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P. Hughett

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Paul Hughett

Life Sciences Division
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

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A Discrete-time Linear Shift-invariant System Not Representable as a Convolution

Paul Hughett

Life Sciences Division, Lawrence Berkeley National Laboratory, One Cyclotron Road, Berkeley, CA 94720

A counterexample is presented to the claim that every discrete-time linear shift-invariant system can be represented as a convolution. We will say that a linear operator is σ-linear if superposition holds for an infinite number of terms; σ-linearity is stronger than linearity. Every σ-linear shift-invariant operator can be represented as a convolution.

1 Introduction

A widely used result in discrete-time linear system theory is that the action of every discrete-time linear shift-invariant system can be represented as the convolution of the input signal with the impulse response of the system. This result does not appear to have a commonly accepted name, and we shall refer to it as the representation theorem. This result appears, for example, in the textbooks by Oppenheim and Schafer[1], Rabiner and Gold[2], and Jackson[3]. This paper gives a counterexample to demonstrate that this representation is not always valid. The usual proof of the result makes the assumption that, if superposition holds for a finite number of terms, it necessarily holds for an infinite number of terms as well; in fact, infinite superposition is a stronger property which does not necessarily follow from linearity. Since this stronger property extends linearity from finite sums to countably infinite sums, we will call it σ-linearity by analogy to the way that a σ-field allows countably infinite sums where an ordinary field allows only finite sums.

Kailath[4] has pointed out that linearity does not imply (in our terms) σ-linearity but did not explore the consequences of this fact, perhaps because the only examples known to him were rather pathological. Steiglitz[5] has observed that σ-linearity is required for the usual proof of the representation theorem but again did not examine the consequences. Sandberg and Ball[6,7] have investigated the conditions under which a continuous-time system can be represented as a convolution.

A correct theorem can be obtained in two different ways. The first is to weaken its conclusion to that which can validly be proven from ordinary linearity; the weaker conclusion is that representability holds for every sequence with only a finite number of non-zero elements. The second is to strengthen the hypothesis of the theory by requiring that the LSI system be σ-linear rather than simply linear; then the system can be represented as a convolution for every input sequence. This paper does not attempt to determine criteria for the σ-linearity of an LSI system, and more research will be needed to do so.

2 Notation and Previous Results

We denote by $\mathbb{C}^{\mathbb{Z}}$ the set of all functions from $\mathbb{Z}$ to $\mathbb{C}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{C}$ is the set of complex numbers. That is, $\mathbb{C}^{\mathbb{Z}}$ is the set of all two-sided infinite sequences of complex numbers. Members of $\mathbb{C}^{\mathbb{Z}}$ will be denoted by lower case italic letters. The notation $x[n]$ will be used to denote the nth element in the sequence x; the notation x_n the nth of a set of sequences. Unless otherwise specified, the indices j, k, and n range over all integer values; this applies in particular to summations such as $\sum_n x[n]$. The unit sample sequence δ is the sequence defined by $\delta[0] = 1$ and $\delta[n] = 0$ for all other n. The shifted unit sample sequence δ_n is the sequence defined by $\delta_n[k] = 1$ if $n = k$ and 0 overwise.

The convolution of two sequences x and y is the complex sequence $x * y$ with elements

$$(x * y)[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k] \tag{1}$$

provided that the sum converges for all values of n. Otherwise the convolution $x * y$ is undefined. This definition is stricter than the usual definition in that it requires that the sum converge for all output values.

The set $\mathbb{C}^{\mathbb{Z}}$ is a vector space over the complex numbers. Let $\mathcal{A}$ be a linear subspace of $\mathbb{C}^{\mathbb{Z}}$. An operator F defined on $\mathcal{A}$ is a mapping $\mathsf{F} : \mathcal{A} \to \mathcal{A}$; note that this means that $\mathsf{F}x$ exists and is in $\mathcal{A}$ for every $x \in \mathcal{A}$. An operator F defined on $\mathcal{A}$ is linear if $\mathsf{F}(\alpha x + \beta y) = \alpha(\mathsf{F}x) + \beta(\mathsf{F}y)$ for all sequences $x, y \in \mathcal{A}$ and all (complex) scalars α, β. Linear operators will be denoted by sans serif letters. The shift operator z^p is defined by $(\mathsf{z}^p y)[n] = y[n-p]$ for any integer p. A linear operator F on $\mathcal{A}$ is shift-invariant if $\mathsf{z}^p\mathsf{F}x = \mathsf{F}\mathsf{z}^p x$ for every sequence $x \in \mathcal{A}$ and every integer p. For brevity, we will refer to a linear shift-invariant operator on $\mathbb{C}^{\mathbb{Z}}$ as an LSI operator. The impulse response of an LSI operator F is the sequence $f = \mathsf{F}\delta$; we will assume that $\delta \in \mathcal{A}$ and so $\mathsf{F}\delta$ is always defined.

An infinite sum $\sum_{n=-\infty}^{\infty} x_n$ of sequences $x_n \in \mathcal{A}$ converges pointwise to some $x \in \mathbb{C}^{\mathbb{Z}}$ if the sum $\sum_{n=-\infty}^{\infty} x_n[k]$ converges for every k. More explicitly, for every integer k and every $\epsilon > 0$, there exists an integer N such that $|x[k] - \sum_{n=-N}^{N} x_n[k]| < \epsilon$. The infinite sum $\sum_{n=-\infty}^{\infty} x_n$ converges uniformly to x if, for every $\epsilon > 0$, there exists an integer N such that $|x[k] - \sum_{n=-N}^{N} x_n[k]| < \epsilon$ for every k. Uniform convergence is stronger than pointwise convergence in that it requires that N be independent of the value of k.

3 A Counterexample

The representation theorem is the claim that every LSI operator can be represented as a convolution with its impulse response. That is, for every LSI operator F and every sequence $x \in \mathbb{C}^{\mathbb{Z}}$, $\mathsf{F}x = (\mathsf{F}\delta) * x$.

The representation theorem as just stated is false, and we now present an LSI operator which cannot be represented as a convolution. Let F be the operator defined by

$$(\mathsf{F}x)[n] = x[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[n+k] \tag{2}$$

for any sequence x such that the limit exists. We will call this the ideal DC-blocking filter, since what it does is to remove the DC or average component of the input signal without otherwise modifying the signal. To see that this is true, observe that the term involving the limit just computes the average value of the signal, which is then subtracted from the signal itself.

Now we claim that the operator F is linear and shift-invariant. To see that F is linear, choose any integer n and consider

$$\begin{aligned}
(\mathsf{F}(\alpha x + \beta y))[n] &= (\alpha x + \beta y)[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} (\alpha x + \beta y)[n+k] \\
&= (\alpha x[n] + \beta y[n]) \\
&\qquad - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} (\alpha x[n+k] + \beta y[n+k]) \\
&= \alpha \left(x[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[n+k] \right) \\
&\qquad + \beta \left(y[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} y[n+k] \right)
\end{aligned}$$

$$= \alpha(\mathsf{F}x)[n] + \beta(\mathsf{F}y)[n] \tag{3}$$

and it follows that $\mathsf{F}(\alpha x + \beta y) = \alpha \mathsf{F}x + \beta \mathsf{F}y$, or that F is linear. To see that F is shift-invariant, let z be the unit-shift operator and consider

$$\begin{aligned}
(\mathsf{F}(\mathsf{z}^p x))[n] &= (\mathsf{z}^p x)[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} (\mathsf{z}^p x)[n+k] \\
&= x[n-p] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[n+k-p] \\
&= (\mathsf{F}x)[n-p] = (\mathsf{z}^p(\mathsf{F}x))[n]
\end{aligned} \tag{4}$$

and it follows that $\mathsf{F}\mathsf{z}^p x = \mathsf{z}^p \mathsf{F}x$, or that F is shift-invariant.

The impulse response of F has elements

$$\begin{aligned}
f[n] &= (\mathsf{F}\delta)[n] = \delta[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \delta[n+k] = \delta[n] - 0 \\
&= \delta[n]
\end{aligned} \tag{5}$$

and so $f = \mathsf{F}\delta = \delta$.

Now let u be the unit step sequence defined by $u[n] = 0$ for $n < 0$ and $u[n] = 1$ for $n \geq 0$. Then $\mathsf{F}u$ has elements

$$\begin{aligned}
(\mathsf{F}u)[n] &= u[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} u[n+k] \\
&= u[n] - \lim_{N\to\infty} \frac{N+1}{2N+1} \\
&= u[n] - \tfrac{1}{2} \\
&= \begin{cases} -\frac{1}{2} & n < 0 \\ +\frac{1}{2} & \text{otherwise} \end{cases} .
\end{aligned} \tag{6}$$

However, $(\mathsf{F}\delta) * u = f * u = \delta * u = u \neq \mathsf{F}u$ and so the representation theorem is false for this LSI operator.

4 The False Representation Theorem

If the theorem is false, then there must be an error somewhere in its proof. Let us examine the proof more carefully.

Theorem 1 *(False) Every LSI operator* F *can be represented as a convolution with its impulse response. That is, for every sequence* x, $\mathsf{F}x = (\mathsf{F}\delta) * x$.

PROOF. Fix any sequence x and observe that $\sum_n x[n]\,\delta_n = x$. Then it follows by linearity that

$$\mathsf{F}x = \mathsf{F}\left(\sum_n x[n]\,\delta_n\right) = \sum_n x[n]\,(\mathsf{F}\delta_n) \quad . \tag{7}$$

Now let $f = \mathsf{F}\delta$ be the impulse response of F and observe that for any k, $(\mathsf{F}\delta_n)[k] = f[k-n]$. Then choose any k and compute

$$(\mathsf{F}x)[k] = \sum_n x[n] \cdot (\mathsf{F}\delta_n)[k] = \sum_n x[n]\,f[k-n] = (x * f)[k] \quad . \tag{8}$$

Since this equality holds for any k, we have shown that $\mathsf{F}x = (\mathsf{F}\delta) * x$. □

To see where the proof goes wrong, let's substitute the definition of the counterexample into equation (7) of the proof to obtain

$$\mathsf{F}\left(\sum_n x[n]\,\delta_n\right) = \sum_n x[n]\,\delta_n - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \sum_n x[n+k]\,\delta_{n+k} \tag{9}$$

which the proof claims is always equal to

$$\sum_n x[n]\,(\mathsf{F}\delta_n) = \sum_n x[n] \left(\delta_n[n] - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \delta_n[n+k]\right) \quad . \tag{10}$$

That is, it is necessary to interchange the order of the limit and the summation in n. This is always valid if the sum involves any finite number of terms and is valid for an infinite number of terms if the summation in n is uniformly convergent. But, for the counterexample, the infinite sum

$$\sum_n x[n+k]\,\delta_{n+k} = \sum_n u[n+k]\,\delta_{n+k} = \sum_{n=0}^{\infty} \delta_{n+k} \tag{11}$$

is not uniformly convergent, and so it is not valid to interchange the sum and limit.

In effect, the proof assumes that for any linear operator F, the equality

$$\mathsf{F}\left(\sum_n x[n]\,\delta_n\right) = \sum_n x[n]\,(\mathsf{F}\delta_n) \tag{12}$$

holds even if the summation involves an infinite number of terms. But as we have just seen, this is not always true. That is, linearity over an infinite number of terms is a stronger property than linearity over a finite number of terms. We will call this stronger property σ-linearity, and devote the rest of this paper to precisely defining σ-linearity and proving a correct version of the representation theorem.

5 Two Correct Representation Theorems

But before we investigate σ-linearity, there is another way to correct the theorem and that is to weaken the conclusion to one which is implied by linearity alone. To do this, we observe that the interchange of sum and limit is valid for any finite sum, or more generally, for an infinite sum containing only a finite number of non-zero terms.

Theorem 2 *Let F be a linear shift-invariant operator on $\mathbb{C}^{\mathbb{Z}}$. Then $\mathsf{F}x = (\mathsf{F}\delta) * x$ for every sequence x which contains only finitely many non-zero elements.*

PROOF. Fix any sequence x with finitely many non-zero elements and observe that $\sum_n x[n]\,\delta_n = x$. Since this sum contains only finitely many non-zero terms, it follows by linearity that

$$\mathsf{F}x = \mathsf{F}\left(\sum_n x[n]\,\delta_n\right) = \sum_n x[n]\,(\mathsf{F}\delta_n) \quad . \tag{13}$$

Now let $f = \mathsf{F}\delta$ be the impulse response of F and observe that for any k, $(\mathsf{F}\delta_n)[k] = f[k-n]$. Now fix k and compute

$$(\mathsf{F}x)[k] = \sum_n x[n]\cdot(\mathsf{F}\delta_n)[k] = \sum_n x[n]\,f[k-n] = (x * f)[k] \quad . \tag{14}$$

Since this equality holds for any k, we have the desired result that $\mathsf{F}x = (\mathsf{F}\delta)*x$.
□

The definition of σ-linearity involves a few subtleties that were absent in the definition of linearity. The first is that not every infinite sum converges, and so we cannot define σ-linearity to require superposition for every infinite sum, but only for convergent infinite sums. The second is that there are several possible definitions of convergence for infinite sums of functions; since pointwise convergence allows us to prove representability, that is what we will use. Finally, it appears, from some preliminary investigations into criteria for σ-linearity, that the more we restrict the set of sequences over which σ-linearity must hold, the fewer restrictions we must impose for an operator to be σ-linear; thus it is useful to define σ-linearity for an operator defined over some subspace $\mathcal{A}$ of $\mathbb{C}^{\mathbb{Z}}$. Given these considerations, we define σ-linearity as follows.

Definition 3 *An LSI operator* F *defined on* $\mathcal{A} \subset \mathbb{C}^{\mathbb{Z}}$ *is pointwise* σ*-linear if (1) for every complex number* α *and every sequence* $x \in \mathcal{A}$, $\mathsf{F}(\alpha x) = \alpha\,\mathsf{F}(x)$*; and (2) for every infinite sum* $\sum_n x_n$ *of sequences* $x_n \in \mathcal{A}$ *that converges pointwise to a sequence* $x \in \mathcal{A}$*, the series* $\sum_n \mathsf{F}x_n$ *converges pointwise to* $\mathsf{F}x$.

Note that a σ-linear operator is necessarily a linear operator, since all but a finite number of terms in the sum may be set to zero. We will use the abbreviation σ-LSI to mean pointwise σ-linear and shift-invariant.

As with σ-linearity, it is useful to define representability over subspaces of $\mathbb{C}^{\mathbb{Z}}$.

Definition 4 *An LSI operator* F *defined on* $\mathcal{A} \subset \mathbb{C}^{\mathbb{Z}}$ *is representable if* $\mathsf{F}x = (\mathsf{F}\delta) * x$ *for every* $x \in \mathcal{A}$.

Now we can state and prove another version of the representation theorem.

Theorem 5 *Suppose that an operator* F *defined on* $\mathcal{A} \subset \mathbb{C}^{\mathbb{Z}}$ *is shift-invariant and pointwise* σ*-linear. Then* F *is representable. That is,* $\mathsf{F}x = (\mathsf{F}\delta) * x$ *for every* $x \in \mathcal{A}$.

PROOF. Fix any sequence $x \in \mathcal{A}$ and observe that $\sum_n x[n]\,\delta_n$ converges pointwise to x. Then it follows by σ-linearity that

$$\mathsf{F}x = \mathsf{F}\left(\sum_n x[n]\,\delta_n\right) = \sum_n x[n]\,(\mathsf{F}\delta_n) \tag{15}$$

where the last sum converges pointwise. Now define $f = \mathsf{F}\delta$ and observe that for any k, $(\mathsf{F}\delta_n)[k] = f[k-n]$. Now fix k and compute

$$(\mathsf{F}x)[k] = \sum_n x[n] \cdot (\mathsf{F}\delta_n)[k] = \sum_n x[n]\,f[k-n] = (x * f)[k] \quad . \tag{16}$$

Since this equality holds for any k, we have that $\mathsf{F}x = (\mathsf{F}\delta) * x$. $\square$

6 Conclusions

We have seen that not every LSI operator can be represented as a convolution. This means that any result in discrete-time linear systems theory that relies on this representation is valid only for the subset of LSI operators that can be so represented. More work is needed to determine which LSI operators are representable. Since σ-linearity implies representability, one approach is to investigate criteria for σ-linearity in various sequence spaces. For example, one preliminary result[8] for operators defined on the set of bounded sequences (ℓ_∞) is that an LSI operator is representable if and only if it satisfies a certain locality property. More precisely, an LSI operator F defined on ℓ_∞ is semilocal if for every $\epsilon > 0$ there exists an integer N such that $|(\mathsf{F}y)[0]| < \epsilon \, ||x||_\infty$, where $y[n] = 0$ for all $|n| < N$ and $y[n] = x[n]$ otherwise. Then F is representable if and only if it is semilocal; furthermore, either of these implies (but not conversely) that $\mathsf{F}\delta$ is absolutely summable.

The representation theorem is used to prove several other results in linear system theory, including the claims that: (1) The transfer function of an LSI operator is the Fourier transform of its impulse response. (2) An LSI operator is stable if and only if its impulse response is absolutely summable. (3) The output of a cascade of LSI operators is independent of the order of the operators. Unless these results can be proven without using the representation theorem, their application must be restricted to representable or σ-linear operators.

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