eigenvalue problems,

$$2L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^3_b v(u,g)^m = (\Delta^g_{(u)}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^3))v(u,g)^m \tag{3.15a}$$

$$2L^{ab}_{g,\infty}\mathcal{T}^2_a\mathcal{T}^3_b v(t,g)^m = (\Delta^g_{(t)}(m) - \Delta^g(\mathcal{T}^2) - \Delta^g(\mathcal{T}^3))v(t,g)^m \tag{3.15b}$$

$$(\sum_{i=1}^{4}\mathcal{T}^i_a)v(u,g)^m = (\sum_{i=1}^{4}\mathcal{T}^i_a)v(t,g)^m = 0 \quad , \quad a = 1\ldots\dim g \tag{3.15c}$$

$$\bar{v}(u,g)_m v(u,g)^n = \bar{v}(t,g)_m v(t,g)^n = \delta^n_m \tag{3.15d}$$

$$v(u,g)^m\bar{v}(u,g)_m = v(t,g)^m\bar{v}(t,g)_m = I_g \quad . \tag{3.15e}$$

Here

$$\Delta^g_{(u)}(m) \equiv \Delta^g(\mathcal{T}^m) \quad , \quad \Delta^g_{(t)}(m') \equiv \Delta^g(\mathcal{T}^{m'}) \tag{3.16}$$

are the high-level (affine-primary) conformal weights under the affine-Sugawara construction of irreps $\mathcal{T}^m$ and $\mathcal{T}^{m'}$ in $\mathcal{T}^1 \otimes \mathcal{T}^3$ and $\mathcal{T}^2 \otimes \mathcal{T}^3$ respectively. Explicit forms of the u and t-channel invariants are obtained formally by a $2 \leftrightarrow 3$ and a $2 \leftrightarrow 4$ interchange respectively in eq.(3.9).

In analogy to the s-channel blocks $\mathcal{F}^{(s)}_g$ in eq.(3.11), we define the u-channel blocks $\mathcal{F}^{(u)}_g$ using the corresponding u-channel invariants.

$$Y_g(y) = \sum_{m,n} d(u)^m \mathcal{F}^{(u)}_g(y)_m{}^n v(u,g)_n \tag{3.17a}$$

$$\mathcal{F}^{(u)}_g(y)_m{}^n = \bar{v}(u,g)_m[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(u,g)^n + \mathcal{O}(k^{-2}) \tag{3.17b}$$

$$= (1-y)^{\Delta^g_{(u)}(m) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^3)}\left[\delta^n_m - c(u,g)_m{}^n \sum_{p=1}^{\infty}\frac{(1-y)^p}{p}\right] + \mathcal{O}(k^{-2}) \tag{3.17c}$$

$$c(u,g)_m{}^n = \bar{v}(u,g)_m 2L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^2_b v(u,g)^n \quad . \tag{3.17d}$$

The expansion (3.17c) is obtained from (3.17b) following steps analogous to those in (3.12). The limiting behavior of the u-channel blocks

$$\mathcal{F}^{(u)}_g(y)_m{}^n \underset{y\to 1}{\sim} \Gamma^{(u)}_g(m,n)(1-y)^{\Delta^g_{(u)}(m,n) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^3)} + \mathcal{O}(k^{-2}) \tag{3.18a}$$

$$\Delta^g_{(u)}(m,n) = \begin{cases} \Delta^g_{(u)}(m) + \mathcal{O}(k^{-2}) & , \ n = m \\ 1 + \mathcal{O}(k^{-1}) & , \ n \neq m \end{cases} \tag{3.18b}$$

$$\Gamma^{(u)}_g(m,n) = \begin{cases} 1 + \mathcal{O}(k^{-2}) & , \ n = m \\ -c(u,g)_m{}^n + \mathcal{O}(k^{-2}) & , \ n \neq m \end{cases} \tag{3.18c}$$

(followed by integer-spaced secondaries) is easily read from (3.17c). As seen above for the s-channel blocks, the diagonal u-channel blocks show affine primary conformal weights with residue $\mathcal{O}(k^0)$, while the off-diagonal blocks show integer descendants of affine primaries, and we are again unable to determine the $\mathcal{O}(k^{-1})$ part of the off-diagonal conformal weights.

Analytic blocks

It is clear from the discussion above that the s and u-channel blocks $\mathcal{F}^{(s)}_g$ and $\mathcal{F}^{(u)}_g$ are high-level forms of analytic blocks, but identification of the analytic t-channel blocks is more subtle. We begin by defining the preliminary t-channel blocks $\mathcal{F}^{(t)}_g$ as the coefficients in the t-channel eigenbasis

$$Y_g(y) = \sum_{m,n} d(t)^m \mathcal{F}^{(t)}_g(y)_m{}^n \bar{v}(t,g)_n \tag{3.19a}$$

$$\mathcal{F}^{(t)}_g(y)_m{}^n = \bar{v}(t,g)_m[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(t,g)^n + \mathcal{O}(k^{-2}) \tag{3.19b}$$

in parallel with our expansions above for the s and u-channels.

We must next consider continuation of the logarithms to the t-channel, for which we use the following two rules

$$\ln(1-y) = \ln(-y) + \ln\left(1 - \frac{1}{y}\right) \quad , \quad |\arg(-y)| < \pi \tag{3.20a}$$

$$\ln y = \ln(-y) - i\pi\,\mathrm{sign}(\arg(-y)) \tag{3.20b}$$

throughout this paper. The left side of (3.20b) is defined for $|\arg(y)| < \pi$ (so that $\ln y$ has its cut is to the left), while the right side of (3.20b) is defined for $|\arg(-y)| < \pi$ (so that $\ln(-y)$ has its cut to the right).

In the finite-level example of Appendix B, the relation (3.20b) is used in the equivalent form

$$y^{\nu} = (-y)^{\nu} \exp[-i\pi\nu \mathrm{sign}(\arg(-y))] \tag{3.21}$$

to continue singular s-channel factors to the t-channel. The non-analytic phase in (3.20b) and (3.21) is therefore associated to operator ordering in the four-point Green function, as discussed in Ref.[21]. The continuation (3.20a) is also seen in the example of Appendix B as the high-level limit of well-known continuation formulae for hypergeometric functions.

We must therefore factor the non-analytic phase out of the preliminary t-channel blocks to obtain the analytic t-channel blocks. More precisely, we define

$$\mathcal{F}_g^{(t)}(y)_m{}^n = \hat{\mathcal{F}}_g^{(t)}(y)_m{}^p U_g(y)_p{}^n \quad , \quad \hat{\mathcal{F}}_g^{(t)}(y)_m{}^n = \mathcal{F}_g^{(t)}(y)_m{}^p (U_g(y)^{-1})_p{}^n \tag{3.22a}$$

$$\begin{aligned} U_g(y)_p{}^n &= \bar{v}(t,g)_p \exp[-2\pi i L_{g,\infty}^{ab} \mathcal{T}_a^1 \mathcal{T}_b^2 \mathrm{sign}(\arg(-y))] v(t,g)^n + \mathcal{O}(k^{-2}) \\ &= \delta_p^n - i\pi c'(t,g)_p{}^n \mathrm{sign}(\arg(-y)) + \mathcal{O}(k^{-2}) \end{aligned} \tag{3.22b}$$

$$\begin{aligned} (U_g(y)^{-1})_p{}^n &= \bar{v}(t,g)_p \exp[2\pi i L_{g,\infty}^{ab} \mathcal{T}_a^1 \mathcal{T}_b^2 \mathrm{sign}(\arg(-y))] v(t,g)^n + \mathcal{O}(k^{-2}) \\ &= \delta_p^n + i\pi c'(t,g)_p{}^n \mathrm{sign}(\arg(-y)) + \mathcal{O}(k^{-2}) \end{aligned} \tag{3.22c}$$

$$c'(t,g)_p{}^n = \bar{v}(t,g)_p 2L_{g,\infty}^{ab} \mathcal{T}_a^1 \mathcal{T}_b^2 v(t,g)^n \tag{3.22d}$$

$$U_g(y^*) = U_g(y)^{-1} \quad , \quad U_g(y)^\dagger = U_g(y)^{-1} \tag{3.22e}$$

where $\hat{\mathcal{F}}_g^{(t)}(y)$ are the analytic t-channel blocks and $U_g(y)_m{}^n$ is the non-analytic unitary phase matrix of the affine-Sugawara constructions (unitary because the sign function is real). Then we find the explicit form of the analytic t-channel blocks,

$$\hat{\mathcal{F}}_g^{(t)}(y)_m{}^n = \bar{v}(t,g)_m [\mathbb{1} + 2L_{g,\infty}^{ab} (\mathcal{T}_a^1 [\mathcal{T}_b^2 + \mathcal{T}_b^3] \ln(-y) + \mathcal{T}_a^1 \mathcal{T}_b^3 \ln\left(1 - \frac{1}{y}\right))] v(t,g)^n + \mathcal{O}(k^{-2}) \tag{3.23a}$$

$$\begin{aligned} &= \bar{v}(t,g)_m [\mathbb{1} - (2L_{g,\infty}^{ab} \mathcal{T}_a^2 \mathcal{T}_b^3 + \sum_{i=1}^{3} \Delta^g(\mathcal{T}^i) - \Delta^g(\mathcal{T}^4)) \ln(-y)] \\ &\qquad \times [\mathbb{1} + 2L_{g,\infty}^{ab} \mathcal{T}_a^1 \mathcal{T}_b^3 \ln\left(1 - \frac{1}{y}\right)] v(t,g)^n + \mathcal{O}(k^{-2}) \end{aligned} \tag{3.23b}$$

$$= (-y)^{-\Delta_{(t)}^g(m) - \Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^4)} \left[\delta_m^n - c(t,g)_m{}^n \sum_{p=1}^{\infty} \left(\frac{1}{y}\right)^p \frac{1}{p} \right] + \mathcal{O}(k^{-2}) \tag{3.23c}$$

$$c(t,g)_m{}^n = v(t,g)_m 2L_{g,\infty}^{ab} \mathcal{T}_a^1 \mathcal{T}_b^3 v(t,g)^n \quad . \tag{3.23d}$$

To obtain (3.23b), we have used the identity

$$v(t,g)_m [2L_{g,\infty}^{ab} (\mathcal{T}_a^1 \mathcal{T}_b^2 + \mathcal{T}_a^2 \mathcal{T}_b^3 + \mathcal{T}_a^3 \mathcal{T}_b^1) - \gamma_g \mathbb{1}] = 0 \tag{3.24a}$$

$$\gamma_g = \Delta^g(\mathcal{T}^4) - \Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^2) - \Delta^g(\mathcal{T}^3) \tag{3.24b}$$

which follows from the g-global Ward identity (3.15c).

From (3.23c) we read the limiting behavior of the analytic t-channel blocks

$$\hat{\mathcal{F}}_g^{(t)}(y)_m{}^n \underset{y\to\infty}{\sim} \Gamma_g^{(t)}(m,n)(-y)^{-\Delta_{(t)}^g(m,n) - \Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^4)} + \mathcal{O}(k^{-2}) \tag{3.25a}$$

$$\Delta_{(t)}^g(m,n) = \begin{cases} \Delta_{(t)}^g(m) + \mathcal{O}(k^{-2}) & , \ n = m \\ 1 + \mathcal{O}(k^{-1}) & , \ n \neq m \end{cases} \tag{3.25b}$$

$$\Gamma_g^{(t)}(m,n) = \begin{cases} 1 + \mathcal{O}(k^{-2}) & , \ n = m \\ -c(t,g)_m{}^n + \mathcal{O}(k^{-2}) & , \ n \neq m \end{cases} \tag{3.25c}$$

and the remarks below (3.13) apply in this case as well. In particular, one might guess the exact u and t-channel results

$$\Delta_{(u)}^g(m,n) = \Delta_{(u)}^g(m) + 1 - \delta_{m,n} \quad , \quad \forall\, m,n \tag{3.26a}$$

$$\Delta_{(t)}^g(m,n) = \Delta_{(t)}^g(m) + 1 - \delta_{m,n} \quad , \quad \forall\, m,n \tag{3.26b}$$

which are in agreement with the KZ example in Appendix B.

In what follows, we introduce a unified notation $\rho = \mathrm{s, t, u}$ for the three channels and their corresponding blocks $(\mathcal{F}_g^{(\rho)})_m{}^n$,

$$\bar{v}(\rho,g)_m v(\rho,g)^n = \delta_m^n \quad , \quad v(\rho,g)_\alpha^m \bar{v}(\rho,g)_m^\beta = (I_g)_\alpha^\beta \tag{3.27a}$$

$$Y_g(y) = \sum_{m,n} d(\rho)^m \mathcal{F}_g^{(\rho)}(y)_m{}^n \bar{v}(\rho,g)_n \quad , \quad \rho = \mathrm{s,t,u} \tag{3.27b}$$

$$\mathcal{F}_g^{(\rho)}(y)_m{}^n = \bar{v}(\rho,g)_m [\mathbb{1} + 2L_{g,\infty}^{ab} (\mathcal{T}_a^1 \mathcal{T}_b^2 \ln y + \mathcal{T}_a^1 \mathcal{T}_b^3 \ln(1-y))] v(\rho,g)^n + \mathcal{O}(k^{-2}) \tag{3.27c}$$

$$(\mathcal{F}_g^{(\rho)}(y)_m{}^n)^* = \mathcal{F}_g^{(\rho)}(y^*)_n{}^m \tag{3.27d}$$

where the last relation follows by unitarity, that is, hermiticity of the basic matrices in the correlators.

We finally note that the number B_g of affine-Sugawara blocks in each of the channels,

$$B_g = (d_g)^2 \tag{3.28}$$

is equal to the square of the dimension d_g of the g-invariants in any channel.

For the special case of the 3333 correlator on $SU(3)$, Appendix B provides a check of our high-level blocks against the finite-level blocks obtained by Knizhnik and Zamolodchikov [4] in this case.

Crossing relations

Using completeness of the three sets of eigenvectors, one finds that the three sets of blocks are related by the crossing relations.

$$\mathcal{F}_g^{(\rho)}(y)_m{}^n = [X_g(\rho\sigma) + \mathcal{O}(k^{-2})]_m{}^p \mathcal{F}_g^{(\sigma)}(y)_p{}^q ([X_g(\rho\sigma) + \mathcal{O}(k^{-2})]^{-1})_q{}^n \tag{3.29a}$$

$$= X_g(\rho\sigma)_m{}^p \mathcal{F}_g^{(\sigma)}(y)_p{}^q X_g^{-1}(\rho\sigma)_q{}^n + \mathcal{O}(k^{-2}) \quad . \quad \rho,\sigma = \mathrm{s,t,u} \tag{3.29b}$$

$$X_g(\rho\sigma)_m{}^n = v(\rho,g)_m v(\sigma,g)^n \tag{3.29c}$$

$$X_g^{-1}(\rho\sigma)_m{}^n = X_g(\sigma\rho)_m{}^n = (X_g(\rho\sigma)_n{}^m)^* \tag{3.29d}$$

where $\rho \neq \sigma$ and we call $X_g(\rho\sigma)$ in (3.29c) the crossing matrix from channel σ to channel ρ. The last relation (3.29d) says that the crossing matrices $X_g(\rho\sigma)_m{}^n$ are unitary $X_g^\dagger = X_g^{-1}$ for each $\rho \neq \sigma$, and the crossing matrices explicitly satisfy the consistency relations

$$X_g(\rho\sigma)X_g(\sigma\tau)X_g(\tau\rho) = X_g(\rho\tau)X_g(\tau\sigma)X_g(\sigma\rho) = 1 \tag{3.30a}$$

$$(1)_m{}^n = \delta_m^n \tag{3.30b}$$

which says that we return to the same blocks when we go around an s,t,u cycle.

In the special case when $\mathcal{T}^2 \sim \mathcal{T}^3$, the conformal weights exchanged in the u-channel are the same as in the s-channel. In further detail, we have

$$L^{ab}_{g,\infty}(\mathcal{T}_a^1\mathcal{T}_b^3)_\alpha{}^\beta = L^{ab}_{g,\infty}(\mathcal{T}_a^1\mathcal{T}_b^2)_{\alpha'}{}^{\beta'} \tag{3.31}$$

in this case, where $\alpha' = (\alpha_1\alpha_3\alpha_2\alpha_4)$ and similarly for β'. Then we may identify the g-invariants of the u-channel in terms of those of the s-channel

$$v(\mathrm{u},g)_\alpha^m = v(\mathrm{s},g)_{\alpha'}^m \quad , \quad \bar{v}(\mathrm{u},g)_m^\alpha = \bar{v}(\mathrm{s},g)_m^{\alpha'} \tag{3.32}$$

where $m = (r,\xi,\xi')$ is the same irrep $\mathcal{T}^r$ in both channels. It follows from (3.17b), (3.31) and (3.29c) that

$$X_g(\mathrm{su})^{-1} = X_g(\mathrm{su}) \quad , \quad X_g(\mathrm{su})^2 = 1 \tag{3.33a}$$

$$X_g(\mathrm{us})^{-1} = X_g(\mathrm{us}) \quad , \quad X_g(\mathrm{us})^2 = 1 \tag{3.33b}$$

$$\mathcal{F}_g^{(\mathrm{u})}(y)_m{}^n = \mathcal{F}_g^{(\mathrm{s})}(1-y)_m{}^n \quad . \tag{3.33c}$$

Then, using (3.33) in (3.29b), one finds that the s-channel affine-Sugawara blocks close under s-u crossing,

$$\mathcal{F}_g^{(\mathrm{s})}(1-y)_m{}^n = [X_g(\mathrm{su}) + \mathcal{O}(k^{-2})]_m{}^p\, \mathcal{F}_g^{(\mathrm{s})}(y)_p{}^q\, [X_g(\mathrm{su}) + \mathcal{O}(k^{-2})]_q{}^n \tag{3.34}$$

as they should in this case.

In the special case when all four representations are the same, one finds that the unitary crossing matrices are also idempotent $X(\rho\sigma)^2 = 1$ and hence $X(\rho\sigma) = X(\sigma\rho)$ for all $\rho \neq \sigma$: then, the Yang-Baxter-like relation

$$X_g(\rho\sigma)X_g(\sigma\tau)X_g(\tau\rho) = X_g(\tau\rho)X_g(\sigma\tau)X_g(\rho\sigma) = 1 \tag{3.35}$$

follows from the consistency relations (3.30).

Using (3.29b) and (3.22a), we finally write down the crossing relations among all three sets $\mathcal{F}_g^{(\mathrm{s})}$, $\mathcal{F}_g^{(\mathrm{u})}$, $\hat{\mathcal{F}}_g^{(\mathrm{t})}$ of analytic affine-Sugawara blocks.

$$\mathcal{F}_g^{(\mathrm{s})} = [X_g(\mathrm{su}) + \mathcal{O}(k^{-2})]\, \mathcal{F}_g^{(\mathrm{u})}\, [X_g(\mathrm{su}) + \mathcal{O}(k^{-2})]^{-1} \tag{3.36a}$$

$$\mathcal{F}_g^{(\mathrm{u})} = [X_g(\mathrm{us}) + \mathcal{O}(k^{-2})]\, \mathcal{F}_g^{(\mathrm{s})}\, [X_g(\mathrm{us}) + \mathcal{O}(k^{-2})]^{-1} \tag{3.36b}$$

$$\mathcal{F}_g^{(\mathrm{s})} = [X_g(\mathrm{st}) + \mathcal{O}(k^{-2})]\, \hat{\mathcal{F}}_g^{(\mathrm{t})}\, [X_g(\mathrm{st})U_g^{-1} + \mathcal{O}(k^{-2})]^{-1} \tag{3.36c}$$

$$\hat{\mathcal{F}}_g^{(\mathrm{t})} = [X_g(\mathrm{ts}) + \mathcal{O}(k^{-2})]\, \mathcal{F}_g^{(\mathrm{s})}\, [U_g X_g(\mathrm{ts}) + \mathcal{O}(k^{-2})]^{-1} \tag{3.36d}$$

$$\mathcal{F}_g^{(\mathrm{u})} = [X_g(\mathrm{ut}) + \mathcal{O}(k^{-2})]\, \hat{\mathcal{F}}_g^{(\mathrm{t})}\, [X_g(\mathrm{ut})U_g^{-1} + \mathcal{O}(k^{-2})]^{-1} \tag{3.36e}$$

$$\hat{\mathcal{F}}_g^{(\mathrm{t})} = [X_g(\mathrm{tu}) + \mathcal{O}(k^{-2})]\, \mathcal{F}_g^{(\mathrm{u})}\, [U_g X_g(\mathrm{tu}) + \mathcal{O}(k^{-2})]^{-1} \tag{3.36f}$$

where U_g is the non-analytic unitary phase matrix (3.22b) of the affine-Sugawara constructions. It is known [21] that the crossing matrices of analytic blocks involve non-analytic factors, and we remark that, according to eqs. (3.22b) and (3.29c), the phase matrix provides the entire $\mathcal{O}(k^{-1})$ corrections to the full crossing matrices in (3.36).

3.2 Non-chiral WZW correlators

To construct a set of high-level non-chiral WZW correlators from the affine-Sugawara blocks, we take the diagonal construction in the s-channel blocks (3.11c),

$$Y_g(y^*,y)_\alpha{}^\beta = \sum_{m,n,p} (\mathcal{F}_g^{(\mathrm{s})}(y)_p{}^m)^* \mathcal{F}_g^{(\mathrm{s})}(y)_p{}^n\, v(\mathrm{s},g)_\alpha^m \bar{v}(\mathrm{s},g)_n^\beta + \mathcal{O}(k^{-2}) \tag{3.37a}$$

$$= \sum_{m,n} v(\mathrm{s},g)_\alpha^m\, [\mathcal{F}_g^{(\mathrm{s})}(y^*)\mathcal{F}_g^{(\mathrm{s})}(y)]_m{}^n \bar{v}(\mathrm{s},g)_n^\beta + \mathcal{O}(k^{-2}) \tag{3.37b}$$

which shows trivial monodromy around $y = 0$. These correlators satisfy the high-level forms of the holomorphic and anti-holomorphic KZ equations, and the corresponding g-global conditions on the left and the right. In the special case of the $n\bar{n}\bar{n}n$ correlator on $SU(n)$, they also agree with the diagonal construction studied by Knizhnik and Zamolodchikov in [4].

To see that these correlators have trivial monodromy around $y = 1$ and $y = \infty$, one uses the crossing relations (3.29) of the affine-Sugawara blocks to rewrite the correlator (3.37) in the two alternate forms

$$Y_g(y^*,y)_\alpha{}^\beta = \sum_{m,n} v(\mathrm{u},g)_\alpha^m\, [\mathcal{F}_g^{(\mathrm{u})}(y^*)\mathcal{F}_g^{(\mathrm{u})}(y)]_m{}^n \bar{v}(\mathrm{u},g)_n^\beta + \mathcal{O}(k^{-2}) \tag{3.38a}$$

$$= \sum_{m,n} v(\mathrm{t},g)_\alpha^m\, [\mathcal{F}_g^{(\mathrm{t})}(y^*)\mathcal{F}_g^{(\mathrm{t})}(y)]_m{}^n \bar{v}(\mathrm{t},g)_n^\beta + \mathcal{O}(k^{-2}) \tag{3.38b}$$

where the u and t-channel blocks are given in (3.17b) and (3.19b).

We can also express the t-channel form (3.38b) in terms of the analytic t-channel blocks (3.23),

$$Y_g(y^*,y)_\alpha{}^\beta = \sum_{m,n} v(\mathrm{t},g)_\alpha^m\, [\hat{\mathcal{F}}_g^{(\mathrm{t})}(y^*)U_g(y^*)\hat{\mathcal{F}}_g^{(\mathrm{t})}(y)U_g(y)]_m{}^n\, v(\mathrm{t},g)_n^\beta + \mathcal{O}(k^{-2}) \tag{3.39a}$$

$$= \sum_{m,n} v(\mathrm{t},g)_\alpha^m\, [\hat{\mathcal{F}}_g^{(\mathrm{t})}(y)U_g(y^*)U_g(y)\hat{\mathcal{F}}_g^{(\mathrm{t})}(y)]_m{}^n\, v(\mathrm{t},g)_n^\beta + \mathcal{O}(k^{-2}) \tag{3.39b}$$

$$= \sum_{m,n} v(\mathrm{t},g)^m_\alpha \, [\hat{\mathcal{F}}^{(\mathrm{t})}_g(y^*)\hat{\mathcal{F}}^{(\mathrm{t})}_g(y)]_m{}^n \, v(\mathrm{t},g)^\beta_n + \mathcal{O}(k^{-2}) \tag{3.39c}$$

where we have used the fact that

$$[A,B] = \mathcal{O}(k^{-2}) \quad \text{when} \quad A,B = \mathbb{1} + \mathcal{O}(k^{-1}) \tag{3.40}$$

and the first property in (3.22e) of the phase matrix U_g.

Using completeness and the form (3.11c) of the affine-Sugawara blocks, we also find the summed form of the non-chiral WZW correlators

$$\begin{aligned} Y_g(y^*,y)_\alpha{}^\beta &= \Big\{[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y^* + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y^*))]I_g \\ &\quad \times[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]\Big\}_\alpha{}^\beta + \mathcal{O}(k^{-2}) \end{aligned} \tag{3.41a}$$

$$= \{[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln|y|^2 + \mathcal{T}^1_a\mathcal{T}^3_b \ln|1-y|^2)]I_g\}_\alpha{}^\beta + \mathcal{O}(k^{-2}) \tag{3.41b}$$

where I_g is the projector (3.7) onto the G-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$, and we have used eq.(3.8) to obtain the second form, which explicitly shows two of the trivial monodromies. The third trivial monodromy, around $y = \infty$, also follows immediately because both terms in (3.41b) are proportional to $|y|$ at large y. The correct t-channel singularities are then obtained by an application of the g-global Ward identity (3.24), using I_g in the form (3.15e).

Using the g-crossing matrices (3.29c), Appendix A gives alternate expressions for the g-blocks (3.27), the analytic t-channel g-blocks (3.23) and the g-correlators (3.37)

4 The Coset Constructions

4.1 The coset blocks

The next simplest, and next most symmetric, set of conformal field theories are the g/h coset constructions [1,2,7,8], whose chiral correlators are defined by (2.1) with

$$L^{ab}_{g/h,\infty} = \frac{P^{ab}_{g/h}}{2k} \quad , \quad P_{g/h} = P_g - P_h \tag{4.1a}$$

$$Y_{g/h}(y)\sum_{i=1}^{4}\mathcal{T}^i_a = 0 \quad , \quad a = 1\ldots\dim h \tag{4.1b}$$

where $h \subset g$. These correlators are the high-level solutions of the general coset equations of Refs.[15,16,14] on simple g, and the results below are the high-level form of the general coset blocks studied in [22,15,16,14].

We begin by reorganizing the high-level coset correlators (2.1) as,

$$\begin{aligned} Y^\alpha_{g/h}(y) &= \{v_g[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))] \\ &\quad \times[\mathbb{1} - 2L^{ab}_{h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]\Big\}^\alpha + \mathcal{O}(k^{-2}) \end{aligned} \tag{4.2}$$

where we have used (4.1a) and moved the terms of the h theory to the right.

To define the ρ=s, t and u-channel coset blocks, we need the g-invariant eigenvectors $v(\rho,g)^m$, $v(\rho,g)_m$ of Section 3, and also the corresponding h-invariant eigenvectors $\psi(\rho,h)$,

$$2L^{ab}_{h,\infty}\mathcal{T}^1_a\mathcal{T}^2_b\psi(\mathrm{s},h)^M = (\Delta^h_{(\mathrm{s})}(M) - \Delta^h_{M_1}(\mathcal{T}^1) - \Delta^h_{M_2}(\mathcal{T}^2))\psi(\mathrm{s},h)^M \tag{4.3a}$$

$$2L^{ab}_{h,\infty}\mathcal{T}^1_a\mathcal{T}^3_b\psi(\mathrm{u},h)^M = (\Delta^h_{(\mathrm{u})}(M) - \Delta^h_{M_1}(\mathcal{T}^1) - \Delta^h_{M_3}(\mathcal{T}^3))\psi(\mathrm{u},h)^M \tag{4.3b}$$

$$2L^{ab}_{h,\infty}\mathcal{T}^2_a\mathcal{T}^3_b\psi(\mathrm{t},h)^M = (\Delta^h_{(\mathrm{t})}(M) - \Delta^h_{M_2}(\mathcal{T}^2) - \Delta^h_{M_3}(\mathcal{T}^3))\psi(\mathrm{t},h)^M \tag{4.3c}$$

$$L^{ab}_{h,\infty}\mathcal{T}^i_a\mathcal{T}^i_b\psi(\rho,h)^M = \Delta^h_{M_i}(\mathcal{T}^i)\psi(\rho,h)^M \quad , \quad i = 1\ldots 4 \quad , \quad \rho = \mathrm{s},\mathrm{t},\mathrm{u} \tag{4.3d}$$

$$(\sum_{i=1}^{4}\mathcal{T}^i_a)\psi(\rho,h)^M = 0 \quad , \quad a = 1\ldots\dim h \quad , \quad \rho = \mathrm{s},\mathrm{t},\mathrm{u} \tag{4.3e}$$

whose properties parallel those of the g-invariants. In particular, the eigenvalue problems (4.3a-c) are compatible with the diagonalization of the h conformal weights in (4.3d) because the matrices $L^{ab}_h\mathcal{T}^i_a\mathcal{T}^j_b$ and $L^{ab}_h\mathcal{T}^i_a\mathcal{T}^i_b$ commute. The h-global Ward identities (4.3e) are also compatible with the eigenvalue problems, whose matrices are h-invariant. It then follows from the high-level form of the relation

$$2L^{ab}_h\mathcal{T}^i_a\mathcal{T}^j_b = L^{ab}_h(\mathcal{T}^i_a + \mathcal{T}^j_a)(\mathcal{T}^i_b + \mathcal{T}^j_b) - (L^{ab}_h\mathcal{T}^i_a\mathcal{T}^i_b + L^{ab}_h\mathcal{T}^j_a\mathcal{T}^j_b) \quad , \quad 1 \le i < j \le 4 \tag{4.4}$$

that the quantities $\Delta^h_{(\rho)}(M)$ in (4.3a-c) are the high-level forms of the broken conformal weights of h-irreps in the ρ-channel (that is, the decomposition of $\mathcal{T} \otimes \mathcal{T}'$ into h-irreps).

The h-invariant eigenvectors also satisfy completeness and orthonormality,

$$\psi(\rho,h)_M\psi(\rho,h)^N = \delta^N_M \quad , \quad \psi(\rho,h)^M\psi(\rho,h)_M = I_h \quad , \quad \rho = \mathrm{s},\mathrm{t},\mathrm{u} \tag{4.5a}$$

$$[L^{ab}_{h,\infty}\mathcal{T}^i_a\mathcal{T}^j_b, I_h] = 0 \quad , \quad 1 \le i,j \le 4 \tag{4.5b}$$

where I_h is the projection operator onto the h-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$.

As an explicit example, we give the solution for the $U(1)$-invariant s-channel eigenvectors of the coset correlator

$$(\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3,\mathcal{T}^4) = (j_1,j_2,j_3,j_4) \quad \text{in} \quad \frac{SU(2)}{U(1)} \quad . \tag{4.6}$$

In this case we need

$$L^{ab}_{U(1),\infty} = \frac{\delta^a_3\delta^b_3}{2k} \quad , \quad \mathcal{T}^i_3 = \sqrt{\psi^2_g}\begin{pmatrix} j_i & & 0 \\ & \ddots & \\ 0 & & -j_i \end{pmatrix} \tag{4.7}$$

where ψ^2_g is the $SU(2)$ root length squared and we have taken the usual magnetic quantum number basis for the matrices, with $\alpha_i = M_i$, $|M_i| \le j_i$. The solution of the eigenvalue problem (4.3a) is then

$$\psi(\mathrm{s},U(1))^M_\alpha = \delta^M_\alpha\delta(\sum_{i=1}^{4}M_i = 0) \quad , \quad M = (M_1,M_2,M_3,M_4) \tag{4.8a}$$

$$\Delta^{U(1)}_{(s)}(M) = \frac{(M_1+M_2)^2}{x} \quad , \quad \Delta^{U(1)}_{M_i}(\mathcal{T}^i) = \frac{M_i^2}{x} \quad , \; i = 1\ldots 4 \tag{4.8b}$$

where $x = 2k/\psi_g^2$ is the invariant level of $g = SU(2)$. For more general coset correlators the eigenvectors $\psi(s,h)$ are squares of products of Clebsch-Gordan coefficients times Clebsch-Gordan coefficients for branching of g-irreps into h-irreps [23].

Using completeness of $v(g), \bar{v}(g)$ and $\psi(h), \bar{\psi}(h)$, we have [15,16,14]

$$v_g^\alpha = \sum_m d(\rho)^m \bar{v}(\rho,g)^\alpha_m \tag{4.9a}$$

$$Y^\alpha_{g/h}(y) = \sum_{m,M} d(\rho)^m C^{(\rho)}_{g/h}(y)_m{}^M \bar{\psi}(\rho,h)^\alpha_M \tag{4.9b}$$

where $C^{(\rho)}_{g/h}(y)$ are the coset blocks. Further use of completeness gives the explicit form of the high-level coset blocks

$$C^{(\rho)}_{g/h}(y)_m{}^M = \mathcal{F}^{(\rho)}_g(y)_m{}^n e(\rho,g/h)_n{}^N (\mathcal{F}^{(\rho)}_h(y)^{-1})_N{}^M \quad , \quad \rho = \mathrm{s,t,u} \tag{4.10a}$$

$$\mathcal{F}^{(\rho)}_h(y)_N{}^M = \bar{\psi}(\rho,h)_N[\mathbb{1} + 2L^{ab}_{h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]\psi(\rho,h)^M + \mathcal{O}(k^{-2}) \tag{4.10b}$$

$$(\mathcal{F}^{(\rho)}_h(y)^{-1})_N{}^M = \bar{\psi}(\rho,h)_N[\mathbb{1} - 2L^{ab}_{h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]\psi(\rho,h)^M + \mathcal{O}(k^{-2}) \tag{4.10c}$$

$$e(\rho,g/h)_n{}^N = \bar{v}(\rho,g)_n \psi(\rho,h)^N \tag{4.10d}$$

where $\mathcal{F}^{(\rho)}_g$ are the ρ-channel g-blocks (of the affine-Sugawara construction on g) given in eq.(3.27), and $e(\rho,g/h)$ is the embedding matrix of the g-invariants $v(g)$ in the h-invariants $\psi(h)$. The inverse h-blocks $\mathcal{F}^{-1}_h$ are the inverse of the h-blocks $\mathcal{F}_h$. In Ref.[15], the exact coset blocks were written as $(C_{g/h})_m{}^M = (\mathcal{F}_g)_m{}^n(\mathcal{F}^{-1}_h)_n{}^M$, where $(\mathcal{F}^{-1}_h)_n{}^M = e(g/h)_n{}^N(\mathcal{F}^{-1}_h)_N{}^M$ in the present notation.

The s and u-channel coset blocks in (4.10) are high-level forms of analytic blocks, as above. To obtain the analytic t-channel coset blocks, we first use the continuation formulae (3.20) to find the analytic t-channel h-blocks $\hat{\mathcal{F}}^{(t)}_h$ and their inverse,

$$\hat{\mathcal{F}}^{(t)}_h(y)_N{}^M = \bar{\psi}(t,h)_N[\mathbb{1} + 2L^{ab}_{h,\infty}(\mathcal{T}^1_a[\mathcal{T}^2_b + \mathcal{T}^3_b]\ln(-y) + \mathcal{T}^1_a\mathcal{T}^3_b \ln\left(1-\frac{1}{y}\right))]\psi(t,h)^M + \mathcal{O}(k^{-2}) \tag{4.11a}$$

$$(\hat{\mathcal{F}}^{(t)}_h(y)^{-1})_N{}^M = \bar{\psi}(t,h)_N[\mathbb{1} - 2L^{ab}_{h,\infty}(\mathcal{T}^1_a[\mathcal{T}^2_b + \mathcal{T}^3_b]\ln(-y) + \mathcal{T}^1_a\mathcal{T}^3_b \ln\left(1-\frac{1}{y}\right))]\psi(t,h)^M + \mathcal{O}(k^{-2}) \tag{4.11b}$$

$$\mathcal{F}^{(t)}_h(y) = \hat{\mathcal{F}}^{(t)}_h(y)U_h(y) \quad , \quad \hat{\mathcal{F}}^{(t)}_h(y) = \mathcal{F}^{(t)}_h(y)U_h(y)^{-1} \tag{4.11c}$$

$$U_h(y)_M{}^N = \bar{\psi}(t,h)_M \exp[-2\pi i L^{ab}_{h,\infty}\mathcal{T}^1_a\mathcal{T}^2_b \mathrm{sign}(\arg(-y))]\psi(t,h)^N + \mathcal{O}(k^{-2}) \tag{4.11d}$$

$$U_h(y^*) = U_h(y)^{-1} \quad , \quad U_h(y)^\dagger = U_h(y)^{-1} \tag{4.11e}$$

whose form closely parallels that of the analytic t-channel blocks $\hat{\mathcal{F}}^{(t)}_g$ in (3.23a). Here $U_h(y)$ is the non-analytic unitary phase matrix of the h-blocks embedded in g. Then we may rearrange the t-channel coset blocks as follows,

$$C^{(t)}_{g/h}(y) = \hat{\mathcal{F}}^{(t)}_g(y)U_g(y)e(t,g/h)U_h(y)^{-1}\hat{\mathcal{F}}^{(t)}_h(y)^{-1} \tag{4.12a}$$

$$= \hat{\mathcal{F}}^{(t)}_g(y)e(t,g/h)U_g(y)U_h(y)^{-1}\hat{\mathcal{F}}^{(t)}_h(y)^{-1} \tag{4.12b}$$

$$= [\hat{\mathcal{F}}^{(t)}_g(y)e(t,g/h)\hat{\mathcal{F}}^{(t)}_h(y)^{-1}][U_g(y)U_h(y)^{-1}] \tag{4.12c}$$

where we have used the fact that

$$U_g(y)_m{}^P e(t,g/h)_P{}^M = e(t,g/h)_m{}^P U_g(y)_P{}^M \tag{4.13a}$$

$$U_g(y)_P{}^M \equiv \bar{\psi}(t,h)_P \exp[-2\pi i L^{ab}_{g,\infty}\mathcal{T}^1_a\mathcal{T}^2_b \mathrm{sign}(\arg(-y))]\psi(t,h)^M + \mathcal{O}(k^{-2}) \tag{4.13b}$$

in the second step and the commutation identity (3.40) in the last step.

From (4.12c), we read the form and properties of the analytic t-channel coset blocks $\hat{C}^{(t)}_{g/h}(y)$,

$$\hat{C}^{(t)}_{g/h}(y) = \hat{\mathcal{F}}^{(t)}_g(y)e(t,g/h)\hat{\mathcal{F}}^{(t)}_h(y)^{-1} \tag{4.14a}$$

$$C^{(t)}_{g/h}(y) = \hat{C}^{(t)}_{g/h}(y)U_{g/h}(y) \quad , \quad \hat{C}^{(t)}_{g/h}(y) = C^{(t)}_{g/h}(y)U_{g/h}(y)^{-1} \tag{4.14b}$$

$$\begin{aligned} U_{g/h}(y)_M{}^N &= U_g(y)_M{}^P(U_h(y)^{-1})_P{}^N \\ &= \bar{\psi}(t,h)_M \exp[-2\pi i L^{ab}_{g/h,\infty}\mathcal{T}^1_a\mathcal{T}^2_b \mathrm{sign}(\arg(-y))]\psi(t,h)^N + \mathcal{O}(k^{-2}) \end{aligned} \tag{4.14c}$$

$$U_{g/h}(y^*) = U_{g/h}(y)^{-1} \quad , \quad U_{g/h}(y)^\dagger = U_{g/h}(y)^{-1} \tag{4.14d}$$

where $U_{g/h}(y)$ is the non-analytic unitary phase matrix of the g/h coset constructions.

The limiting behavior of the analytic coset blocks $C^{(s)}_{g/h}$, $C^{(u)}_{g/h}$, $\hat{C}^{(t)}_{g/h}$ follows from their form in (4.10a) and (4.14a), together with the results above for g and h,

$$C^{(s)}_{g/h}(y)_m{}^M \underset{y\to 0}{\sim} \Gamma^{(s)}_{g/h}(m,M)y^{\Delta^{g/h}_{(s)}(m,M) - \Delta^{g/h}_{M_1}(\mathcal{T}^1) - \Delta^{g/h}_{M_2}(\mathcal{T}^2)} + \mathcal{O}(k^{-2}) \tag{4.15a}$$

$$C^{(u)}_{g/h}(y)_m{}^M \underset{y\to 1}{\sim} \Gamma^{(u)}_{g/h}(m,M)(1-y)^{\Delta^{g/h}_{(u)}(m,M) - \Delta^{g/h}_{M_1}(\mathcal{T}^1) - \Delta^{g/h}_{M_3}(\mathcal{T}^3)} + \mathcal{O}(k^{-2}) \tag{4.15b}$$

$$\hat{C}^{(t)}_{g/h}(y)_m{}^M \underset{y\to\infty}{\sim} \Gamma^{(t)}_{g/h}(m,M)(-y)^{-\Delta^{g/h}_{(t)}(m,M) - \Delta^{g/h}_{M_1}(\mathcal{T}^1) + \Delta^{g/h}_{M_4}(\mathcal{T}^4)} + \mathcal{O}(k^{-2}) \tag{4.15c}$$

$$\Delta^{g/h}_{M_i}(\mathcal{T}^i) = \Delta^g(\mathcal{T}^i) - \Delta^h_{M_i}(\mathcal{T}^i) \quad , \; i = 1\ldots 4 \tag{4.15d}$$

$$\Delta^{g/h}_{(\rho)}(m,M) = \begin{cases} \Delta^g_{(\rho)}(m) - \Delta^h_{(\rho)}(M) + \mathcal{O}(k^{-2}) & , \; e(\rho,g/h)_m{}^M \neq 0 \quad (4.15e) \\ 1 + \mathcal{O}(k^{-1}) & , \; e(\rho,g/h)_m{}^M = 0 \quad (4.15f) \end{cases}$$

$$\Gamma^{(\rho)}_{g/h}(m,M) = \begin{cases} e(\rho,g/h)_m{}^M + \mathcal{O}(k^{-2}) & , \; e(\rho,g/h)_m{}^M \neq 0 \quad (4.15g) \\ -e(\rho,g/h)_m{}^N e(\rho,g/h)_N{}^M + \mathcal{O}(k^{-2}) & , \; e(\rho,g/h)_m{}^M = 0 \quad (4.15h) \end{cases}$$

where the matrices $c(\rho, g/h)$

$$c(\mathrm{s}, g/h)_N{}^M = \psi(\mathrm{s}, h)_N 2L^{ab}_{g/h,\infty} \mathcal{T}^1_a \mathcal{T}^3_b \psi(\mathrm{s}, h)^M \tag{4.16a}$$

$$c(\mathrm{u}, g/h)_N{}^M = \psi(\mathrm{u}, h)_N 2L^{ab}_{g/h,\infty} \mathcal{T}^1_a \mathcal{T}^2_b \psi(\mathrm{u}, h)^M \tag{4.16b}$$

$$c(\mathrm{t}, g/h)_N{}^M = \psi(\mathrm{t}, h)_N 2L^{ab}_{g/h,\infty} \mathcal{T}^1_a \mathcal{T}^3_b \psi(\mathrm{t}, h)^M \tag{4.16c}$$

are defined in analogy to those of the g theory.

The g/h conformal weights in (4.15d) and (4.15e) for $\epsilon(\rho, g/h)_m{}^M \neq 0$ are the correct conformal weights of the external and intermediate coset-broken affine primary fields, and the intermediate broken affine primary states contribute with residue $\mathcal{O}(k^0)$, in accord with the general OPE (2.17).

The $(1 + \mathcal{O}(k^{-1}))$ conformal weights in (4.15f) are broken affine secondaries (with residue $\mathcal{O}(k^{-1})$ in (4.15h)) which are not necessarily integer descendants of broken affine primaries; see for example the exact conformal blocks

$$nn\bar{n}\bar{n} \quad \text{in} \quad \frac{SU(n)_{x_1} \times SU(n)_{x_2}}{SU(n)_{x_1+x_2}} \tag{4.17}$$

obtained in Ref.[15]. All the conformal weights in (4.15d-f) check against the large $x_1 = x_2 = x$ form of these blocks.

Appendix D studies a coset example on simple g

$$3\bar{3}\bar{3}3 \quad \text{in} \quad \frac{SU(3)}{SU(2)_{\mathrm{irr}}} \tag{4.18}$$

in some detail. This case shows a block which begins at $\mathcal{O}(k^{-2})$.

We finally note that the number $B_{g/h}$ of coset blocks in each of the channels,

$$B_{g/h} = d_g \cdot d_h \tag{4.19}$$

is the product of the dimensions d_g and d_h of the g- and h-invariants in any channel. In fact $d_h \geq d_g$ because $h \subset g$, so that the inequality

$$B_{g/h} \geq B_g \tag{4.20}$$

is obtained for comparison of correlators with fixed external g-irreps, where B_g in (3.28) is the number of affine-Sugawara blocks in each of the channels. The result (4.20) is in accord with the intuitive expectation that the number of blocks grows with increased symmetry breaking.

Crossing relations

Following the development of the previous section we find the crossing relations for the embedding matrix and the (inverse) h-blocks,

$$\epsilon(\rho, g/h)_m{}^M = X_g(\rho\sigma)_m{}^n \epsilon(\sigma, g/h)_n{}^N X_h^{-1}(\rho\sigma)_N{}^M \tag{4.21a}$$

$$(\mathcal{F}_h^{(\rho)}(y)^{-1})_M{}^N = [X_h(\rho\sigma) + \mathcal{O}(k^{-2})]_M{}^P (\mathcal{F}_h^{(\sigma)}(y)^{-1})_P{}^Q ([X_h(\rho\sigma) + \mathcal{O}(k^{-2})]^{-1})_Q{}^N \tag{4.21b}$$

where $X_g(\rho\sigma)$ are the g-crossing matrices (3.29c) and $X_h(\rho\sigma)$ are the corresponding h-crossing matrices,

$$X_h(\rho\sigma)_M{}^N = \psi(\rho, h)_M \psi(\sigma, h)^N \tag{4.22a}$$

$$X_h^{-1}(\rho\sigma)_M{}^N = X_h(\sigma\rho)_M{}^N = (X_h(\rho\sigma)_N{}^M)^* \tag{4.22b}$$

which are also unitary. Using (3.29b) and (4.21) we obtain the crossing relations of the coset blocks,

$$\mathcal{C}^{(\rho)}_{g/h}(y)_m{}^M = [X_g(\rho\sigma) + \mathcal{O}(k^{-2})]_m{}^n \mathcal{C}^{(\sigma)}_{g/h}(y)_n{}^N ([X_h(\rho\sigma) + \mathcal{O}(k^{-2})]^{-1})_N{}^M \tag{4.23}$$

which involve, as expected, the crossing matrices X_g and X_h of g and of h.

The h-crossing matrices satisfy the same consistency relations,

$$X_h(\rho\sigma)X_h(\sigma\tau)X_h(\tau\rho) = X_h(\rho\tau)X_h(\tau\sigma)X_h(\sigma\rho) = 1 \tag{4.24a}$$

$$(1)_M{}^N = \delta^N_M \tag{4.24b}$$

which were seen for the g-crossing matrices in (3.30).

When the external g-irreps satisfy $\mathcal{T}^2 \sim \mathcal{T}^3$, we find that $X_h(\mathrm{us})^2 = 1$ and $\mathcal{F}_h^{(\mathrm{u})}(y) = \mathcal{F}_h^{(\mathrm{s})}(1-y)$, as for the g-blocks. Together with the corresponding relations for the g-quantities in this case, this implies

$$\epsilon(\mathrm{u}, g/h) = \epsilon(\mathrm{s}, g/h) \quad , \quad \mathcal{C}^{(\mathrm{u})}_{g/h}(y) = \mathcal{C}^{(\mathrm{s})}_{g/h}(1-y) \tag{4.25}$$

and then,

$$\mathcal{C}^{(\mathrm{s})}_{g/h}(1-y)_m{}^M = [X_g(\mathrm{su}) + \mathcal{O}(k^{-2})]_m{}^n \mathcal{C}^{(\mathrm{s})}_{g/h}(y)_n{}^N [X_h(\mathrm{su}) + \mathcal{O}(k^{-2})]_N{}^M \tag{4.26}$$

so that the s-channel coset blocks are closed under crossing in this case, as expected.

Using (4.23) and (4.14b), we finally write down the crossing relations among all three sets $\mathcal{C}^{(\mathrm{s})}_{g/h}$, $\mathcal{C}^{(\mathrm{u})}_{g/h}$, $\hat{\mathcal{C}}^{(\mathrm{t})}_{g/h}$ of analytic coset blocks,

$$\mathcal{C}^{(\mathrm{s})}_{g/h} = [X_g(\mathrm{su}) + \mathcal{O}(k^{-2})]\mathcal{C}^{(\mathrm{u})}_{g/h}[X_h(\mathrm{su}) + \mathcal{O}(k^{-2})]^{-1} \tag{4.27a}$$

$$\mathcal{C}^{(\mathrm{u})}_{g/h} = [X_g(\mathrm{us}) + \mathcal{O}(k^{-2})]\mathcal{C}^{(\mathrm{s})}_{g/h}[X_h(\mathrm{us}) + \mathcal{O}(k^{-2})]^{-1} \tag{4.27b}$$

$$\mathcal{C}^{(\mathrm{s})}_{g/h} = [X_g(\mathrm{st}) + \mathcal{O}(k^{-2})]\hat{\mathcal{C}}^{(\mathrm{t})}_{g/h}[X_h(\mathrm{st})U^{-1}_{g/h} + \mathcal{O}(k^{-2})]^{-1} \tag{4.27c}$$

$$\hat{\mathcal{C}}^{(\mathrm{t})}_{g/h} = [X_g(\mathrm{ts}) + \mathcal{O}(k^{-2})]\mathcal{C}^{(\mathrm{s})}_{g/h}[U_{g/h}X_h(\mathrm{ts}) + \mathcal{O}(k^{-2})]^{-1} \tag{4.27d}$$

$$\mathcal{C}^{(\mathrm{u})}_{g/h} = [X_g(\mathrm{ut}) + \mathcal{O}(k^{-2})]\hat{\mathcal{C}}^{(\mathrm{t})}_{g/h}[X_h(\mathrm{ut})U^{-1}_{g/h} + \mathcal{O}(k^{-2})]^{-1} \tag{4.27e}$$

$$\hat{\mathcal{C}}^{(\mathrm{t})}_{g/h} = [X_g(\mathrm{tu}) + \mathcal{O}(k^{-2})]\mathcal{C}^{(\mathrm{u})}_{g/h}[U_{g/h}X_h(\mathrm{tu}) + \mathcal{O}(k^{-2})]^{-1} \tag{4.27f}$$

where $U_{g/h}$ is the non-analytic unitary phase matrix (4.14c) of the g/h coset constructions.

As seen above for the affine-Sugawara constructions, the phase matrix provides the entire $\mathcal{O}(k^{-1})$ corrections to the full coset crossing matrices in (4.27).

Fixed external h representations

The crossing relations (4.23) of the coset blocks mix the internal h-irreps (M) which arise from different external irreps of h (that is, the h-irreps which arise from the h-decomposition of the g-irreps $\mathcal{T}^i$).

To obtain blocks characterized by fixed external irreps of h, we introduce a hermitean projection operator $\mathcal{P}_h = \mathcal{P}(T^{h1}, T^{h2}, T^{h3}, T^{h4})$ to select any four external h-irreps of interest,

$$\psi(\rho,h)_\alpha^{\tilde{M}}\psi(\rho,h)_{\tilde{M}}^\beta = (\mathcal{P}_h)_\alpha{}^\beta \tag{4.28a}$$

$$\mathcal{P}_h\psi(\rho,h)^M = \psi(\rho,h)^{\tilde{M}}\delta_{\tilde{M}}^M \tag{4.28b}$$

$$[L_{h,\infty}^{ab}\mathcal{T}_a^i\mathcal{T}_b^j, \mathcal{P}_h] = [L_{g/h,\infty}^{ab}\mathcal{T}_a^i\mathcal{T}_b^j, \mathcal{P}_h] = 0 \quad , \quad 1 \le i,j \le 4 \tag{4.28c}$$

where $\tilde{M}$ runs over the eigenvectors associated to the fixed external set of h-irreps. The inverse h-blocks are block diagonal under this decomposition

$$\begin{aligned}
&(\mathcal{F}_h^{(\rho)}(y)^{-1})_M{}^{\tilde{N}} \\
&\quad = \psi(\rho,h)_M[\mathbb{1} - 2L_{h,\infty}^{ab}(\mathcal{T}_a^1\mathcal{T}_b^2\ln y + \mathcal{T}_a^1\mathcal{T}_b^3\ln(1-y))]\psi(\rho,h)^{\tilde{N}} + \mathcal{O}(k^{-2}) \\
&\quad = \psi(\rho,h)_M[\mathbb{1} - 2L_{h,\infty}^{ab}(\mathcal{T}_a^1\mathcal{T}_b^2\ln y + \mathcal{T}_a^1\mathcal{T}_b^3\ln(1-y))]\mathcal{P}_h\psi(\rho,h)^{\tilde{N}} + \mathcal{O}(k^{-2}) \\
&\quad = \psi(\rho,h)_M\mathcal{P}_h[\mathbb{1} - 2L_{h,\infty}^{ab}(\mathcal{T}_a^1\mathcal{T}_b^2\ln y + \mathcal{T}_a^1\mathcal{T}_b^3\ln(1-y))]\psi(\rho,h)^{\tilde{N}} + \mathcal{O}(k^{-2}) \\
&\quad = \delta_M^{\tilde{M}}(\mathcal{F}_h^{(\rho)}(y)^{-1})_{\tilde{M}}{}^{\tilde{N}}
\end{aligned} \tag{4.29}$$

where we have used the first relation in (4.28c). Then the corresponding subset of coset blocks is

$$(\mathcal{C}_{g/h})_m{}^{\tilde{M}} = (\mathcal{F}_g)_m{}^n\epsilon(g/h)_n{}^N(\mathcal{F}_h^{-1})_N{}^{\tilde{M}} = (\mathcal{F}_g)_m{}^n\epsilon(g/h)_n{}^{\tilde{N}}(\mathcal{F}_h^{-1})_{\tilde{N}}{}^{\tilde{M}} \quad . \tag{4.30}$$

Similarly, the h-crossing matrices are block diagonal under this decomposition,

$$X_h^{-1}(\rho\sigma)_M{}^{\tilde{N}} = \psi(\sigma,h)_M\psi(\rho,h)^{\tilde{N}} = \psi(\sigma,h)_M\mathcal{P}_h\psi(\rho,h)^{\tilde{N}} = \delta_M^{\tilde{M}}X_h^{-1}(\rho\sigma)_{\tilde{M}}{}^{\tilde{N}} \quad . \tag{4.31}$$

Then, it follows from (4.23) and (4.31) that

$$\mathcal{C}_{g/h}^{(\sigma)}(y)_m{}^{\tilde{M}} = [X_g(\rho\sigma) + \mathcal{O}(k^{-2})]_m{}^n\,\mathcal{C}_{g/h}^{(\sigma)}(y)_n{}^{\tilde{N}}\,([X_h(\rho\sigma) + \mathcal{O}(k^{-2})]^{-1})_{\tilde{N}}{}^{\tilde{M}} \tag{4.32}$$

which shows that the selected subset of coset blocks is closed under crossing

The selected subset of analytic coset blocks $\mathcal{C}_{g/h}^{(s)}(y)_m{}^{\tilde{M}}$, $\mathcal{C}_{g/h}^{(u)}(y)_m{}^{\tilde{M}}$ and $\hat{\mathcal{C}}_{g/h}^{(t)}(y)_m{}^{\tilde{M}}$ is also closed under crossing. To see this we need the fact the non-analytic coset phase matrix (4.14c) is also block diagonal,

$$U_{g/h}(y)_M{}^{\tilde{N}} = \delta_M^{\tilde{M}}U_{g/h}(y)_{\tilde{M}}{}^{\tilde{N}} \tag{4.33a}$$

$$\mathcal{C}_{g/h}^{(t)}(y)_m{}^{\tilde{M}} = \hat{\mathcal{C}}_{g/h}^{(t)}(y)_m{}^{\tilde{N}}U_{g/h}(y)_{\tilde{N}}{}^{\tilde{M}} \quad , \quad \hat{\mathcal{C}}_{g/h}^{(t)}(y)_m{}^{\tilde{M}} = \mathcal{C}_{g/h}^{(t)}(y)_m{}^{\tilde{N}}(U_{g/h}(y)^{-1})_{\tilde{N}}{}^{\tilde{M}} \tag{4.33b}$$

which follows from (4.14c) and the second relation in (4.28c). The restricted phase matrix $U_{g/h}(y)_{\tilde{M}}{}^{\tilde{N}}$ is unitary in each subspace. Then, we have for example that

$$\mathcal{C}_{g/h}^{(s)}(y)_m{}^{\tilde{M}} = X_g(\mathrm{st})_m{}^n\hat{\mathcal{C}}_{g/h}^{(t)}(y)_n{}^{\tilde{R}}U_{g/h}(y)_{\tilde{R}}{}^{\tilde{N}}X_h^{-1}(\mathrm{st})_{\tilde{N}}{}^{\tilde{M}} + \mathcal{O}(k^{-2}) \tag{4.34a}$$

$$= X_g(\mathrm{st})_m{}^n[\hat{\mathcal{C}}_{g/h}^{(t)}(y)_n{}^{\tilde{R}}]U_{g/h}(y)_{\tilde{R}}{}^{\tilde{N}}X_h^{-1}(\mathrm{st})_{\tilde{N}}{}^{\tilde{M}} + \mathcal{O}(k^{-2}) \tag{4.34b}$$

where the last step follows from (4.33a).

The explicit form of these projection operators can be quite complicated in the general case, but there are some simple, highly symmetric cases where the form of $\mathcal{P}_h$ is very simple. As an example, consider the situation when each of the four external g-irreps branches into a single h-irrep, so that the g/h-broken conformal weights of g-irrep $\mathcal{T}^i$ are degenerate,

$$(L_{g/h,\infty}^{ab}\mathcal{T}_a^i\mathcal{T}_b^i)_\alpha{}^\beta = \Delta^{g/h}(\mathcal{T}^i)\delta_\alpha^\beta \quad , \quad i = 1\ldots 4 \quad . \tag{4.35}$$

In this case, all the coset-broken components of the g-irrep $\mathcal{T}^i$ are on an equal footing, and one may choose the trivial projector

$$\mathcal{P}_h = \mathbb{1} \quad . \tag{4.36}$$

This is the situation, e.g., in

$$\mathcal{T} = (\mathcal{T}_1, 1) \quad \text{in} \quad \frac{g_{x_1}\times g_{x_2}}{g_{x_1+x_2}} \tag{4.37}$$

examples of which were studied in Ref.[15]. Examples on simple g include

$$\mathcal{T} = n \text{ or } n \quad \text{in} \quad \frac{SU(n)_x}{SO(n)_{2x}} = \begin{cases} \frac{SU(3)_x}{SU(2)_{4x}} & , \quad n = 3 \\ \frac{SU(n)_x}{SO(n)_{2x}} & , \quad n \ge 4 \end{cases} \tag{4.38a}$$

$$\mathcal{T} = 2n \quad \text{in} \quad \frac{SO(2n)_x}{SO(n)_x \times SO(n)_x} \tag{4.38b}$$

and the case $n = 3$ of (4.38a) will be considered in detail in Appendix D. In (4.38a) the n of $SU(n)$ is the n of $SO(n) \subset SU(n)$, while in (4.38b) the $2n$ of $SO(2n)$ is the (n,n) of $(SO(n)\times SO(n)) \subset SO(2n)$. As we will discuss below, these simple cases are examples of a more general situation in ICFT (see Section 5).

4.2 Non-chiral coset correlators

To construct a set of high-level non-chiral correlators for the coset constructions, we take the s-channel diagonal construction,

$$Y_{g/h}(\mathcal{P}_h|y^*,y) = \sum_{m,\tilde{M}}|\mathcal{C}_{g/h}^{(s)}(y)_m{}^{\tilde{M}}|^2 + \mathcal{O}(k^{-2}) \tag{4.39}$$

which shows trivial monodromy around $y = 0$. To see that (4.39) has trivial monodromy around $y = 1$ and $y = \infty$, one uses the crossing relations (4.32) of the coset blocks to rewrite the coset correlator (4.39) in the two alternate forms.

$$Y_{g/h}(\mathcal{P}_h|y^*, y) = \sum_{m,\tilde{M}} |C^{(u)}_{g/h}(y)_m{}^{\tilde{M}}|^2 + \mathcal{O}(k^{-2}) \tag{4.40a}$$

$$= \sum_{m,\tilde{M}} |C^{(t)}_{g/h}(y)_m{}^{\tilde{M}}|^2 + \mathcal{O}(k^{-2}) \quad . \tag{4.40b}$$

We can also use (4.33b) to express the t-channel form (4.40b) of the correlator in terms of the analytic t-channel coset blocks,

$$Y_{g/h}(\mathcal{P}_h|y^*, y) = \sum_{m,\tilde{N},\tilde{M}} |\hat{C}^{(t)}_{g/h}(y)_m{}^{\tilde{N}} U_{g/h}(y)_{\tilde{N}}{}^{\tilde{M}}|^2 + \mathcal{O}(k^{-2}) = \sum_{m,\tilde{N}} |\hat{C}^{(t)}_{g/h}(y)_m{}^{\tilde{N}}|^2 + \mathcal{O}(k^{-2}) \tag{4.41}$$

where the last step follows from the unitarity of the restricted coset phase matrix.

Using completeness and the explicit form (4.10) of the coset blocks, the summed form of these coset correlators is

$$\begin{aligned} Y_{g/h}(\mathcal{P}_h|y^*, y) =& \mathrm{Tr}\left\{[\mathbb{1} + 2L^{ab}_{g/h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y^* + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y^*))]I_g \right. \\ & \left. \times[\mathbb{1} + 2L^{ab}_{g/h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]\mathcal{P}_h\right\} + \mathcal{O}(k^{-2}) \end{aligned} \tag{4.42a}$$

$$= \mathrm{Tr}[(\mathbb{1} + 2L^{ab}_{g/h,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln|y|^2 + \mathcal{T}^1_a\mathcal{T}^3_b \ln|1-y|^2))I_g\mathcal{P}_h] + \mathcal{O}(k^{-2}) \tag{4.42b}$$

where I_g is the projector onto the G-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$ and $\mathcal{P}_h$ is the projector onto the desired subset of external h representations. To obtain the second form, which explicitly shows two of the trivial monodromies, we used the second relation in (4.28c). Trivial monodromy around $y = \infty$ is also easily seen following the discussion below eq.(3.41).

5 A Simple Class of Correlators in ICFT

5.1 $L(g;H)$-degenerate states and correlators

In this section, we use the intuition gained in our discussion of the affine-Sugawara and coset constructions above to identify what we believe to be the simplest, most highly symmetric processes in ICFT.

In the first place , we restrict our attention to the ICFTs with a symmetry, that is, to the H-invariant CFTs on g, whose inverse inertia tensors L_H satisfy

$$\omega(H)L_H\,\omega(H)^{-1} = L_H \quad , \quad \omega(H) \in H \tag{5.1}$$

where $H \subset G$ is any subgroup of G, including finite groups and the Lie groups. The matrix $\omega(H)_a{}^b$ is in the adjoint of g. For the H-invariant CFTs, the conformal weight matrix of irrep $\mathcal{T}$ of g and hence the broken conformal weights $\Delta^H_\alpha(\mathcal{T})$ are H-invariant,

$$\Omega(H,\mathcal{T})L^{ab}_H\mathcal{T}_a\mathcal{T}_b\,\Omega^{-1}(H,\mathcal{T}) = L^{ab}_H\mathcal{T}_a\mathcal{T}_b \quad , \quad \Omega(H,\mathcal{T}) \in H \tag{5.2a}$$

$$\Omega(H,\mathcal{T})_\alpha{}^\beta[\Delta^H_\alpha(\mathcal{T}) - \Delta^H_\beta(\mathcal{T})] = 0 \tag{5.2b}$$

where $\Omega(H,\mathcal{T})_\alpha{}^\beta$ is in irrep $\mathcal{T}$ and we have used (2.9) to obtain (5.2b).

In the H-invariant CFTs, we further restrict ourselves to the most symmetric broken affine primary fields, that is, to the irreps $\mathcal{T}$ of g whose L^{ab}-broken conformal weights $\Delta^H_\alpha(\mathcal{T}) = \Delta^H(\mathcal{T})$, $\alpha = 1 \ldots \dim\mathcal{T}$ are completely degenerate

$$(L^{ab}_H\mathcal{T}_a\mathcal{T}_b)_\alpha{}^\beta = \Delta^H(\mathcal{T})\delta^\beta_\alpha \tag{5.3}$$

at all levels. In what follows, such irreps of g are called the $L(g;H)$-degenerate states because, in these cases, the irrep of g decomposes into a unique irrep of H. Finally, we restrict the discussion to the $L(g;H)$-degenerate processes, which are those correlators all of whose external states are $L(g;H)$-degenerate. In this sense, the $L(g;H)$-degenerate processes are the most symmetric correlators in ICFT.

Although they are by no means generic, it is easy to find examples of $L(g;H)$-degenerate states in the H-invariant CFTs. The simplest cases of $L(g;H)$-degenerate states are all the affine primary states of all the affine-Sugawara constructions, which are in fact $L(g;G)$-degenerate.

Examples of $L(g;h)$-degenerate states in the g/h coset constructions include those mentioned in (4.37) and (4.38). These are RCFT examples in the Lie h-invariant CFTs, and in principle many irrational examples, beyond the coset constructions, can be found among the Lie h-invariant CFTs.

Irrational examples in the much larger set of H-invariant CFTs, beyond the Lie h-invariant CFTs, are already known, including the irrational cases [24]

$$\mathcal{T} = n \text{ or } \bar{n} \quad \text{in} \quad (SU(n)_x)^{\#}_M \tag{5.4a}$$

$$\mathcal{T} = 2n \quad \text{in} \quad (SO(2n)_x)^{\#}_M \tag{5.4b}$$

where H is a finite subgroup of $SO(n) \subset SU(n)$ and $(SO(n) \times SO(n)) \subset SO(2n)$ in (5.4a) and (5.4b) respectively. The case $n = 3$ in (5.4a) will be considered in detail in Section 6.

We should also remark that the $L(g;H)$-degenerate conformal weights of the coset examples in (4.38) and the irrational examples in (5.4) all obey the unified conformal weight formula,

$$\Delta^H_\alpha(\mathcal{T}) = \Delta^H(\mathcal{T}) = \frac{c}{2xn} \tag{5.5}$$

where x is the invariant level of g and c is the central charge, which is rational for the coset constructions and irrational for $SU(n)^{\#}_M$ and $SO(2n)^{\#}_M$. The occurence of

a) $L(g;H)$-degenerate states
b) a unified form of the conformal weights
for these rational and irrational families is not totally surprising, since both families of constructions are contained in the same (maximally-symmetric) ansatz [24] of the Virasoro master equation.

In what follows, we will find uniform formulae for the high-level conformal blocks and correlators of all possible $L(g;H)$-degenerate processes in ICFT.

5.2 Conformal blocks in ICFT

We study only the class of $L(g;H)$-degenerate correlators in ICFT. Fig.2 shows these correlators generically, with one degenerate conformal weight $\Delta_i^H \equiv \Delta^H(\mathcal{T}^i)$, $i = 1\ldots 4$ for each external state.

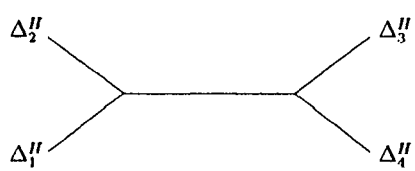

Fig. 2. The $L(g;H)$-degenerate correlators.

In this case, the chiral correlators (2.1) take the form,

$$Y_H^\alpha(y) = \bar{v}_g^\beta \Lambda_H(y)_\beta{}^\alpha + \mathcal{O}(k^{-2}) \tag{5.6a}$$

$$\Lambda_H(y) = \mathbb{1} + 2L_{H,\infty}^{ab}[\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y)] \tag{5.6b}$$

$$\begin{aligned} &= \mathbb{1} + [L_{H,\infty}^{ab}(\mathcal{T}_a^1 + \mathcal{T}_a^2)(\mathcal{T}_b^1 + \mathcal{T}_b^2) - (\Delta_1^H + \Delta_2^H)\mathbb{1}] \ln y \\ &\quad + [L_{H,\infty}^{ab}(\mathcal{T}_a^1 + \mathcal{T}_a^3)(\mathcal{T}_b^1 + \mathcal{T}_b^3) - (\Delta_1^H + \Delta_3^H)\mathbb{1}] \ln(1-y) \end{aligned} \tag{5.6c}$$

$$Y_H\Omega(H) = Y_H \quad , \quad \Omega(H) = \prod_{i=1}^{4} \Omega(H, \mathcal{T}^i) \quad . \tag{5.6d}$$

Here we have used the high-level forms of the identities

$$L_H^{ab}\mathcal{T}_a^i\mathcal{T}_b^i = \Delta_i^H \mathbb{1} \quad , \quad i = 1\ldots 4 \tag{5.7a}$$

$$2L_H^{ab}\mathcal{T}_a^i\mathcal{T}_b^j = L_H^{ab}(\mathcal{T}_a^i + \mathcal{T}_a^j)(\mathcal{T}_b^i + \mathcal{T}_b^j) - (\Delta_i^H + \Delta_j^H)\mathbb{1} \quad , \quad 1 \le i < j \le 4 \tag{5.7b}$$

to obtain the alternate form in (5.6c). The statement in (5.7a) is the $L(g;H)$-degeneracy of each external state. The condition (5.6d), which enforces the H-symmetry of the system, follows from the H-invariance of the relevant matrices

$$[\Lambda_H, \Omega(H)] = 0 \tag{5.8}$$

and the fact that v_g, being g-invariant, is also invariant under $\Omega(H)$.

To find ρ = s, t and u-channel block bases for the conformal blocks, we introduce the H-invariant eigenvectors $\psi(\rho, H)$ of the ρ-channel,

$$2L_{H,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^2\psi(\mathrm{s}, H)^M = (\Delta_{(\mathrm{s})}^H(M) - \Delta_1^H - \Delta_2^H)\psi(\mathrm{s}, H)^M \tag{5.9a}$$

$$2L_{H,\infty}^{ab}\mathcal{T}_a^1\mathcal{T}_b^3\psi(\mathrm{u}, H)^M = (\Delta_{(\mathrm{u})}^H(M) - \Delta_1^H - \Delta_3^H)\psi(\mathrm{u}, H)^M \tag{5.9b}$$

$$2L_{H,\infty}^{ab}\mathcal{T}_a^2\mathcal{T}_b^3\psi(\mathrm{t}, H)^M = (\Delta_{(\mathrm{t})}^H(M) - \Delta_2^H - \Delta_3^H)\psi(\mathrm{t}, H)^M \tag{5.9c}$$

$$\psi(\rho, H)_M\psi(\rho, H)^N = \delta_M^N \quad , \quad \psi(\rho, H)_\alpha^M\bar{\psi}(\rho, H)_M^\beta = (I_H)_\alpha^\beta \tag{5.9d}$$

$$\Omega^{-1}(H)\psi(\rho, H)^M = \psi(\rho, H)^M \quad , \quad \bar{\psi}(\rho, H)_M\Omega(H) = \bar{\psi}(\rho, H)_M \tag{5.9e}$$

where $(I_H)_\alpha^\beta$ is the projector onto the H-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$. According to the identity (5.7b), the quantities $\Delta_{(\rho)}^H(M)$ are the L^{ab}-broken high-level conformal weights of the broken affine primary states in the ρ-channel.

We remind the reader that the correlators (5.6) include all the correlators in H-invariant CFTs with $L(g;H)$-degenerate external states. This includes in particular all the correlators of all the affine-Sugawara constructions, in which case the eigenvectors $\psi(\rho, H)$ may be taken as the g-invariants $v(\rho, g)$ of Section 3, and all the coset correlators whose external states are $L(g;h)$-degenerate, in which case the eigenvectors $\psi(\rho, H)$ may be identified as the h-invariants $\psi(\rho, h)$ of Section 4.

The ρ = s, t and u-channel conformal blocks $\mathcal{B}_H^{(\rho)}$ are then obtained by inserting completeness sums in (5.6), according to

$$\Lambda_H = \Lambda_H I_H = \Lambda_H\psi(\rho, H)^M\bar{\psi}(\rho, H)_M \quad , \quad \forall\, \rho \quad . \tag{5.10}$$

In this way, we obtain the three expansions,

$$Y_H^\alpha(y) = \sum_{m,M} d(\mathrm{s})^m \mathcal{B}_H^{(\mathrm{s})}(y)_m{}^M\psi(\mathrm{s}, H)_M^\alpha \tag{5.11a}$$

$$= \sum_{m,M} d(\mathrm{u})^m \mathcal{B}_H^{(\mathrm{u})}(y)_m{}^M\psi(\mathrm{u}, H)_M^\alpha \tag{5.11b}$$

$$= \sum_{m,M} d(\mathrm{t})^m \mathcal{B}_H^{(\mathrm{t})}(y)_m{}^M\psi(\mathrm{t}, H)_M^\alpha \tag{5.11c}$$

where the ρ-channel blocks $\mathcal{B}_H^{(\rho)}(y)$ are

$$\mathcal{B}_H^{(\rho)}(y)_m{}^M = v(\rho, g)_m\Lambda_H(y)\bar{\psi}(\rho, H)^M + \mathcal{O}(k^{-2}) \tag{5.12a}$$

$$= c(\rho, H)_m{}^N\bar{\psi}(\rho, H)_N\Lambda_H(y)\bar{\psi}(\rho, H)^M + \mathcal{O}(k^{-2}) \quad , \quad \rho = \mathrm{s}, \mathrm{t}, \mathrm{u} \tag{5.12b}$$

$$\Lambda_H(y) = \mathbb{1} + 2L_{H,\infty}^{ab}[\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y)] \tag{5.12c}$$

$$c(\rho, H)_m{}^M = v(\rho, g)_m\bar{\psi}(\rho, H)^M \quad . \tag{5.12d}$$

Here $\epsilon(\rho, H)$ is the embedding matrix of the g-invariants in the H-invariants.

The s and u-channel blocks $\mathcal{B}_H^{(s)}$ and $\mathcal{B}_H^{(u)}$ are analytic blocks, as above, and the analytic t-channel blocks $\hat{\mathcal{B}}_H^{(t)}$.

$$\hat{\mathcal{B}}_H^{(t)}(y)_m{}^M = v(t,g)_m \hat{A}_H(y)\psi(t,H)^M + \mathcal{O}(k^{-2}) \tag{5.13a}$$

$$= \epsilon(t,H)_m{}^N \psi(t,H)_N \hat{A}_H(y)\psi(t,H)^M + \mathcal{O}(k^{-2}) \tag{5.13b}$$

$$\hat{A}_H(y) = \mathbb{1} + 2L^{ab}_{H,\infty}(\mathcal{T}_a^1[\mathcal{T}_b^2 + \mathcal{T}_b^3]\ln(-y) + \mathcal{T}_a^1\mathcal{T}_b^3 \ln\left(1 - \frac{1}{y}\right)) \tag{5.13c}$$

$$\mathcal{B}_H^{(t)}(y) = \hat{\mathcal{B}}_H^{(t)}(y)U_H(y) \quad , \quad \hat{\mathcal{B}}_H^{(t)}(y) = \mathcal{B}_H^{(t)}(y)U_H(y)^{-1} \tag{5.13d}$$

$$U_H(y)_M{}^N = \psi(t,H)_M \exp[-2\pi i L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^2 \mathrm{sign}(\arg(-y))]\psi(t,H)^N + \mathcal{O}(k^{-2}) \tag{5.13e}$$

$$U_H(y^*) = U_H(y)^{-1} \quad , \quad U_H(y)^\dagger = U_H(y)^{-1} \tag{5.13f}$$

are also obtained by now-familiar steps, including the continuation rules (3.20). The quantity $U_H(y)$ in (5.13e) is the non-analytic unitary phase matrix of the $L(g;H)$-degenerate correlators in ICFT.

The expressions (5.12a,b) and (5.13a,b) for the high-level analytic blocks $\mathcal{B}_H^{(s)}$, $\mathcal{B}_H^{(u)}$, $\hat{\mathcal{B}}_H^{(t)}$ of the $L(g;H)$-degenerate correlators in ICFT are among the central results of this paper.

To study the limiting behavior of the analytic blocks, we use the eigenvalue problems (5.9) to rearrange the blocks in each of the channels as follows,

$$\mathcal{B}_H^{(s)}(y)_m{}^M = \epsilon(s,H)_m{}^N \psi(s,H)_N[\mathbb{1} + 2L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^3 \ln(1-y)]\psi(s,H)^M$$
$$\times [1 + (\Delta^H_{(s)}(M) - \Delta_1^H - \Delta_2^H)\ln y] + \mathcal{O}(k^{-2}) \tag{5.14a}$$

$$= \epsilon(s,H)_m{}^N \left[\delta_N^M - c(s,H)_N{}^M \sum_{p=1}^{\infty} \frac{y^p}{p}\right] y^{\Delta^H_{(s)}(M) - \Delta_1^H - \Delta_2^H} + \mathcal{O}(k^{-2}) \tag{5.14b}$$

$$c(s,H)_N{}^M = \psi(s,H)_N 2L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^3\psi(s,H)^M \tag{5.14c}$$

$$\mathcal{B}_H^{(u)}(y)_m{}^M = \epsilon(u,H)_m{}^N \left[\delta_N^M - c(u,H)_N{}^M \sum_{p=1}^{\infty} \frac{(1-y)^p}{p}\right](1-y)^{\Delta^H_{(u)}(M) - \Delta_1^H - \Delta_3^H} + \mathcal{O}(k^{-2}) \tag{5.14d}$$

$$c(u,H)_N{}^M = \bar{\psi}(u,H)_N 2L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^2\psi(u,H)^M \tag{5.14e}$$

$$\hat{\mathcal{B}}_H^{(t)}(y)_m{}^M = v(t,g)_m\{\mathbb{1} + 2L^{ab}_{H,\infty}\mathcal{T}_a^1[\mathcal{T}_b^2 + \mathcal{T}_b^3]\ln(-y)]$$
$$\times [\mathbb{1} + 2L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^3 \ln\left(1 - \frac{1}{y}\right)]\psi(t,H)^M + \mathcal{O}(k^{-2}) \tag{5.14f}$$

$$= \epsilon(t,H)_m{}^N \psi(t,H)_N[\mathbb{1} + 2L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^3 \ln\left(1 - \frac{1}{y}\right)]$$
$$\times [\mathbb{1} - (2L^{ab}_{H,\infty}\mathcal{T}_a^2\mathcal{T}_b^3 + \sum_{i=1}^{3}\Delta_i^H - \Delta_4^H)\ln(-y)]\psi(t,H)^M + \mathcal{O}(k^{-2}) \tag{5.14g}$$

$$= \epsilon(t,H)_m{}^N \left[\delta_N^M - c(t,H)_N{}^M \sum_{p=1}^{\infty}\left(\frac{1}{y}\right)^p \frac{1}{p}\right](-y)^{-\Delta^H_{(t)}(M) - \Delta_1^H + \Delta_4^H} + \mathcal{O}(k^{-2}) \tag{5.14h}$$

$$c(t,H)_N{}^M = \psi(t,H)_N 2L^{ab}_{H,\infty}\mathcal{T}_a^1\mathcal{T}_b^3\psi(t,H)^M \quad . \tag{5.14i}$$

To obtain the form (5.14g) of the analytic t-channel blocks we also used the g-global Ward identity on the g-invariants $v(\rho,g)_m$,

$$v(\rho,g)_m[2L^{ab}_H(\mathcal{T}_a^1\mathcal{T}_b^2 + \mathcal{T}_a^2\mathcal{T}_b^3 + \mathcal{T}_a^3\mathcal{T}_b^1) - \gamma_H\mathbb{1}] = 0 \tag{5.15a}$$

$$\gamma_H = \Delta_4^H - \Delta_1^H - \Delta_2^H - \Delta_3^H \tag{5.15b}$$

applied here at high level for the case ρ=t.

Using the expressions in (5.14), we find the limiting behavior of the conformal blocks

$$\mathcal{B}_H^{(s)}(y)_m{}^M \underset{y\to 0}{\sim} \Gamma_H^{(s)}(m,M)y^{\Delta^H_{(s)}(m,M) - \Delta_1^H - \Delta_2^H} + \mathcal{O}(k^{-2}) \tag{5.16a}$$

$$\mathcal{B}_H^{(u)}(y)_m{}^M \underset{y\to 1}{\sim} \Gamma_H^{(u)}(m,M)(1-y)^{\Delta^H_{(u)}(m,M) - \Delta_1^H - \Delta_3^H} + \mathcal{O}(k^{-2}) \tag{5.16b}$$

$$\hat{\mathcal{B}}_H^{(t)}(y)_m{}^M \underset{y\to\infty}{\sim} \Gamma_H^{(t)}(m,M)(-y)^{-\Delta^H_{(t)}(m,M) - \Delta_1^H + \Delta_4^H} + \mathcal{O}(k^{-2}) \tag{5.16c}$$

$$\Delta^H_{(\rho)}(m,M) = \begin{cases} \Delta^H_{(\rho)}(M) + \mathcal{O}(k^{-2}) & , \quad \epsilon(\rho,H)_m{}^M \neq 0 \quad (5.16d) \\ 1 + \mathcal{O}(k^{-1}) & , \quad \epsilon(\rho,H)_m{}^M = 0 \quad (5.16e) \end{cases}$$

$$\Gamma_H^{(\rho)}(m,M) = \begin{cases} \epsilon(\rho,H)_m{}^M + \mathcal{O}(k^{-2}) & , \quad \epsilon(\rho,H)_m{}^M \neq 0 \\ -\epsilon(\rho,H)_m{}^N c(\rho,H)_N{}^M + \mathcal{O}(k^{-2}) & , \quad \epsilon(\rho,H)_m{}^M = 0 \end{cases} \tag{5.16f}$$

followed by integer-spaced secondaries. The blocks with $\epsilon(\rho,H)_m{}^M \neq 0$ begin at $\mathcal{O}(k^0)$ and exhibit leading singularities (with $\mathcal{O}(k^0)$ residues) whose high-level conformal weights $\Delta^H_{(\rho)}(M)$ in (5.9) are those of the correct broken affine-primary states in each of the three channels. The remaining blocks, which begin at $\mathcal{O}(k^{-1})$, show leading singularities which are broken affine secondaries. As noted for the affine-Sugawara and coset constructions in Sections 3 and 4, this pattern is in agreement with the general OPE in (2.17).

Further discussion of these conformal weights follows that given for the coset constructions below (4.16). In particular, as noted for the cosets, the $(1+\mathcal{O}(k^{-1}))$ ρ-channel conformal weights in (5.16e) are broken affine secondaries which need not be integer descendants of broken affine primary states. Identification of these states is therefore an important open problem in ICFT.

<u>Number of blocks</u>

We finally note that, for an $L(g;H)$-degenerate process, the number B_H of blocks in each of the channels

$$B_H = d_g \cdot d_H \tag{5.17}$$

is the product of the dimension d_g of g-invariants and the dimension d_H of H-invariants in any channel. We know that $d_H \geq d_h \geq d_g$ when H is a finite subgroup of the Lie group generated by $h \subset g$, and hence we obtain the double inequality

$$B_H \geq B_{g/h} \geq B_g \tag{5.18}$$

for comparison of correlators with fixed external g-irreps, where $B_{g/h}$ and B_g in (4.19) and (3.28) are the number of coset and affine-Sugawara blocks respectively in any channel. This double inequality summarizes the symmetry hierarchy within the $L(g;H)$-degenerate processes, and is in accord with the expectation that the number of blocks increases with increased symmetry breakdown in ICFT.

In Appendices B and D and Section 6, we study the $L(g;H)$-degenerate correlator 3333 under the three constructions,

- the affine-Sugawara construction on $SU(3)$
- the coset construction $SU(3)/SU(2)_{\rm irr}$
- the irrational construction $SU(3)^{\#}_M$

to illustrate the double inequality (5.18). As discussed below, the symmetry hierarchy for these three constructions is $SU(3) \supset SU(2)_{\rm irr} \supset O$, where $SU(2)_{\rm irr}$ is the irregular embedding of $SU(2)$ in $SU(3)$ and O is the octahedral group symmetry of the irrational construction.

5.3 Crossing relations

Using the completeness relations (3.27a) and (5.9d) of the g-invariant and H-invariant eigenvectors respectively, we verify the crossing relations among the blocks,

$$\mathcal{B}^{(\rho)}_H(y)_m{}^M = [X_g(\rho\sigma) + \mathcal{O}(k^{-2})]_m{}^n\, \mathcal{B}^{(\sigma)}_H(y)_n{}^N\, ([X_H(\rho\sigma) + \mathcal{O}(k^{-2})]^{-1})_N{}^M \tag{5.19a}$$

$$X_H(\rho\sigma)_M{}^N = \psi(\rho, H)_M \psi(\sigma, H)^N \tag{5.19b}$$

$$X_H^{-1}(\rho\sigma)_M{}^N = X_H(\sigma\rho)_M{}^N = (X_H(\rho\sigma)_N{}^M)^* \tag{5.19c}$$

where $X_g(\rho\sigma)$ is the affine-Sugawara crossing matrix defined in (3.29) and $X_H(\rho\sigma)$ in (5.19b) is another set of unitary crossing matrices, called the H-crossing matrices, from the σ-channel to the ρ-channel.

The H-crossing matrices satisfy the same consistency relations

$$X_H(\rho\sigma)X_H(\sigma\tau)X_H(\tau\rho) = X_H(\rho\tau)X_H(\tau\sigma)X_H(\sigma\rho) = 1 \tag{5.20a}$$

$$(1)_M{}^N = \delta^N_M \tag{5.20b}$$

found for g and h in (3.30) and (4.24).

When the external g-irreps satisfy $\mathcal{T}^2 \sim \mathcal{T}^3$, we may take

$$\psi(\mathrm{u}, H)^M_\alpha = \psi(\mathrm{s}, H)^M_{\alpha'} \quad , \quad \psi(\mathrm{u}, H)^\alpha_M = \psi(\mathrm{s}, H)^{\alpha'}_M \tag{5.21}$$

and then one finds that,

$$\Lambda_H(y)_{\alpha'}{}^{\beta'} = \Lambda_H(1-y)_\alpha{}^\beta \tag{5.22a}$$

$$\mathcal{B}^{(\mathrm{u})}_H(y) = \mathcal{B}^{(\mathrm{s})}_H(1-y) \tag{5.22b}$$

$$X_H(\mathrm{su}) = X_H^{-1}(\mathrm{su}) = X_H(\mathrm{us}) \tag{5.22c}$$

where $\alpha' = (\alpha_1\alpha_3\alpha_2\alpha_4)$. It follows that the set of s-channel blocks is closed under crossing

$$\mathcal{B}^{(\mathrm{s})}_H(1-y)_m{}^M = [X_g(\mathrm{us}) + \mathcal{O}(k^{-2})]_m{}^n\, \mathcal{B}^{(\mathrm{s})}_H(y)_n{}^N\, [X_H(\mathrm{us}) + \mathcal{O}(k^{-2})]_N{}^M \tag{5.23}$$

as it should be in this case. Similar relations hold when any two external states are the same.

Using (5.19a) and (5.13d), we finally write down the crossing relations of the three sets $\mathcal{B}^{(\mathrm{s})}_H$, $\mathcal{B}^{(\mathrm{u})}_H$, $\hat{\mathcal{B}}^{(\mathrm{t})}_H$ of analytic blocks,

$$\mathcal{B}^{(\mathrm{s})}_H = [X_g(\mathrm{su}) + \mathcal{O}(k^{-2})]\, \mathcal{B}^{(\mathrm{u})}_H\, [X_H(\mathrm{su}) + \mathcal{O}(k^{-2})]^{-1} \tag{5.24a}$$

$$\mathcal{B}^{(\mathrm{u})}_H = [X_g(\mathrm{us}) + \mathcal{O}(k^{-2})]\, \mathcal{B}^{(\mathrm{s})}_H\, [X_H(\mathrm{us}) + \mathcal{O}(k^{-2})]^{-1} \tag{5.24b}$$

$$\mathcal{B}^{(\mathrm{s})}_H = [X_g(\mathrm{st}) + \mathcal{O}(k^{-2})]\, \hat{\mathcal{B}}^{(\mathrm{t})}_H\, [X_H(\mathrm{st})U_H^{-1} + \mathcal{O}(k^{-2})]^{-1} \tag{5.24c}$$

$$\hat{\mathcal{B}}^{(\mathrm{t})}_H = [X_g(\mathrm{ts}) + \mathcal{O}(k^{-2})]\, \mathcal{B}^{(\mathrm{s})}_H\, [U_H X_H(\mathrm{ts}) + \mathcal{O}(k^{-2})]^{-1} \tag{5.24d}$$

$$\mathcal{B}^{(\mathrm{u})}_H = [X_g(\mathrm{ut}) + \mathcal{O}(k^{-2})]\, \hat{\mathcal{B}}^{(\mathrm{t})}_H\, [X_H(\mathrm{ut})U_H^{-1} + \mathcal{O}(k^{-2})]^{-1} \tag{5.24e}$$

$$\hat{\mathcal{B}}^{(\mathrm{t})}_H = [X_g(\mathrm{tu}) + \mathcal{O}(k^{-2})]\, \mathcal{B}^{(\mathrm{u})}_H\, [U_H X_H(\mathrm{tu}) + \mathcal{O}(k^{-2})]^{-1} \tag{5.24f}$$

where U_H is the non-analytic unitary phase matrix (5.13e) of the $L(g;H)$-degenerate correlators. As seen above for the affine-Sugawara and coset constructions, the phase matrix provides the entire $\mathcal{O}(k^{-1})$ corrections to the full crossing matrices in (5.24).

For the special case of the $L(g;h)$-degenerate coset correlators (with $L_H = L_{g/h}$ and $\psi(H) = \psi(h)$) the general high-level blocks (5.12), (5.13) reduce precisely to the $L(g;H)$-degenerate subset of high-level coset blocks computed in (4.10), (4.14). In the same way, the crossing relations (5.24) reduce in this case to the coset crossing relations in (4.27).

5.4 Non-chiral correlators in ICFT

For the general $L(g;H)$-degenerate process, we construct a set of high-level non-chiral correlators using the diagonal construction in the s-channel blocks (5.12),

$$Y_H(y^*, y) = \sum_{m,M} |\mathcal{B}^{(\mathrm{s})}_H(y)_m{}^M|^2 + \mathcal{O}(k^{-2}) \tag{5.25}$$

which shows trivial monodromy around $y = 0$. Using the crossing relations (5.19) we can also express this correlator in terms of u or t-channel blocks

$$Y_H(y^*, y) = \sum_{m,M} |\mathcal{B}_H^{(u)}(y)_m{}^M|^2 + \mathcal{O}(k^{-2}) \tag{5.26a}$$

$$= \sum_{m,M} |\mathcal{B}_H^{(t)}(y)_m{}^M|^2 + \mathcal{O}(k^{-2}) \tag{5.26b}$$

which show trivial monodromy around $y = 1$ and $y = \infty$ respectively.

The non-chiral correlator can also be expressed in terms of the analytic t-channel blocks (5.13a),

$$Y_H(y^*, y) = \sum_{m,N,M} |\hat{\mathcal{B}}_H^{(t)}(y)_m{}^N U_H(y)_N{}^M|^2 + \mathcal{O}(k^{-2}) = \sum_{m,N} |\hat{\mathcal{B}}_H^{(t)}(y)_m{}^N|^2 + \mathcal{O}(k^{-2}) \tag{5.27}$$

where the last step uses the unitarity (5.13f) of the phase matrix U_H.

Using completeness and the explicit form of the conformal blocks, we also obtain the summed form of the non-chiral correlators

$$\begin{aligned} Y_H(y^*, y) = &\mathrm{Tr}\{[\mathbb{1} + 2L^{ab}_{H,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y^* + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y^*))]I_g \\ &\times [\mathbb{1} + 2L^{ab}_{H,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]\} + \mathcal{O}(k^{-2}) \end{aligned} \tag{5.28a}$$

$$= \mathrm{Tr}[(\mathbb{1}+2L^{ab}_{H,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln|y|^2+\mathcal{T}^1_a\mathcal{T}^3_b \ln|1-y|^2))I_g]+\mathcal{O}(k^{-2}) \tag{5.28b}$$

where I_g is the projector onto the G-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$. The last form explicitly shows two of the trivial monodromies, and trivial monodromy around $y = \infty$ is easily seen following the discussion below eq.(3.41). One also sees the expected crossing symmetry

$$Y_H(1-y^*, 1-y) = Y_H(y^*, y) \tag{5.29}$$

when $\mathcal{T}^2 \sim \mathcal{T}^3$. We finally note that the general $L(g;H)$-degenerate correlators (5.28) correctly include the $L(g;h)$-degenerate coset correlators obtained from (4.42) when $\mathcal{P}_h = 1$.

Using the embedding matrices (5.12d) and the H-crossing matrices (5.19b), Appendix A gives alternate expressions for the blocks and correlators of the $L(g;H)$-degenerate processes in ICFT.

6 Blocks and Correlators in $\mathrm{SU}(3)^{\#}_{\mathrm{M}}$

As an explicit example in irrational conformal field theory, we work out here the high-level conformal blocks and non-chiral correlators for a particular $L(g;H)$-degenerate process in the unitary irrational level-family [24]

$$(SU(3)_x)^{\#}_M \tag{6.1}$$

where x is the invariant level of $SU(3)$. For simplicity below, this construction is often called $SU(3)^{\#}_M$. The construction is included in the larger maximally-symmetric ansatz for all simply-laced g, which was in fact the first set of ICFTs found in the Virasoro master equation. The closely related coset construction $SU(3)_x/SU(2)_{4x}$, which also resides in the maximally-symmetric ansatz, is studied in Appendix D.

The exact forms of the central charge and the conformal weights of the 3 and $\bar{3}$ representations under $(SU(3)_x)^{\#}_M$ are

$$c[(SU(3)_x)^{\#}_M] = \frac{2x}{x+3}\left[2 - \frac{x^2-8x+17}{\sqrt{4x^4-28x^3+17x^2+160x-128}}\right] \tag{6.2a}$$

$$\Delta(\mathcal{T}_{(3)}) = \Delta(\mathcal{T}_{(\bar{3})}) = \frac{c}{6x} \tag{6.2b}$$

where the 3-fold degenerate conformal weights in (6.2b) strongly suggest that the 3 and $\bar{3}$ are $L(g;H)$-degenerate representations.

As discussed further in Appendix C, the level-family $(SU(3)_x)^{\#}_M$ has a finite group symmetry

$$H(SU(3)^{\#}_M) = O \subset SU(2)_{\mathrm{irr}} \tag{6.3}$$

where O is the octahedral group and $SU(2)_{\mathrm{irr}}$ is the irregularly embedded $SU(2) \subset SU(3)$. The degeneracy of the 3 and $\bar{3}$ is due to the octahedral symmetry of the construction, which mixes all three components of each representation. Thus the 3 and $\bar{3}$ are $L(SU(3);O)$-degenerate representations in $(SU(3)_x)^{\#}_M$, as desired.

For the high-level computations in $(SU(3)_x)^{\#}_M$ below, we need only the high-level forms of the inverse inertia tensor (in the Gell-Mann basis) and the degenerate conformal weights,

$$L^{ab}_{O,\infty} = \frac{1}{x\psi_g^2}\theta_a\delta_{ab} \quad , \quad \theta_a = \begin{cases} 1 & a = 1,4,6 \\ 0 & a = 3,8,2,5,7 \end{cases} \tag{6.4a}$$

$$c = 3 + \mathcal{O}(x^{-1}) \tag{6.4b}$$

$$\Delta^O(\mathcal{T}_{(3)}) = \Delta^O(\mathcal{T}_{(\bar{3})}) = \frac{1}{2x} + \mathcal{O}(x^{-2}) \tag{6.4c}$$

which identifies $P^{ab} = \theta_a\delta_{ab}$ as the high-level projector of $SU(3)^{\#}_M$. Moreover, we will consider only the $L(SU(3);O)$-degenerate process $3\bar{3}\bar{3}3$ in $SU(3)^{\#}_M$,

$$\mathcal{T}^1 = \mathcal{T}^4 = \mathcal{T}_{(3)} \quad , \quad \mathcal{T}^2 = \mathcal{T}^3 = \mathcal{T}_{(\bar{3})} \tag{6.5}$$

shown schematically in Fig.3. The matrix irrep of the 3 and $\bar{3}$ in the Gell-Mann basis are given by,

$$\mathcal{T}_{(3)} = \frac{\sqrt{\psi_g^2}}{2}\lambda_a \quad , \quad \mathcal{T}_{(\bar{3})} = \frac{\sqrt{\psi_g^2}}{2}\lambda_a \tag{6.6a}$$

$$\lambda_a = -\lambda_a^T = \begin{cases} -\lambda_a & a = 3,8,1,4,6 \\ \lambda_a & a = 2,5,7 \end{cases} \tag{6.6b}$$

where λ_a, $a = 1\ldots8$ are the Gell-Mann matrices.

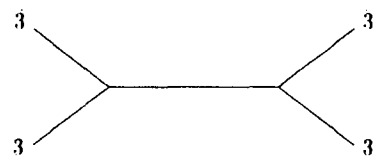

Fig. 3. An $L(SU(3);O)$-degenerate correlator in $SU(3)^{\#}_M$.

To compute the high-level blocks in the s-channel, we need to solve the eigenvalue problem (5.9a) for the s-channel O-invariant eigenvectors $\psi(\mathrm{s},O)$, which reads in this case,

$$[-\frac{1}{2x}\sum_{a=1,4,6}\lambda^1_a\lambda^2_a+\frac{1}{x}\mathbb{1}]_\alpha{}^\beta\,\psi(\mathrm{s},O)^M_\beta=\Delta^O_{(\mathrm{s})}(M)\psi(\mathrm{s},O)^M_\alpha \tag{6.7a}$$

$$\prod_{i=1}^{4}(\omega^i_l)_{\alpha_i}{}^{\beta_i}\psi(\mathrm{s},O)^M_\beta=\psi(\mathrm{s},O)^M_\alpha \quad , \quad l=1,2 \tag{6.7b}$$

$$\omega_1=\exp(i\pi\lambda_2/2)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix} \tag{6.7c}$$

$$\omega_2=\exp(i\pi\lambda_5/2)\exp(i\pi\lambda_7/2)=\begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix} \quad . \tag{6.7d}$$

The matrices ω_1 and ω_2 which appear in the O-invariance condition (6.7b) may be taken as the generators of O.

After some algebra, one finds the following orthonormal set of s-channel eigenvectors $\psi(\mathrm{s},O)^M$ and their eigenvalues $\Delta^O_{(\mathrm{s})}(M)$,

$$\psi(\mathrm{s},O)^1_\alpha=\frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \quad , \quad \Delta^O_{(\mathrm{s})}(1)=0 \tag{6.8a}$$

$$\psi(\mathrm{s},O)^2_\alpha=\frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}+\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}-2\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\delta_{\alpha_1\alpha_3}] \quad , \quad \Delta^O_{(\mathrm{s})}(2)=\frac{1}{2x} \tag{6.8b}$$

$$\psi(\mathrm{s},O)^3_\alpha=\frac{1}{3\sqrt{2}}[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}-3\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\delta_{\alpha_1\alpha_3}] \quad , \quad \Delta^O_{(\mathrm{s})}(3)=\frac{3}{2x} \tag{6.8c}$$

$$\psi(\mathrm{s},O)^4_\alpha=\frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}-\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}] \quad , \quad \Delta^O_{(\mathrm{s})}(4)=\frac{3}{2x} \tag{6.8d}$$

$$\psi(\mathrm{s},O)^\alpha_M=(\psi(\mathrm{s},O)^M_\beta)^*\eta^{\beta\alpha}=\psi(\mathrm{s},O)^M_\alpha \tag{6.8e}$$

where the last relation says that the left and right eigenvectors coincide in this case.

In ICFT, the high-level fusion rules [16,14] of the broken affine primaries follow the Clebsch-Gordan coefficients of their corresponding matrix irreps, so the s-channel should

show the exchange of broken affine primary states corresponding to the vacuum and the adjoint representation,

$$3\otimes 3=1\oplus 8+\mathcal{O}(k^{-1}) \quad . \tag{6.9}$$

Indeed, the first conformal weight in (6.8a) is the conformal weight of the vacuum, and the other three high-level conformal weights in (6.8b-d) are precisely the high-level form of the three degenerate subsets of broken conformal weights of the adjoint (see Appendix C).

Similarly, we can solve for the u and t-channel eigenvectors, which are given by

$$\psi(\mathrm{u},O)^M=\psi(\mathrm{s},O)^M|_{2\leftrightarrow 3} \quad , \quad \Delta^O_{(\mathrm{u})}(M)=\Delta^O_{(\mathrm{s})}(M) \tag{6.10a}$$

$$\psi(\mathrm{t},O)^M=\psi(\mathrm{s},O)^M|_{2\leftrightarrow 4} \quad , \quad \Delta^O_{(\mathrm{t})}(M)=\frac{2}{x}-\Delta^O_{(\mathrm{s})}(M) \tag{6.10b}$$

where $2\leftrightarrow 3$ and $2\leftrightarrow 4$ mean respectively $\alpha_2\leftrightarrow\alpha_3$ and $\alpha_2\leftrightarrow\alpha_4$ in the explicit expressions of the s-channel eigenvectors (6.8). The result in (6.10a) is in accord with (5.21) since $T^2\sim T^3$, so that the u-channel conformal weights are identical to the ones in the s-channel. The conformal weights found in the t-channel,

$$\Delta^O_{(\mathrm{t})}(M)=(\frac{2}{x},\frac{3}{2x},\frac{1}{2x},\frac{1}{2x}) \tag{6.11}$$

are also in agreement with the conformal weights of broken affine primaries in the known high-level fusion rule

$$3\otimes 3=6\oplus\bar{3}+\mathcal{O}(k^{-1}) \quad . \tag{6.12}$$

In particular, the last value in (6.11) is the completely degenerate conformal weight of the $\bar{3}$ and the first three coincide with the three degenerate subsets (C.11b) of the 6, according to the high-level form (C.13a).

Using eq.(5.19b), the high-level s-u and s-t O-crossing matrices are computed from the eigenvectors as

$$X_O(\mathrm{us})_M{}^N=\psi(\mathrm{u},O)^M\psi(\mathrm{s},O)^N=\frac{1}{6}\begin{pmatrix}2&2\sqrt{3}&-2\sqrt{2}&2\sqrt{3}\\2\sqrt{3}&3&\sqrt{6}&-3\\-2\sqrt{2}&\sqrt{6}&4&\sqrt{6}\\2\sqrt{3}&-3&\sqrt{6}&3\end{pmatrix} \tag{6.13a}$$

$$X_O(\mathrm{ts})_M{}^N=\psi(\mathrm{t},O)^M\psi(\mathrm{s},O)^N=\frac{1}{6}\begin{pmatrix}2&2\sqrt{3}&-2\sqrt{2}&-2\sqrt{3}\\2\sqrt{3}&3&\sqrt{6}&3\\-2\sqrt{2}&\sqrt{6}&4&-\sqrt{6}\\-2\sqrt{3}&3&-\sqrt{6}&3\end{pmatrix} \tag{6.13b}$$

which are orthogonal and idempotent matrices in this case. The third O-crossing matrix

$$X_O(\mathrm{ut})=X_O(\mathrm{us})X_O(\mathrm{ts}) \tag{6.14}$$

follows from the consistency relation (5.20).

For the crossing of the blocks one also needs the high-level affine-Sugawara crossing matrices (3.29c) for $g = SU(3)$. The ρ-channel $SU(3)$-invariant eigenvectors and the corresponding crossing matrices are

$$v(\mathrm{s}, SU(3))^V_\alpha = \frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \tag{6.15a}$$

$$v(\mathrm{s}, SU(3))^A_\alpha = \frac{1}{2\sqrt{2}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} - \frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}] \tag{6.15b}$$

$$v(\mathrm{u}, SU(3)) = v(\mathrm{s}, SU(3))|_{2\leftrightarrow 3} \tag{6.15c}$$

$$v(\mathrm{t}, SU(3))^6_\alpha = \frac{1}{2\sqrt{6}}[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} + \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}] \tag{6.15d}$$

$$v(\mathrm{t}, SU(3))^{\bar{3}}_\alpha = \frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} - \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}] \tag{6.15e}$$

$$X_{SU(3)}(\mathrm{us})_m{}^n = v(\mathrm{u}, SU(3))^m v(\mathrm{s}, SU(3))^n = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & -1 \end{pmatrix} \tag{6.15f}$$

$$X_{SU(3)}(\mathrm{ts})_m{}^n = v(\mathrm{t}, SU(3))^m v(\mathrm{s}, SU(3))^n = \frac{1}{3}\begin{pmatrix} \sqrt{6} & \sqrt{3} \\ \sqrt{3} & -\sqrt{6} \end{pmatrix} \tag{6.15g}$$

where the labels V, A stand for vacuum and adjoint irrep, and $6, \bar{3}$ for symmetric and antisymmetric irrep. The third g-crossing matrix is given by $X_{SU(3)}(\mathrm{ut}) = X_{SU(3)}(\mathrm{us})\, X^{-1}_{SU(3)}(\mathrm{ts})$.

Finally, we write down the 8 high-level s-channel conformal blocks (5.12) of the $3\bar{3}\bar{3}3$ correlator in $SU(3)^\#_M$,

$$\mathcal{B}^{(\mathrm{s})}_O(y)_m{}^M = \epsilon(\mathrm{s}, O)_m{}^N[1 + (\Delta^O_{(\mathrm{s})} - \frac{1}{x}\cdot 1)\ln y + (Q^O_{(\mathrm{su})} - \frac{1}{x}\cdot 1)\ln(1-y)]_N{}^M + \mathcal{O}(x^{-2}) \tag{6.16}$$

where $(1)_N{}^M = \delta^M_N$ and

$$\epsilon(\mathrm{s}, O)_m{}^M = v(\mathrm{s}, SU(3))^m \psi(\mathrm{s}, O)^M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}\sqrt{6} & -\frac{1}{2} & \frac{1}{4}\sqrt{6} \end{pmatrix} \tag{6.17a}$$

$$(\Delta^O_{(\mathrm{s})})_N{}^M = \Delta^O_{(\mathrm{s})}(M)\delta^M_N \quad , \quad \Delta^O_{(\mathrm{s})}(M) = \Delta^O_{(\mathrm{u})}(M) = (0, \frac{1}{2x}, \frac{3}{2x}, \frac{3}{2x}) \tag{6.17b}$$

$$(Q^O_{(\mathrm{su})})_N{}^M = \sum_L X_O(\mathrm{us})_N{}^L \Delta^O_{(\mathrm{u})}(L) X_O(\mathrm{us})_L{}^M = \frac{1}{12x}\begin{pmatrix} 12 & -4\sqrt{3} & 0 & 0 \\ -4\sqrt{3} & 9 & \sqrt{6} & -3 \\ 0 & \sqrt{6} & 12 & 3\sqrt{6} \\ 0 & -3 & 3\sqrt{6} & 9 \end{pmatrix} . \tag{6.17c}$$

Here we have used the alternate expression (A.9) for the $L(g;H)$-degenerate blocks in Appendix A. The u and t-channel blocks can be computed from the s-channel blocks

above using the crossing relation (5.19) and the explicit forms of the crossing matrices $X_{SU(3)}(\mathrm{us}), X_{SU(3)}(\mathrm{ts})$ in (6.15), and $X_O(\mathrm{us}), X_O(\mathrm{ts})$ in (6.13). Moreover, using the explicit form of the non-analytic phase matrix (5.13e) for this process,

$$U_O(y)_M{}^N = \sum_L X_O(\mathrm{ts})_M{}^L \exp\left(-\pi i[\Delta^O_{(\mathrm{s})}(L) - \frac{1}{x}]\mathrm{sign}(\arg(-y))\right) X_O(\mathrm{ts})_L{}^N + \mathcal{O}(x^{-2}) \tag{6.18}$$

the analytic t-channel blocks follow from the crossing relation (5.24f)

Using (5.16a,d-f) we obtain the following limiting behavior as $y \to 0$ for the 8 s-channel blocks (6.16) of this correlator,

$$\mathcal{B}^{(\mathrm{s})}_O(y)_m{}^M \underset{y\to 0}{\sim} \Gamma^{(\mathrm{s})}_O(m, M) y^{\Delta^O_{(\mathrm{s})}(m,M) - 1/x} + \mathcal{O}(x^{-2}) \quad , \quad m = V, A \quad , \quad M = 1,2,3,4 \tag{6.19a}$$

$$\Delta^O_{(\mathrm{s})}(V,1) = 0 + \mathcal{O}(x^{-2}) \quad , \quad \Delta^O_{(\mathrm{s})}(A,2) = \frac{1}{2x} + \mathcal{O}(x^{-2}) \tag{6.19b}$$

$$\Delta^O_{(\mathrm{s})}(A,3) = \frac{3}{2x} + \mathcal{O}(x^{-2}) \quad , \quad \Delta^O_{(\mathrm{s})}(A,4) = \frac{3}{2x} + \mathcal{O}(x^{-2}) \tag{6.19c}$$

$$\Delta^O_{(\mathrm{s})}(V,2) = 1 + \mathcal{O}(x^{-1}) \quad , \quad \Delta^O_{(\mathrm{s})}(A,1) = 1 + \mathcal{O}(x^{-1}) \tag{6.19d}$$

$$\Delta^O_{(\mathrm{s})}(V,3) = \mathcal{O}(x^0) \quad , \quad \Delta^O_{(\mathrm{s})}(V,4) = \mathcal{O}(x^0) \tag{6.19e}$$

$$\Gamma^{(\mathrm{s})}_O(m, M) = \begin{pmatrix} 1 & \frac{1}{3x}\sqrt{3} & 0 & 0 \\ \frac{1}{4x}\sqrt{2} & \frac{1}{4}\sqrt{6} & -\frac{1}{2} & \frac{1}{4}\sqrt{6} \end{pmatrix} + \mathcal{O}(x^{-2}) \quad . \tag{6.19f}$$

The explicit form of these residues was obtained using (5.16d), the embedding matrix (6.17a) and the relation $\epsilon(\mathrm{s}, O) = Q^O_{(\mathrm{su})} - \frac{1}{x}\cdot 1$.

The four conformal weights in (6.19b,c) are the broken affine primary states in (6.9), whose residues are $\mathcal{O}(x^0)$ in accord with the general OPE in (2.17). The two conformal weights in (6.19d) are broken affine secondary states (with residues which are $\mathcal{O}(x^{-1})$) which are not necessarily integer descendants of broken affine primary states. The conformal weights in (6.19e) cannot be determined through this order because their residues $\Gamma^{(\mathrm{s})}_O$ are zero through $\mathcal{O}(x^{-1})$, and indeed these entire blocks begin at order $\mathcal{O}(x^{-2})$,

$$\mathcal{B}_O(y)_V{}^3 \ , \ \mathcal{B}_O(y)_V{}^4 \ = \ \mathcal{O}(x^{-2}) \tag{6.20}$$

a phenomenon also encountered in the coset example of Appendix D. To see (6.20) directly from (6.16) note that, for these blocks, $\epsilon(\mathrm{s}, O)_m{}^M = 0$ and $\epsilon(\mathrm{s}, O)_m{}^N (Q^O_{(\mathrm{su})})_N{}^M = 0$. In the u-channel we also find two blocks which begin at $\mathcal{O}(x^{-2})$, while in the t-channel there is one such block.

In agreement with (5.17), the number of blocks for this $L(SU(3); O)$-degenerate process is

$$B_O = 2 \cdot 4 = 8 \quad . \tag{6.21}$$

Because of the increasing symmetry breakdown,

$$O \subset SU(2)_{\mathrm{irr}} \subset SU(3) \tag{6.22}$$

the number (6.21) is larger than the number of blocks

$$B_{SU(3)} = 2 \cdot 2 = 4 \quad , \quad B_{SU(3)/SU(2)} = 2 \cdot 3 = 6 \tag{6.23}$$

for the same correlator under the affine-Sugawara construction (see Appendix B) and the closely related coset construction studied in Appendix D. Taken together, (6.21) and (6.23) are an illustration of the double inequality (5.18).

Using eqs.(A.13), (A.14) we also find the following expression for the high-level non-chiral correlators of $SU(3)^{\#}_{M}$,

$$Y_O(y^*, y) = \sum_M E(\mathrm{s}, O)_M{}^M [1 + (\Delta^O_{(\mathrm{s})} - \frac{1}{x} \cdot 1) \ln |y|^2 + (\Delta^O_{(\mathrm{s})} - \frac{1}{x} \cdot 1) \ln |1-y|^2]_M{}^M + \mathcal{O}(x^{-2}) \tag{6.24a}$$

$$E(\mathrm{s}, O)_M{}^N = \sum_m (\epsilon(\mathrm{s}, O)_m{}^M)^* \epsilon(\mathrm{s}, O)_m{}^N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{8} & -\frac{1}{8}\sqrt{6} & \frac{3}{8} \\ 0 & -\frac{1}{8}\sqrt{6} & \frac{1}{4} & -\frac{1}{8}\sqrt{6} \\ 0 & \frac{3}{8} & -\frac{1}{8}\sqrt{6} & \frac{3}{8} \end{pmatrix} \tag{6.24b}$$

where we have used $X_O(\mathrm{us}) E(\mathrm{s}, O) X_O(\mathrm{us}) = E(\mathrm{s}, O)$ and the diagonal s-channel conformal weight matrix $\Delta^O_{(\mathrm{s})}$ is given in (6.17b). This result explicitly shows the crossing symmetry (5.29), as it should since $\mathcal{T}^2 \sim \mathcal{T}^3$ in this case.

We finally remark that the high-level blocks and correlators of the K-conjugate theory

$$SU(3)/SU(3)^{\#}_{M} \quad , \quad \tilde{L} = L_{SU(3)} - L \tag{6.25}$$

can be easily obtained from the results above, by substituting everywhere the K-conjugate conformal weights $\tilde{\Delta}(\mathcal{T}) = \Delta^g(\mathcal{T}) - \Delta(\mathcal{T})$ for the conformal weights $\Delta(\mathcal{T})$. Moreover, the results above can easily be extended to the $L(g; H)$-degenerate correlators $nnnn$ in the larger family of ICFTs called $SU(n)^{\#}_{M}$ [24]; in this case, the number of H-invariant tensors stays the same, with closely analogous forms for all the more general results.

7 Conclusions

The generalized KZ equations of ICFT provide a uniform description of the chiral correlators of rational and irrational conformal field theory, and the solution of these equations is known at high level on simple g. The apparent simplicity of this result is deceptive, however, because the solution describes a vast variety of generically irrational conformal field theories ranging from the most symmetric (the RCFTs) to totally asymmetric (the generic ICFT).

In this paper, we have begun the resolution of the high-level chiral correlators into high-level conformal blocks and non-chiral correlators, beginning with the simplest and most symmetric classes.

In particular, we began by working out the high-level blocks and correlators of all the

- affine-Sugawara constructions on simple g
- coset constructions on simple g.

Both results are new, and the results for the cosets are apparently inaccessible by other methods.

Based on this analysis, we then identified what we believe to be the simplest and most symmetric class of correlators in ICFT. These are the

- $L(g; H)$-degenerate processes in H-invariant CFTs on simple g

which are those correlators whose external states have entirely degenerate conformal weights $\Delta_\alpha = \Delta$. This class of correlators includes all the affine-Sugawara correlators, a highly-symmetric subset of coset correlators and a presumably large set of irrational correlators, examples of which are known.

For this simple class of correlators we were able to find the general expression for the high-level blocks and non-chiral correlators, and we worked out an irrational example with octahedral symmetry on $SU(3)$.

Our results emphasize that the $L(g; H)$-degenerate correlators are a very special class of correlators indeed, since they have a finite number of conformal blocks (at least in the semi-classical approximation), whereas the generic correlator in ICFT is expected to involve an infinite number of blocks. We are intrigued to find that ICFT resembles RCFT in this simple domain, and we are optimistic that the simplicity of the $L(g; H)$-degenerate correlators can provide a foothold for further exploration.

Additional information is needed, however, to go beyond the leading orders of the $L(g;H)$-degenerate processes in ICFT. The central question here is whether the number of conformal blocks remains finite, as we found in the semi-classical approximation, or increases with the order of k^{-1}. At finite values of the level, one will also need to consider the roles of the affine cutoff [4,15] and fixed-point resolution [25].

The more immediate open direction is to find the high-level conformal blocks of irrational correlators beyond the set of $L(g; H)$-degenerate processes. An ever-increasing number of blocks is expected here as one confronts the progressively larger symmetry breakdown of ICFT, signalled by the L^{ab}-broken conformal weights Δ_α.

In this direction, we remind the reader of the known singularities of the invariant flat connections W which govern the exact (finite level) correlators of ICFT. For example, it is known that [9]

$$W(\tilde{u}, u)_\alpha{}^\beta \underset{\tilde{u}, u \to 0}{=} \left(\frac{u}{\tilde{u}}\right)^{\Delta_{\alpha_1}(\mathcal{T}^1) + \Delta_{\alpha_2}(\mathcal{T}^2) - \Delta_{\mathfrak{s}_1}(\mathcal{T}^1) - \Delta_{\mathfrak{s}_2}(\mathcal{T}^2)} \frac{(2L^{ab}\mathcal{T}^1_a \mathcal{T}^2_b)_\alpha{}^\beta}{u} \tag{7.1a}$$

$$= \begin{cases} \frac{(2L^{ab}\mathcal{T}^1_a \mathcal{T}^2_b)_\alpha{}^\beta}{u} + \mathcal{O}(k^{-2}) & (\text{high } k) \quad (7.1b) \\ \frac{(2L^{ab}\mathcal{T}^1_a \mathcal{T}^2_b)_\alpha{}^\beta}{u} & (L(g; H)\text{-degenerate}) \quad (7.1c) \end{cases}$$

where u and $\hat{u}$ are the variables of the theory and its K-conjugate theory respectively. The result (7.1a) shows the apparently non-Fuchsian α, β dependent shielding factor, which is hidden in the high-level limit (7.1b), and which simplifies to unity at all levels, shown in (7.1c), for the $L(g;H)$-degenerate processes. We believe that this phenomenon underlies the simplicity of the class of $L(g;H)$-degenerate processes in ICFT, and it may be necessary to consider this factor in the physical interpretation of the high-level logarithmic singularities of correlators beyond the simple class we have considered here.

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Appendix A: Alternate expressions for blocks and correlators

In this appendix, we use the relevant crossing matrices to give alternate expressions for the conformal blocks and correlators of any set of external states in the affine-Sugawara constructions (see Section 3) and of any $L(g;H)$-degenerate process in the more general H-invariant CFTs (see Section 5).

Affine-Sugawara constructions

We begin with the ρ-channel affine-Sugawara blocks in (3.27),

$$\mathcal{F}_g^{(\rho)}(y)_m{}^n = \ddot{v}(\rho,g)_m[\mathbb{1} + 2L^{ab}_{g,\infty}(\mathcal{T}^1_a\mathcal{T}^2_b \ln y + \mathcal{T}^1_a\mathcal{T}^3_b \ln(1-y))]v(\rho,g)^n + \mathcal{O}(k^{-2}) \quad . \tag{A.1}$$

Using the definitions (3.2), (3.15) of the g-invariant ρ-channel eigenvectors $v(\rho,g)$ and the g-crossing matrices X_g in (3.29c), we have the g-crossing relations,

$$v(\rho,g)_m = X_g(\rho\sigma)_m{}^n v(\sigma,g)_n \quad , \quad v(\rho,g)^m = v(\sigma,g)^n X_g(\sigma\rho)_n{}^m \quad . \tag{A.2}$$

Using these relations, we obtain the alternate form of the affine-Sugawara blocks,

$$\begin{aligned}\mathcal{F}_g^{(\rho)}(y)_m{}^n = [1 &+ (Q^g_{(\rho s)} - (\Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^2))\cdot 1)\ln y \\ &+ (Q^g_{(\rho u)} - (\Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^3))\cdot 1)\ln(1-y)]_m{}^n + \mathcal{O}(k^{-2})\end{aligned} \tag{A.3}$$

where

$$(1)_m{}^n = \delta^n_m \quad . \quad (Q^g_{(\rho\sigma)})_m{}^n = \begin{cases} (\Delta^g_{(\rho)})_m{}^n = \Delta^g_{(\rho)}(m)\delta^n_m & , \quad \rho = \sigma \\ \sum_l X_g(\rho\sigma)_m{}^l \Delta^g_{(\sigma)}(l) X_g(\sigma\rho)_l{}^n & , \quad \rho \neq \sigma \end{cases} \tag{A.4}$$

and $\Delta^g_{(\rho)}(m)$ are the ρ-channel affine-Sugawara conformal weights in (3.2) and (3.15).

We also give the corresponding alternate form of the analytic t-channel affine-Sugawara blocks (3.23a),

$$\begin{aligned}\hat{\mathcal{F}}_g^{(t)}(y)_m{}^n &= [1 + (Q^g_{(ts)} + Q^g_{(tu)} - (2\Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^2) + \Delta^g(\mathcal{T}^3))\cdot 1)\ln(-y) \\ &\quad + (Q^g_{(tu)} - (\Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^3))\cdot 1)\ln\left(1 - \frac{1}{y}\right)]_m{}^n + \mathcal{O}(k^{-2})\end{aligned} \tag{A.5a}$$

$$\begin{aligned}&= [1 - (Q^g_{(tt)} + (\Delta^g(\mathcal{T}^1) - \Delta^g(\mathcal{T}^4))\cdot 1)\ln(-y) \\ &\quad + (Q^g_{(tu)} - (\Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^3))\cdot 1)\ln\left(1 - \frac{1}{y}\right)]_m{}^n + \mathcal{O}(k^{-2})\end{aligned} \tag{A.5b}$$

where $Q^g_{(t\sigma)}$, $\sigma = s,t,u$ is given in (A.4). Here, the second form (A.5b) follows from (A.5a) using the ρ=t form of the conformal weight sum rule,

$$\Delta^g_{(\rho)} + X_g(\rho\sigma)\Delta^g_{(\sigma)}X_g(\sigma\rho) + X_g(\rho\tau)\Delta^g_{(\tau)}X_g(\tau\rho) = \left(\sum_{i=1}^{4}\Delta^g(\mathcal{T}^i)\right)\cdot 1 \quad , \quad \rho \neq \sigma \neq \tau \neq \rho \tag{A.6}$$

which is itself a direct consequence of the g-global Ward identity (3.24).

Substitution of the alternate forms (A.3) of the affine-Sugawara blocks in the expression (3.37) for the affine-Sugawara correlators then gives the corresponding alternate form for the non-chiral correlators,

$$\begin{aligned}Y_g(y^*,y)_\alpha{}^\beta = \sum_{m,n}[1 &+ (Q^g_{(\rho s)} - (\Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^2))\cdot 1)\ln|y|^2 \\ &+ (Q^g_{(\rho u)} - (\Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^3))\cdot 1)\ln|1-y|^2]_m{}^n\, v(\rho,g)^m_\alpha \ddot{v}(\rho,g)^\beta_n + \mathcal{O}(k^{-2})\end{aligned} \tag{A.7}$$

which explicitly shows the trivial monodromies around $y = 0$, 1 and ∞.

$L(g;H)$-degenerate processes

Following the development for the affine-Sugawara constructions above, we may find similar alternate forms for the blocks and correlators of the general $L(g;H)$-degenerate process.

Using the definitions (5.9) of the H-invariant eigenvectors $\ddot{v}(\rho,H)$ and the H-crossing matrices X_H in (5.19), we have the H-crossing relations,

$$\ddot{v}(\rho,H)_M = X_H(\rho\sigma)_M{}^N \ddot{v}(\sigma,H)_N \quad . \quad \ddot{v}(\rho,H)^M = \ddot{v}(\sigma,H)^N X_H(\sigma\rho)_N{}^M \quad . \tag{A.8}$$

Using these relations in the expressions (5.12b) for the blocks, we obtain the following alternate form of the $L(g;H)$-degenerate blocks,

$$\begin{aligned}\mathcal{B}^{(\rho)}_H(y)_m{}^M = \epsilon(\rho,H)_m{}^N[1 &+ (Q^H_{(\rho s)} - (\Delta^H_1 + \Delta^H_2)\cdot 1)\ln y \\ &+ (Q^H_{(\rho u)} - (\Delta^H_1 + \Delta^H_3)\cdot 1)\ln(1-y)]_N{}^M + \mathcal{O}(k^{-2})\end{aligned} \tag{A.9}$$

where $\epsilon(\rho,H)$ are the ρ-channel embedding matrices (5.12d) and

$$(1)_M{}^N = \delta^N_M \quad , \quad (Q^H_{(\rho\sigma)})_M{}^N = \begin{cases} (\Delta^H_{(\rho)})_M{}^N = \Delta^H_{(\rho)}(M)\delta^N_M & , \quad \rho = \sigma \\ \sum_L X_H(\rho\sigma)_M{}^L \Delta^H_{(\sigma)}(L) X_H(\sigma\rho)_L{}^N & , \quad \rho \neq \sigma \end{cases} \tag{A.10}$$

with $\Delta^H_{(\rho)}(M)$ the ρ-channel conformal weights in (5.9). These results include all the correlators of the affine-Sugawara constructions and all the $L(g;h)$-degenerate processes of the g/h coset constructions.

We also give the corresponding alternate form for the analytic t-channel blocks (5.13a),

$$\begin{aligned}\hat{\mathcal{B}}^{(t)}_H(y)_m{}^M &= \epsilon(t,H)_m{}^N[1 + (Q^H_{(ts)} + Q^H_{(tu)} - (2\Delta^H_1 + \Delta^H_2 + \Delta^H_3)\cdot 1)\ln(-y) \\ &\qquad + (Q^H_{(tu)} - (\Delta^H_1 + \Delta^H_3)\cdot 1)\ln\left(1 - \frac{1}{y}\right)]_N{}^M + \mathcal{O}(k^{-2}) \qquad \text{(A.11a)} \\ &= \epsilon(t,H)_m{}^N[1 - (Q^H_{(tt)} + (\Delta^H_1 - \Delta^H_4)\cdot 1)\ln(-y) \\ &\qquad + (Q^H_{(tu)} - (\Delta^H_1 + \Delta^H_3)\cdot 1)\ln\left(1 - \frac{1}{y}\right)]_N{}^M + \mathcal{O}(k^{-2}) \qquad \text{(A.11b)}\end{aligned}$$

where $Q^H_{(t\sigma)}$, $\sigma = s,t,u$ is given in (A.10). Here, the second form (A.11b) follows from (A.11a) using the $\rho = t$ form of the conformal weight sum rule in $L(g;H)$-degenerate processes,

$$\begin{aligned}&\epsilon(\rho,H)[\Delta^H_{(\rho)} + X_H(\rho\sigma)\Delta^H_{(\sigma)}X_H(\sigma\rho) + X_H(\rho\tau)\Delta^H_{(\tau)}X_H(\tau\rho)] \\ &\qquad = \epsilon(\rho,H)\sum_{i=1}^4 \Delta^H_i \quad , \quad \rho \neq \sigma \neq \tau \neq \rho\end{aligned} \tag{A.12}$$

which is itself a direct consequence of the g-global Ward identity (5.15).

Finally, we give the corresponding alternate form of the non-chiral correlators (5.25), using the expression (A.9) for the blocks,

$$\begin{aligned}Y(y^*,y) = \sum_{M,N} E(\rho,H)_M{}^N[1 &+ (Q^H_{(\rho s)} - (\Delta^H_1 + \Delta^H_2)\cdot 1)\ln|y|^2 \\ &+ (Q^H_{(\rho u)} - (\Delta^H_1 + \Delta^H_3)\cdot 1)\ln|1-y|^2]_N{}^M + \mathcal{O}(k^{-2})\end{aligned} \tag{A.13}$$

where

$$E(\rho,H)_M{}^N = \sum_m (\epsilon(\rho,H)_m{}^M)^* \epsilon(\rho,H)_m{}^N \quad . \tag{A.14}$$

This form of the correlator explicitly shows the trivial monodromies around $y = 0$, 1 and ∞.

Appendix B: Comparison with the blocks of Knizhnik and Zamolodchikov

In this appendix, we check our high-level affine-Sugawara blocks (3.12), (3.17) and (3.23) against the exact blocks obtained in Ref.[4] for the 3̂3̂3̂3 correlator on $SU(3)$.

To find the explicit form of our high-level blocks in this case, we need first the high-level form of the affine-Sugawara construction on $g = SU(3)$,

$$L^{ab}_{g,\infty} = \frac{1}{x\psi_g^2}\delta_{ab} \tag{B.1}$$

where ψ_g is the highest root of $SU(3)$, x is the invariant level of affine $SU(3)$ and we have used the Gell-Mann basis. The matrix irreps of the 3 and $\bar{3}$ are given in (6.6) and the corresponding high-level conformal weights are

$$\Delta^g(\mathcal{T}_{(3)}) = \Delta^g(\mathcal{T}_{(\bar{3})}) = \frac{4}{3x} + \mathcal{O}(x^{-2}) \quad . \tag{B.2}$$

Using the ρ-channel invariants in (6.15a-e) in the eigenvalue problems (3.2a) and (3.15a,b), we also obtain the high-level intermediate ρ-channel affine-primary conformal weights,

$$\Delta^g_{(\rho)}(m) = \begin{cases} \Delta^g(\mathcal{T}_{(1)}) = 0 + \mathcal{O}(x^{-2}) & , \quad m = V \\ \Delta^g(\mathcal{T}_{(8)}) = \frac{3}{x} + \mathcal{O}(x^{-2}) & , \quad m = A \end{cases} \quad , \quad \rho = s,u \tag{B.3a}$$

$$\Delta^g_{(t)}(m) = \begin{cases} \Delta^g(\mathcal{T}_{(6)}) = \frac{10}{3x} + \mathcal{O}(x^{-2}) & , \quad m = 6 \\ \Delta^g(\mathcal{T}_{(\bar{3})}) = \frac{4}{3x} + \mathcal{O}(x^{-2}) & , \quad m = \bar{3} \end{cases} \tag{B.3b}$$

where $m = (V,A)$ labels the vacuum and adjoint representations in the s and u-channels and $m = (6,\bar{3})$ labels the symmetric and antisymmetric representations in the t-channel.

Finally, we use the corresponding crossing matrices (6.15f,g) to compute the matrices $c(\rho,g)$ in (3.12d), (3.17d) and (3.23d),

$$c(s,g) = c(u,g) = X_g(su)[\Delta^g_{(u)} - \frac{8}{3x}\cdot 1]X_g(us) = \frac{1}{3x}\begin{pmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & -7 \end{pmatrix} \tag{B.4a}$$

$$c(t,g) = X_g(tu)[\Delta^g_{(u)} - \frac{8}{3x}\cdot 1]X_g(ut) = \frac{1}{3x}\begin{pmatrix} -5 & 3\sqrt{2} \\ 3\sqrt{2} & -2 \end{pmatrix} \tag{B.4b}$$

where 1 and $\Delta^g_{(u)}$ are defined in (A.4). With these data we obtain the explicit form of our high-level blocks for the 3̂3̂3̂3 correlator,

$$\mathcal{F}^{(s)}_g(y)_m{}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3x}\begin{pmatrix} -8 & 0 \\ 0 & 1 \end{pmatrix}\ln y + \frac{1}{3x}\begin{pmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & -7 \end{pmatrix}\ln(1-y) + \mathcal{O}(x^{-2}) \tag{B.5a}$$

$$\mathcal{F}^{(u)}_g(y)_m{}^n = \mathcal{F}^{(s)}_g(1-y)_m{}^n \tag{B.5b}$$

$$\hat{\mathcal{F}}_g^{(t)}(y)_m{}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{3x}\begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix}\ln(-y) + \frac{1}{3x}\begin{pmatrix} -5 & 3\sqrt{2} \\ 3\sqrt{2} & -2 \end{pmatrix}\ln\left(1-\frac{1}{y}\right) + \mathcal{O}(x^{-2}) \tag{B.5c}$$

where $m = (V, A)$ in (B.5a,b) and $m = (6, \bar{3})$ in (B.5c). The relation (B.5b) is in accord with (3.33c), since $T^2 \sim T^3$ in this case.

We wish to check our high-level blocks (B.5) against the exact results obtained by KZ, who, however, use a different basis for the $SU(3)$-invariant tensors,

$$v_1^{\mathrm{KZ}}(g) = \delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \quad , \quad v_2^{\mathrm{KZ}}(g) = \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} \quad . \tag{B.6}$$

Comparing these invariants with our Clebsch basis (6.15a-e) of $SU(3)$-invariants, we learn that the basis transformation of our ρ-channel blocks (B.5) to the KZ basis is,

$$d_m \mathcal{F}_g^{(\rho)}(y)_m{}^\mu = \mathcal{F}_g^{(\rho)}(y)_m{}^n L_n{}^\mu \quad , \quad L_n{}^\mu = \frac{1}{12}\begin{pmatrix} 4 & 0 \\ -\sqrt{2} & 3\sqrt{2} \end{pmatrix} \quad , \quad m = V, A \quad , \quad \rho = \mathrm{s}, \mathrm{u} \tag{B.7a}$$

$$d_m \hat{\mathcal{F}}_g^{(t)}(y)_m{}^\mu = \hat{\mathcal{F}}_g^{(t)}(y)_m{}^n M_n{}^\mu \quad , \quad M_n{}^\mu = \frac{1}{12}\begin{pmatrix} \sqrt{6} & \sqrt{6} \\ 2\sqrt{3} & -2\sqrt{3} \end{pmatrix} \quad , \quad m = 6, \bar{3} \tag{B.7b}$$

where $\mu = 1, 2$ labels the KZ invariants, the normalization constants d_m are arbitrary and the blocks $\mathcal{F}_g^{(s,u)}(y)_m{}^\mu$, $\hat{\mathcal{F}}_g^{(t)}(y)_m{}^\mu$ are the KZ blocks. More explicitly, our prediction for the high-level analytic KZ blocks is then,

$$\mathcal{F}_g^{(s)}(y)_m{}^\mu = \begin{pmatrix} 1 & 0 \\ 1 & -3 \end{pmatrix} + \frac{1}{3x}\begin{pmatrix} -8 & 0 \\ 1 & -3 \end{pmatrix}\ln y + \frac{1}{3x}\begin{pmatrix} 1 & -3 \\ 1 & 21 \end{pmatrix}\ln(1-y) + \mathcal{O}(x^{-2}) \tag{B.8a}$$

$$\mathcal{F}_g^{(u)}(y)_m{}^\mu = \mathcal{F}_g^{(s)}(1-y)_m{}^\mu \tag{B.8b}$$

$$\hat{\mathcal{F}}_g^{(t)}(y)_m{}^\mu = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \frac{1}{3x}\begin{pmatrix} 10 & 10 \\ 4 & -4 \end{pmatrix}\ln(-y) + \frac{1}{3x}\begin{pmatrix} 1 & -11 \\ 1 & 5 \end{pmatrix}\ln\left(1-\frac{1}{y}\right) + \mathcal{O}(x^{-2}) \tag{B.8c}$$

where we have chosen the particular values of the normalization constants

$$d_V = \frac{1}{3} \quad , \quad d_A = -\frac{1}{6\sqrt{2}} \quad , \quad d_6 = \frac{1}{2\sqrt{6}} \quad , \quad d_{\bar{3}} = \frac{1}{2\sqrt{3}} \tag{B.9}$$

with some pedagogical foresight.

To check that these blocks are precisely the high-level limit of the KZ blocks on $SU(3)$, we recall the exact form of the s-channel KZ blocks $\mathcal{F}_g^{(s)}(y)_m{}^\mu$ on $SU(3)$ [4],

$$\mathcal{F}_g^{(s)}(y)_V{}^1 = y^{-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(\lambda, -\lambda, 1-3\lambda; y) \tag{B.10a}$$

$$\mathcal{F}_g^{(s)}(y)_V{}^2 = \frac{1}{x}y^{1-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(1+\lambda, 1-\lambda, 2-3\lambda; y) \tag{B.10b}$$

$$\mathcal{F}_g^{(s)}(y)_A{}^1 = y^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(2\lambda, 4\lambda, 1+3\lambda; y) \tag{B.10c}$$

$$\mathcal{F}_g^{(s)}(y)_A{}^2 = -3y^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(2\lambda, 4\lambda, 3\lambda; y) \tag{B.10d}$$

$$\Delta^g(\mathcal{T}_{(3)}) = \frac{4}{3(x+3)} \quad , \quad \Delta^g(A) = \Delta^g(\mathcal{T}_{(8)}) = \frac{3}{x+3} \quad , \quad \lambda = \frac{1}{x+3} \tag{B.10e}$$

where $m = V, A$ label the vacuum and adjoint blocks, $\Delta^g(\mathcal{T}_{(3)})$ is the conformal weight of the 3, $\Delta^g(A)$ is the conformal weight of the adjoint representation and $F(a, b, c; y)$ is the hypergeometric function. Using the high-level expansions,

$$F\left(\frac{a}{x+d}, \frac{b}{x+e}, \frac{c}{x+f}; y\right) = 1 - \frac{ab}{cx}\ln(1-y) + \mathcal{O}(x^{-2}) \tag{B.11a}$$

$$F\left(\frac{a}{x+d}, \frac{b}{x+e}, 1+\frac{c}{x+f}; y\right) = 1 + \mathcal{O}(x^{-2}) \tag{B.11b}$$

$$F\left(\frac{a}{x+d}, 1+\frac{b}{x+e}, 1+\frac{c}{x+f}; y\right) = 1 - \frac{a}{x}\ln(1-y) + \mathcal{O}(x^{-2}) \tag{B.11c}$$

$$F\left(1+\frac{a}{x+d}, 1+\frac{b}{x+e}, 2+\frac{c}{x+f}; y\right) = -\frac{\ln(1-y)}{y} + \mathcal{O}(x^{-1}) \tag{B.11d}$$

one finds that the high-level limit of (B.10) agrees precisely with the predicted form in (B.8a). For the u-channel the check follows the same steps, with the replacement $y \to 1-y$ everywhere.

To continue the s-channel KZ blocks (B.10) to the t-channel, one uses the standard continuation formula [26]

$$\begin{aligned} F(a,b,c;y) = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-y)^{-a}F(a, 1-c+a, 1-b+a; \frac{1}{y}) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-y)^{-b}F(b, 1-c+b, 1-a+b; \frac{1}{y}) \quad , \quad |\arg(-y)| < \pi \quad . \end{aligned} \tag{B.12}$$

According to the expansions (B.11), the high-level limit of this formula in the cases of all four blocks in (B.10) is

$$\ln(1-y) = \ln(-y) + \ln\left(1-\frac{1}{y}\right) \quad , \quad |\arg(-y)| < \pi \tag{B.13}$$

and this limit is identical to the first continuation rule (3.20a) used in the text. To continue the factors in front of the hypergeometric functions, we use the relations

$$(1-y)^\alpha = (-y)^\alpha\left(1-\frac{1}{y}\right)^\alpha \tag{B.14a}$$

$$y^\beta = (-y)^\beta \exp[-i\pi\beta \mathrm{sign}(\arg(-y))] \tag{B.14b}$$

which are equivalent, finite forms of the continuation rules (3.20a,b).

Factoring out the non-analytic phases generated by (B.14b) (which then appear in the crossing matrices to the t-channel), we find the finite-level form of the analytic t-channel KZ blocks,

$$\hat{\mathcal{F}}_g^{(t)}(y)_6{}^1 = (-y)^{-\Delta^g(\mathcal{T}_{(6)})}\left(1-\frac{1}{y}\right)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})} F(4\lambda,\lambda,1+2\lambda;\frac{1}{y}) \tag{B.15a}$$

$$\hat{\mathcal{F}}_g^{(t)}(y)_6{}^2 = (-y)^{-\Delta^g(\mathcal{T}_{(6)})}\left(1-\frac{1}{y}\right)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})} F(4\lambda,1+\lambda,1+2\lambda;\frac{1}{y}) \tag{B.15b}$$

$$\hat{\mathcal{F}}_g^{(t)}(y)_3{}^1 = (-y)^{-\Delta^g(\mathcal{T}_{(3)})}\left(1-\frac{1}{y}\right)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})} F(2\lambda,-\lambda,1-2\lambda;\frac{1}{y}) \tag{B.15c}$$

$$\hat{\mathcal{F}}_g^{(t)}(y)_{\bar{3}}{}^2 = -(-y)^{-\Delta^g(\mathcal{T}_{(3)})}\left(1-\frac{1}{y}\right)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})} F(2\lambda,1-\lambda,1-2\lambda;\frac{1}{y}) \tag{B.15d}$$

$$\Delta^g(\mathcal{T}_{(6)}) = \frac{10}{3(x+3)} \quad , \quad \Delta^g(\mathcal{T}_{(3)}) = \Delta^g(\mathcal{T}_{(\bar{3})}) = \frac{4}{3(x+3)} \quad , \quad \lambda = \frac{1}{x+3} \quad . \tag{B.15e}$$

The high-level forms of these blocks agree precisely with our prediction (B.5c).

For completeness, we finally give the finite-level forms of the s-channel blocks $\mathcal{F}_g^{(s)}(y)_m{}^n$ in our Clebsch basis,

$$\mathcal{F}_g^{(s)}(y)_V{}^V = y^{-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(\lambda,-\lambda,-3\lambda;y) \tag{B.16a}$$

$$\mathcal{F}_g^{(s)}(y)_V{}^A = \frac{2\sqrt{2}\lambda}{3(1-3\lambda)}y^{1-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(1+\lambda,1-\lambda,2-3\lambda;y) \tag{B.16b}$$

$$\mathcal{F}_g^{(s)}(y)_A{}^A = y^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(2\lambda,4\lambda,3\lambda;y) \tag{B.16c}$$

$$\mathcal{F}_g^{(s)}(y)_A{}^V = \frac{2\sqrt{2}\lambda}{3(1+3\lambda)}y^{1+\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}(1-y)^{\Delta^g(A)-2\Delta^g(\mathcal{T}_{(3)})}F(1+2\lambda,1+4\lambda,2+3\lambda;y) \tag{B.16d}$$

$$\Delta^g(\mathcal{T}_{(3)}) = \frac{4}{3(x+3)} \quad , \quad \Delta^g(A) = \Delta^g(\mathcal{T}_{(8)}) = \frac{3}{x+3} \quad , \quad \lambda = \frac{1}{x+3} \tag{B.16e}$$

which are easily obtained from the s-channel KZ blocks (B.10) and the basis transformation (B.7a). The basis transformation gives two of these blocks as linear combinations of two hypergeometric functions, which we have then combined into a single hypergeometric function using Gauss' contiguous relations.

The exact blocks (B.16) also show quite explicitly the high-level pattern discussed in the text for the general affine-Sugawara blocks in our Clebsch basis: the diagonal blocks begin at order $\mathcal{O}(k^0)$ with leading singularities which are affine primary states, while the off-diagonal blocks begin at $\mathcal{O}(k^{-1})$ with leading singularities which are affine secondary states. Moreover, one sees that the conjectured result (3.14) for the exact conformal weights of the general blocks is indeed correct in this case.

Appendix C: The level-families $SU(3)^{\#}_M$ and $SU(3)/SU(2)_{\rm irr}$

In this appendix we review [24,17] various results for the unitary irrational level-family

$$(SU(3)_x)^{\#}_M \tag{C.1}$$

and the closely-related level-family of the coset construction

$$\frac{SU(3)}{SU(2)_{\rm irr}} = \frac{SU(3)_x}{SU(2)_{4x}} \tag{C.2}$$

both of which occur in the maximally-symmetric ansatz on $SU(3)$. $SU(2)_{\rm irr}$ denotes the irregularly embedded $SU(2)$ subgroup of $SU(3)$ generated by $J_{2,5,7}$. The results given here are used in Section 6 and Appendix D.

In the (Cartesian) Gell-Mann basis (6.6), the maximally-symmetric construction $(SU(3)_x)^{\#}_M$ has the form [24]

$$L^{ab} = \frac{1}{\psi_g^2}\ell_a\delta_{ab} \quad , \quad \ell_a = \begin{cases} \ell_c & a=3,8 \\ \ell_h & a=2,5,7 \\ \ell_r & a=1,4,6 \end{cases} \tag{C.3a}$$

$$T = \frac{1}{\psi_g^2}:[\ell_c(J_3^2+J_8^2)+\ell_h(J_2^2+J_5^2+J_7^2)+\ell_r(J_1^2+J_4^2+J_6^2)]: \tag{C.3b}$$

$$c = x(2\ell_c+3\ell_h+3\ell_r) \tag{C.3c}$$

where T is the stress tensor, ψ_g is the highest root of $SU(3)$, x is the affine level and c is the central charge. The exact form of c is given in (6.2a), but we refer to [24] for the exact[d] forms of the coefficients $\ell_{c,h,r}$. The construction above includes the coset construction $SU(3)/SU(2)_{\rm irr}$ as a special case when the further symmetry relation

$$\ell_c = \ell_r \equiv \ell_{g/h} \tag{C.4}$$

is obeyed.

$SU(3)^{\#}_M$ is an H-invariant CFT with symmetry group [17]

$$H(SU(3)^{\#}_M) = O = \text{ octahedral group} \subset SU(2)_{\rm irr} \tag{C.5}$$

where O is the octahedral group (rotational symmetry group of the cube, with order 24) and $SU(2)_{\rm irr}$ is the irregular embedding of $SU(2)$ in $SU(3)$. The octahedral group includes the elements

$$\Omega_{(2)} = \exp(i\frac{\pi}{\sqrt{\psi_g^2}}J_2(0)) \quad . \quad \Omega_{(5)} = \exp(i\frac{\pi}{\sqrt{\psi_g^2}}J_5(0)) \quad , \quad \Omega_{(7)} = \exp(i\frac{\pi}{\sqrt{\psi_g^2}}J_7(0)) \tag{C.6}$$

where $J_a(0)$ are the zero modes of the currents J_a, and in particular we may take the two elements ω_1 and ω_2.

$$\omega_1 = \Omega_{(2)} \quad . \quad \omega_2 = \Omega_{(5)}\Omega_{(7)} \tag{C.7}$$

[d] The relation to the notation of Ref.[24] is $\ell_c = 3\lambda$, $\ell_h = (L_- - L_+)/2$ and $\ell_r = (L_- + L_+)/2$.

which satisfy

$$\omega_1^4 = 1 \quad . \quad \omega_2^3 = 1 \tag{C.8a}$$

$$\omega_1\omega_2^2\omega_1 = \omega_2 \quad . \quad \omega_1\omega_2\omega_1 = \omega_2\omega_1^2\omega_2 \tag{C.8b}$$

as the generators of the octahedral group.

The coset construction $SU(3)/SU(2)_{\rm irr}$ has the larger Lie group symmetry

$$H(SU(3)/SU(2)_{\rm irr}) = SU(2)_{\rm irr} \tag{C.9}$$

because of the symmetry relation (C.4).

The 3 and $\bar{3}$ are $L(g;H)$-degenerate irreps of $SU(3)^{\#}_M$ and $SU(3)/SU(2)_{\rm irr}$ with completely degenerate conformal weights,

$$\Delta(\mathcal{T}_{(3)}) = \Delta(\mathcal{T}_{(\bar{3})}) = \frac{c}{6x} \qquad (3) \tag{C.10}$$

where the number in parentheses denotes the degeneracy.

For $(SU(3)_x)^{\#}_M$ one also finds the L^{ab}-broken conformal weights of the 8 (adjoint) and 6 (symmetric),

$$\Delta(\mathcal{T}_{(8)}) = \begin{cases} \ell_c + \frac{1}{2}(3\ell_h + \ell_r) & (3) \\ \frac{3}{2}(\ell_h + \ell_r) & (2) \\ \ell_c + \frac{1}{2}(\ell_h + 3\ell_r) & (3) \end{cases} \tag{C.11a}$$

$$\Delta(\mathcal{T}_{(6)}) = \Delta(\mathcal{T}_{(\bar{6})}) = \begin{cases} \frac{2}{3}(2\ell_c + 3\ell_r) & (1) \\ \frac{1}{3}\ell_c + \frac{3}{2}(\ell_h + \ell_r) & (3) \\ \frac{4}{3}\ell_c + \frac{1}{2}(3\ell_h + \ell_r) & (2) \end{cases} \quad . \tag{C.11b}$$

These forms show that the 8 and the 6 each split into three subsets of degenerate weights, in agreement with the block analysis of Section 6.

For $SU(3)_x/SU(2)_{4x}$ the splitting is reduced to two subsets,

$$\Delta_{g/h}(\mathcal{T}_{(8)}) = \begin{cases} \frac{3}{2}(\ell_h + \ell_{g/h}) & (5) \\ \frac{1}{2}(\ell_h + 5\ell_{g/h}) & (3) \end{cases} \tag{C.12a}$$

$$\Delta_{g/h}(\mathcal{T}_{(6)}) = \Delta_{g/h}(\mathcal{T}_{(\bar{6})}) = \begin{cases} \frac{10}{3}\ell_{g/h} & (1) \\ \frac{3}{2}\ell_h + \frac{11}{6}\ell_{g/h} & (5) \end{cases} \tag{C.12b}$$

according to (C.4) and (C.11). These forms are in agreement with the coset block analysis of Appendix D.

For the computations of Section 6 and Appendix D, we need the high-level forms of the two constructions.

$$(SU(3)_x)^{\#}_M \quad : \ell_r = \frac{1}{x} + \mathcal{O}(x^{-2}) \quad . \quad \ell_c = \ell_h = \mathcal{O}(x^{-2}) \quad . \quad c = 3 + \mathcal{O}(x^{-1}) \tag{C.13a}$$

$$\frac{SU(3)_x}{SU(2)_{4x}} \quad : \ell_{g/h} = \frac{1}{x} + \mathcal{O}(x^{-2}) \quad . \quad \ell_h = \mathcal{O}(x^{-2}) \quad . \quad c = 5 + \mathcal{O}(x^{-1}) \tag{C.13b}$$

which can be used with (C.10), (C.11) and (C.12) to obtain the high-level forms of all the quantities discussed in this appendix.

Appendix D: Blocks and correlators in SU(3)/SU(2)$_{\rm irr}$

As an explicit example in rational conformal field theory, we work out in this appendix the high-level conformal blocks and correlators of a particular $L(g;h)$-degenerate process in the level-family of the coset construction

$$\frac{g}{h} = \frac{SU(3)_x}{SU(2)_{4x}} = \frac{SU(3)}{SU(2)_{\rm irr}} \tag{D.1}$$

which is included in the family of coset examples (4.38a).

This level-family has the Lie symmetry $SU(2)_{\rm irr}$, and the 3 and $\bar{3}$ representations are $L(SU(3);SU(2)_{\rm irr})$-degenerate.

For the high-level computations in $SU(3)/SU(2)_{\rm irr}$ below, we need the high-level form of the inverse inertia tensor (in the Gell-Mann basis) and the degenerate conformal weights,

$$L^{ab}_{g/h,\infty} = \frac{1}{x\psi_g^2}\theta_a\delta_{ab} \quad , \quad \theta_a = \begin{cases} 1 & a = 3,8,1,4,6 \\ 0 & a = 2,5,7 \end{cases} \tag{D.2a}$$

$$\Delta^{g/h}(\mathcal{T}_{(3)}) = \Delta^{g/h}(\mathcal{T}_{(\bar{3})}) = \frac{5}{6x} + \mathcal{O}(x^{-2}) \tag{D.2b}$$

and we will consider here the same process, that is $3\bar{3}\bar{3}3$, which we studied for the affine-Sugawara construction and the irrational construction $SU(3)^{\#}_M$ in Appendix B and Section 6 respectively.

To compute the high-level blocks in the s-channel, we need to determine the s-channel eigenvectors $\psi(s,SU(2))$ from the eigenvalue problem (5.9), which reads in this case,

$$[-\frac{1}{2x}\sum_{a=\substack{3,8\\1,4,6}}\lambda^1_a\lambda^2_a + \frac{5}{3x}\mathbb{1}]_\alpha{}^\beta\,\psi(s,SU(2))^M_\beta = \Delta^{g/h}_{(s)}(M)\psi(s,SU(2))^M_\alpha \tag{D.3a}$$

$$\sum_{i=1}^{4}(\lambda^i_a)_\alpha{}^\beta\psi(s,SU(2))^M_\beta = 0 \quad , \quad a = 2,5,7 \quad . \tag{D.3b}$$

Here we have used the properties (6.6b) of the Gell-Mann matrices, and the global condition (D.3b) enforces the $SU(2)_{\rm irr}$-invariance of the coset construction.

After some algebra, the following orthonormal set of s-channel eigenvectors is found

$$\psi(s,SU(2))^1_\alpha = \frac{1}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \quad . \quad \Delta^{g/h}_{(s)}(1) = 0 \tag{D.4a}$$

$$\psi(s,SU(2))^2_\alpha = \frac{1}{2\sqrt{5}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} + \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3} - \frac{2}{3}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}] \quad . \quad \Delta^{g/h}_{(s)}(2) = \frac{3}{2x} \tag{D.4b}$$

$$\psi(\mathrm{s}, SU(2))^3_\alpha = \frac{1}{2\sqrt{3}}[\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} - \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}] \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(3) = \frac{5}{2x} \tag{D.4c}$$

$$\psi(\mathrm{s}, SU(2))^\alpha_M = (\psi(\mathrm{s}, SU(2))^M_\beta)^* \eta^{\beta\alpha} = \psi(\mathrm{s}, SU(2))^M_\alpha \tag{D.4d}$$

where the last relation says that the left and right eigenvectors coincide in this case.

The results (D.4) are in agreement with the high-level fusion rule (6.9); in particular, the conformal weight in (D.4a) corresponds to the vacuum, while the remaining two weights in (D.4b,c) are the high-level form of the two degenerate subsets of the coset-broken conformal weights of the adjoint (see eqs.(C.12a) and (C.13b)).

Similarly, we can solve for the u and t-channel eigenvectors, which are given by

$$\psi(\mathrm{u}, SU(2))^M = \psi(\mathrm{s}, SU(2))^M|_{2\to 3} \quad , \quad \Delta^{g/h}_{(\mathrm{u})}(M) = \Delta^{g/h}_{(\mathrm{s})}(M) \tag{D.5a}$$

$$\psi(\mathrm{t}, SU(2))^M = \psi(\mathrm{s}, SU(2))^M|_{2\leftrightarrow 4} \quad , \quad \Delta^{g/h}_{(\mathrm{t})}(M) = \frac{10}{3x} - \Delta^{g/h}_{(\mathrm{s})}(M) \tag{D.5b}$$

where $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$ mean $\alpha_2 \leftrightarrow \alpha_3$ and $\alpha_2 \leftrightarrow \alpha_4$ in the explicit expressions (D.4) for the s-channel eigenvectors. Since $\mathcal{T}^2 \sim \mathcal{T}^3$, the result in (D.5a) is a special case of (5.21) and the u-channel conformal weights are identical to those in the s-channel. The conformal weights of the t-channel,

$$\Delta^{g/h}_{(\mathrm{t})}(M) = (\frac{10}{3x}, \frac{11}{6x}, \frac{5}{6x}) \tag{D.6}$$

are also in agreement with the coset-broken conformal weights of the high-level fusion rule (6.12). In particular, the last value is the completely degenerate conformal weight of the $\bar{3}$ and the first two coincide with the two degenerate subsets (C.12b) of the 6, according to the high-level form (C.13b).

Using (5.19b), the $SU(2)$-crossing matrices are computed from the eigenvectors as

$$X_{SU(2)}(\mathrm{us})_M{}^N = \psi(\mathrm{u}, SU(2))^M \psi(\mathrm{s}, SU(2))^N = \frac{1}{6}\begin{pmatrix} 2 & 2\sqrt{5} & 2\sqrt{3} \\ 2\sqrt{5} & 1 & -\sqrt{15} \\ 2\sqrt{3} & -\sqrt{15} & 3 \end{pmatrix} \tag{D.7a}$$

$$X_{SU(2)}(\mathrm{ts})_M{}^N = \psi(\mathrm{t}, SU(2))^M \psi(\mathrm{s}, SU(2))^N = \frac{1}{6}\begin{pmatrix} 2 & 2\sqrt{5} & -2\sqrt{3} \\ 2\sqrt{5} & 1 & \sqrt{15} \\ -2\sqrt{3} & \sqrt{15} & 3 \end{pmatrix} \tag{D.7b}$$

which are orthogonal and idempotent matrices in this case. The third crossing matrix $X_{SU(2)}(\mathrm{ut}) = X_{SU(2)}(\mathrm{us})X_{SU(2)}(\mathrm{ts})$ follows from the consistency relations (5.20).

Finally, we use the $SU(3)$ eigenvectors (6.15) and the alternate expression (A.9) for the general $L(q;H)$-degenerate blocks to write down the 6 s-channel coset blocks of the 3333 correlator in $SU(3)/SU(2)_{\mathrm{irr}}$.

$$C^{(\mathrm{s})}_{g/h}(y)_m{}^M = c(\mathrm{s}, g/h)_m{}^N[1 + (\Delta^{g/h}_{(\mathrm{s})} - \frac{5}{3x}\cdot 1)\ln y + (Q^{g/h}_{(\mathrm{su})} - \frac{5}{3x}\cdot 1)\ln(1-y)]_N{}^M + \mathcal{O}(x^{-2}) \tag{D.8}$$

where $(1)_N{}^M = \delta^M_N$ and

$$c(\mathrm{s}, g/h)_m{}^M = v(\mathrm{s}, SU(3))^m \psi(\mathrm{s}, SU(2))^M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4}\sqrt{10} & \frac{1}{4}\sqrt{6} \end{pmatrix} \tag{D.9a}$$

$$(\Delta^{g/h}_{(\mathrm{s})})_N{}^M = \Delta^{g/h}_{(\mathrm{s})}(M)\delta^M_N \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(M) = \Delta^{g/h}_{(\mathrm{u})}(M) = (0, \frac{3}{2x}, \frac{5}{2x}) \tag{D.9b}$$

$$(Q^{g/h}_{(\mathrm{su})})_N{}^M = \sum_L X_{SU(2)}(\mathrm{us})_N{}^L \Delta^{g/h}_{(\mathrm{u})}(L) X_{SU(2)}(\mathrm{us})_L{}^M = \frac{1}{12x}\begin{pmatrix} 20 & -4\sqrt{5} & 0 \\ -4\sqrt{5} & 13 & -3\sqrt{15} \\ 0 & -3\sqrt{15} & 15 \end{pmatrix} . \tag{D.9c}$$

The u and t-channel blocks can be computed from the s-channel blocks above using the crossing relation (5.19a) and the explicit forms of the crossing matrices $X_{SU(3)}(\mathrm{us})$, $X_{SU(3)}(\mathrm{ts})$ in (6.15) and $X_{SU(2)}(\mathrm{us})$, $X_{SU(2)}(\mathrm{ts})$ in (D.7). Moreover, using the explicit form of the non-analytic phase matrix (5.13e) for this process,

$$U_{g/h}(y)_M{}^N = \sum_L X_{SU(2)}(\mathrm{ts})_M{}^L \exp\left(-\pi i[\Delta^{g/h}_{(\mathrm{s})}(L) - \frac{5}{3x}]\mathrm{sign}(\arg(-y))\right) X_{SU(2)}(\mathrm{ts})_L{}^N + \mathcal{O}(x^{-2}) \tag{D.10}$$

the analytic t-channel blocks follow from the crossing relation (5.24f)

Using (5.16a,d) we obtain the following limiting behavior as $y \to 0$ for the 6 s-channel blocks (D.8) of this correlator,

$$C^{(\mathrm{s})}_{g/h}(y)_m{}^M \underset{y\to 0}{\sim} \Gamma^{(\mathrm{s})}_{g/h}(m, M) y^{\Delta^{g/h}_{(\mathrm{s})}(m,M) - 5/3x} + \mathcal{O}(x^{-2}) \quad , \quad m = V, A \quad , \quad M = 1,2,3 \tag{D.11a}$$

$$\Delta^{g/h}_{(\mathrm{s})}(V,1) = 0 + \mathcal{O}(x^{-2}) \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(A,2) = \frac{3}{2x} + \mathcal{O}(x^{-2}) \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(A,3) = \frac{5}{2x} + \mathcal{O}(x^{-2}) \tag{D.11b}$$

$$\Delta^{g/h}_{(\mathrm{s})}(V,2) = 1 + \mathcal{O}(x^{-1}) \quad , \quad \Delta^{g/h}_{(\mathrm{s})}(A,1) = 1 + \mathcal{O}(x^{-1}) \tag{D.11c}$$

$$\Delta^{g/h}_{(\mathrm{s})}(V,3) = \mathcal{O}(x^0) \tag{D.11d}$$

$$\Gamma^{(\mathrm{s})}_{g/h}(m, M) = \begin{pmatrix} 1 & \frac{1}{3x}\sqrt{5} & 0 \\ \frac{5}{12x}\sqrt{2} & \frac{1}{4}\sqrt{10} & \frac{1}{4}\sqrt{6} \end{pmatrix} + \mathcal{O}(x^{-2}) \quad . \tag{D.11e}$$

The explicit form of the residues was obtained using (5.16d), the embedding matrix (D.9a) and the relation $c(\mathrm{s}, g/h) = Q^{g/h}_{(\mathrm{su})} - \frac{5}{3x}\cdot 1$.

The three conformal weights in (D.11b) are broken affine primary states and the two in (D.11c) are broken affine secondary states which are not necessarily integer descendants of broken affine primary states. The conformal weight in (D.11d) cannot be determined through this order because the residue of the corresponding block $C_{g/h}(y)_V{}^3$, and the block itself, is zero through order $\mathcal{O}(x^{-1})$, so this block begins at $\mathcal{O}(x^{-2})$. To see this directly from (D.8) note that for this block $c(\mathrm{s}, g/h)_m{}^M = 0$ and $c(\mathrm{s}, g/h)_m{}^N(Q^{g/h}_{(\mathrm{su})})_N{}^M = 0$. We also find one block which begins at $\mathcal{O}(x^{-2})$ in the u-channel and in the t-channel.

In accord with (4.19), the number of blocks in this process is

$$B_{SU(3)/SU(2)} = 2 \cdot 3 = 6 \tag{D.12}$$

while the same process under the affine-Sugawara construction on $SU(3)$ and the irrational construction $SU(3)^{\#}_{M}$ showed 4 and 8 blocks respectively. This is in accord with the double inequality (5.18) and the increasing symmetry breakdown $O \subset SU(2)_{\rm irr} \subset SU(3)$ of the three constructions.

Using eqs.(A.13), (A.14), we also find the following expression for the high-level non-chiral correlators of $SU(3)/SU(2)_{\rm irr}$,

$$Y_{g/h}(y^*, y) = \sum_M E(\mathrm{s}, g/h)_M{}^M [1 + (\Delta^{g/h}_{(\mathrm{s})} - \frac{5}{3x} \cdot 1) \ln |y|^2 + (\Delta^{g/h}_{(\mathrm{s})} - \frac{5}{3x} \cdot 1) \ln |1-y|^2]_M{}^M + \mathcal{O}(x^{-2}) \tag{D.13a}$$

$$E(\mathrm{s}, g/h)_M{}^N = \sum_m (\epsilon(\mathrm{s}, g/h)_m{}^M)^* \epsilon(\mathrm{s}, g/h)_m{}^N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{8} & \frac{1}{8}\sqrt{15} \\ 0 & \frac{1}{8}\sqrt{15} & \frac{3}{8} \end{pmatrix} \tag{D.13b}$$

where we have used $X_{SU(2)}(\mathrm{us}) E(\mathrm{s}, g/h) X_{SU(2)}(\mathrm{us}) = E(\mathrm{s}, g/h)$ and where the diagonal s-channel conformal weight matrix $\Delta^{g/h}_{(\mathrm{s})}$ is given in (D.9b). This result explicitly shows the crossing symmetry (5.29), as it should since $T^2 \sim T^3$ in this case.

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