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Forces and Stored Energy in Thin Cosine(nθ) Accelerator magnets.*

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Abstract

We wish to compute Lorentz forces, equilibrium stress and stored energy in thin multipole magnets (Fig. 1), that are proportional to cos(nθ) and whose strength varies purely as a Fourier sinusoidal series of the longitudinal coordinate z (say proportional to $\cos\frac{(2m-1)\pi z}{L}$ where L denotes the *half-period* and m=1,2,3 ...). We shall demonstrate that in cases where the current is situated on such a surface of discontinuity at r=R (i.e. J=f(θ,z)), by computing the Lorentz force and solving the state of equilibrium on that surface, a closed form solution can be obtained for single function magnets as well as for any combination of interacting nested multi function magnets.

The results that have been obtained, indicate that the total axial force on the end of a single multipole magnet n is independent (orthogonal) to any other multipole magnet i as long as $n \neq i$. The same is true for the stored energy, the total energy of a nested set of multipole magnets is equal to the some of the energy of the individual magnets (of the same period length 2L). Finally we demonstrate our results on a nested set of magnets a dipole (n=1) and a quadrupole (n=2) that have an identical single periodicity ω_1.

We show that in the limiting 2D case (period 2L tends to infinity), the force reduces to the commonly known 2D case.

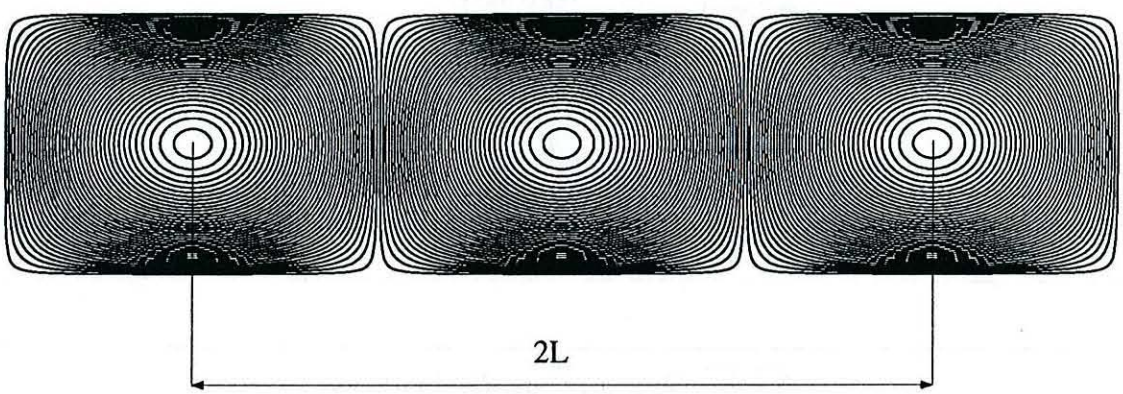

Figure 1 View of a full period array of a single function (m=1) quad.

Lorentz Force on a Surface of Discontinuity

The Lorentz force density on a thin surface of discontinuity[b] (per unit area s) may be expressed as given by

$$\frac{d\vec{F}}{dS} = \vec{f}_s = \vec{J}_s \times < \vec{B} >$$

where $< \vec{B} >$ denotes the average magnetic field on the surface $< \vec{B} >= \frac{\vec{B}_1+\vec{B}_2}{2}$ and $\vec{J}_s$ corresponds to the surface current density. In a previous note[cd] we derived the magnetic field components inside and outside a current sheet for an ideal current density that is proportional to cosine nθ. We shall evaluate $< B >$, compute the Lorentz force f, express the equilibrium condition on a surface element and solve for the total force as a function of position (to simplify the analysis we have not included contributions from a highly permeable iron yoke).

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b Utility of the Maxwell Stress Tensor for Computing Magnetic Forces — L.Jackson Laslett, Lawerence Berkeley Laboratory, report ERAN-160, August 24 1971.

c Combined Right and Left Hand Helical Function Magnets, SC-MAG-534, January 1996.

d Magnetic Field Components in a Sinusoidally Varying Helical Wiggler, LBL-35928, SC-MAG-464, July 1994.

The field on the inside, outside and on a current sheet is,

r≤R :

$$B_r = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n^{'}(\omega_m r)\sin n\theta \cos\omega_m z$$
$$B_\theta = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{1}{r}I_n(\omega_m r)\cos n\theta \cos\omega_m z$$
$$B_z = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n(\omega_m r)\sin n\theta \sin\omega_m z$$

r≥R :

$$B_r = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I_n^{'}(\omega_m R)}{K_n^{'}(\omega_m R)}K_n^{'}(\omega_m r)\sin n\theta \cos\omega_m z$$
$$B_\theta = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{I_n^{'}(\omega_m R)}{K_n^{'}(\omega_m R)}\frac{1}{r}K_n(\omega_m r)\cos n\theta \cos\omega_m z$$
$$B_z = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I_n^{'}(\omega_m R)}{K_n^{'}(\omega_m R)}K_n(\omega_m r)\sin n\theta \sin\omega_m z$$

r=R :

$$< B_r > = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n^{'}(\omega_m R)\sin n\theta \cos\omega_m z$$
$$< B_\theta > = -\frac{1}{2}\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{1}{R}\frac{[I_n(\omega_m R)K_n(\omega_m R)]^{'}}{K_n^{'}(\omega_m R)}\cos n\theta \cos\omega_m z$$
$$< B_z > = \frac{1}{2}\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{[I_n(\omega_m R)K_n(\omega_m R)]^{'}}{K_n^{'}(\omega_m R)}\sin n\theta \sin\omega_m z$$

We substitute

$$\omega_m = \frac{(2m-1)\pi}{L} \qquad and \qquad G_{n,m} = n!R^n\left(\frac{2}{\omega_m R}\right)^n B_{n,m}$$

where n=1,2,3.... denotes a magnet type such as a dipole, quadrupole etc, $nB_{n,m}$ denotes the dipole, gradient etc and m=1,2,3...., corresponds to a given periodicity where L is the half period. We consider the term $(\omega_n R)$ to be the argument of all Modified Bessel functions I_n and K_n, and all derivatives of such functions taken to be with respect to that argument.

We may express the field directly in terms of current density (see following page)

r≤R :

$$B_r = \mu_0 \sum_{n=1}\sum_{m=1} J_{0n,m} \frac{(\omega_m R)^2}{n} K_n^{'}(\omega_m R) I_n^{'}(\omega_m r) \sin n\theta \cos \omega_m z$$

$$B_\theta = \mu_0 \sum_{n=1}\sum_{m=1} J_{0n,m} \omega_m R^2 K_n^{'}(\omega_m R) \frac{I_n(\omega_m r)}{r} \cos n\theta \cos \omega_m z$$

$$B_z = -\mu_0 \sum_{n=1}\sum_{m=1} J_{0n,m} \frac{(\omega_m R)^2}{n} K_n^{'}(\omega_m R) I_n(\omega_m r) \sin n\theta \sin \omega_m z$$

r≥R :

$$B_r = \mu_0 \sum_{n=1}\sum_{m=1} J_{0n,m} \frac{(\omega_m R)^2}{n} I_n^{'}(\omega_m R) K_n^{'}(\omega_m r) \sin n\theta \cos \omega_m z$$

$$B_\theta = \mu_0 \sum_{n=1}\sum_{m=1} J_{0n,m} \omega_m R^2 I_n^{'}(\omega_m R) \frac{K_n(\omega_m r)}{r} \cos n\theta \cos \omega_m z$$

$$B_z = -\mu_0 \sum_{n=1}\sum_{m=1} J_{0n,m} \frac{(\omega_m R)^2}{n} I_n^{'}(\omega_m R) K_n(\omega_m r) \sin n\theta \sin \omega_m z$$

r=R :

$$< B_r >= \mu_0 \sum_{n=1}\sum_{m=1} J_{0n,m} \frac{(\omega_m R)^2}{n} I_n^{'}(\omega_m R) K_n^{'}(\omega_m R) \sin n\theta \cos \omega_m z$$

$$< B_\theta >= \frac{\mu_0}{2} \sum_{n=1}\sum_{m=1} J_{0n,m} \omega_m R [I_n(\omega_m R) K_n(\omega_m R)]^{'} \cos n\theta \cos \omega_m z$$

$$< B_z >= -\frac{\mu_0}{2} \sum_{n=1}\sum_{m=1} J_{0n,m} \frac{(\omega_m R)^2}{n} [I_n(\omega_m R) K_n(\omega_m R)]^{'} \sin n\theta \sin \omega_m z$$

Current Density

In Reference[c] we have expressed the current density as $\vec{J} = J_\theta \hat{e}_\theta + J_z \hat{e}_z$ and its components as,

$$\vec{J}(\theta, z)|_{r=R} = -\frac{1}{\mu_0 R} \left\{ \begin{array}{l} 0\hat{e}_r \\ \sum_{n=1}\sum_{m=1} \frac{G_{n,m}}{K_n'(\omega_m R)} \sin n\theta \sin \omega_m z \hat{e}_\theta \\ \sum_{n=1}\sum_{m=1} \frac{nG_{n,m}}{\omega_m R} \frac{1}{K_n'(\omega_m R)} \cos n\theta \cos \omega_m z \hat{e}_z \end{array} \right\}$$

Such a pair satisfies the conservation condition $\nabla \cdot \vec{J}_s = \frac{\partial J_z}{\partial z} + \frac{1}{R}\frac{\partial J_\theta}{\partial \theta} = 0$ as required.

We may wish to express $G_{n,m}$ in terms of the current density or the total current per pole (Amp-turn) a procedure first introduced in Reference[c,e]

The component of current density at θ=0 and z=0 is purely in the z direction $\vec{J} = J_z \hat{e}_z = J_0$, therefore

$$G_{n,m} = -\frac{J_{0n,m}}{n} \mu_0 \omega_m R^2 K_n'(\omega_m R)$$

and the current density,

$$\vec{J}(\theta, z)|_{r=R} = \left\{ \begin{array}{l} 0\hat{e}_r \\ \sum_{n=1}\sum_{m=1} \frac{J_{0n,m}}{n} \omega_m R \sin n\theta \sin \omega_m z \hat{e}_\theta \\ \sum_{n=1}\sum_{m=1} J_{0n,m} \cos n\theta \cos \omega_m z \hat{e}_z \end{array} \right\}$$

Integrating the current density, the total Amp-turn $\boxed{\text{per pole}}$ is,

$$I_{n,m} = J_{0n,m} \int_0^{\frac{\pi}{2n}} \cos n\theta R d\theta = J_{0n,m} \frac{R}{n} = -\frac{G_{n,m}}{\mu_0 \omega_m R K_n'(\omega_m R)}$$

or,

$$\begin{aligned} G_{n,m} &= -I_{n,m} \mu_0 \omega_m R K_n'(\omega_m R) \\ J_{0n,m} &= \frac{n}{R} I_{n,m} \end{aligned}$$

[e] Forces in a Thin Cosine(nθ) Helical Wiggler, LBL-36988, SC-MAG-495, March 1995.

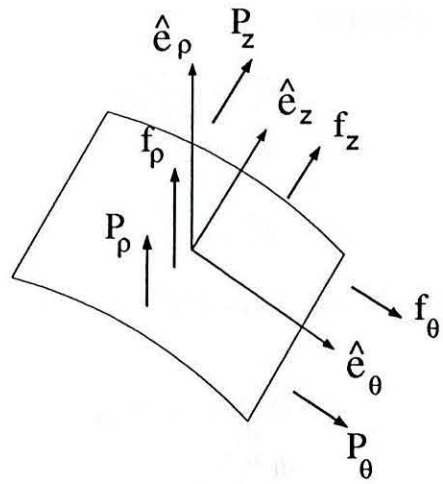

Figure 2 Forces on a current sheet. The f's are the local Lorentz forces and the P's are the local equilibrium forces.

Lorentz Force

The Lorentz force on a current sheet is,

$$\vec{f} = \frac{d\vec{F}}{ds} = \vec{J} \times < \vec{B} > = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\theta & \hat{e}_z \\ 0 & J_\theta & J_z \\ < B_\rho > & < B_\theta > & < B_z > \end{vmatrix}$$

$$\vec{f} = (J_\theta < B_z > - J_z < B_\theta >)\hat{e}_\rho + J_z < B_\rho > \hat{e}_\theta - J_\theta < B_\rho > \hat{e}_z$$

We compute the Lorentz force components by substituting the average field and current density,

$$f_\rho = -\frac{1}{2\mu_0 R^2} \sum_n \sum_m \sum_i \sum_j G_{n,m} G_{i,j} \frac{[I_i(\omega_j R) K_i(\omega_j R)]'}{K_n'(\omega_m R) K_i'(\omega_j R)} \times$$
$$\times \left(\omega_j R \sin n\theta \sin \omega_m z \sin i\theta \sin \omega_j z + \frac{ni}{\omega_m R} \cos n\theta \cos \omega_m z \cos i\theta \sin \omega_j z \right)$$

$$f_\theta = \frac{1}{\mu_0 R^2} \sum_n \sum_m \sum_i \sum_j n G_{n,m} G_{i,j} \left(\frac{\omega_j}{\omega_m} \right) \frac{I_i'(\omega_j R)}{K_n'(\omega_m R)} \cos n\theta \cos \omega_m z \sin i\theta \cos \omega_j z$$

$$f_z = -\frac{1}{\mu_0 R^2} \sum_n \sum_m \sum_i \sum_j G_{n,m} G_{i,j} \omega_j R \frac{I_i'(\omega_j R)}{K_n'(\omega_m R)} \sin n\theta \sin \omega_m z \sin i\theta \cos \omega_j z$$

Equilibrium

The equilibrium of a current carrying surface element $\delta\theta\delta z$ requires that, (Fig. 2)

$$P_\rho \hat{e}_\rho + f_\rho R \delta\theta \delta z \hat{e}_\rho + d(P_\theta \hat{e}_\theta) + f_\theta R \delta\theta \delta z \hat{e}_\theta + d(P_z \hat{e}_z) + f_z R \delta\theta \delta z \hat{e}_z = 0$$

where the f's are the local Lorentz body forces and the P's are the total equilibrium forces.

We shall make the following substitutions :

$$P_\rho^{''} = \frac{P_\rho}{R\delta\theta\delta z}$$
$$P_\theta^{'} = \frac{dP_\theta}{dz}$$
$$P_z^{'} = \frac{dP_z}{Rd\theta}$$

where P' is a force per unit length and P'' is a force per unit area.

since :

$$\frac{d(P_\theta \hat{e}_\theta)}{dz} = P_\theta^{'}\hat{e}_\theta$$
$$\frac{d(P_z \hat{e}_z)}{Rd\theta} = P_z^{'}\hat{e}_z$$
$$\frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_\rho$$

the resulting three differential equations are :

$$\boxed{P_\rho^{''}\hat{e}_\rho + f_\rho\hat{e}_\rho - \frac{P_\theta^{'}}{R}\hat{e}_\rho + \frac{dP_\theta^{'}}{Rd\theta}\hat{e}_\theta + f_\theta\hat{e}_\theta + \frac{dP_z^{'}}{dz}\hat{e}_z + f_z\hat{e}_z = 0}$$

or

$$\boxed{\begin{aligned} P_\rho^{''} &= -f_\rho + \frac{P_\theta^{'}}{R} \\ \frac{dP_\theta^{'}}{d\theta} &= -Rf_\theta \\ \frac{dP_z^{'}}{dz} &= -f_z \end{aligned}}$$

Solution to $P_\theta^{'}$

In solving $P_\theta^{'}$

$$\boxed{\frac{dP_\theta^{'}}{d\theta} = -Rf_\theta}$$

we shall make use of,

$$\int_{-\frac{\pi}{2n}}^{\theta} \cos n\theta' \sin i\theta' d\theta' = \left\{ \begin{array}{ll} \frac{\cos(n-i)\theta'}{2(n-i)} - \frac{\cos(n+i)\theta'}{2(n+i)}\Big|_{-\frac{\pi}{2n}}^{\theta} & n \neq i \\ -\frac{\cos 2n\theta'}{4n}\Big|_{-\frac{\pi}{2n}}^{\theta} & n = i \end{array} \right\}$$

$$\int_{-\frac{\pi}{2n}}^{\theta} \cos n\theta' \sin i\theta' d\theta' = \left\{ \begin{array}{ll} \dfrac{\cos(n-i)\theta}{2(n-i)} - \dfrac{\cos(n+i)\theta}{2(n+i)} - \dfrac{n}{n^2-i^2}\sin\dfrac{i\pi}{2n} & n \neq i \\ -\dfrac{\cos^2 n\theta}{2n} & n = i \end{array} \right\}$$

so that

$$P_\theta' = -R \int_{-\frac{\pi}{2n}}^{\theta} f_\theta d\theta$$

$$= -\frac{1}{\mu_0 R} \sum_n \sum_m \sum_i \sum_j n G_{n,m} G_{i,j} \left(\frac{\omega_j}{\omega_m}\right) \frac{I_i'(\omega_j R)}{K_n'(\omega_m R)} \cos\omega_m z \cos\omega_j z \times$$

$$\times \left\{ \begin{array}{ll} \dfrac{\cos(n-i)\theta}{2(n-i)} - \dfrac{\cos(n+i)\theta}{2(n+i)} - \dfrac{n}{n^2-i^2}\sin\dfrac{i\pi}{2n} & n \neq i \\ -\dfrac{\cos^2 n\theta}{2n} & n = i \end{array} \right\}$$

Expressing the force in terms of current density,

$$G_{n,m} G_{i,j} = \mu_0^2 R^4 \frac{\omega_m \omega_j}{ni} J_{0n,m} J_{0i,j} K_n'(\omega_m R) K_i'(\omega_j R)$$

$$\boxed{\begin{aligned} P_\theta' = -\mu_0 R \sum_n \sum_m \sum_i \sum_j J_{0n,m} J_{0i,j} \frac{(\omega_j R)^2}{i} K_i'(\omega_j R) I_i'(\omega_j R) \cos\omega_m z \cos\omega_j z \times \\ \times \left\{ \begin{array}{ll} \dfrac{\cos(n-i)\theta}{2(n-i)} - \dfrac{\cos(n+i)\theta}{2(n+i)} - \dfrac{n}{n^2-i^2}\sin\dfrac{i\pi}{2n} & n \neq i \\ -\dfrac{\cos^2 n\theta}{2n} & n = i \end{array} \right\} \end{aligned}}$$

<u>Solution to</u> P_z'

In solving P_z',

$$\boxed{\frac{dP_z'}{dz} = -f_z}$$

we shall make use of the following integration,

$$\int_{-L}^{z} \sin\omega_m z' \cos\omega_j z' dz = \left\{ \begin{array}{ll} \dfrac{\cos(\omega_m - \omega_j)z'}{2(\omega_m - \omega_j)} - \dfrac{\cos(\omega_m + \omega_j)z'}{2(\omega_m + \omega_j)}\Big|_{-L}^{z} & m \neq j \\ -\dfrac{\cos 2\omega_m z'}{4\omega_m}\Big|_{-L}^{z} & m = j \end{array} \right\}$$

$$\int_{-L}^{z} \sin \omega_m z' \cos \omega_j z' dz = \left\{ \begin{array}{ll} -\dfrac{\cos(\omega_m - \omega_j)z}{2(\omega_m - \omega_j)} - \dfrac{\cos(\omega_m + \omega_j)z}{} + \dfrac{\omega_m}{\omega_m^2 - \omega_j^2} & m \neq j \\ \dfrac{\sin^2 \omega_m z}{2\omega_m} & m = j \end{array} \right\}$$

Integrating the force between the limits $z' = -L$ and $z' = z$ the axial force is therefore,

$$\begin{aligned} P_z' &= -\int_{-L}^{z} f_z dz \\ &= \frac{1}{2\mu_0 R} \sum_n \sum_m \sum_i \sum_j G_{n,m} G_{i,j} \omega_j \frac{I_i'(\omega_j R)}{K_n'(\omega_m R)} \sin n\theta \sin i\theta \times \\ &\quad \times \left\{ \begin{array}{ll} -\dfrac{\cos(\omega_m - \omega_j)z}{(\omega_m - \omega_j)} - \dfrac{\cos(\omega_m + \omega_j)z}{(\omega_m + \omega_j)} + \dfrac{2\omega_m}{\omega_m^2 - \omega_j^2} & m \neq j \\ \dfrac{\sin^2 \omega_m z}{\omega_m} & m = j \end{array} \right\} \end{aligned}$$

and by replacing $G_{n,m}$ with the expression for current density,

$$\boxed{\begin{aligned} P_z' &= \frac{\mu_0}{2} \sum_n \sum_m \sum_i \sum_j \frac{(\omega_j R)^2 (\omega_m R)}{ni} J_{0n,m} J_{0i,j} I_i'(\omega_j R) K_i'(\omega_j R) \sin n\theta \sin i\theta \times \\ &\quad \times \left\{ \begin{array}{ll} -\dfrac{\cos(\omega_m - \omega_j)z}{(\omega_m - \omega_j)} - \dfrac{\cos(\omega_m + \omega_j)z}{(\omega_m + \omega_j)} + \dfrac{2\omega_m}{\omega_m^2 - \omega_j^2} & m \neq j \\ \dfrac{\sin^2 \omega_m z}{\omega_m} & m = j \end{array} \right\} \end{aligned}}$$

The maximum axial force is at the magnet ends i.e. at $z = \frac{L}{2}$,

$$P_z' = \frac{1}{2\mu_0 R} \sum_n \sum_m \sum_i \sum_j G_{n,m} G_{i,j} \omega_j \frac{I_i'(\omega_j R)}{K_n'(\omega_m R)} \sin n\theta \sin i\theta \left\{ \begin{array}{ll} \dfrac{2\omega_m}{\omega_m^2 - \omega_j^2} & m \neq j \\ \dfrac{1}{\omega_m} & m = j \end{array} \right\}$$

The <u>total axial force</u> over the end is, $P_{z_{z=\frac{L}{2}}} = \int_0^{2\pi} P'_{z_{z=\frac{L}{2}}} R d\theta$

For i=n

$$P_z = \frac{\pi}{2\mu_0} \sum_n \sum_m \sum_j G_{n,m} G_{n,j} \frac{I_n'(\omega_j R)}{K_n'(\omega_m R)} \begin{Bmatrix} \dfrac{2\omega_m \omega_j}{\omega_m^2 - \omega_j^2} & m \neq j \\ 1 & m = j \end{Bmatrix}$$

or

$$P_z = \frac{\pi R^2 \mu_0}{2} \sum_n \sum_m \sum_j J_{0n,m} J_{0n,j} \frac{(\omega_m R)(\omega_j R)}{n^2} I_n'(\omega_j R) K_n'(\omega_j R) \begin{Bmatrix} \dfrac{2\omega_m \omega_j}{\omega_m^2 - \omega_j^2} & m \neq j \\ 1 & m = j \end{Bmatrix}$$

and for $n \neq i$, $P_z = 0$.

We note that the total axial force on the end of magnet type n is independent (orthogonal) to any nested magnet type i as long as $i \neq n$.

In the above integration we have used,

$$\int_0^{2\pi} \sin n\theta \sin i\theta d\theta = \begin{Bmatrix} 0 & n \neq i \\ \pi & n = i \end{Bmatrix}$$

Solution to P_ρ''

We express the local radial pressure P_ρ'' , which is a reactive compressive pressure directed inwards in the $-\rho$ direction, as :

$$\begin{aligned} P_\rho'' &= -f_\rho + \frac{P_\theta'}{R} \\ &= \frac{1}{2\mu_0 R^2} \sum_n \sum_m \sum_i \sum_j G_{n,m} G_{i,j} \frac{(I_i(\omega_j R) K_i(\omega_j R))'}{K_n'(\omega_m R) K_i'(\omega_j R)} \begin{bmatrix} \omega_j R \sin n\theta \sin i\theta \sin \omega_m z \sin \omega_j z + \\ + \dfrac{ni}{\omega_m R} \cos n\theta \cos i\theta \cos \omega_m z \cos \omega_j z \end{bmatrix} \\ &- \frac{1}{\mu_0 R^2} \sum_n \sum_m \sum_i \sum_j n G_{n,m} G_{i,j} \left(\frac{\omega_j}{\omega_m}\right) \frac{I_i'(\omega_j R)}{K_n'(\omega_m R)} \cos \omega_m z \cos \omega_j z \times \\ &\qquad \times \begin{Bmatrix} \dfrac{\cos(n-i)\theta}{2(n-i)} - \dfrac{\cos(n+i)\theta}{2(n+i)} - \dfrac{n}{n^2 - i^2} \sin \dfrac{i\pi}{2n} & n \neq i \\ -\dfrac{\cos^2 n\theta}{2n} & n = i \end{Bmatrix} \end{aligned}$$

in terms of current density,

$$P_\rho'' = \frac{\mu_0 R^2}{2} \sum_n \sum_m \sum_i \sum_j J_{0n,m} J_{0i,j} \frac{\omega_m \omega_j}{ni} (I_i(\omega_j R) K_i(\omega_j R))' \begin{bmatrix} \omega_j R \sin n\theta \sin i\theta \sin \omega_m z \sin \omega_j z + \\ + \dfrac{ni}{\omega_m R} \cos n\theta \cos i\theta \cos \omega_m z \cos \omega_j z \end{bmatrix}$$

$$- \mu_0 R^2 \sum_n \sum_m \sum_i \sum_j J_{0n,m} J_{0i,j} \frac{\omega_j^2}{i} I_i'(\omega_j R) K_i'(\omega_j R) \cos \omega_m z \cos \omega_j z \times$$

$$\times \left\{ \begin{array}{ll} \dfrac{\cos(n-i)\theta}{2(n-i)} - \dfrac{\cos(n+i)\theta}{2(n+i)} - \dfrac{n}{n^2 - i^2} \sin \dfrac{i\pi}{2n} & n \neq i \\ -\dfrac{\cos^2 n\theta}{2n} & n = i \end{array} \right\}$$

The limiting 2D case

The results of the general force equations can be reduced to the more familiar 2D case by extending the period $2L \to \infty$. We note that for such a limit $s = \omega_m R \to 0$ (as well as $s = \omega_j R \to 0$) we can make use of,

$$s \to 0$$

$$I_n(s) \sim \frac{1}{n!} \left(\frac{s}{2}\right)^n$$

$$K_n(s) \sim \frac{(n-1)!}{2} \left(\frac{s}{2}\right)^{-n}$$

$$I_n'(s) \sim \frac{1}{2(n-1)!} \left(\frac{s}{2}\right)^{n-1}$$

$$K_n'(s) \sim -\frac{n!}{4} \left(\frac{s}{2}\right)^{-(n+1)}$$

$$I_n'(s) K_n'(s) \sim -\frac{n}{2s^2}$$

$$\frac{I_n'(s)}{K_n'(s)} \sim -\frac{2}{n!(n-1)!} \left(\frac{s}{2}\right)^{2n}$$

$$lims_{s \to 0} \frac{G_{n,m}^2 I_n'(s)}{K_n'(s)} \to -2nR^{2n} B_n^2$$

$$lims_{s \to 0} \frac{G_{n,m} G_{i,j} I_i'(\omega_j R)}{K_n'(\omega_m R)} \to -2iR^{n+i} \frac{\omega_m}{\omega_j} B_n B_i$$

$$lims_{s \to 0} [I_n K_n]' \to 0$$

$$lims_{s \to 0} n B_{n,m} \to \frac{\mu_0 J_{0n,m}}{2R^{n-1}}$$

reducing the 2D forces to :

$$P'_{\theta_{2d}} = \frac{2}{\mu_0 R} \sum_n \sum_m \sum_i \sum_j n B_{n,m} i B_{i,j} R^{n+i} \left\{ \begin{array}{ll} \frac{\cos(n-i)\theta}{2(n-i)} - \frac{\cos(n+i)\theta}{2(n+i)} - \frac{n}{n^2-i^2} \sin\frac{i\pi}{2n} & n \neq i \\ -\frac{\cos^2 n\theta}{2n} & n = i \end{array} \right\}$$

$$P'_{\theta_{2d}} = \frac{\mu_0 R}{2} \sum_n \sum_m \sum_i \sum_j J_{0n,m} J_{0i,j} \left\{ \begin{array}{ll} \frac{\cos(n-i)\theta}{2(n-i)} - \frac{\cos(n+i)\theta}{2(n+i)} - \frac{n}{n^2-i^2} \sin\frac{i\pi}{2n} & n \neq i \\ -\frac{\cos^2 n\theta}{2n} & n = i \end{array} \right\}$$

The total axial force for n=i (m=j) reduces to,

$$P_{z_{2d}} = -\frac{\pi}{\mu_0} \sum_n \frac{R^{2n} (n B_{n,m})^2}{n}$$

$$P_{z_{2d}} = -\frac{\mu_0 \pi R^2}{4} \sum_n \frac{J_{0n,m}^2}{n}$$

(the case $n \neq i$ $P_{z_{2d}} = 0$).
and the radial pressure is,

$$P''_{\rho,2d} = \frac{2}{\mu_0 R^2} \sum_n \sum_m \sum_i \sum_j n B_{n,m} i B_{i,j} R^{n+i} \left\{ \begin{array}{ll} \frac{\cos(n-i)\theta}{2(n-i)} - \frac{\cos(n+i)\theta}{2(n+i)} - \frac{n}{n^2-i^2} \sin\frac{i\pi}{2n} & n \neq i \\ -\frac{\cos^2 n\theta}{2n} & n = i \end{array} \right\}$$

or

$$P''_{\rho,2d} = \frac{\mu_0}{2} \sum_n \sum_m \sum_i \sum_j J_{0n,m} J_{0i,j} \left\{ \begin{array}{ll} \frac{\cos(n-i)\theta}{2(n-i)} - \frac{\cos(n+i)\theta}{2(n+i)} - \frac{n}{n^2-i^2} \sin\frac{i\pi}{2n} & n \neq i \\ -\frac{\cos^2 n\theta}{2n} & n = i \end{array} \right\}$$

For a single function magnet (n=i, m=j), the maximum field occurs at r=R is,

$$B_{max,n,m} = n B_{n,m} R^{n-1} = \frac{\mu_0 J_{0n,m}}{2}$$

and the forces reduce to[f],

$$P'_{\theta_{2d}} = -\frac{R}{\mu_0}\sum_n\sum_m \frac{B^2_{max,n,m}}{n}\cos^2 n\theta$$

or

$$P'_{\theta_{2d}} = -\frac{\mu_0 R}{4}\sum_n\sum_m \frac{J^2_{0n,m}}{n}\cos^2 n\theta$$

The total axial force,

$$P_{z_{2d}} = -\frac{\pi R^2}{\mu_0}\sum_n \frac{B^2_{max,n}}{n}$$

and

$$P''_{\rho,2d} = -\frac{1}{\mu_0}\sum_n\sum_m \frac{B^2_{max,n,m}}{n}\cos^2 n\theta$$

or

$$P''_{\rho,2d} = -\frac{\mu_0}{4}\sum_n\sum_m \frac{J^2_{0n,m}}{n}\cos^2 n\theta$$

The stored energy in multipole windings

Calculating the stored energy from

$$E = \frac{1}{2}\int\int\int \vec{J}\cdot\vec{A}dv$$

we need to integrate the vector product over the current surface only :

$$E = \int_0^{2\pi}\int_{-L}^{L} \vec{J}\cdot\vec{A}d\sigma = \int_0^{2\pi}\int_{-L}^{L} \vec{J}\cdot\vec{A}Rd\theta dz$$

(the current density is generally per unit area but when applied to thin windings is per unit length and the units of energy is $J = T\cdot A\cdot m^2$).

f Forces in a Thin Cosine nθ Winding — R.Meuser, Engineering Note M5266, November 15, 1978.

The product of the current density and the vector potential[c] is :

$$J \cdot A = \begin{aligned} &\frac{1}{2\mu_0 R}\sum_n\sum_m\sum_i\sum_j \frac{G_{n,m}}{K_n'(\omega_m R)}\frac{G_{i,j}}{K_i'(\omega_j R)}[I_{i+1}(\omega_j R)K_{i+1}(\omega_j R) + I_{i-1}(\omega_j R)K_{i-1}(\omega_j R)]\times \\ &\times \sin n\theta \sin i\theta \sin \omega_m z \sin \omega_j z + \\ &+\frac{1}{\mu_0 R}\sum_n\sum_m\sum_i\sum_j \frac{G_{n,m}}{K_n'(\omega_m R)}\frac{G_{i,j}}{K_i'(\omega_j R)}\frac{niI_i(\omega_j R)K_i(\omega_j R)}{\omega_m R\omega_j R}\cos n\theta \cos i\theta \cos \omega_m z \cos \omega_j z \end{aligned}$$

In performing the integration we shall make use of the following integrals :

$$\int_0^{2\pi} \sin n\theta \sin i\theta d\theta = \begin{Bmatrix} 0 & n \neq i \\ \pi & n = i \end{Bmatrix}$$

$$\int_0^{2\pi} \cos n\theta \cos i\theta d\theta = \begin{Bmatrix} 0 & n \neq i \\ \pi & n = i \end{Bmatrix}$$

$$\int_{-L}^{L} \sin \omega_m z \sin \omega_j z dz = \begin{Bmatrix} 0 & m \neq j \\ L & m = j \end{Bmatrix}$$

$$\int_{-L}^{L} \cos \omega_m z \cos \omega_j z dz = \begin{Bmatrix} 0 & m \neq j \\ L & m = j \end{Bmatrix}$$

As a result of orthogonality the stored energy reduces from a 4 sum product to a 2 sum, indicating that there is NO MUTUAL INDUCTANCE between <u>different</u> multipole coils:

$$E = \frac{1}{2}\int_0^{2\pi}\int_{-L}^{L} J \cdot ARd\theta dz = \begin{aligned} &\frac{\pi L}{4\mu_0}\sum_n\sum_m \frac{G_{n,m}^2}{K_n'^2(\omega_m R)}[I_{i+1}(\omega_m R)K_{i+1}(\omega_m R) + I_{i-1}(\omega_m R)K_{i-1}(\omega_m R)] + \\ &+\frac{\pi L}{4\mu_0}\sum_n\sum_m \frac{G_{n,m}^2}{K_n'^2(\omega_m R)}\frac{2n^2}{(\omega_m R)^2}I_n(\omega_m R)K_n(\omega_m R) \end{aligned}$$

and since

$$I_{n+1}K_{n+1} + I_{n-1}K_{n-1} + \frac{2n^2}{(\omega_m R)^2}I_n K_n = -2K_n' I_n'$$

the stored energy is :

$$\boxed{E = -\frac{\pi L}{2\mu_0}\sum_n\sum_m G_{n,m}^2 \frac{I_n'(\omega_m R)}{K_n'(\omega_m R)}}$$

or dividing by the volume the energy density is :

$$\boxed{e = -\frac{1}{2\mu_0 R^2}\sum_n\sum_m G_{n,m}^2 \frac{I_n'(\omega_m R)}{K_n'(\omega_m R)}}$$

Finally in terms of current density :

$$e = -\frac{\mu_0}{2}\sum_n\sum_m \frac{J^2_{0_z n,m}(\omega_m R)^2}{n^2} I_n'(\omega_m R) K_n'(\omega_m R)$$

resulting in the same expression as for helical multipole magnets[g]

EXAMPLE

Combined Dipole n=1 and Quadrupole n=2

The combined magnetic field of a nested dipole (n=1) and quadrupole (n=2) magnets with the same period 2Land single periodicity $\omega_1 = \frac{\pi}{L}$ is,

r≤R :

$$\begin{aligned}
B_r &= -\Big(G_{1,1}\omega_1 I_1'(\omega_1 r)\sin\theta + G_{2,1}\omega_1 I_2'(\omega_1 r)\sin 2\theta\Big)\cos\omega_1 z \\
B_\theta &= -\left(G_{1,1}\frac{1}{r}I_1(\omega_1 r)\cos\theta + 2G_{2,1}\frac{1}{r}I_2(\omega_1 r)\cos 2\theta\right)\cos\omega_1 z \\
B_z &= \quad (G_{1,1}\omega_1 I_1(\omega_1 r)\sin\theta + G_{2,1}\omega_1 I_2(\omega_1 r)\sin 2\theta)\sin\omega_1 z
\end{aligned}$$

r≥R :

$$\begin{aligned}
B_r &= -\left(G_{1,1}\frac{I_1'(\omega_1 R)}{K_1'(\omega_1 R)}K_1'(\omega_1 r)\sin\theta + G_{2,1}\frac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}K_2'(\omega_1 r)\sin 2\theta\right)\omega_1\cos\omega_1 z \\
B_\theta &= -\left(G_{1,1}\frac{I_1'(\omega_1 R)}{K_1'(\omega_1 R)}\frac{1}{r}K_1(\omega_1 r)\cos\theta + 2G_{2,1}\frac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}\frac{1}{r}K_2(\omega_1 r)\cos 2\theta\right)\cos\omega_1 z \\
B_z &= \quad \left(G_{1,1}\frac{I_1'(\omega_1 R)}{K_1'(\omega_1 R)}K_1(\omega_1 r)\sin\theta + G_{2,1}\frac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}K_2(\omega_1 r)\sin 2\theta\right)\omega_1\sin\omega_1 z
\end{aligned}$$

r=R :

$$\begin{aligned}
< B_r > &= -\Big(G_{1,1}I_1'(\omega_1 R)\sin\theta + G_{2,1}I_2'(\omega_1 R)\sin 2\theta\Big)\omega_1\cos\omega_1 z \\
< B_\theta > &= -\frac{1}{2}\left(G_{1,1}\frac{[I_1(\omega_1 R)K_1(\omega_1 R)]'}{K_1'(\omega_1 R)}\cos\theta + 2G_{2,1}\frac{[I_2(\omega_1 R)K_2(\omega_1 R)]'}{K_2'(\omega_1 R)}\cos 2\theta\right)\frac{1}{R}\cos\omega_1 z \\
< B_z > &= \frac{1}{2}\left(G_{1,1}\frac{[I_1(\omega_1 R)K_1(\omega_1 R)]'}{K_1'(\omega_1 R)}\sin\theta + G_{2,1}\frac{[I_2(\omega_1 R)K_2(\omega_1 R)]'}{K_2'(\omega_1 R)}\sin 2\theta\right)\omega_1\sin\omega_1 z
\end{aligned}$$

g The Vector Potential and Stored Energy of Thin Cosine(nθ) Helical Wiggler Magnet, LBL-38075, SC-MAG-529, December 1995.

Where,

$$G_{1,1} = \frac{2}{\omega_1} B_{1,1} = -\mu_0 R^2 \omega_1 K_1^{'}(\omega_1 R) J_{01,1} = -\mu_0 R \omega_1 K_1^{'}(\omega_1 R) I_{1,1}$$
$$G_{2,1} = \frac{8}{\omega_1^2} B_{2,1} = -\frac{\mu_0}{2} R^2 \omega_1 K_2^{'}(\omega_1 R) J_{02,1} = -\mu_0 R \omega_1 K_2^{'}(\omega_1 R) I_{2,1}$$

and

$$\omega_1 = \frac{\pi}{L}$$
$$B_{1,1} = \text{dipole} \quad \text{field}$$
$$2B_{2,1} = \text{gradient}$$

The forces acting on the dipole (incorporating the relation $\cos\theta + \frac{\cos 3\theta}{3} = \frac{4}{3}\cos^3\theta$) are,

$$P_\theta^{'} = \frac{1}{2\mu_0 R} \begin{bmatrix} G_{1,1}^2 \dfrac{I_1^{'}(\omega_1 R)}{K_1^{'}(\omega_1 R)} \cos^2 \omega_1 z \cos^2\theta + \\ + G_{1,1} G_{2,1} \dfrac{4}{3} \dfrac{I_2^{'}(\omega_1 R)}{K_1^{'}(\omega_1 R)} \cos^2 \omega_1 z \cos^3\theta \end{bmatrix}$$

$$P_\theta^{'} = \frac{\mu_0 R}{2} \begin{bmatrix} J_{01,1}^2 (\omega_1 R)^2 K_1^{'}(\omega_1 R) I_1^{'}(\omega_1 R) \cos^2\theta + \\ + J_{01,1} J_{02,1} \dfrac{(\omega_1 R)^2}{2} \dfrac{4}{3} K_2^{'}(\omega_1 R) I_2^{'}(\omega_1 R) \cos^3\theta \end{bmatrix} \cos^2 \omega_1 z$$

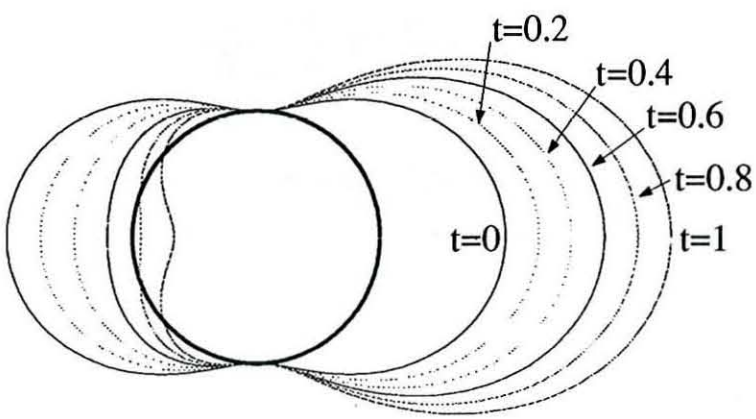

Figure 3 A polar plot of the magnitude of $P_\theta^{'}$ at z=0 in a dipole n=1 according to $P_\theta^{'} = \cos^2\theta + t\left[\cos\theta + \frac{\cos(3\theta)}{3}\right]$, t=0 corresponds to a single function dipole (no quad)

and

$$P_z' = \frac{1}{2\mu_0 R}\left[\begin{array}{l} G_{1,1}^2 \dfrac{I_1'(\omega_1 R)}{K_1'(\omega_1 R)} + \\ + 2G_{1,1}G_{2,1}\dfrac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}\cos\theta \end{array}\right]\sin^2\theta\sin^2\omega_1 z$$

$$P_z' = \frac{\mu_0 R}{2}\left[\begin{array}{l} J_{01,1}^2 I_1'(\omega_1 R)K_1'(\omega_1 R) + \\ + J_{01,1}J_{02,1}I_2'(\omega_1 R)K_2'(\omega_1 R)\cos\theta \end{array}\right](\omega_1 R)^2\sin^2\theta\sin^2\omega_1 z$$

The total force acting on the dipole end is,

$$P_z = \frac{\pi}{2\mu_0}G_{1,1}^2\frac{I_1'(\omega_1 R)}{K_1'(\omega_1 R)}$$

$$P_z = \frac{\pi R^2\mu_0}{2}(\omega_1 R)^2 I_1'(\omega_1 R)K_1'(\omega_1 R)J_{01,1}^2$$

and the radial stress is,

$$P_\rho'' = \frac{1}{2\mu_0 R^2}\left[\begin{array}{l} G_{1,1}^2 \dfrac{(I_1(\omega_1 R)K_1(\omega_1 R))'}{K_1'^2(\omega_1 R)}\left[\begin{array}{l}\omega_1 R\sin^2\theta\sin^2\omega_1 z+\\ +\dfrac{1}{\omega_1 R}\cos^2\theta\cos^2\omega_1 z\end{array}\right]+ \\ +G_{1,1}G_{2,1}\dfrac{(I_2(\omega_1 R)K_2(\omega_1 R))'}{K_1'(\omega_1 R)K_2'(\omega_1 R)}\left[\begin{array}{l}\omega_1 R\sin\theta\sin 2\theta\sin^2\omega_1 z+\\ +\dfrac{2}{\omega_1 R}\cos\theta\cos 2\theta\cos^2\omega_1 z\end{array}\right]+ \\ +G_{1,1}^2\dfrac{I_1'(\omega_1 R)}{K_1'(\omega_1 R)}\cos^2\theta\cos^2\omega_1 z+ \\ +G_{1,1}G_{2,1}\dfrac{4}{3}\dfrac{I_2'(\omega_1 R)}{K_1'(\omega_1 R)}\cos^3\theta\cos^2\omega_1 z \end{array}\right]$$

$$P_\rho'' = \frac{\mu_0 R^2}{2}\left[\begin{array}{l} J_0^2\omega_1^2(I_1(\omega_1 R)K_1(\omega_1 R))'\left[\begin{array}{l}\omega_1 R\sin^2\theta\sin^2\omega_1 z+\\ +\dfrac{1}{\omega_1 R}\cos^2\theta\cos^2\omega_1 z\end{array}\right]+ \\ +J_{01,1}J_{02,1}\dfrac{\omega_1^2}{2}(I_2(\omega_1 R)K_2(\omega_1 R))'\left[\begin{array}{l}\omega_1 R\sin\theta\sin 2\theta\sin^2\omega_1 z+\\ +\dfrac{2}{\omega_1 R}\cos\theta\cos 2\theta\cos^2\omega_1 z\end{array}\right]+ \\ +J_{01,1}^2\omega_1^2 I_1'(\omega_1 R)K_1'(\omega_1 R)\cos^2\theta\cos^2\omega_1 z+ \\ +J_{01,1}J_{02,1}\dfrac{\omega_1^2}{2}\dfrac{4}{3}I_2'(\omega_1 R)K_2'(\omega_1 R)\cos^3\theta\cos^2\omega_1 z \end{array}\right]$$

and the stored energy density,

$$e = -\frac{1}{2\mu_0 R^2}G_{1,1}^2\frac{I_1'(\omega_1 R)}{K_1'(\omega_1 R)}$$

$$e = -\frac{\mu_0}{2}J_{0_z 1,1}^2(\omega_1 R)^2 I_1'(\omega_1 R)K_1'(\omega_1 R)$$

The forces acting on the quadrupole n=2 are,
The azimuthal force,

$$P_\theta^{'} = \frac{1}{\mu_0 R}\left[\begin{array}{l} -G_{2,1}G_{1,1}\dfrac{I_1^{'}(\omega_1 R)}{K_2^{'}(\omega_1 R)}\left(\cos\theta - \dfrac{\cos 3\theta}{3} - \dfrac{2\sqrt{2}}{3}\right) + \\ + \dfrac{G_{2,1}^2}{2}\dfrac{I_2^{'}(\omega_1 R)}{K_2^{'}(\omega_1 R)}\cos^2 2\theta \end{array}\right]\cos^2\omega_1 z$$

$$P_\theta^{'} = \frac{\mu_0 R}{2}\left[\begin{array}{l} -J_{02,1}J_{01,1}(\omega_1 R)^2 K_1^{'}(\omega_1 R)I_1^{'}(\omega_1 R)\left(\cos\theta - \dfrac{\cos 3\theta}{3} - \dfrac{2\sqrt{2}}{3}\right) + \\ + J_{02,1}^2\left(\dfrac{\omega_1 R}{2}\right)^2 K_2^{'}(\omega_1 R)I_2^{'}(\omega_1 R)\cos^2 2\theta \end{array}\right]\cos^2\omega_1 z$$

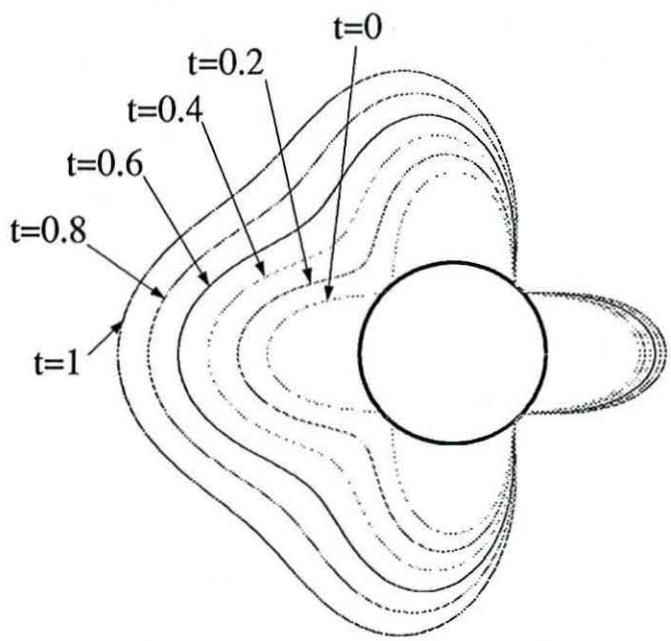

Figure 4 A polar plot of the magnitude of $P_\theta^{'}$ at z=0 for a quad n=2, according to $P_\theta^{'} = \cos^2 2\theta - t\left[\cos\theta - \frac{\cos(3\theta)}{3} - \frac{2\sqrt{2}}{3}\right]$, t=0 corresponds to a single function quad (no dipole)

The axial force,

$$P_z' = \frac{1}{2\mu_0 R}\begin{bmatrix} G_{2,1}G_{1,1}\dfrac{I_1'(\omega_1 R)}{K_2'(\omega_1 R)}\sin 2\theta \sin\theta + \\ + G_{2,1}^2\dfrac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}\sin^2 2\theta \end{bmatrix}\sin^2\omega_1 z$$

$$P_z' = \frac{\mu_0 R}{4}\begin{bmatrix} J_{02,1}J_{01,1}I_1'(\omega_1 R)K_1'(\omega_1 R)\sin 2\theta\sin\theta + \\ + \dfrac{J_{02,1}^2}{2}I_2'(\omega_1 R)K_2'(\omega_1 R)\sin^2 2\theta \end{bmatrix}(\omega_1 R)^2\sin^2\omega_1 z$$

The total force on the quad end,

$$P_z = \frac{\pi}{2\mu_0}G_{2,1}^2\frac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}$$

$$P_z = \frac{\pi R^2\mu_0}{8}(\omega_1 R)^2 I_2'(\omega_1 R)K_2'(\omega_1 R)J_{02,1}^2$$

The radial pressure is,

$$P_\rho'' = \frac{1}{2\mu_0 R^2}\begin{bmatrix} G_{2,1}G_{1,1}\dfrac{(I_1(\omega_1 R)K_1(\omega_1 R))'}{K_2'(\omega_1 R)K_1'(\omega_1 R)}\begin{bmatrix}\omega_1 R\sin 2\theta\sin\theta\sin^2\omega_1 z + \\ + \dfrac{2}{\omega_1 R}\cos 2\theta\cos\theta\cos^2\omega_1 z\end{bmatrix} + \\ + G_{2,1}^2\dfrac{(I_2(\omega_1 R)K_2(\omega_1 R))'}{K_2'^2(\omega_1 R)}\begin{bmatrix}\omega_1 R\sin^2 2\theta\sin^2\omega_1 z + \\ + \dfrac{4}{\omega_1 R}\cos^2 2\theta\cos^2\omega_1 z\end{bmatrix} - \\ - 2G_{2,1}G_{1,1}\dfrac{I_1'(\omega_1 R)}{K_2'(\omega_1 R)}\left(\cos\theta - \dfrac{\cos 3\theta}{3} - \dfrac{2\sqrt{2}}{3}\right)\cos^2\omega_1 z + \\ + G_{2,1}^2\dfrac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}\cos^2 2\theta\cos^2\omega_1 z \end{bmatrix}$$

$$P_\rho'' = \frac{\mu_0 R^2}{2}\left[\begin{array}{l} J_{02,1}J_{01,1}\dfrac{\omega_1^2}{2}(I_1(\omega_1 R)K_1(\omega_1 R))'\left[\begin{array}{l}\omega_1 R\sin 2\theta\sin\theta\sin^2\omega_1 z+\\ +\dfrac{2}{\omega_1 R}\cos 2\theta\cos\theta\cos^2\omega_1 z\end{array}\right]+ \\ +J_{02,1}^2\dfrac{\omega_1^2}{4}(I_2(\omega_1 R)K_2(\omega_1 R))'\left[\begin{array}{l}\omega_1 R\sin^2 2\theta\sin^2\omega_1 z+\\ +\dfrac{4}{\omega_1 R}\cos^2 2\theta\cos^2\omega_1 z\end{array}\right]- \\ -J_{02,1}J_{01,1}\omega_1^2 I_1'(\omega_1 R)K_1'(\omega_1 R)\left(\cos\theta-\dfrac{\cos 3\theta}{3}-\dfrac{2\sqrt{2}}{3}\right)\cos^2\omega_1 z+ \\ +J_{02,1}^2\dfrac{\omega_1^2}{4}I_2'(\omega_1 R)K_2'(\omega_1 R)\cos^2 2\theta\cos^2\omega_1 z \end{array}\right]$$

and the stored energy density,

$$e = -\frac{1}{2\mu_0 R^2}G_{2,1}^2\frac{I_2'(\omega_1 R)}{K_2'(\omega_1 R)}$$

$$e = -\frac{\mu_0}{8}J_{0_z 2,1}^2(\omega_1 R)^2 I_2'(\omega_1 R)K_2'(\omega_1 R)$$

Simulation of Current density and flow lines

To generate flow lines we make use of a technic first demonstrated by J.Lasslet and W. Fawley of this laboratory. The character of the flow lines for a single function magnet n, will follow from the differential equation,

$$\frac{Rd\theta}{dz} = \frac{J_\theta}{J_z}$$

Assuming the current density for magnet type n as,

$$\vec{J} = -\frac{1}{\mu_0 R}\left[\sum_{m=1} \frac{G_{n,m}}{K_n'(\omega_m R)} \sin n\theta \sin \omega_m z \hat{e}_\theta + \sum_{m=1} \frac{nG_{n,m}}{\omega_m R} \frac{1}{K_n'(\omega_m R)} \cos n\theta \cos \omega_m z \hat{e}_z\right]$$

so that,

$$\frac{Rd\theta}{dz} = \frac{\sum\limits_{m=1} \frac{G_{n,m}}{K_n'(\omega_m R)} \sin n\theta \sin \omega_m z}{\sum\limits_{m=1} \frac{nG_{n,m}}{\omega_m R} \frac{1}{K_n'(\omega_m R)} \cos n\theta \cos \omega_m z}$$

and

$$\frac{n \cos n\theta}{\sin n\theta} d\theta - \frac{\sum\limits_{m=1} \frac{G_{n,m}}{K_n'(\omega_m R)} \sin \omega_m z}{R \sum\limits_{m=1} \frac{1}{\omega_m R} \frac{G_{n,m}}{K_n'(\omega_m R)} \cos \omega_m z} dz = 0$$

or,

$$\ln(\sin \mathrm{n}\theta) + \ln\left(\sum_{m=1} \frac{1}{\omega_m R} \frac{G_{n,m}}{K_n'(\omega_m R)} \cos \omega_m z\right) = \text{const.}$$

So that,

$$\sin n\theta = \frac{const.}{\sum\limits_{m=1} \frac{1}{\omega_m R} \frac{G_{n,m}}{K_n'(\omega_m R)} \cos \omega_m z}$$

and the flow lines are therefore,

$$\sin n\theta = \frac{\sum\limits_{m=1} \frac{1}{\omega_m R} \frac{G_{n,m}}{K_n'(\omega_m R)}}{\sum\limits_{m=1} \frac{1}{\omega_m R} \frac{G_{n,m}}{K_n'(\omega_m R)} \cos \omega_m z} \sin n\theta_0$$

$$\sin n\theta = \frac{\sum\limits_{m=1} J_{0n,m}}{\sum\limits_{m=1} J_{0n,m} \cos \omega_m z} \sin n\theta_0$$

where θ_0 denotes the value of θ at z=0.

In a special case, we may choose special values for $J_{0n,m}$ such that,

$$J_{0n,m} = \frac{J_{0n}}{2^{2(M-1)}} \binom{2M-1}{M-m} = J_{0n} \frac{1}{2^{2(M-1)}} \frac{(2M-1)!}{(M+m-1)!(M-m)!}$$

where M is the number of m terms used in a particular case and J_{0n} is a constant.

We note that in this particular case

$$\frac{1}{2^{2(M-1)}}\sum_{m=1}^{M}\frac{(2M-1)!}{(M+m-1)!(M-m)!}\cos\omega_m z = \cos^{2M-1}\omega_1 z$$
$$\frac{1}{2^{2(M-1)}}\sum_{m=1}^{M}\frac{(2M-1)!}{(M+m-1)!(M-m)!} = 1$$

and therefore the current density is,

$$\sum_{m=1} J_{0n,m}\cos\omega_m z = J_{0n}\sum_{m=1}^{M}\frac{1}{2^{2(M-1)}}\frac{(2M-1)!}{(M+m-1)!(M-m)!}\cos\omega_m z = J_{0n}\cos^{2M-1}\omega_1 z$$

With that the flow lines reduce to the simple expression,

$$\boxed{\sin n\theta = \frac{1}{\cos^{2M-1}\omega_1 z}\sin n\theta_0}$$

and the components of current density are,

$$\boxed{\vec{J}(\theta,z)|_{r=R} = J_{0n}\left\{\begin{array}{l} 0\hat{e}_r \\ \frac{\omega_1 R}{n}(2M-1)\cos^{2(M-1)}\omega_1 z\sin\omega_1 z\sin n\theta\hat{e}_\theta \\ \cos^{2M-1}\omega_1 z\cos n\theta\hat{e}_z \end{array}\right\}}$$

On the following page we demonstrate 3 cases of flow lines for M=1, M=2 and M=3 used for a quad n=2.

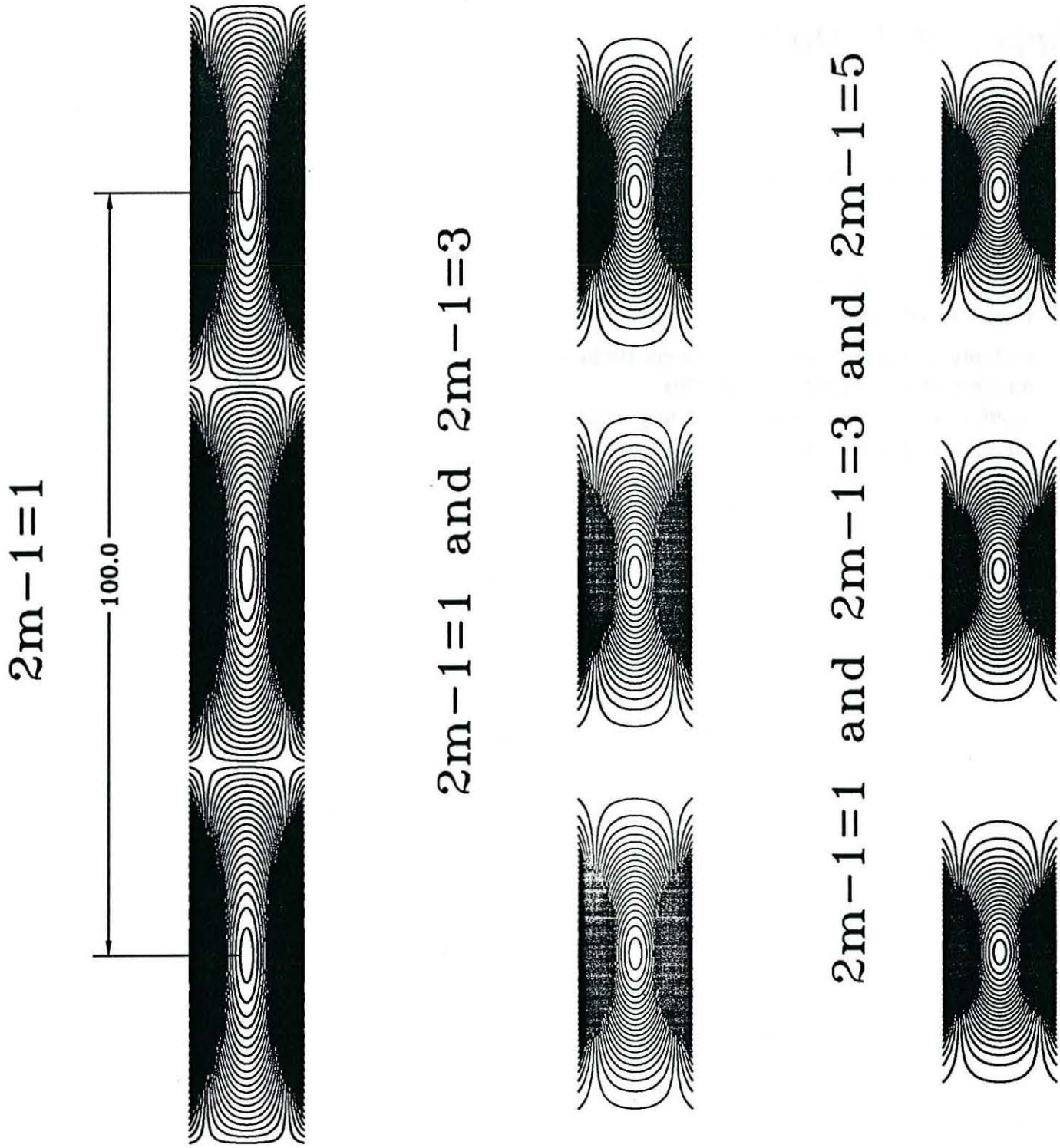

Figure 5 View of a full period array of a quad with m=1 only, a summation over m=1,2 , and m=1,2,3. These special cases revel the reduction in crowding between magnets at the expense of an increased non-linear field.

Appendix A Units

In MKS units :

I : amp
B : Tesla (or Weber/meter^2)
L : meter
F : newton

$\frac{N}{m} = T \cdot A \quad , \quad \frac{J}{m^3} = \frac{T \cdot A}{m}$

Useful conversions :

multiply (N/m) by 5.710174e-3 to get (lb/inch)
multiply (N) by 0.22481 to get (lb)
multiply (N/m^2) by 1.450384e—4 to get (psi)
multiply (psi) by 6.8947 to get (MPascal)