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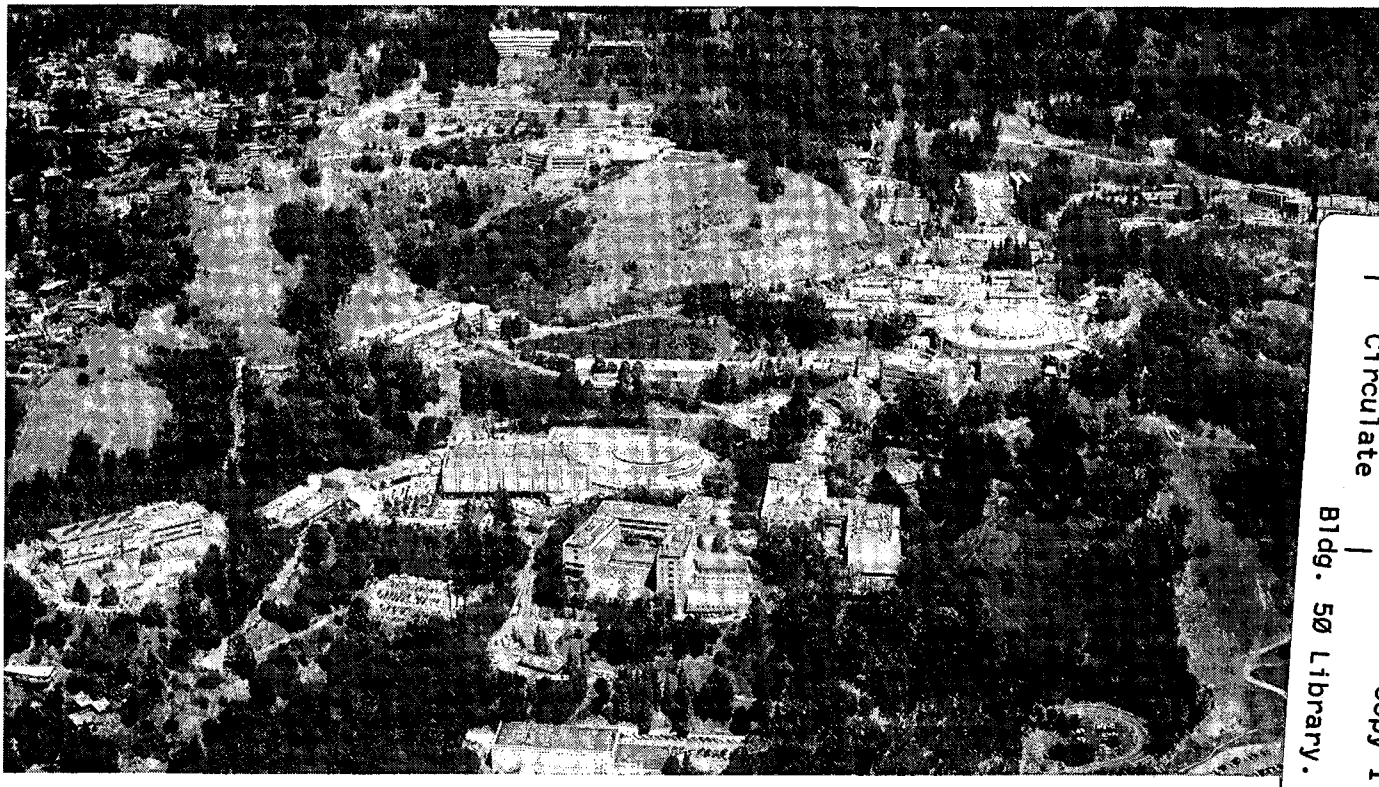

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A Grand Unified Supersymmetric Theory of Flavor *

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Abstract

A grand unified $SU(5)$ theory is constructed with a hierarchical breaking of a $U(2)$ flavor symmetry. The small parameters of the squark and slepton mass matrices, necessary to solve the supersymmetric flavor-changing problem, and the inter-generational quark and lepton mass hierarchies are both generated from the $U(2)$ symmetry breaking parameters. The flavor interactions of the theory are tightly constrained, with just 10 free real parameters for both the fermion and scalar sectors. All but one of the 8 small fermion mass ratios, and all of the 3 small Cabibbo-Kobayashi-Maskawa mixing angles, can be understood without introducing small dimensionless Yukawa parameters. Predictions are made for 2 of the Cabibbo-Kobayashi-Maskawa

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mixing angles and for 2 of the fermion masses. The six flavor mixing matrices which appear at the neutralino vertices, and which in general are arbitrary unitary matrices, are determined in terms of just a single free parameter.

1. The flavor group U(2)

The fermion mass puzzle arose with the discovery of the muon, and has become more pressing with the discovery of each new quark and lepton. In terms of the standard model, the question is: what is the origin of the small dimensionless parameters in the Yukawa coupling matrices? In supersymmetric extensions of the standard model, the spectrum of squarks and sleptons possess a second puzzle. Although none of these particles have masses much less than the weak scale, the scalar mass matrices are highly constrained by flavor-changing processes [1], and must involve a second set of small dimensionless parameters.

The fermion and scalar mass matrices are different aspects of the supersymmetric flavor problem, so that it is attractive to consider these two sets of small parameters to be related. The key to such a relationship is provided by flavor symmetries.

A flavor group G_f, which commutes with supersymmetry, treats quarks and squarks identically. In the G_f symmetric limit the squarks acquire masses, but have mass matrices with a high degree of flavor conservation, while the quarks are massless, except possibly the heaviest ones. The lighter quark masses are generated when G_f is broken hierarchically by a set of vevs, v_i, so that the small parameters of the Yukawa matrices involve $v_i/M_f \equiv \epsilon_i$, where M_f is a flavor mass scale. Such breakings also introduce corrections to the squark mass matrices, some of which violate flavor. However, these flavor-changing effects are proportional to ϵ_i, and are suppressed for the same reason that some quarks are light. Such a mechanism deserves the title "super-GIM" [2].

The power and simplicity of this use of approximate flavor symmetries was first illustrated using $G_f = U(3)^5$, the maximal flavor group of the standard model, with the ϵ_i taken to be the three Yukawa matrices [3]. Such a scheme, called effective weak scale supersymmetry, provides a framework for the soft operators which is greatly preferable to the universality assumption. However, this scheme treated the Yukawa matrices as phenomenological

symmetry breaking parameters, and did not provide a theory for their origin. Several such models have been constructed over the last three years [4-15], based on flavor groups which are Abelian or non-Abelian, continuous or discrete, and gauged or global.

We consider this development – the ability to construct supersymmetric theories of flavor – to be of great importance. For quark and lepton masses it provides a symmetry basis for textures, which need no longer be postulated purely on grounds of phenomenological simplicity. Not only can these theories solve the flavor-changing problem, but the coupling to the fermion mass problem produces a very constrained framework. In the present paper, we continue our attempt to develop a theory with a simple believable symmetry structure, which solves the flavor-changing problem, provides an economical description of the quark and lepton spectrum, and is able to make experimentally testable predictions, both in the fermion and scalar sectors.

Three requirements provide a guide in choosing the flavor group, G_f.

1. *G_f must solve the flavor-changing problem.*

   The minimal, most straightforward and compelling flavor symmetry solution to the flavor-changing problem is for G_f to be non-Abelian, with the lightest two generations in doublets

$$q_a = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad u_a = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad d_a = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad \ell_a = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \quad e_a = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \quad (1)$$

   If this symmetry is sufficiently weakly broken, the resulting near degeneracy of the scalars solves the flavor-changing puzzle. † We find it surprising that this elegant idea was not studied prior to 1993, when $G_f = SU(2)$ was considered [4].

†Flavor changing amplitudes are also induced by a non-degeneracy between the scalars of the third generation and those of the lighter two generations. These effects, although close to the limits of what experiments allow, are not problematic if the relevant mixing angles are similar to the corresponding CKM mixings and/or the amount of fractional mass splitting is somewhat less than maximal.

2. *G_f must be compatible with gauge unification.*

   There are many groups which could have the representation structure of (1). The choice can be greatly reduced by requiring that the group acts identically on $(q_a, u_a, d_a, \ell_a, e_a) \equiv \psi_a$, as results from a theory in which the components of a generation are unified.

3. *In the symmetric limit, fermions of the first two generations must be massless.*

   The flavor group $G_f = SU(2)$ allows the interaction $q_a \epsilon^{ab} d_b h$, giving unacceptable, large, degenerate masses to d and s quarks. We are therefore led to consider $G_f = U(2)$, which can be written as $SU(2) \times U(1)$ with ψ_a transforming as (2,1). The tensor ϵ^{ab} is a non-trivial singlet of $U(2)$ carrying charge -2, so that $U(2)$ invariance allows Yukawa couplings only for the third generation, which is taken to transform as a trivial $U(2)$ singlet.

A discrete subgroup of $U(2)$ might provide an acceptable alternative choice for G_f. We prefer the continuous groups, however, because $U(2)$ contains a $U(1)$ subgroup with a color anomaly. The Peccei-Quinn solution to the strong CP problem [16] arises as an automatic consequence of the above three requirements, which led us to choose $G_f = U(2)$. The strong CP problem involves the phase of the determinant of the quark mass matrix, and hence is clearly an aspect of the flavor problem. The Peccei-Quinn symmetry naturally finds a home as a subgroup of a more comprehensive flavor group. This solution of the strong CP problem would be lost if $U(2)$ were gauged. Gauging a continuous flavor group is problematic, however, as the D^2 contribution to the scalar masses reintroduces the flavor-changing problem [17]. We are therefore led to a non-Abelian, continuous, global flavor group: $G_f = U(2)$.

While we believe the choice of $G_f = U(2)$ is very well motivated, it is obviously not unique. For example, $U(2)$ could be extended to $U(3)$, with

the three generations forming a triple (ψ_a, ψ_3). We view $U(2)$ as a stage of partial flavor unification. We prefer to study $U(2)$ first: the top quark mass strongly breaks $U(3)$ to $U(2)$, and hence it is the weakly broken $U(2)$ which must solve the flavor-changing and fermion mass hierarchy problems. It is important to establish whether $U(2)$ theories can solve these problems. While the representation structure (1) appears promising, a general low energy effective $U(2)$ theory does not solve the flavor changing problem [10].

A complete $U(3)$ flavor-unified theory would not only be elegant, but it also offers the prospect of a flavor symmetry origin for R parity, which $U(2)$ alone is unable to provide, since matter parity is a parity of $U(3)$ triality [14].

Although Abelian symmetries can constrain the mass matrices to solve the flavor-changing problem [5], we find the necessary group structure to be less compelling than that of $U(2)$ or $U(3)$, due to a large freedom in the choice of charge quantum numbers. For example, the rank 2 case of $G_f = U(1)^2$ contains two symmetry breaking parameters, ϵ_1 and ϵ_2, which can appear in a mass matrix element as $\epsilon_1^n \epsilon_2^m$, where n and m are positive integers which can be freely chosen by suitable charge assignments. Compare this to the rank 2, non-Abelian care of $G_f = U(2)$, which also has 2 symmetry breaking parameters, ϵ and ϵ', which we find appear only linearly in the Yukawa matrices. Indeed, while the small parameters ϵ and ϵ' solve the flavor-changing problem and account for the two intergenerational fermion mass hierarchies, they are unable to describe all the features of the quark and lepton mass matrices. Nevertheless, we find that the highly constrained group theory, and the resulting testable predictions, are an important virtue of the $U(2)$ theory. In this paper we seek to understand several other features of the quark and lepton mass matrices from the SU(5) unified gauge symmetry.

2. The Structure of $U(2)$ Theories.

In the next sections we discuss in detail the simplest $U(2)$ models and their predictions. In this section we discuss general aspects of the construction of models with $G_f = U(2)$.

The generations are assigned to $\psi_a(2)+\psi_3(1)$, where ψ represents q, u, d, ℓ, or e, and does not imply any particular choice of gauge group. We choose the two light Higgs doublets, h, to be G_f singlets, both for simplicity and because, in the $U(3)$ extension of the flavor group, this allows for a flavor symmetry origin of matter parity. The renormalizable superpotential contains Yukawa couplings only for the third generation, $\psi_3\psi_3 h$, and the first question is therefore how $U(2)$ breaking can lead to a 23 entry for the Yukawa matrices.

The only known way of generating small dimensionless parameters is from perturbative loop factors or from ratios of mass scales. A radiative origin for m_e/m_μ in a theory with $G_f = U(2)$ has been discussed elsewhere [18], in this paper we consider the fermion hierarchies to arise from a set of flavon vevs which break the flavor group at scales beneath some flavor scale M_f. From the viewpoint of an effective theory beneath M_f, it is clear that the 23 entry of the Yukawa matrices must come from an interaction of the form

$$\frac{1}{M_f}\,[\psi_3\phi^a\psi_a h]_F \tag{2}$$

where ϕ^a is a doublet flavon, with opposite $U(1)$ charge to ψ_a, taking a vev $\langle\phi^a\rangle = (O, V)$. The most general effective theory would also contain interactions quadratic in ϕ^a:

$$\frac{1}{M_f^2}\,[\psi_a\phi^a\phi^b\psi_b\ h]_F \tag{3}$$

and

$$\frac{1}{M_f^2}\,[\psi^{\dagger a}\phi_a^\dagger\phi^b\psi_b\ z^\dagger z]_D, \tag{4}$$

where z is a supersymmetry breaking spurion, taken dimensionless, $z = m\theta^2$. Operators (2) and (3) lead to masses for second generation fermions at order ϵ^2, where $\epsilon = V/M$, while (4) leads to a non-degeneracy between the scalars of the first two generations which is also of order ϵ^2. Hence in the lepton sector

$$\frac{m_{\tilde{e}}^2 - m_{\tilde{\mu}}^2}{m_{\tilde{e}}^2 + m_{\tilde{\mu}}^2} \approx O\left(\tfrac{m_\mu}{m_\tau}\right) \tag{5}$$

and in the down quark sector

$$\frac{m_{\tilde{d}}^2 - m_{\tilde{s}}^2}{m_{\tilde{d}}^2 + m_{\tilde{s}}^2} \approx O\left(\tfrac{m_s}{m_b}\right). \tag{6}$$

When combined with rotations in the 1/2 space to diagonalize the fermion mass matrix, these non-degeneracies are extremely problematic for $\mu \to e\gamma$ and ϵ_K [10].

The general effective field theory based on $G_f = U(2)$ leads to difficulties. However, an important point with regard to constructing supersymmetric theories of flavor is that *specific models, especially if they are simple, typically do not lead to the most general set of G_f invariant operators of the low energy effective theory.* This result has been crucial in several models which have been constructed [8, 19, 14, 15]. For supersymmetric theories of flavor, low energy effective theories are useful only if they can be used to demonstrate that certain symmetry schemes are safe from flavor-changing problems. If a general effective theory has problematic flavor-changing properties, it simply tells us which operators should be avoided in constructing explicit models.

In this paper we generate small Yukawa couplings, from $\langle\phi\rangle / M_f$, by rotating from flavor to mass eigenstates [20]. Let ψ represent the light matter of q, u, d, ℓ, e, where for now we omit flavor indices. Suppose that it has a Yukawa coupling $\chi h \psi$ to a Higgs doublet h and some heavy matter $\chi = Q, U, D, L, E$. The heavy generations are vector-like, with mass terms $M\bar{\chi}\chi$. Finally, mass mixing between light and heavy matter is induced by $\langle\phi\rangle = \epsilon M_f$ via the interaction $\bar{\chi}\phi\psi$ (as always in this paper, we assume that the flavons, ϕ, are gauge singlets). The theory is described by the superpotential

$$W = \bar{\chi}(M_f \chi + \phi\psi) + \chi h \psi \tag{7}$$

where coupling constants of order unity are understood.[‡] The vev $\langle\phi\rangle$ implies

[‡]Since ϕ is non-trivial under G_f, χ and ψ are typically distinguished by G_f. In the cases where they have the same G_f transformation, χ is defined as the linear combination which has a bare mass coupling to $\bar{\chi}$.

that the heavy state is $\chi' = \chi + \epsilon\psi$ while the light matter is $\psi' = \psi - \epsilon\chi$ rather than ψ, so that when the heavy mass eigenstate is decoupled the interaction $\chi h \psi$ contains a small Yukawa coupling for $\psi' : \epsilon\psi' h \psi'$. The small parameter ϵ arises because G_f is broken at a scale less than M_f.

This mass mixing of states introduces a similar non-trivial effect in the soft supersymmetry breaking interactions. If ψ and χ have different G_f transformation properties the soft $\mathbf{m}^2$ matrix is diagonal, with entries m_ψ^2, m_χ^2. Rotating from the flavor basis (ψ, χ) to the mass basis (ψ', χ'), one finds:

$$\begin{pmatrix} m_\psi^2 & O \\ O & m_\chi^2 \end{pmatrix} \rightarrow \begin{pmatrix} m_\psi^2 + \epsilon^2 m_\chi^2 & \epsilon(m_\chi^2 - m_\psi^2) \\ \epsilon(m_\chi^2 - m_\psi^2) & m_\chi^2 + \epsilon^2 m_\psi^2 \end{pmatrix}. \tag{8}$$

On decoupling the heavy eigenstate χ', only the $m_\psi^2 + \epsilon^2 m_\chi^2$ entry of this matrix is of interest. When flavor indices are reintroduced, this entry is a 3×3 matrix, and the $\epsilon^2 m_\chi^2$ terms can lead to non-degeneracies and flavor-changing entries at order ϵ^2 [21]. If χ and ψ have the same G_f transformation, there are additional order ϵ contributions to the $\psi'^\dagger \psi'$ mass matrix, which arise from an initial $\chi^\dagger \psi$ operator.

The generation of interactions involving light eigenstates, suppressed by powers of ϵ, from interactions that involved the initial χ flavor eigenstate, can be summarized by

$$[\chi\psi h]_F \rightarrow \epsilon[\psi\psi h]_F \tag{9a}$$

$$[\chi\psi h\ z]_F \rightarrow \epsilon[\psi\psi h\ z]_F \tag{9b}$$

$$[\chi^\dagger\chi\ z^\dagger z]_D \rightarrow \epsilon^2[\psi^\dagger\psi\ z^\dagger z]_D \tag{9c}$$

where (9b) yields soft trilinear scalar interactions. An immediate consequence of this picture is that there are no scalar mass terms linear in ϵ. For example, the operator $[\psi_3^\dagger \phi^a \psi_a]_D$ can never be generated by this mechanism.

It is frequently useful to use an approximate diagrammatic technique to perform the generation of the operators 9a, 9b, 9c from diagonalization of heavy mass matrices. This is especially true for models more complicated

than the simplest example discussed here. The three diagrams for 9a, 9b and 9c are shown in Figures 1a, 1b, 1c. If χ^2 contains a G_f singlet, additional $O(\epsilon^2)$ contributions to the Yukawa couplings amongst the light states result from:

$$[\chi\chi h]_F \rightarrow \epsilon^2[\psi\psi h]_F \tag{10}$$

as illustrated in Figure 2. Such $O(\epsilon^2)$ contributions to Yukawa matrices are more dangerous than the $O(\epsilon)$ contributions of (9a) from Figure 1a: to get a particular value for a Yukawa coupling, they require a larger value of ϵ and hence the scalar mass operators of (9c) lead to larger flavor-changing effects.

In this paper we consider only "first order" Froggatt-Nielsen mixing, as described above. In this case the mixing from a χ state, which has a coupling to the Higgs, to an external ψ state is linear in flavon fields. Theories in which more powers of ϕ appear between Higgs and external states are possible, by having a chain of internal heavy states of differing G_f quantum numbers. In this paper we do not consider theories with higher order mixings: generally they are expected to be more dangerous than theories with just first order mixing because the higher the order of the mixing the larger the ϵ necessary to give the observed fermion masses.

We now consider the case of $U(2)$ where the external ψ states are ψ_a and ψ_3, and the Higgs field h is a $U(2)$ singlet. The 23 and 22 entries of the Yukawa coupling matrices cannot arise from the diagram of Figure 2, because then the contributions of Figure 1c to the scalar masses lead to the disastrous splittings of (5) and (6). This result is independent of the U(2) representation choices for the χ and ϕ fields.

The 23 entry of the Yukawa matrices must be generated by Figure 1a, so that a U(2) doublet flavon, ϕ^a is necessary and the operator in (9a) is $[\psi_3\phi^a\psi_a h]_F$. What are the $U(2)$ properties of χ? There are just two possibilities, either it is a singlet, χ, or a doublet χ^a. The choice is critical, from the diagram of Fig. 1c it is immediately clear that the singlet χ exchange generates the dangerous operator (4), while the doublet χ^a exchange generates a

harmless contribution to the third generation scalar mass: $[\psi_3^\dagger\psi_3\phi_a^\dagger\phi_a z^\dagger z]_D$. A solution to the flavor-changing problem, based on the flavor group $U(2)$ alone, dictates that there should be no singlet χ states. Given the necessity of the doublet flavon, ϕ^a, there can similarly be no χ_b^a states.

A 22 entry for the Yukawa matrices can only be generated from Figure 1a, which requires $(\phi,\chi) = (\phi^{ab},\chi^a)$, where $\phi^{ab} = +\phi^{ba}$, $\langle\phi^{22}\rangle \neq 0$. In this case the splitting in mass of the scalars of the first two generations is quadratic in the second generation fermion mass:

$$\frac{m_{\tilde{e}}^2 - m_{\tilde{\mu}}^2}{m_{\tilde{e}}^2 + m_{\tilde{\mu}}^2} \approx O\left(\frac{m_\mu^2}{m_\tau^2}\right) \tag{11}$$

and similarly for the up and down sectors. This gives contributions to $\mu \to e\gamma$ and ϵ_K which are acceptable, although close to the limit of what experiments allow. In this paper we construct the minimal $U(2)$ model, in which there is no two index symmetric tensor ϕ^{ab}.

Finally we consider generating Yukawa matrix elements which involve the lightest generation. In principle these could originate from the diagram of Figure 2, which involves χ states with zero $U(1)$ charge: $\chi, \chi_b^a, \chi_{bd}^{ac}$.... However, the large vev of ϕ^a, necessary for V_{cb}, implies that χ and χ_b^a should be absent, so such diagrams would necessarily involve χ states with at least four tensor indices, and therefore ϕ states with at least three tensor indices. Ignoring such complicated possibilities, all contributions to the Yukawa matrices arise from Figure 1a, and therefore from the exchange of doublet χ states: χ^a. Hence, assuming no second order Froggatt-Nielsen mixing, the only question is how many such χ^a states there are. Even this is only relevant in the case of a unified gauge group where gauge breaking enters the masses of the χ^a states non-trivially. In this paper we consider a single χ^a state.

The most general contributions to Yukawa matrices from Figure 1a therefore involve $(\phi;\chi) = (\phi^a, S^{ab}, A^{ab}; \chi^a)$ where $S^{ba} = +S^{ab}$ and $A^{ba} = -A^{ab}$. The corresponding mixing of states is described by

$$\overline{\chi}_a(M_f\chi^a + \phi^a\psi_3 + S^{ab}\psi_b + A^{ab}\psi_b). \tag{12}$$

Allowing for the most general possible vevs of these flavons, this leads to Yukawa matrices of the form

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{M_f} \begin{pmatrix} S^{11} & S_+ & \phi^1 \\ S_- & S^{22} & \phi^2 \\ \phi^1 & \phi^2 & 0 \end{pmatrix} \tag{13}$$

and scalar mass matrices, from Figure 1c, of the form

$$\mathbf{m}^2 = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_1^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + \frac{1}{M_f^2} \begin{pmatrix} S_+^2 + (S^{11})^2 & S^{11}S_- + S^{22}S_+ & \phi^1 S^{22} + \phi^2 S_+ \\ S^{11}S_- + S^{22}S_+ & S_-^2 + (S^{22})^2 & \phi^2 S^{22} + \phi^1 S_- \\ \phi^1 S^{11} + \phi^2 S_+ & \phi^2 S^{22} + \phi^1 S_- & (\phi^1)^2 + (\phi^2)^2 \end{pmatrix} \tag{14}$$

where the fields stand for their vevs, and $S_\pm = S^{12} \pm A^{12}$. The trilinear soft scalar interactions from Figure 1b take the form of (13). The flavor-changing effects from this general scheme, which invokes only χ^a states, are acceptable: the exchange of scalars of the lighter two generations give effects which are automatically well below expermental limits. Flavor changing amplitudes are also induced by a non-degeneracy between the scalars of the third generation and those of the lighter two generations. These effects, although close to the limits of what experiments allow, are not problematic if the relevant mixing angles are similar to the corresponding CKM mixings and/or the amount of fractional mass splitting is somewhat less than maximal.

In this paper, rather than studying the most general doublet χ^a scheme given by (12), (13), and (14), we study the very simplest such scheme, in which S^{ab} is absent. Several interesting phenomenological features follow from the vanishing of the 22 entry. [§] In this case, since $\langle A^{12} \rangle$ preserves $SU(2)$, $\langle \phi^a \rangle$ can be chosen to lie in the $a = 2$ direction. The Yukawa matrices and scalar mass matrices then depend on only two flavor vevs; $\epsilon = \langle \phi^2 \rangle / M_f$

[§] The case of $S^{22} \neq 0$ will be discussed elsewhere.

and $\epsilon' = \langle A^{12} \rangle / M_f$, and take the forms

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & \epsilon' & 0 \\ -\epsilon' & 0 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix} \tag{15}$$

and

$$\mathbf{m}^2 = \begin{pmatrix} m_1^2 + \epsilon'^2 m^2 & 0 & \epsilon\epsilon' m^2 \\ 0 & m_1^2 + \epsilon'^2 m^2 & 0 \\ \epsilon\epsilon' m^2 & 0 & m_3^2 + \epsilon^2 m^2 \end{pmatrix}. \tag{16}$$

In (13) - (16) it is understood that each mass mixing entry involves an unknown 0(1) coefficient. However, the ϵ' terms of (15) are antisymmetric, and the two $\epsilon'^2 m^2$ terms of (16) are identical since they do not violate $SU(2)$, ¶ hence

$$\frac{m_{\tilde{e}}^2 - m_{\tilde{\mu}}^2}{m_{\tilde{e}}^2 + m_{\tilde{\mu}}^2} \approx O\left(\frac{m_e m_\mu^2}{m_\tau^3} \right) \tag{17}$$

A $U(2)$ flavor symmetry which solves the flavor-changing problem of supersymmetry provides a powerful tool for constraining the flavor sector of supersymmetric theories. Assuming only that the Higgs doublets are trivial under $U(2)$, and that more complicated higher order mixings are irrelevant, we have shown that the entire flavor structure is generated from doublet χ^a exchange, as shown in (12), (13) and (14). Furthermore, the assumption that S^{ab} is absent leads to the remarkably simple theory of (15) and (16). It is this theory that was introduced in [15], and in this paper we study further consequences of this theory in the case that the gauge group is grand unified.

3. The Minimal $U(2)$ Symmetric Model.

In this section we review the minimal $U(2)$ flavor structure in the case that the gauge group is $SU(3) \times SU(2) \times U(1)$. These results were obtained in reference [15].

¶The coefficients of S_+ and S_- of (13) are also equal, as are the coefficients of S_+^2 and S_-^2 of (14).

The theory is defined by the interactions of (12) and (13), with the S^{ab} tensor absent:

$$\overline{\chi}_a(M_f\chi^a + \phi^a\psi_3 + A^{ab}\psi_b) + h(\psi_3\psi_3 + \chi^a\psi_a). \tag{18}$$

Each of the matter fields $(\psi_3, \psi_a, \chi^a, \overline{\chi}_a)$ contains all components of a generations: q, u^c, d^c, ℓ, e^c, or the conjugate representations in the case of $\overline{\chi}_a$, which we represent by the index i, and h represents both light Higgs doublets. In (18), the coupling constants and their i dependence are left understood; hence $\overline{\chi}_a\phi^a\psi_3 = \lambda_i\overline{\chi}_{ai}\phi^a\psi_{3i}$, $h\chi^a\psi_a = \lambda'_{ij}h\chi^a_i\psi_{aj}$ $(ij = qu^c, u^cq, qd^c, d^cq, le^c, e^cl)$, etc.

The texture of the Yukawa and scalar trilinear matrices, $\boldsymbol{\lambda}$ and $\boldsymbol{\xi}$, is given in (16), and that of the scalar masses in (17). The off-diagonal $\epsilon\epsilon' m^2$ entries of (17) are numerical insignificant, and can be dropped. The diagonal correction terms, $\epsilon'^2 m^2$ and $\epsilon^2 m^2$, can be reabsorbed into the definition of the m_1^2 and m_3^2 parameters, so that the scalar mass matrices are

$$\mathbf{m}_i^2 = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_1^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}_i . \tag{19}$$

The scalars of the first two generations are accurately degenerate, and the $\mathbf{m}^2$ matrices involve 10 free parameters m_{1i}^2 and m_{3i}^2.

The Yukawa interactions are $\psi_I^T\boldsymbol{\lambda}_I\psi_I^c$, involving coupling matrices

$$\boldsymbol{\lambda}_I = \begin{pmatrix} 0 & -De^{i\phi_D} & 0 \\ De^{i\phi_D} & 0 & Ce^{i\phi_C} \\ 0 & Be^{i\phi_B} & Ae^{i\phi_A} \end{pmatrix}_I \tag{20}$$

where $I = U, D, E$ labels up, down and charged lepton sectors, and A_I, B_I, C_I and D_I are real and positive. The phases of these matrices can be factored into diagonal phase matrices $\mathbf{P}$ and $\mathbf{P}^c$:

$$\boldsymbol{\lambda}_I = \mathbf{P}_I \begin{pmatrix} 0 & -D & 0 \\ D & 0 & C \\ 0 & B & A \end{pmatrix}_I \mathbf{P}_I^c \tag{21}$$

where

$$\mathbf{P}_I = \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}_I \tag{22a}$$

and

$$\mathbf{P}_I^c = \begin{pmatrix} e^{i(\phi_D-\beta)} & 0 & 0 \\ 0 & e^{i\phi_B} & 0 \\ 0 & 0 & e^{i\phi_A} \end{pmatrix}_I \tag{22b}$$

where $\alpha_I = (\phi_B - \phi_D)_I$ and $\beta_I = (\phi_C - \phi_A)_I$. Superfield phase rotations can remove all phases, except $\alpha = \alpha_U - \alpha_D$ and $\beta = \beta_U - \beta_D$, which appear only in charged current interactions.

The Yukawa matrices can be diagonalized by orthogonal rotations

$$\begin{pmatrix} 0 & -D & 0 \\ D & 0 & C \\ 0 & B & A \end{pmatrix}_I = \mathbf{R}_{23_I}\mathbf{R}_{12_I} \begin{pmatrix} -\frac{AD^2}{BC} & 0 & 0 \\ 0 & -\frac{BC}{A} & 0 \\ 0 & 0 & A \end{pmatrix}_I \mathbf{R}_{12_I}^{c^T}\mathbf{R}_{23_I}^{c^T} \tag{23}$$

so that the flavor mixing matrices, $\mathbf{W}_I$ and $\mathbf{W}_I^c$, appearing at neutral gaugino ($\tilde{\lambda}$) vertices, $\tilde{\psi}_I^\dagger \mathbf{W}_I \psi_I \tilde{\lambda}$ and $\tilde{\psi}_I^{c\dagger} \mathbf{W}_I^c \psi_I^c \tilde{\lambda}$, are given, in the mass basis, by

$$\begin{aligned} \mathbf{W}_I^{(c)} &= \mathbf{R}_{23_I}^{(c)}\mathbf{R}_{12_I}^{(c)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s_{23} \\ 0 & -s_{23} & 1 \end{pmatrix}_I^{(c)} \begin{pmatrix} 1 & s_{12} & 0 \\ -s_{12} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_I^{(c)} \\ &= \begin{pmatrix} 1 & s_{12} & 0 \\ -s_{12} & 1 & s_{23} \\ s_{12}s_{23} & -s_{23} & 1 \end{pmatrix}_I^{(c)} . \end{aligned} \tag{24}$$

We have assumed that $(B/A)_I, (C/A)_I \approx 0(\epsilon)$ and $(D/A)_I \approx 0(\epsilon')$ with $\epsilon' \ll \epsilon \ll 1$ so that the small angle approximation is always valid. We will find later that this is not necessarily always true. The minimal $U(2)$ theory, in this approximation, has the interesting feature that $W_{I_{13}}^{(c)} = 0$. Thus, for example, the photino vertex contains $\tilde{\tau}^* e$ but not $\tilde{e}^* \tau$; staus can be made in electron collisions, but selectrons will not decay to taus.

The antisymmetry of the 12 entry of the Yukawa matrices implies

$$s^c_{12_I} = -s_{12_I}. \tag{25}$$

The angles of the mixing matrices arise from the diagonalization of the fermion mass matrices, and depend on the fermion mass eigenvalues and the three free parameters $r_I = (C/B)_I$:

$$(s_{12})_I = \left(\sqrt{\frac{m_1}{m_2}}\right)_I \tag{26a}$$

$$(s_{23})_I = \left(\sqrt{r\frac{m_2}{m_3}}\right)_I \tag{26b}$$

$$(s^c_{23})_I = \left(\sqrt{\frac{1}{r}\frac{m_2}{m_3}}\right)_I \tag{26c}$$

where $(m_{1,2,3})_I$ are the fermion mass eigenvalues of generations (1,2,3), renormalized at the flavor scale M_f. Choosing A, B, C, D positive allows θ_{12}, θ_{23} and θ^c_{23} to be taken in the first quadrant.

The trilinear scalar matrices, $\boldsymbol{\xi}_I$, also have the texture (20). By comparing Figures (1a) and (1b), one discovers that the difference between $\boldsymbol{\lambda}_I$ and $\boldsymbol{\xi}_I$ originates from the difference between the supersymmetric interactions of h and the trilinear scalar interactions of h. After the superfield phase redefinitions of (22)

$$\boldsymbol{\xi}_I = \begin{pmatrix} 0 & -A_4 D & 0 \\ A_4 D & 0 & A_3 C \\ 0 & A_2 B & A_1 A \end{pmatrix}_I \tag{27}$$

where, in general $A_{1\ldots4}$ are four complex parameters. This pattern, like that of $\mathbf{m}^2_i$, does not lead to flavor-changing difficulties. If A_i are all real, then the theory still possesses just two phases, α and β. If A_i are universal then the $\boldsymbol{\xi}_I$ and $\boldsymbol{\lambda}_I$ are simultaneously diagonalized.

The CKM matrix is given by

$$\mathbf{V} = \mathbf{W}^\dagger_U \mathbf{P}_U \mathbf{P}^*_D \mathbf{W}_D \tag{28a}$$

or

$$\mathbf{V} = \begin{pmatrix} 1 & s_1 - s_2 e^{i\phi} & s_2 s_3 \\ s_2 - s_1 e^{i\phi} & e^{i\phi} & -s_3 \\ -s_1 s_3 & s_3 & e^{-i\phi} \end{pmatrix} \tag{28b}$$

where further phase redefinitions have been performed to go from (28a) to (28b) and

$$\phi = \alpha + \beta = (\alpha_U - \alpha_D) + (\beta_U - \beta_D), \tag{29a}$$

$$s_1 = s_{12_D} \tag{29b}$$

$$s_2 = s_{12_U} \tag{29c}$$

$$s_3 = |s_{23_D} e^{i\beta} - s_{23_U}|. \tag{29d}$$

The angles $\theta_{1,2,3}$ can all be taken in the first quadrant. The CP invariant J is given by

$$J = Im V_{ud} V_{tb} V_{ub}^* V_{td}^* = s_1 s_2 s_3^2 s_\phi. \tag{30}$$

Assuming that the observed CP violation in K decays is described by the standard model box diagrams, the measurement of Re ϵ in CP violation in semileptonic K meson decays implies that $s_\phi > 0$, so that ϕ is in the first or second quadrant, depending on the sign of c_ϕ which is determined from $|V_{us}|$. The form (28b) for $\mathbf{V}$ has been obtained in another context [22] and its consequences explored elsewhere [23, 24]. We stress that, in the present theory, it is a consequence of a symmetry: the $U(2)$ flavor group.

After superfield rotations to diagonalize the fermion masses, and phase rotations on scalars to make the neutralino vertices real, as in (24), the charged wino interactions are

$$\begin{aligned} &[\tilde{u}^\dagger(\mathbf{P}_U \mathbf{P}_D^* \mathbf{W}_D) d \ + \tilde{\nu}^\dagger \mathbf{W}_E e]\tilde{\omega}^+ \\ &+[\tilde{d}^\dagger(\mathbf{P}_D \mathbf{P}_U^* \mathbf{W}_U) u \ + \tilde{e}^\dagger \mathbf{W}_E \nu]\tilde{\omega}^-. \end{aligned} \tag{31}$$

The $U(2)$ symmetry alone has solved the flavor-changing problem, and produced a significant economy of parameters in the flavor sector, allowing

many predictions. Any supersymmetric extension of the standard model must involve[‖]

- 9 quark and lepton masses.
- 15 squark and slepton masses.
- 1 quark mixing matrix, $\mathbf{V}$
- 6 neutralino mixing matrices, $\mathbf{W}_I$ and $\mathbf{W}_I^c$. The 4 chargino mixing matrices are not independent: $\mathbf{W}_q^+ = \mathbf{W}_U\mathbf{V}, \mathbf{W}_q^- = \mathbf{W}_D\mathbf{V}^\dagger$ and $\mathbf{W}_e^\pm = \mathbf{W}_E$.

While the hierarchical breaking of $U(2)$ by $\epsilon' \ll \epsilon \ll 1$ provides an origin for the hierarchy between the fermion masses of the three generations, the 9 quark and lepton masses remain free parameters. On the other hand there are only 10 independent squark and slepton masses, since $U(2)$ forces $m_{2i}^2 = m_{1i}^2$. The economical achievements of $U(2)$ are mainly in the mixing matrices, however, and we discuss this below by considering the number of parameters which enter the quark and lepton masses, and all the mixing matrices.

The lepton sector involves just 4 parameters, $(A, B, C, D)_E$, because the four phases $(\phi_{A,B,C,D})_E$ can be eliminated. Once $\tan\beta$ is known, three combinations of these $(A, BC/A, AD^2/BC)_E$ are determined by (m_τ, m_μ, m_e), leaving just one free parameter $r_E = (C/B)_E$ for the 4 leptonic gaugino mixing matrices.

In the quark sector there are 10 free parameters: $(A, B, C, D)_{U,D}, \alpha$ and β. The quark masses and CKM matrix involve precisely 10 independent observables, so one might guess that these could be used to determine the free parameters. However, this is not correct. The quark masses do determine 6 linear combinations of the free parameters: $(A, BC/A, AD^2/BC)_{U,D}$, leaving four free parameters: $r_{U,D} = (C/B)_{U,D}, \alpha$ and β. The CKM matrix, $\mathbf{V}$, is parameterized by s_1, s_2, s_3, ϕ of (29). Of these, $s_1 = \sqrt{m_d/m_s}$ and $s_2 = \sqrt{m_u/m_c}$ depend only on the same combinations of parameters that are determined by the quark masses. The parameters $\phi = \alpha + \beta$ and $s_3 = s_3(r_U, r_D, \beta)$ are determined from V_{us} and V_{cb}, and depend on two com-

[‖]We omit the trilinear parameters in this discussion.

binations of $(r_U, r_D, \alpha, \beta)$. Hence, the quark masses and $\mathbf{V}$ depend on only 8 of the original 10 parameters. The two predictions in $\mathbf{V}$ are

$$\left|\frac{V_{td}}{V_{ts}}\right| = s_1 = \sqrt{\frac{m_d}{m_s}} = 0.230 \pm 0.008 \tag{32a}$$

$$\left|\frac{V_{ub}}{V_{cb}}\right| = s_2 = \sqrt{\frac{m_u}{m_c}} = 0.063 \pm 0.009 \tag{32b}$$

to be compared with the experimental values of 0.2 ± 0.1 and 0.08 ± 0.02 respectively.

The 4 neutralino matrices $\mathbf{W}_{U,D}$ and $\mathbf{W}^c_{U,D}$, of (24), depend only on the two free parameters $r_{U,D}$, which enter the angles as shown in (26). Similarly the two quark chargino mixing matrices, $\mathbf{W}^{\pm}_q$, shown in (31), depend only on $r_{U,D}, \alpha$ and β.

Hence we can summarize the achievements made possible by the introduction of $U(2)$ and its minimal breaking.

• The supersymmetric flavor-changing problem is solved and the Yukawa matrices are forced to have a simple texture, leading to the predictions (32).

• Two small parameters, ϵ and ϵ', describe both the hierarchy of intergenerational fermion masses, and the smallness of flavor-changing effects induced by superpartner exchange; a structure summarized by (16) and (17).

• Any supersymmetric extension of the standard model necessarily involves 6 new independent flavor mixing matrices, which can be taken as those appearing at neutral gaugino vertices, $\mathbf{W}^{(c)}_I$. In the $U(2)$ theory described above, these 6 new matrices depend on only three free parameters, r_I.

While these results are considerable, the limits to the achievements of $U(2)$ are also apparent. There are free parameters for each fermion mass, V_{cb} and for s_{23_I}.

The standard model has 12 flavor observables, ignoring CP violation. Of these, the hierarchy $m_u : m_c : m_t$ can be understood as $1 : \epsilon^2 : \epsilon'^2/\epsilon^2$, and 2 parameters of the CKM matrix are predicted, leaving 7 observables for

which $U(2)$ provides no understanding. These 7 remaining pieces of the flavor puzzle can be described in terms of the parameters $(A, B, C, D)_I$, defined by the Yukawa matrices in (21):

$$\frac{m_b}{m_\tau} \approx 1 \;\Rightarrow\; \frac{A_D}{A_E} \approx 1 \tag{33a}$$

$$\frac{m_s}{m_b} \approx \frac{1}{3}\frac{m_\mu}{m_\tau} \approx \frac{1}{50} \;\Rightarrow\; \frac{B_D C_D}{A_D^2} \approx \frac{1}{3}\frac{B_E C_E}{A_E^2} \approx \frac{1}{50} \tag{33b}$$

$$\frac{m_e m_\mu}{m_\tau^2} \approx \frac{m_d m_s}{m_b^2} \;\Rightarrow\; \frac{D_D}{A_D} \approx \frac{D_E}{A_E} \tag{33c}$$

$$\frac{m_t}{m_b} \gg 1 \;\Rightarrow\; \frac{A_U v_2}{A_D v_1} \gg 1 \tag{33d}$$

$$\frac{m_c}{m_t} \approx \frac{1}{10}\frac{m_s}{m_b} \;\Rightarrow\; \frac{B_U C_U}{A_U^2} \approx \frac{1}{10}\frac{B_D C_D}{A_D^2} \tag{33e}$$

$$\frac{m_u m_c}{m_t^2} \approx 5.10^{-4}\frac{m_d m_s}{m_b^2} \;\Rightarrow\; \frac{D_U}{A_U} \approx \frac{1}{50}\frac{D_D}{A_D} \tag{33f}$$

$$V_{cb} \approx \frac{1}{25} \;\Rightarrow\; \left|\frac{C_D}{A_D}e^{i\beta} - \frac{C_U}{A_U}\right| \approx \frac{1}{25} \tag{33g}$$

where the approximate equalities hold to better than a factor of 2, and all parameters and masses are renormalized at the high flavor scale, M_f. A comparison of (33b) and (33g) shows that $B_D \gg C_D$.

As an example, the mass matrices may be given, at the factor of 2 level and ignoring phases, by

$$\mathbf{m}_U = \begin{pmatrix} 0 & 10^{-4} & 0 \\ -10^{-4} & 0 & \frac{x}{30} \\ 0 & \frac{1}{30x} & 1 \end{pmatrix} 175GeV \tag{34a}$$

$$\mathbf{m}_D = \begin{pmatrix} 0 & 10^{-3} & 0 \\ -10^{-3} & 0 & \frac{y}{150} \\ 0 & \frac{1}{30y} & \frac{1}{10} \end{pmatrix} 25GeV \tag{34b}$$

$$\mathbf{m}_E = \begin{pmatrix} 0 & 10^{-3} & 0 \\ -10^{-3} & 0 & \frac{z}{50} \\ 0 & \frac{1}{30z} & \frac{1}{10} \end{pmatrix} 25GeV. \tag{34c}$$

In section 5 we study the consequences of a $U(2)$ flavor symmetry in an $SU(5)$ grand unified theory. Is such a unified extension possible? If so, can the $SU(5)$ unification shed light on any of the patterns and hierarchies of (33) and (34)? Before addressing these questions, in the next section we extend the analysis for fermion masses and mixing matrices in the minimal $U(2)$ model to the case that the rotations in the 23 sector are large.

4. Large 23 Mixing.

In $U(2)$ theories, with the minimal texture given in (20), the 23 mixing angle in the right-handed down sector, s^c_{23D}, is expected to be large. This follows from the observation that V_{cb} and m_s/m_b are of comparable magnitude. More precisely, if we forbid V_{cb} from resulting from a cancellation of large terms in (33g), then $C_D/A_D \lesssim 1/10$. From m_s/m_b of (33b) we deduce that $B_D/D_D \gtrsim 1/5$. Thus this 23 mixing in the right-handed down sector is expected to be larger than Cabibbo mixing. A naive estimate gives $s^c_{23D} \approx (m_s/m_b)/V_{cb} \approx 0.5$. In both $SU(3)\times SU(2)\times U(1)$ and $SU(5)$ theories discussed in this paper, there are acceptable fits to the data with $s^c_{23D} \approx 0.3$ so that the small angle approximation of the previous sector is not a bad first approximation. However, in both theories there are also good fits to the data with $s^c_{23D} \approx 0.7$, which can only be discovered with the analysis of this section . In this section we derive expressions for mass eigenvalues and mixing matrices which treat the θ^c_{23} diagonalization exactly, while still using small angle approximations for θ_{23}, θ_{12} and θ^c_{12}. Rotations in the 23 space yield:**

$$R^T_{23}(s_{23}) \begin{pmatrix} 0 & -D & 0 \\ D & 0 & C \\ 0 & B & A \end{pmatrix} R^c_{23}(s^c_{23}) = \begin{pmatrix} 0 & -Dc^c_{23} & -Ds^c_{23} \\ D & -\frac{BC}{A\xi} & 0 \\ 0 & 0 & A\xi \end{pmatrix} \tag{35}$$

where $\xi = \sqrt{1+y^2}$ and $y = B/A$ is not necessarily small. The right-handed

**This analysis applies to $I = U, D$ or E, but for clarity the subscript I is dropped.

mixing angle has

$$s^c_{23} = \frac{y}{\xi} \qquad c^c_{23} = \frac{1}{\xi} \tag{36}$$

while the left-handed mixing angle is

$$s_{23} = \frac{1}{y}\frac{m_2}{m_3} \tag{37}$$

which is comparable to m_2/m_3 for y near unity. The only small parameter of the heavy 2×2 sector of the Yukawa matrix is C/A, and both $s_{23} = C/\xi^2 A$ and $m_2/m_3 = yC/\xi^2 A$ are linear in C/A. The product

$$s_{23}s^c_{23} = \frac{1}{\xi}\frac{m_2}{m_3} \tag{38}$$

which plays an important role in flavor changing phenomenology, is reduced by $1/\xi$ compared to the small angle result. In the limit that y is small and $\xi = 1 + y^2 \to 1$, these formulae reduce to the small angle versions of the previous section. However, even if $y = 1/3$, the y^2 correction terms must be kept if predictions, for example for V_{ub}/V_{cb}, are to be accurate at the 10% level.

The right-hand side of (35) shows that the large θ^c_{23} rotation has had two further important consequences: a non-negligible 13 entry has been generated, requiring an additional rotation, R_{13}, and the 21 and 12 entries are no longer equal in magnitude, implying that θ_{12} and θ^c_{12} will have differing magnitudes. The required diagonalization now has the form

$$R^T_{13}R^T_{12}R^T_{23}\begin{pmatrix} 0 & -D & 0 \\ D & 0 & C \\ 0 & B & A \end{pmatrix} R^c_{23}R^c_{12} = \begin{pmatrix} \frac{-AD^2}{BC} & 0 & 0 \\ 0 & \frac{-BC}{A\xi} & 0 \\ 0 & 0 & A\xi \end{pmatrix} \tag{39}$$

where R_{13} is defined with opposite sign to the other rotations

$$R_{13} = \begin{pmatrix} 1 & 0 & -s_{13} \\ 0 & 1 & 0 \\ s_{13} & 0 & 1 \end{pmatrix} \tag{40}$$

so that s_{13}, like s_{23}, s_{23}^c and s_{12}, is positive. We choose all angles to be in the first quadrant, except s_{12}^c which is in the second. We find

$$s_{12} = \frac{1}{\xi^{1/2}}\sqrt{\frac{m_1}{m_2}} \qquad s_{12}^c = -\xi^{1/2}\sqrt{\frac{m_1}{m_2}} \tag{41}$$

and

$$s_{13} = y^2 s_{12} s_{23} = \frac{y}{\xi^{1/2}}\sqrt{\frac{m_1 m_2}{m_3^2}}. \tag{42}$$

This last result shows that the 13 mixing in the up sector is irrelevant even if y_U is of order unity. Such 13 rotations, however, are likely to be important for down and lepton sectors.

The matrix $\mathbf{W}^c$ maintains the same form as (24), except that since θ_{23}^c is now large, c_{23}^c cannot be put to unity:

$$\mathbf{W}^c = \begin{pmatrix} 1 & s_{12}^c & 0 \\ -c_{23}^c s_{12}^c & c_{23}^c & s_{23}^c \\ s_{12}^c s_{23}^c & -s_{23}^c & c_{23}^c \end{pmatrix} \tag{43}$$

The matrix $\mathbf{W}$ has a form modified by R_{13}

$$\mathbf{W} = \mathbf{R}_{23}\mathbf{R}_{12}\mathbf{R}_{13} = \begin{pmatrix} 1 & s_{12} & -s_{13} \\ -s_{12} & 1 & s_{23} \\ s_{12}s_{23} + s_{13} & -s_{23} & 1 \end{pmatrix} \tag{44}$$

so that the W_{13} entry no longer vanishes. These neutralinos mixing matrices still conserve CP, and are again predicted in terms of just one free parameter in each of the U, D, E sectors.

The CKM matrix is given by $\mathbf{V} = \mathbf{W}_U^\dagger \mathbf{P}_U \mathbf{P}_D^* \mathbf{W}_D$. Since s_{13U} is negligible, $\mathbf{W}_U$ is given by (24). However, s_{13D} is not negligible, so that $\mathbf{W}_D$ has the form of (44), hence

$$\mathbf{V} = \begin{pmatrix} 1 & s_1 - s_2 e^{i\phi} & s_2 s_3 - s_{13D} e^{-i(\alpha+\gamma)} \\ s_2 - s_1 e^{i\phi} & e^{i\phi} & -s_3 \\ -s_1 s_3 + s_{13D} e^{i(\gamma-\beta)} & s_3 & e^{-i\phi} \end{pmatrix} \tag{45}$$

where s_{13D} is given by evaluating (42) in the down sector, and the phase γ is not a new independent phase, but is given by

$$s_3 e^{i\gamma} = s_{23U} - s_{23D} e^{i\beta} \tag{46}$$

and cannot be removed from $\mathbf{V}$ when the $0(y^2)$ corrections are kept. As before, $\phi = \alpha + \beta$, and α and β are the two physical combinations of phases of the original Yukawa matrices, defined in (22). It is important to recall that while $s_2 = \sqrt{m_u/m_c}$, and $s_{23U} = \sqrt{r_U m_c/m_t}$, the definitions of the angles in the down sector have now changed:

$$s_1 = \frac{1}{\xi^{1/2}}\sqrt{\frac{m_d}{m_s}} \tag{47a}$$

$$s_{23D} = \frac{1}{y}\frac{m_s}{m_b} \tag{47b}$$

$$s_{13D} = \frac{y}{\xi^{1/2}}\sqrt{\frac{m_d m_s}{m_b^2}}. \tag{47c}$$

Treating β and ϕ as the two independent phases, the predictions for $|V_{ub}/V_{cb}|$ and $|V_{td}/V_{ts}|$ take the form;

$$\left|\frac{V_{ub}}{V_{cb}}\right| = s_2 \left|1 + y^2 \frac{s_1}{s_2}\frac{s_{23D}}{s_3^2} e^{-i\phi}(s_{23D} - s_{23U}e^{i\beta})\right| \tag{48}$$

$$\left|\frac{V_{td}}{V_{ts}}\right| = s_1 \left|1 + y^2 \frac{s_{23D}}{s_3^2}(s_{23D} - s_{23U}e^{-i\beta})\right| \tag{49}$$

which manifestly display the $O(y^2)$ corrections to the small angle results. The CP invariant is given by

$$J = s_1 s_2 s_3^2 s_\phi + y^2 s_1 s_{23D}(s_2 s_{23D} s_\phi + s_2 s_{23U} s_{\beta-\phi} - s_1 s_{23U} s_\beta). \tag{50}$$

It is useful to take the independent phases as ϕ and β, because c_β is determined to be positive by V_{cb}, and c_ϕ is determined from V_{us}. Furthermore, if the y^2 correction of (50) does not overwhelm the $s_1 s_2 s_3^2 s_\phi$ term, then Reϵ determines s_ϕ to be positive. In this case the only quadrant ambiguity of the theory is the sign of s_β.

5. The Minimal $SU(5) \times U(2)$ Model.

A $U(2)$ flavor symmetry leads to an economical theory of flavor with Yukawa matrices constrained to have a definite texture, and neutralino mixing matrices determined in terms of just three free parameters. Grand unification provides vertical symmetry relations between the U, D and E sectors, reducing further the number of flavor parameters. In this section we study whether the simplest $U(2)$ flavor structure is consistent with $SU(5)$ grand unification, and whether the combination of these symmetries provides further progress in understanding the pattern of quark and lepton masses.

The minimal $SU(5) \times U(2)$ theory is obtained by arranging the light and heavy matter multiplets into $10 + \overline{5}$ representations: $\chi^a = (T^a, \overline{F}^a), \psi_3 = (t_3, \overline{f}_3)$ and $\psi_a = (t_a, \overline{f}_a)$, and explicitly writing all $SU(5)$ invariant interactions of 18:

$$\begin{aligned} \overline{T}_a(M_T T^a + \phi^a t_3 + A^{ab} t_b) &+ F_a(M_F \overline{F}^a + \phi^a \overline{f}_3 + A^{ab} \overline{f}_b) \\ +h(t_3 t_3 + T^a t_a) &+ \overline{h}(t_3 \overline{f}_3 + T^a \overline{f}_a + \overline{F}^a t_a) \end{aligned} \tag{51}$$

where h and $\overline{h}$ are 5 and $\overline{5}$ Higgs multiplets. On integrating out the heavy $T^a, \overline{F}^a$ states, there are 8 contributions to the Yukawa matrices, shown diagrammatically in Figure 3.

Experiment requires that the Yukawa matrices contain significant $SU(5)$ breaking at the grand unification scale, M_G. How can such $SU(5)$ breaking arise? There are three choices for the insertion of $SU(5)$ breaking: ϕ^a or A^{ab} can be $SU(5)$ non-singlets, T^a and $\overline{F}^a$ masses can contain $SU(5)$ breaking, or additional heavy states can be introduced.

We prefer to work in the minimal theory described by (51), with ϕ^a and A^{ab} transforming as $SU(5)$ singlets, but with heavy masses:

$$M_T = M_{T_0}(1 + \epsilon_T Y) \tag{52a}$$

$$M_F = M_{F_0}(1 + \epsilon_F Y) \tag{52b}$$

where M_{T_0} and M_{F_0} are $SU(5)$ invariant masses, which we take to be of order

the unification scale, M_G. The $SU(5)$ breaking masses $\epsilon_T M_{T_0} Y$ and $\epsilon_F M_{F_0} Y$ arise from the vev of a 24-plet, and are proportional to the hypercharge generator, Y. The theory therefore has the tree-level $SU(5)$ breaking of the Yukawa coupling matrices isolated in just two parameters, ϵ_T and ϵ_F.

The Yukawa interactions generated from the 8 diagrams of Figure 8 are $q^T \boldsymbol{\lambda}_U u^c + q^T \boldsymbol{\lambda}_D d^c + e^{cT} \boldsymbol{\lambda}_E \ell$, with

$$\boldsymbol{\lambda}_U = \begin{pmatrix} 0 & -\lambda_4 d_U \epsilon' & 0 \\ \lambda_4 d_U \epsilon' & 0 & \lambda_3 c_U \epsilon \\ 0 & \lambda_3 b_U \epsilon & \lambda_1 \end{pmatrix} \tag{53}$$

$$\boldsymbol{\lambda}_{D,E} = \begin{pmatrix} 0 & -\lambda_8 d_{D,E} \epsilon' & 0 \\ \lambda_8 d_{D,E} \epsilon' & 0 & \lambda_5 r c_{D,E} \epsilon \\ 0 & \lambda_7 b_{D,E} \epsilon & \lambda_2 \end{pmatrix} \tag{54}$$

where $\epsilon = \langle \phi^2 \rangle / M_{T_0}, \epsilon' = \langle A^{12} \rangle / M_{T_0}$ and $\lambda_1 ... \lambda_8$ are the dimensionless products of trilinear Yukawa interactions which appear in the diagrams i) ... viii) of Figure 3, respectively. The parameter $r = M_{T_0}/M_{F_0}$, while the $SU(5)$ breaking effects from the $T^a, \overline{F}^a$ masses are given by the coefficients

$$b_U = b_D = \frac{1}{1 + \frac{1}{6}\epsilon_T} \qquad b_E = \frac{1}{1 + \epsilon_T} \tag{55a}$$

$$c_U = \frac{1}{1 - \frac{2}{3}\epsilon_T} \qquad c_D = \frac{1}{1 + \frac{1}{3}\epsilon_F} \qquad c_E = \frac{1}{1 - \frac{1}{2}\epsilon_F} \tag{55b}$$

$$d_U = \frac{-\frac{5}{6}\epsilon_T}{(1 - \frac{2}{3}\epsilon_T)(1 + \frac{1}{6}\epsilon_T)} \tag{55c}$$

$$d_D = \frac{1}{1 + \frac{1}{6}\epsilon_T} - \frac{\lambda_6}{\lambda_8} r \frac{1}{1 + \frac{1}{3}\epsilon_F} \tag{55d}$$

$$d_E = \frac{1}{1 + \epsilon_T} - \frac{\lambda_6}{\lambda_8} r \frac{1}{1 - \frac{1}{2}\epsilon_F}. \tag{55e}$$

The labelling of the λ parameters allows easy identification of the diagrammatic origin. For example, $\overline{F}^a$ exchange occurs in only diagrams v) and vi),

with λ_5 contributing to the 23 entries of $\boldsymbol{\lambda}_{D,E}$ and λ_6 to the 12 entries of $\boldsymbol{\lambda}_{D,E}$. These contributions are therefore the only ones proportional to r. A close examination of the diagrams of Figure 3 shows that while $\lambda_1 ... \lambda_7$ are independent parameters, $\lambda_8 = \lambda_7 \lambda_4 / \lambda_3$.

In section 3 we argued that $U(2)$ alone did not address 7 pieces of the fermion mass puzzle, as listed in equation (33). The structure of (53) and (54) shows that the addition of $SU(5)$ unification provides an understanding for 4 of these features:

• (33a) The relation $m_b = m_\tau$ at the unification scale is a well-known success of supersymmetric $SU(5)$.

• (33c) If all dimensionless parameters are taken to be or order unity, then $m_e m_\mu / m_\tau^2 \approx m_d m_s / m_b^2$

• (33f) The anomalously small up quark mass can be understood if the $SU(5)$ breaking parameter ϵ_T is small. The vanishing of m_u in the $SU(5)$ limit follows because the TTh interaction gives $\boldsymbol{\lambda}_U$ symmetric, while A^{ab} is antisymmetric and forces the 12 entry to be antisymmetric. This combination of $SU(5)$ and $U(2)$ symmetry breakings to understand the small value of m_u is striking, and we consider it a major achievement of the theory.

• $\frac{m_s}{m_b} \approx V_{cb}$. For textures with vanishing λ_{D22} this requires $\lambda_{D32} \gg \lambda_{D23}$ or $B_D \gg C_D$, as can be seen by comparing (33b) and (33g). From (54) we see that the $SU(5)$ model can give such a hierarchy if r is small, that is if $M_{F_0} \gg M_{T_0}$.

We note that there is an interesting self-consistency among the last three points: in the limits that $\epsilon_T, r \to 0$ the determinantal relation $m_e m_\mu / m_\tau^2 = m_d m_s / m_b^2$ becomes exact.

In the limit of small ϵ_T and r, ϵ_T need only be kept in the 12 and 21 entries of $\boldsymbol{\lambda}_U$ and r only in the 23 entry of $\boldsymbol{\lambda}_{D,E}$. The Yukawa matrices can then be written

$$\boldsymbol{\lambda}_U = \begin{pmatrix} 0 & -\frac{5}{6}\epsilon_T \epsilon' & 0 \\ \frac{5}{6}\epsilon_T \epsilon' & 0 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix} \lambda_1 \qquad (56a)$$

$$\boldsymbol{\lambda}_{D,E} = \begin{pmatrix} 0 & -\epsilon' & 0 \\ \epsilon' & 0 & r_{D,E}\epsilon \\ 0 & \epsilon & \frac{1}{\rho} \end{pmatrix} \lambda_2 \rho \tag{56b}$$

where ϵ and ϵ' have been rescaled:

$$\epsilon = \frac{\lambda_3}{\lambda_1}\frac{\langle\phi^2\rangle}{M_{T_0}} \qquad \epsilon' = \frac{\lambda_4}{\lambda_1}\frac{\langle A^{12}\rangle}{M_{T_0}} \tag{57}$$

$$r_D = r\frac{1}{1+\frac{1}{3}\epsilon_F}\frac{\lambda_5}{\lambda_7} \qquad r_E = r\frac{1}{1-\frac{1}{2}\epsilon_F}\frac{\lambda_5}{\lambda_7} \tag{58}$$

and

$$\rho = \frac{\lambda_1\lambda_7}{\lambda_2\lambda_3}. \tag{59}$$

A posteriori, the small ϵ_T approximation turns out to be good to about 10%. Although hereafter the corrections in ϵ_T are neglected in the explicit analytic formulae, they are kept, as in equations (53) - (55), for numerical purposes. In (51) we have assumed that a single 5 or $\overline{5}$ of Higgs, h and $\overline{h}$, couple to matter. If these contain components of the light Higgs doublets: $h = c_u h_u + ...$, $\overline{h} = c_d h_d + ...$ then c_u and c_d should appear as overall factors in (56a) and (56b) respectively. However, they can be absorbed into λ_1 and λ_2.

In general all parameters appearing in (56a,b) are complex. however, as discussed in section 3, this texture has only two physical phases, α and β. In the $SU(5)$ model, these are given by

$$\alpha = -arg(\epsilon_T) \tag{60a}$$

$$\beta = -arg\left(\frac{1}{1+\frac{1}{3}\epsilon_F}\frac{\lambda_1\lambda_5}{\lambda_2\lambda_3}\right) \tag{60b}$$

in a basis where M_{T_0} and M_{F_0} are real. This shows that CP violation can arise only from the $SU(5)$ breaking masses for T^a and $\overline{F}^a$ or from the λ parameters, not, for example from the vevs of ϕ^a and A^{ab}. If $\epsilon_{T,F}$ were real, we would have $\alpha = 0$ and just a single physical phase $\beta = \phi_{CKM}$. Numerical

fits exclude this possibility [25]. Another simplifying possibility is that CP is violated spontaneously only by the vev of the 24-plet which generates ϵ_T and ϵ_F, which therefore have a common phase, while the λ parameters are real. In this paper we take α and β to be arbitrary.

After performing the phase rotations of (21), we can take all parameters of (56a,b) to be real. The $SU(3) \times SU(2) \times U(1)$ theory of section 3 had 14 flavor parameters: $(A, B, C, D)_I, \alpha$ and β. The $SU(5)$ theory reduces the number of parameters to 10: $\lambda_1, \lambda_2, \epsilon, \epsilon', \epsilon_T, \rho, r_D, r_E, \alpha$ and β. In terms of the $(A, B, C, D)_I$ parameters, $SU(5)$ imposes the 4 relations:

$$A_D = A_E \tag{61a}$$

$$B_D = B_E \tag{61b}$$

$$B_U = C_U \tag{61c}$$

$$D_D = D_E \tag{61d}$$

In the limit of small 23 rotation angles, 11 of the 14 parameters of the $SU(3) \times SU(2) \times U(1)$ model are determined from quark and lepton masses and mixing, giving the two predictions of (32), while the 3 free parameters, $r_I = C_I/B_I$, enter the neutralino mixing matrices, $\mathbf{W}_I$ and $\mathbf{W}_I^c$. In the $SU(5)$ theory, (61a) and (61d) lead to two further predictions: for m_b/m_τ and $m_e m_\mu/m_d m_s$, respectively. The two relations (61b) and (61c) can be viewed as determining two of the free parameters r_I:

$$\frac{r_D}{r_E} = \frac{m_s}{m_\mu} \tag{62a}$$

$$r_U = 1 \tag{62b}$$

respectively, so that the mixing matrices $\mathbf{W}_I$ and $\mathbf{W}_I^c$ depend on only one free parameter.

If the 23 rotation angles of the D, E sectors is large, so that $y \approx 1$, then (48) and (49) are not necessarily predictions of the theory. In the $SU(3) \times SU(2) \times U(1)$ theory these predictions are lost: V_{ub}/V_{cb} and V_{td}/V_{ts}

determine two of the free parameters, so that $\mathbf{W}_I$ and $\mathbf{W}_I^c$ depend on only a single free parameter. In the SU(5) theory, there is only one free parameter, which is therefore determined by V_{ub}/V_{cb}, since it is better measured than V_{td}/V_{ts}, which is predicted from (49). In this case, the $\mathbf{W}_I$ and $\mathbf{W}_I^c$ are completely predicted.

The analysis for the 3rd generation is not new: λ_1 and λ_2 are determined by m_t and m_τ, allowing a prediction for m_b in terms of α_s and $\tan\beta$. For the second generation we obtain the relations at the unification scale M_G:

$$\frac{m_c}{m_t} = \epsilon^2 \tag{63a}$$

$$\frac{m_s}{m_b} = \frac{\rho^2\epsilon^2 r_D}{1+y^2} \tag{63b}$$

$$\frac{m_\mu}{m_\tau} = \frac{\rho^2\epsilon^2 r_E}{1+y^2} \tag{63c}$$

$$|V_{cb}| = \epsilon\left|e^{i\beta}\frac{\rho r_D}{1+y^2} - 1\right| \tag{63d}$$

where $y = \rho\epsilon$, and the y^2 correction terms result from the large angle diagonalization of the 23 space in the D and E sectors, as given in section 4.

The masses of the light generation fermions are obtained from the determinants of the Yukawa matrices

$$\frac{m_u m_c}{m_t^2} = \frac{25}{36}\epsilon_T^2\epsilon'^2 \tag{64a}$$

$$\frac{m_d m_s}{m_b^2} = \frac{m_e m_\mu}{m_\tau^2} = \frac{\rho^2\epsilon'^2}{(1+y^2)^{3/2}} \tag{64b}$$

The equations of (63a,b,c) and (64a,b) provide 6 constraints, which can be viewed as determining all the remaining parameters, except α and β. The CKM matrix is given in (45). The phase $\phi = \alpha + \beta$ is determined from $|V_{us}|$, while a second combination of α and β is determined from $|V_{ub}/V_{cb}|$ via (48). The ratio $|V_{td}/V_{ts}|$, or equivalently J, can then be viewed as a prediction.

The hierarchy of quark and lepton masses in this $SU(5)$ theory can be understood to be due to the small parameters $\epsilon, \epsilon', \epsilon_T$ and r, with all Yukawa couplings, and hence the λ parameters, of order unity. The single exception to this is that ρ is large, as demonstrated from the following simple estimates, which ignore renormalization group scalings and assume that y is not larger than unity. To avoid a precise cancellation between terms on the right-hand side of (63d) we require $\epsilon\rho r_D \lesssim V_{cb}$. This implies from (63b) that $m_s/m_b \lesssim V_{cb}^2/r_D$, which is why r_D must be small, which we obtained by making r small. However from (63a) and (63b), $m_s/m_b = r_D\rho^2 m_c/m_t$, which requires that ρ be large.

The most plausible origin for large ρ is a small value for $\lambda_2 = \lambda_b$. Our inability to understand why ρ is large is nothing other than our lack of understanding of the large m_t/m_b ratio. If we insisted on taking $\lambda_2 \approx \lambda_1$ so that m_t/m_b arises from a large value for $\tan\beta$, we would be forced to make ρ large by taking λ_3 anomalously small. It seems much more natural to us that ρ is large because the large m_t/m_b ratio follows from a large (λ_1/λ_2) ratio. In this case $\tan\beta$ is moderate. Furthermore, since $\lambda_2 = \lambda_b \ll 1$, the renormalization group scalings of the masses and mixing angles from M_G to weak scales need only include contributions from α_s and λ_t. The CKM matrix is easily scaled by noting that the following quantities are 1 loop renormalization group invariants: $V_{us}, V_{cu}, V_{ii}, V_{cb}e^{-I_t}, V_{ub}e^{-I_t}, V_{td}e^{-I_t}, V_{ts}e^{-I_t}$ and Je^{-2I_t}; which all follow from the invariants $s_1, s_2, s_3e^{-I_t}$ and s_ϕ. For the masses, important invariants are: $e^{I_t}m_b/\eta_b m_\tau, e^{-3I_t}m_{u,c}/\eta_{u,c}m_t, e^{-I_t}m_{d,s}\eta_b/\eta_d m_b$ where $I_t = \int \lambda_t^2 dt/4\pi$ and $\eta_i = m_i(m_i)/m_i(m_t)$ for $i = c, b$, whereas for light quarks, $i = u, d, s$, $\eta_i = m_i(1GeV)/m_i(m_t)$. I_t and η_i are plotted in Ref. [19]. A possible origin for small λ_2 is that the Higgs multiplets which couple to $\psi_3\psi_3$ are different from those which couple to $\psi_a\chi^a$. Small λ_2 would result if the Higgs multiplet coupling to $t_3\overline{f}_3$ contains only a small contribution of the light doublet h_d, while other Higgs multiplets contain order unity of the light doublets. This would account for a large value of ρ, but otherwise leave our analysis unchanged.

Above we have described how the 10 free flavor parameters of the $SU(5)$ theory can be determined from data leading to predictions for the three quantities: m_b, $m_e m_\mu / m_d m_s$ and $|V_{td}/V_{ts}|$ (or J). An alternative procedure is to perform a χ^2 fit to see how well the model can account for all the relevant data, which we take to be: the 9 fermion masses, the 3 real CKM mixing angles, ϵ_K, α_s and the $B^0\overline{B}^0$ mixing parameter x_d. The predictions for ϵ_K (x_d) involve the quantities B_K ($\sqrt{B} f_B$), which we take as further observables, "measured" on the lattice. These 17 observables, and their measured values [26, 27, 28] are given in Table 1.

Table 1

m_e	0.511 MeV	
m_μ	105.7 MeV	
m_τ	1777 MeV	
$(m_u/m_d)_{1GeV}$	0.553 ± 0.043	
$(m_s/m_d)_{1GeV}$	18.9 ± 0.8	
$(m_s)_{1GeV}$	(175 ± 55) MeV	*
$(m_c)_{m_c}$	1.27 ± 0.05 GeV	
$(m_b)_{m_b}$	4.25 ± 0.15 GeV	
$(m_t)_{m_t}$	165 ± 10 GeV	
$\lvert V_{us}\rvert$	0.221 ± 0.002	
$\lvert V_{cb}\rvert$	0.038 ± 0.004	*
$\lvert V_{ub}/V_{cb}\rvert$	0.08 ± 0.02	*
$\lvert\epsilon_K\rvert$	$(2.26 \pm 0.02)10^{-3}$	
$\alpha_s(M_Z)$	0.117 ± 0.006	*
x_d	0.71 ± 0.07	*
$\sqrt{B} f_B$	$(180 \pm 30) MeV$	*
B_K	0.8 ± 0.2	*

These 17 observables depend on 14 parameters: the 10 free flavor parameters, the ratio of the two electroweak vevs v_2/v_1, α_s, $\sqrt{B} f_B$ and B_K, so that

the fit has 3 degrees of freedom. Since the uncertainties in the 17 observables are very different, we fix the well measured ones, those without an asterisk in the final column, to their central values. In particular, inputing central values for 8 of the 9 fermion masses, for V_{us} and for ϵ_K allows us to express 9 of the flavor parameters and v_2/v_1 in terms of the other free parameters. The 7 observables labelled in Table 1 by an asterisk, are then fit by varying the 1 remaining independent flavor parameter, which we choose to be y, and the parameters α_s, $\sqrt{B}f_B$ and B_K. The analysis includes the large 23 mixing results of section 4, and is therefore not restricted to small y. The renormalization scalings from grand to weak scales include 1 loop contributions from top and strong coupling constants. For reasons given earlier, we study the case of moderate $\tan\beta$, so the scalings induced by b and τ couplings are negligible.

There are three successful fits in which J, and therefore $Re\epsilon$, are positive, as shown in Table 2. In fits 1 and 2, $y \approx 0.3$ so that the y^2 correction terms are about 10%. For these fits J is dominated by $s_1 s_2 s_3^2 s_\phi$ so that s_ϕ is positive, and they are distinguished by the sign of s_β. In fit 3, $y \approx 1$ and J is dominated by the last term of (50), so that s_β is determined to be negative.

For each of these three fits, Table 2 lists the minimum χ^2 values of the seven observables which were not set to their central values, the value of χ^2_{min} and the corresponding values for 8 of the flavor parameters. (We leave out λ_1, λ_2 and v_2/v_1, which are determined from the standard analysis of the third generation.) Finally, the corresponding values for V_{td}/V_{ts} and J are given. It is clear that each of the fits is extremely good. The analysis of the uncertainties associated with these fits will be discussed in a separate paper [25].

Fits 1 and 2 have small y, and in this limit $\sin\beta$ appears only in the small y^2 correction terms of V_{ub}/V_{cb}, V_{td}/V_{ts} and J, so the fits are very similar. While V_{ub}/V_{cb} and J have about a 10% dependence on the sign of $\sin\beta$, V_{td}/V_{ts} is much less sensitive, as can be understood from (49).

Table 2

	1	2	3
sign $(\sin\phi)$	$+$	$+$	$-$
sign $(\sin\beta)$	$-$	$+$	$-$
$y = \rho\epsilon$	0.305	0.297	1.07
$\alpha_s(M_Z)$	0.117	0.117	0.117
$\lvert V_{cb}\rvert$	0.038	0.040	0.040
$\lvert V_{ub}/V_{cb}\rvert$	0.090	0.071	0.077
m_s /MeV	169	169	164
$f_B\sqrt{B}$ /MeV	173	166	187
x_d	0.730	0.738	0.711
B_K	0.875	0.966	0.855
χ^2_{min}	0.55	1.65	0.55
ϕ	1.373	1.367	-2.008
β	-0.201	0.211	-1.068
ϵ	0.0345	0.0345	0.0359
ϵ' /10^{-4}	4.93	5.04	2.36
ϵ_T	0.172	0.168	0.382
ρ	8.84	8.61	29.8
r_D	0.208	0.219	0.032
r_E	0.659	0.694	0.073
$\lvert V_{td}/V_{ts}\rvert$	0.270	0.267	0.232
J /10^{-5}	2.63	2.14	2.79

In the Yukawa couplings of (56), and in much of section 5, the full ϵ_T dependence of the Yukawa matrices, given in (55), was approximated by taking ϵ_T small and keeping only the ϵ_T dependence in the numerator of (55c). The results of the numerical fit, which included the full ϵ_T dependence, show that this approximation is not very precise, especially for fit 3.

6. Conclusions.

A $U(2)$ flavor group, broken by small parameters ϵ and ϵ', can solve the supersymmetric flavor-changing problem and provide an inter-generational fermion mass hierarchy $1 : \epsilon^2 : \epsilon'^2/\epsilon^2$ [15]. The $U(2)$ symmetry leads to successful predictions for V_{ub}/V_{cb} and V_{td}/V_{ts}, and predicts the 6 flavor mixing matrices at neutralino vertices, $\mathbf{W}_I$ and $\mathbf{W}_I^c$, in terms of just 3 free parameters $r_{U,D,E}$.

In this paper we have shown that such a $U(2)$ flavor group can be successfully imposed on an $SU(5)$ grand unified theory, with the consequences that

- Those small quark and lepton mass hierarchies not understood by ϵ and ϵ', and all 3 small angles of the CKM matrix, can be understood to arise from features of the $SU(5)$ theory.
- The quark and lepton masses, the CKM matrix, and the 6 neutralino mixing matrices $\mathbf{W}_I$ and $\mathbf{W}_I^c$, are described in terms of just 10 flavor parameters (and the ratio of electroweak vevs v_2/v_1).

In addition, the Peccei-Quinn $U(1)$ is a sub group of the $U(2)$ flavor symmetry, and is broken by $\langle A^{12} \rangle = \epsilon' M_G \approx 3 \times 10^{12}$ GeV, so that the axions are of relevance for the astrophysical dark matter problem.

Predictions for the 8 fermion mass ratios at the flavor scale are shown in Table 3, for the cases where the gauge group is $SU(3) \times SU(2) \times U(1)$ and $SU(5)$. The parameters of Table 3 appear in the Yukawa matrices of equation (15) for the $SU(3) \times SU(2) \times U(1)$ theory, and in equation (56) for the $SU(5)$ theory.

For the $SU(5)$ case the predictions are exact, and follow from (56), whereas in the $SU(3) \times SU(2) \times U(1)$ case, "$\approx$" means that ratios of dimensionless couplings are omitted. The $SU(3) \times SU(2) \times U(1)$ theory provides no understanding for many features of the spectrum, for example, for why $m_c/m_t \ll m_s/m_b$ or $m_u m_c/m_t^2 \ll m_d m_s/m_b^2$, and must therefore contain several small dimensionless ratios of Yukawa couplings.

Table 3

	$SU(3) \times SU(2) \times U(1)$	SU(5)
m_b/m_t	≈ 1	λ_2/λ_1
m_b/m_τ	≈ 1	1
m_c/m_t	$\approx \epsilon^2$	ϵ^2
m_s/m_b	$\approx \epsilon^2$	$\rho^2\epsilon^2 r_D$
m_μ/m_τ	$\approx \epsilon^2$	$\rho^2\epsilon^2 r_E$
$m_u m_c/m_t^2$	$\approx \epsilon'^2$	$(25/36)\epsilon'^2\epsilon_T^2$
$m_d m_s/m_b^2$	$\approx \epsilon'^2$	$\epsilon'^2\rho^2$
$m_e m_\mu/m_\tau^2$	$\approx \epsilon'^2$	$\epsilon'^2\rho^2$

On the other hand, the $SU(5)$ theory need contain only one small dimensionless ratio of Yukawa couplings, $\lambda_2/\lambda_1 \ll 1$ to give $m_b/m_t \ll 1$, with all other hierarchies understood. The parameter $\rho = (\lambda_1/\lambda_2)(\lambda_7/\lambda_3)$ is expected to be large (due to the large λ_1/λ_2 ratio), explaining why m_s/m_b and m_μ/m_τ are larger than m_c/m_t, and contributing to the understanding of $m_u m_c/m_t^2 \ll m_d m_s/m_b^2, m_e m_\mu/m_\tau^2$. The anomalously low value for $m_u m_c/m_t^2$ is understood in terms of a small amount of $SU(5)$ breaking, ϵ_T, in the mass of the heavy 10-plet: $M_T = M_{T_0}(1+\epsilon_T Y)$. The vanishing of m_u in the $SU(5)$ symmetric limit is particularly striking: the $T_a A^{ab} T_b h$ coupling is made antisymmetric by $U(2)$ invariance, but symmetric by $SU(5)$ invariance. The only $SU(5)$ breaking in the Yukawa matrices at the unification scale is due to $\epsilon_T \neq 0$ and $r_D \neq r_E$. Since $m_\mu/m_s = r_E/r_D = (1+\epsilon_F/3)/(1-\epsilon_F/2)$ is close to 3, the fractional breaking of $SU(5)$ in the mass of the heavy 5-plet, ϵ_F, is of order unity, where $M_F = M_{F_0}(1+\epsilon_F Y)$.

The consequences of the $U(2)$ flavor symmetry are similar in the $SU(3) \times SU(2) \times U(1)$ and $SU(5)$ theories. In the small 23 rotation angle approximation, valid for fits 1 and 2 of the previous section, the CKM matrix is parameterized by the 4 angles s_1, s_2, s_3 and s_ϕ. The parameters s_1 and s_2 are determined by quark mass ratios $s_1 = \sqrt{m_d/m_s}$ and $s_2 = \sqrt{m_u/m_c}$, so that

the sizes of V_{ub}/V_{cb}, V_{td}/V_{ts} and V_{us} are automatically understood in $U(2)$ theories in terms of quark mass hierarchies. This is not the case for s_3, which also depends on the parameters r_U, r_D and β: $s_3 = |\sqrt{r_D m_s/m_b} e^{i\beta} - \sqrt{r_U m_c/m_t}|$. (The only difference in the expressions for the CKM parameters in the $SU(3) \times SU(2) \times U(1)$ and $SU(5)$ theories, is that, as discussed below, $r_U = 1$ in the $SU(5)$ case.) The observed value of V_{cb} therefore requires that r_D is small. In the $SU(5)$ theory this can be understood as arising from $r = M_{T_0}/M_{F_0} < 1$. [††] Hence, in the $SU(5)$ theory, all small quark and lepton mass ratios, and the small values of all three CKM mixing angles, can be understood in terms of three small symmetry breaking parameters, ϵ, ϵ' and ϵ_T, and the ratio of heavy masses, r. The only exception is the small ratio λ_2/λ_1.

The CP violating phase ϕ is determined to have a large magnitude from $|V_{us}| = |\sqrt{m_d/m_s} - e^{i\phi}\sqrt{m_u/m_c}|$. The size of CP violation can therefore be determined from CP conserving quantities – quark mass ratios and the CKM flavor mixing angles – and is a significant success of the $U(2)$ symmetry.

In going from the $SU(3) \times SU(2) \times U(1)$ theory to the $SU(5)$ theory, the number of independent flavor parameters is reduced from 14 to 10. The parameter relations imposed by $SU(5)$ are shown in (61). They directly give

$$m_b = m_\tau \tag{65a}$$

$$\frac{r_D}{r_E} = \frac{m_s}{m_\mu} \tag{65b}$$

$$r_U = 1 \tag{65c}$$

$$\frac{m_e m_\mu}{m_\tau^2} = \frac{m_d m_s}{m_b^2} \tag{65d}$$

at the unification scale. The success of (65a) is a well-known feature of supersymmetric $SU(5)$. The $SU(5)$ mass relation (65d) is less well-known, but is equally successful. Although such a relation has been obtained before [19], in the present theory it is a consequence of a texture forced by the $U(2)$

[††] It is perhaps surprising that $\epsilon_T \ll \epsilon_F$, given that the T is lighter than the F. However, in practice $r \approx 1/5$, and is not very small.

flavor symmetry. The relations (65b) and (65c) reduce the number of free parameters entering the 6 neutralino mixing matrices, $\mathbf{W}_I$ and $\mathbf{W}_I^c$, from 3 to 1:

$$\mathbf{W}_I = \begin{pmatrix} 1 & \sqrt{\frac{m_1}{m_2}} & 0 \\ -\sqrt{\frac{m_1}{m_2}} & 1 & \sqrt{r\frac{m_2}{m_3}} \\ \sqrt{r\frac{m_1}{m_3}} & -\sqrt{r\frac{m_2}{m_3}} & 1 \end{pmatrix}_I \tag{66a}$$

$$\mathbf{W}_I^c = \begin{pmatrix} 1 & -\sqrt{\frac{m_1}{m_2}} & 0 \\ \sqrt{\frac{m_1}{m_2}} & 1 & \sqrt{\frac{1}{r}\frac{m_2}{m_3}} \\ \sqrt{\frac{1}{r}\frac{m_1}{m_3}} & -\sqrt{\frac{1}{r}\frac{m_2}{m_3}} & 1 \end{pmatrix}_I \tag{66b}$$

with $r_U = 1$ and $r_D/r_E \approx 1/3$. For the case of large 23 rotation angles in the D, E sectors, as in fit 3 of section 5, the forms of the CKM and $\mathbf{W}_I$ and $\mathbf{W}_I^c$ matrices are more complicated. While V_{ub}/V_{cb} can no longer be viewed as a prediction, there are no free parameters at all in $\mathbf{W}_I$ and $\mathbf{W}_I^c$. If supersymmetry is discovered, this theory can be tested by the predictions (66a,b) for $\mathbf{W}_I$ and $\mathbf{W}_I^c$.

The $U(2)$ theory of flavor presented in this paper makes definite predictions for various processes, as will be discussed in a separate paper [25]. However, the $U(2)$ symmetry is insufficient to determine the fractional mass splittings between the scalars of the third generation and the scalars of the lighter two generations, Δ_L and Δ_R for the left and right components respectively. If $\Delta_L = \Delta_R = 1$ in the down sector, then, in the $SU(5)$ theory discussed in this paper, the gluino exchange contribution to ϵ_K exceeds the experimental value by about a factor of 50, for average squark masses and a gluino mass of 1 TeV. Hence, in the down sector of a $U(3)$ theory of flavor, it will be crucial to either suppress Δ_L and/or Δ_R, or to have milder flavor mixings to the third generation than given by (66a,b).

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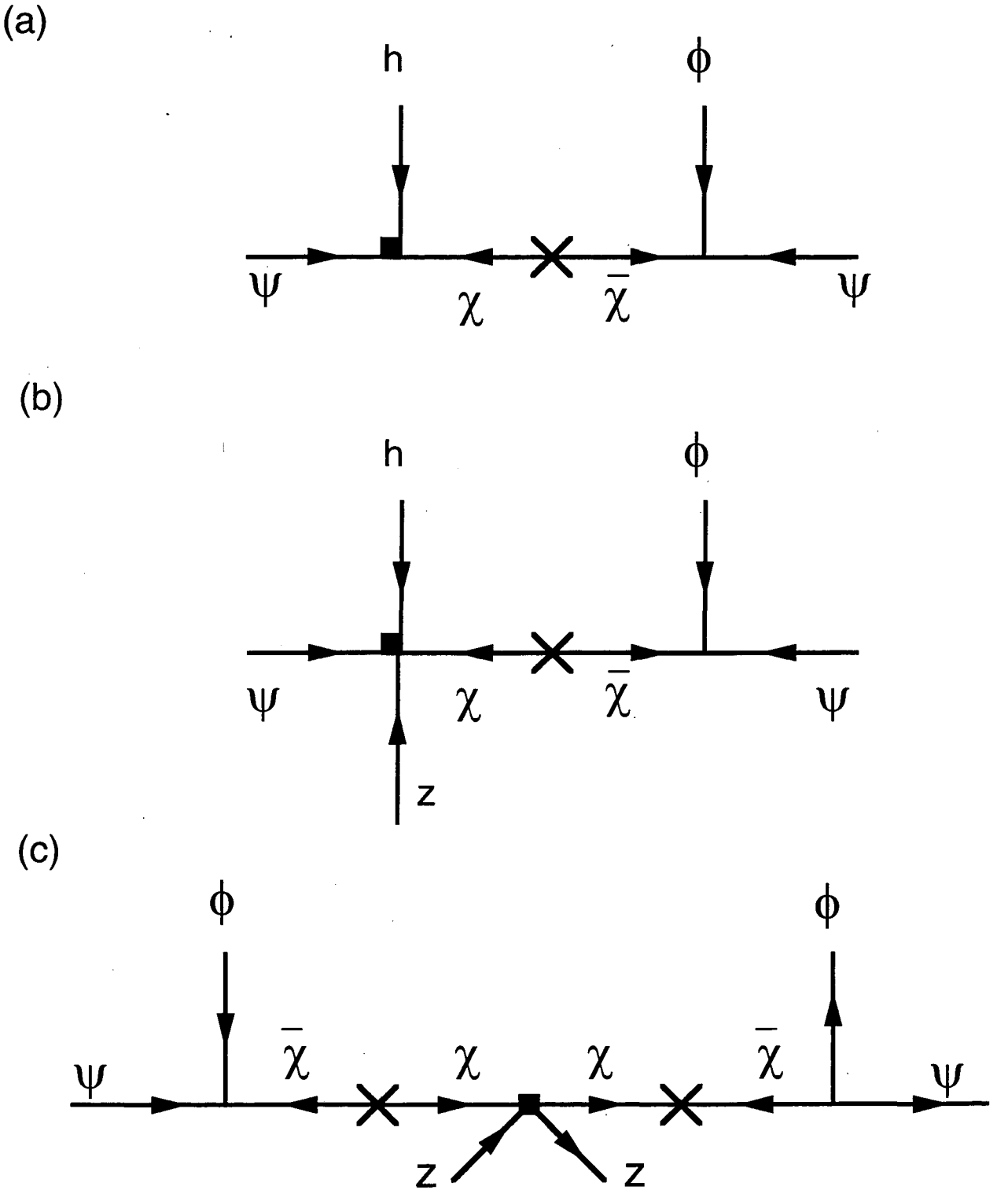
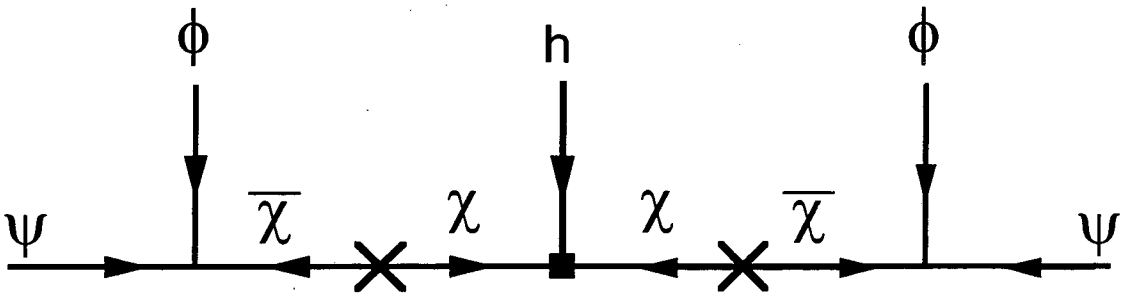
Figure 1



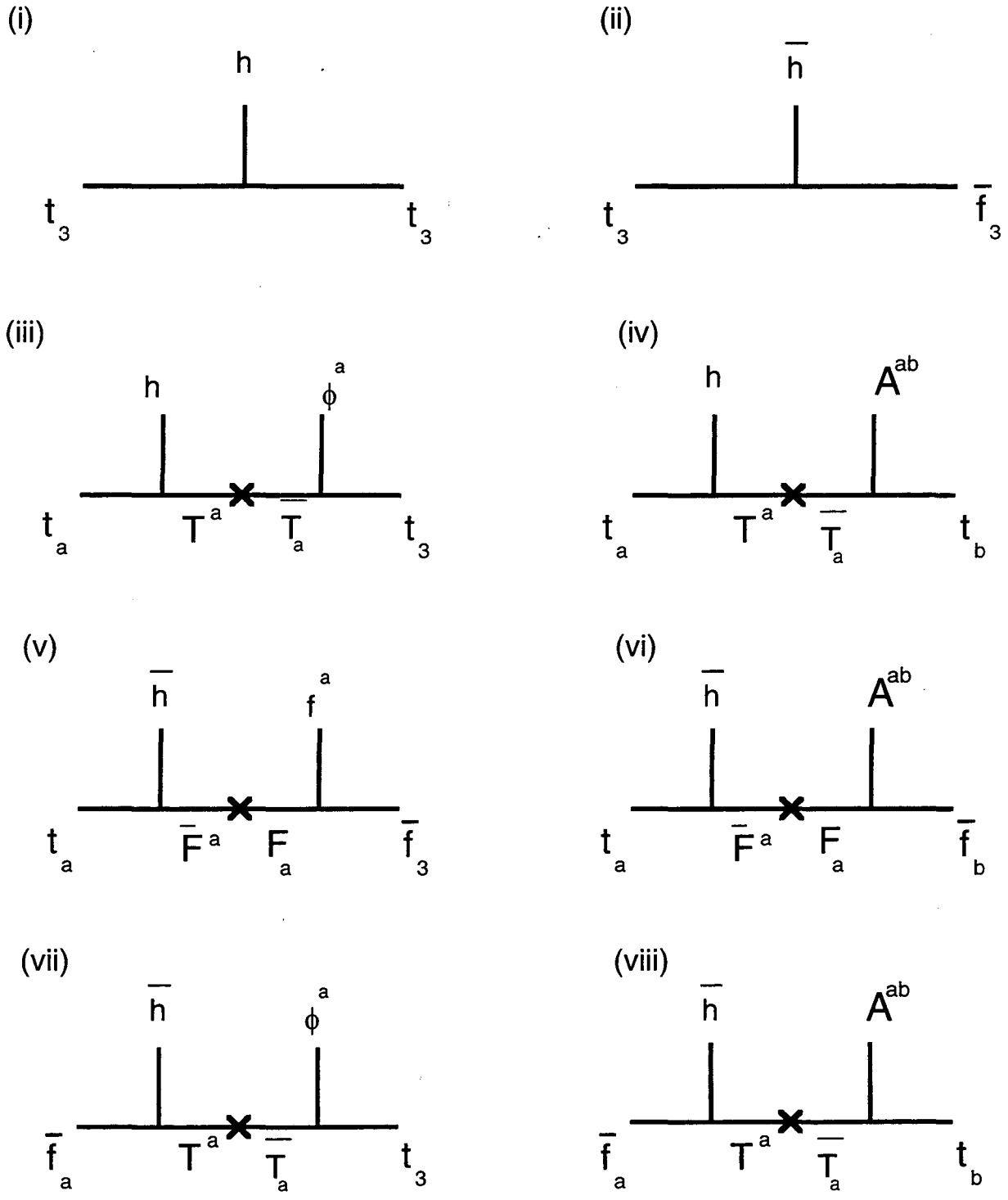
Feynman diagrams which generate operators (9a,b,c) on integrating out the heavy χ states.

Figure 2



Feynman diagram which generates the operator (10) if the flavor symmetry allows the interaction $\chi\chi$ h.

Figure 3



Feynman diagrams which contribute to the Yukawa matrices of (53) and (54) at tree level, (i) and (ii); from integrating out T^a, (iii), (iv), (vii) and (viii); and $\bar{F}^a$, (v) and (vi).

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