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Why Turbulence Theories Cannot Be Like the Kinetic Theory of Gases

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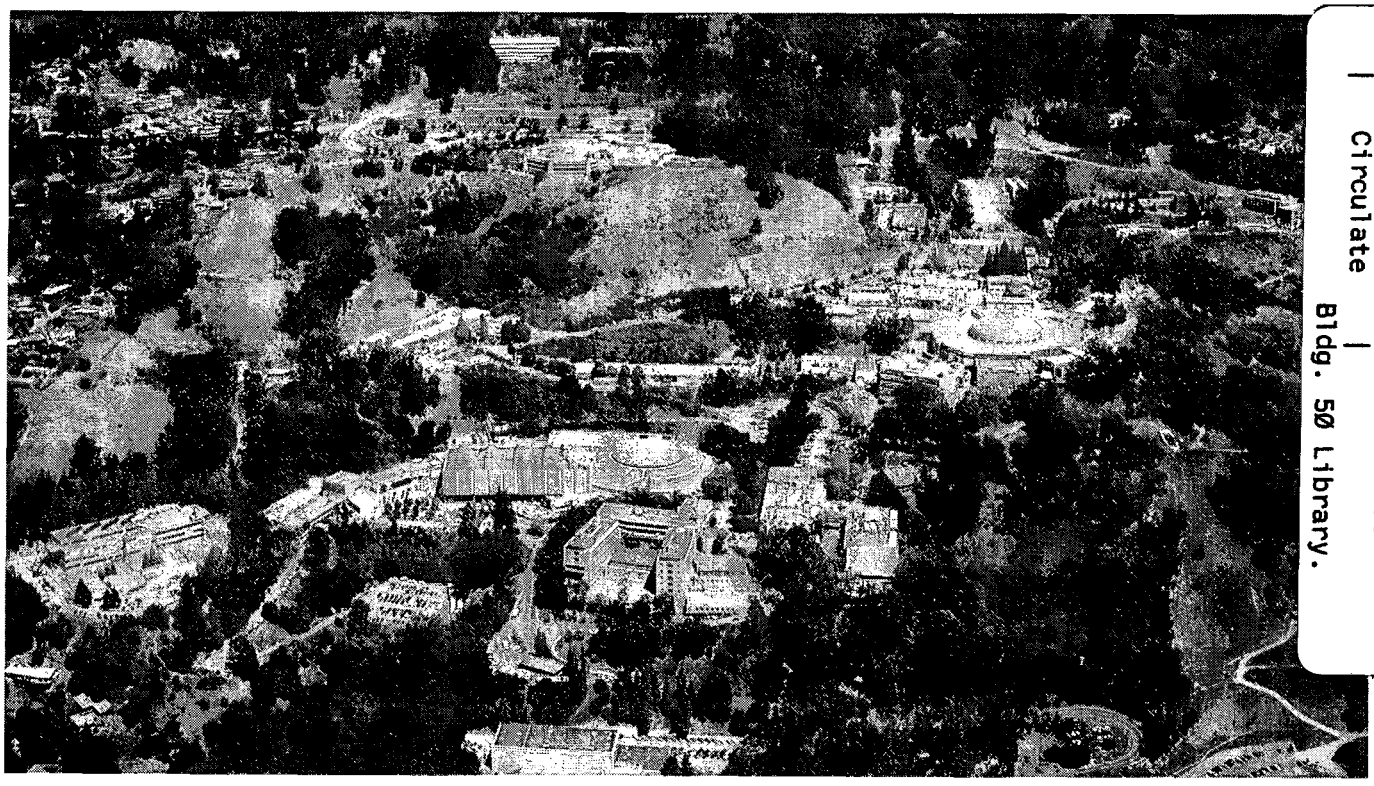

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WHY TURBULENCE THEORIES CANNOT BE LIKE THE KINETIC THEORY OF GASES*

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ABSTRACT

Recent work on random solutions of the Euler and Navier-Stokes equations is summarized, with emphasis on field-theoretical ideas and linear response theory. Implications for practical turbulence modelling are sketched. The main theoretical conclusion is that turbulent states correspond to critical points in a certain phase diagram; this criticality, defined in the text, arises because one must assign probabilities to sets of flows, i.e., to continua, and it is absent in other well-known statistical theories in mechanics; it is the main reason for the qualitative difference between turbulence theory and other statistical theories in mechanics.

INTRODUCTION

You all know the story: Einstein dies and asks the angel Gabriel to explain the structure of the universe. Gabriel swiftly complies with a few equations. then Einstein asks about the structure of turbulence, and Gabriel answers "No one here knows."

A joke often reflects reality, but this one does not. A lot of deep insight about turbulence has accumulated in recent years; I would like to summarize some of this new knowledge from my own idiosyncratic point of view. By "turbulence" I mean "random behavior in fluids and plasmas" (and in this talk I will omit the plasmas). I shall also concentrate on fully-developed turbulence.

The goal of turbulence theory is two-fold: To gain a mathematical understanding of the nature of random solutions of the Navier-Stokes and related equations, and to use this understanding in practical modelling. Good practical modelling is unlikely to come without solid understanding. The reason random solutions are appropriate is the chaotic nature of hydrodynamics; arbitrarily small variation in the data are amplified, no experimental apparatus is identical to any other, and the flow one sees depends on the specific experiment — the very definition of a random solution.

Fully developed turbulence has many degrees of freedom; the natural machinery for dealing with many degrees of freedom is field theory — the realm of functional integrals, gauge invariance, Feynman diagrams and renormalization. Most of us engineers dislike this realm, because it is unfamiliar, and because early messengers from the realm did not bring important news. I believe that the second reason is now obsolete, and therefore the first must be overcome. Field theory offers the right tools for dealing with random functions.

It is natural to focus first on stationary random solutions of the Euler or Navier-Stokes equations, just as it is natural in the kinetic theory of gases to focus first on stationary distributions of the momenta and positions of particles. A stationary random solution in turbulence is the obvious generalization of a statistically steady state in a system of N particles; it is a collection ("space") of functions, in which one has identified subsets, each with an attached probability that a function in the space belong to the subset (see e.g. Gelfand & Vilenkin (1964)). As the functions in the space evolve according to the equations of motion, they move in and out of the various subsets, in such a way that the probability that a function belong to a given subset does not change; those that leave the subset are replaced by others. One difficulty present in turbulence that is absent in the case of N particles is the need for a correct characterization of the space of solutions; if one assigns positive probabilities to functions that cannot appear in natural flows (for example, to collections of wild distributions), the effort is wasted.

Stationary solutions are important because they may attract others — i.e., one may be able to replace long-time averages by averages over a stationary statistical solution (i.e., over the appropriate space of solutions with its time-independent probabilities), and also because non-stationary solutions depend on initial and boundary conditions and few general conclusions can be reached about them. It is understood that stationary solutions may provide only a partial description of real solutions; in turbulence, this partial description usually applies to the small scales.

Stationary flows come in two flavors: equilibrium and non-equilibrium. An equilibrium is what happens after a long-time in an isolated system or a portion of an isolated system. In an isolated system with only energy as an invariant, equilibrium can be characterized by a "micro-canonical" distribution, i.e., equipartition over the set of appropriate equal-energy systems; it can also be characterized by the Gibbs probability distribution, in which the probability of a collection of states is the integral of $Z^{-1}e^{-\beta H}$, with β the "inverse temperature", H the Hamiltonian (or a suitable generalization), and Z, the "partition function", is a normalizing factor, see for example Huang (1963). One often thinks of the Gibbs distribution as describing a system in contact with a "heat bath", but it is also possible to identify the heat bath with the remainder of the system when one considers only a portion of it.

Non-equilibrium steady states are the analogs of what one obtains in kinetic theory when one considers, for

example, the distribution after a long time of velocities and momenta of gas particles between two walls at different temperatures. That distribution of momenta and locations is stationary but not Gibbsian. Unlike a Gibbsian equilibrium, it allows for the irreversible transport of mass, momentum and energy across the system.

The great discovery of Onsager, Callen, and Welton (see e.g. Evans & Morriss (1989)) is that in a system not too far from a Gibbsian equilibrium, non-equilibrium properties (e.g., transport coefficients) can be evaluated on the basis of equilibrium properties. An example is heat capacity, which is perfectly well defined at equilibrium, but measures the response of the system to outside (i.e., non-equilibrium) perturbations. (For an introduction to such "fluctuation-dissipation" theorems, see e.g. Chandler (1987), Forster (1975)). Most of the theory of non-equilibrium processes deals with systems not far from equilibrium. Clearly, stationary turbulence is not in Gibbsian equilibrium, in particular because it features an irreversible energy transfer from large to small scales. The interesting question is: Can turbulence be viewed as a small perturbation of a suitable Gibbsian equilibrium? The key word here is "suitable". The usual answer is "no", but the answer here will be "yes".

Note that the temperature, energy, entropy, etc., to be discussed, refer to the properties of the macroscopic solutions of the Navier-Stokes or Euler equation and are not necessarily related to the temperature, etc., of the molecular motion of the particles that make up the fluid. It is easiest to visualize the difference for incompressible flow, where the macroscopic and molecular degrees of freedom are uncoupled (see e.g. Chorin (1994)).

THE HOPF EQUILIBRIUM, THE KOLMOGOROV SPECTRUM, DIAGRAMMATIC EXPANSIONS

There are two reasons why one usually thinks of turbulence as being far from equilibrium: the identification of equilibrium with the Hopf equilibrium and the historical interpretation of the Kolmogorov spectrum.

In 1952 Hopf and others (Hopf (1952), Lee (1952)) constructed an "equilibrium" for incompressible fluid flow based on properties of Fourier expansions. To save writing, I will present a one-dimensional version of their development. Consider the model equation $\partial_t u + \partial_x(u^2) = 0$, $(\partial_t = \frac{\partial}{\partial t}$, etc.), where u is periodic with period 1. Expand u in Fourier series: $u = \sum \hat{u}_k e^{ikx}$; $\hat{u}_k = \hat{u}_k(t)$ satisfies

$$\frac{d}{dt}\hat{u}_k + ik\sum_{k'} \hat{u}_{k'}\hat{u}_{k-k'} = 0. \tag{1}$$

Assume $\hat{u}_0 = \int_0^1 u dx = 0$, and $\hat{u}_k = 0$ for $|k| \geq K$, where K is a cut-off. One can readily check that $E = \frac{1}{2}\sum |\hat{u}_k|^2$ is invariant under (1). One can further check that the uniform distribution of the set of $\hat{u}_k$'s on the sphere $E =$ constant is also invariant under (1). This distribution can be viewed as a microcanonical distribution. The formal limit $K \to \infty$ produces a probability measure on a function space. Completely analogous constructions can be carried out for the two- and three-dimensional incompressible Euler and even Navier-Stokes equations.

The result is a legitimate probability distribution on a space of functions, which is formally invariant under Euler flow ("formally" means that all questions of existence and convergence are disregarded). A typical "flow" in this collection of flows is a wild distribution, nowhere differentiable. The average energy at a point is infinite. Even more disturbing from the point of view of statistical mechanics, the truncated systems do not have the same constants of motion as the original differential equations. The natural reaction is: If this is equilibrium, real flow must be far from it.

Another source of the belief that turbulence is far from equilibrium is the usual interpretation of Kolmogorov's law; that law states that in the inertial range of scales, across which energy "cascades" from the stirring scales to the dissipation scales, the energy spectrum $E(k)$, i.e., the energy E per wave number k, has the form $E(k) = C\epsilon^{2/3}k^{-5/3}$, where C is an absolute constant and ϵ is the rate of energy dissipation. Neither the idea of a cascade nor the dimensional analysis that leads to this law prejudges the issue of distance from equilibrium. However, the presence of ϵ in the spectral law creates the impression that it is the dissipation that creates the law. An alternate interpretation can be produced, according to which the amount of energy dissipated depends on the amount of energy present, i.e., $\epsilon = (E(k))^{3/2}\, k^{5/2}$, when $E(k)$ may be determined by equilibrium considerations. Examples from polymer theory with power laws similar to Kolmogorov's and where the rate of energy dissipation enters the spectrum in a way similar to what has just been suggested are offered in Chorin (1996b).

Be that as it may, the idea that irreversibility dominates the small scales of turbulence leads naturally to a particular formalism (see e.g. McComb (1989), Lesieur (1990)). The dominant effect is assumed to be the provision of energy at large scales and its removal by viscosity at small scales. Both can be represented by a linear Stokes equation with forcing, which can readily be solved. The nonlinear terms in the Navier-Stokes equations can then be represented as a perturbation expansion ordered by the Reynolds number R. The various terms in this expansion can be represented by Feynman diagrams, and the panoply of perturbative field theory can be used in the attempt to extract useful information. This is an awesome and uncompleted task, as the jump from $R = 0$ to $R = \infty$ is large, and it would be desirable to avoid it by constructing perturbation expansions on other premises.

ALTERNATE EQUILIBRIA IN HYDRODYNAMICS

We now set out to look for more reasonable equilibria for the Euler equations, in the hope that turbulence can be found in their vicinity. Here too we start with a discretization of the equations of motion and plan to take an appropriate limit at the end.

A general procedure for doing so would be as follows: Assume the turbulence lives in a finite volume V; divide V into small pieces of side h and volume h^3; construct a finite number of variables by integrating appropriate continuum variables, for example, the components of the vorticity vector ξ, over the small volumes. (The condition div $\xi = 0$ must be enforced, and there is a machinery for doing that.) The energy E discretizes into a sum E_h over the boxes, and for each h, one can construct an equilibrium Gibbs distribution $Z_h^{-1}\exp(-\beta E_h)$, where β is an "inverse temperature". The question is: what hap-

pens to these equilibria as $h \to 0$?

First note that the question can be asked and answered for the discretized spectral equilibria of the previous section. Given a cut-off K, standard methods show that the temperature $T = \beta^{-1}$ is proportional, for constant energy E, to $E/(K^d)$, where d is the dimension of the space. Thus $T \to 0$ as one approaches the continuum limit.

For the systems discretized by chopping up the vorticity in physical space several things can happen. In two dimensions generally T increases as h decreases, and then T goes beyond into the "negative" (trans-infinite region). (see Chorin (1994, 1996c)). In three space dimensions a richer variety of behaviors may appear.

A reminder of some properties of phase transitions is needed here. The same collection of particles may exist in several phases, for example, H_20 can make up ice, water, or vapor. At 0°C water becomes ice, a "phase transition" of "first order", i.e., one in which the usual thermodynamic functions exhibit a discontinuity. "Higher order" or "critical" phase transitions are less dramatic; an example is the transition of ^{4}He from the normal to the superfluid phase. At a phase transition the correlation length of a physical system is infinite (or else the thermodynamic properties of the system are analytic in parameters such as T). Furthermore, at a critical phase transition a system is "scale invariant", which roughly means the following: If the system is discretized, or is already discrete to start with and its variables are collected into groups in a way that preserves energy, then the properties of the system are invariant under changes in the scale of the discretization or of the grouping. The relation between scale invariance and phase transition comes about because scale invariance can occur only when the correlation length is infinite. If the correlation length is finite, a change of discretization changes the correlation length (for example, if a length that characterizes the discretization is doubled, the correlation length is halved) and thus the system changes.

The question whether alternate equilibria in turbulence can be found now becomes: Does the family of equilibria with parameters β and h have multiple phases with a curve separating them in the (β, h) plane? If yes, the intersection β^* of this phase transition line with the $h = 0$ axis is our candidate for a "reasonable" value of β, and the corresponding equilibrium is our thermal equilibrium.

Note that this is where the fact that we are looking for probability distributions over continuous flows impinges on the analysis. For a discrete collection of particles, invariance under a change of discretization is usually not a relevant consideration; here, however, we have to make sure that our systems have a meaningful continuum limit, and this forces us to consider systems invariant under a refinement of the discretization, and thus forces us towards phase transitions. Phase transitions and critical phenomena have a number of unusual properties that make the use of some standard approximations difficult.

A similar problem has been investigated for superfluid and superconductor vortex systems, where a phase transition is well-known and reasonably well-understood. A heuristic analysis that maps superfluid results onto classical fluid results has been carried out, and will be discussed below by a method that can be generalized to non-equilibrium conditions as long as one does not depart too much from equilibrium. The superfluid vortex system and the Euler/Navier-Stokes fluid systems are not identical (see e.g. Zhou (1996)) and the applicability of the analysis to fluid mechanics is not yet an established fact. However, if the analysis is applicable, certain well-known properties of turbulence are immediately explained, for example, the failure of moment closures — moment expansions always fail near critical points; the feasibility of large-eddy simulation — in other fields, procedures similar to large-eddy simulation require in general a change in the equations for large-scale quantities, except near critical points.

A SIMPLIFIED EQUILIBRIUM MODEL

To give some feeling for the alternate equilibria whose existence has just been postulated, we simplify the description of three-dimensional flow and assume that it consists of a sparse collection of circular vortex loops. This is a far-reaching simplification, but it it leads to relatively simple models. We shall assign probabilities to various arrangements of vortex loops, and the consider what happens when the number of loops increases. The significant parameters of the problem are $\beta = 1/T$ and μ, the "chemical potential" that measures the energy per unit length of a vortex. The chemical potential increases as a vortex is stretched and becomes thinner. Note that this description makes short-shrift of the detailed structure of vortex filaments, though recent work has shown that this structure plays a key role in hydrodynamics. The analysis proceeds via a "dielectric" formalism (Williams (1987)) patterned after work on two-dimensional conductor/insulator transitions (see e.g. Itzykson & Drouffe (1989)). This formalism is not unique (Chorin & Hald, (1996)), and we pick its easiest version (Chorin & Hald, (1995)). The overall strategy described in the previous section will be maintained.

Here I will launch into a technical discussion, so that the paper is not merely a collection of generalities. The gist of this discussion is that fairly standard manipulations that make use of the well-known magnetostatics/hydrodynamics analogy (Chorin (1994)), make possible the construction of appropriate probabilities. They are built up, in a certain approximation, by considering a vortex loop in gas of vortex loops, and making the interaction of the loop and its background self-consistent.

Suppose for a moment that the temperature T of the system of vortex loops is small; there will be very few loops in the system and the impulse they carry will be small. The impulse of a vortex loop is the integral $\frac{1}{2}\kappa \int_{\text{loop}} \mathbf{x} \times d\mathbf{s}$, where κ is the circulation in the loop and $d\mathbf{s}$ is an element of arc length; if the loop is planar, the impulse reduces to κA, where A is the area spanned by the loop (See e.g Chorin (1994)). As the temperature increases, more and larger loops appear in the system. The growth in the number of loops and in the size of the loops are related: If one takes a large loop and places inside it a smaller loop with opposite orientation, the energy of the combined configuration is reduced and its appearance is more likely (this is "polarization"); thus a cloud of small loops allows large loops to form. Eventually, it becomes possible for an infinite loop to form. The result is a phase transition in the vortex system. In the theory of superfluids, this phase transition corresponds to the transition from a superfluid to a normal fluid; we shall argue below that this is also the attract-

ing equilibrium for a classical fluid (i.e., for the set of "excitations", or modes of motion, that make up turbulence in the usual type of fluid). Roughly speaking, classical turbulent systems and superfluid systems are on opposite sides of the transition.

To characterize the phase transition quantitatively, consider a single vortex loop; assume that all the other loops create a polarizable background that modifies the energy of the loop, and look for the range of the parameters β, μ for which this picture is self-consistent. The boundary of that region will be the phase transition line.

Suppose a velocity $\mathbf{u}$ is imposed on the cloud of vortex loops. The loops will orient themselves so as to oppose that velocity. The reduction in energy due to the presence of an impulse $\mathbf{m}$ is $\frac{1}{2}\mathbf{m}\cdot\mathbf{u}$. The average polarizability of the loop, i.e., the average value of $\mathbf{m}\cdot\mathbf{u}$ divided by $u = |\mathbf{u}|$ is $\frac{1}{12}\beta m^2$, where $m = |\mathbf{m}|$. This calculation can be found e.g in Chorin & Hald (1995): One averages over all solid angles, weighing each by the appropriate Gibbs factor which favors lower energies. We already know that $m = \kappa\pi r^2$, where r is the radius of the loop; thus polarizability is a function of r. Next one has to find the density of loops of radius r. We view the number of loops as variable, and it is the grand-canonical ensemble that is relevant (see Huang (1963)).

The grand-canonical partition function is an expansion in powers of the fugacity $y = \exp(-\beta\mu_{\text{loop}})$, where μ_{loop} is the energy needed to create a single loop of radius r. For an isolated circular loop this energy equals $\mu r \log r$, where μ is the energy per unit length of the vortex loop. The coefficient of y is the partition function for a one-loop system, the sum of possible states of that system per unit volume times their Gibbsian weights (Huang (1963)). If the fugacity is small enough one can be content with this single term, which is then the density of the the loops of radius r. (Note that the zero-order term in y does not contribute to the polarization).

To enumerate all states one needs an estimate of the smallest length scale in the problem. For a collection of thin circular filaments it is natural to take the small diameter δ of the filaments as this smallest length scale. In a unit cube there are δ^{-3} possible loop centers. All orientations of the loops are possible, albeit with different probabilities; there are $4\pi r^2 dr\delta^{-3}$ distinct orientations of a loop with radius between r and $r + dr$. Each of these has to be multiplied by the corresponding Gibbs factor $\exp(-\beta E/e)$, where E is the energy of interaction between the loop under discussion and all the others, and $e = e(r)$ is the dielectric "constant", which, in the absence of a scale separation between large and small loops, may well be a function of r. In fact, one must figure out how a loop of radius r is formed and write a history-dependent expression for the potential in the Gibbs factor, but we shall not need this degree of refinement: In a low-fugacity system, E is negligible. The dielectric constant is the sum of all these contributions as r ranges from δ to infinity. It is customary to introduce the function $K = K(r)$ by $\epsilon(r) = \beta/K(r)$. Note that the unknown e or$K$ appears in the exponential. We shall assume for simplicity that K is in fact a constant; the equation for K is nonlinear:

$$K^{-1} = \beta^{-1} + c_1 \int_\delta^\infty r^6 \exp(-Kc_2 r \log r))dr, \qquad (2)$$

where the constant c_1 can be evaluated from the preceding discussion, and is proportional to δ^{-6}. (More precisely, $c_1 = (4/3)\pi^4\kappa^2\delta^{-6}$; The powers of π come about as follows: Two from the formula for polarizability, one from the enumeration of states due the rotations of the loops, and one from the 4π in the relation between loops and the induced velocity.). The expression $c_2 r \log r$ is what we have called μ_{loop}, where c_2 is an appropriate constant. The estimation of c_2 involves some elaborate manipulations. The easiest way to find it is as follows: Assume the energy of a loop can be found as the product of an energy μ per unit length of the filament times the length; this requires dropping $\log r$ from the definition of the energy of a loop — a small error (In Chorin & Hald (1995) it is shown that this simplification is entirely legitimate for fractal loops, but we are not invoking a fractal loop model.). Once this simplification has been carried out, μ can be found from standard arguments about the energy associated with vortex.

Equation (2) can now be rewritten in the form

$$T = \beta^{-1} = K^{-1} - c_1 \int_\delta^\infty r^6 \exp(-K\mu r)dr. \qquad (3)$$

The right-hand side of (3) is a convex function of K with a single maximum T_0 at some K_0. For $T > T_0$ equation (3) cannot be satisfied with K real, and we are outside the region of validity of the "dielectric" approximation, i.e., we have crossed the phase transition line. As μ varies one obtains different values of $\beta_0 = 1/T_0$ which trace out the phase transition line, sketched for a particular choice of parameters in Figure 1. If one now allows vortex stretching to lengthen the vortex lines, or if one tries to approach classical hydrodynamics by lengthening and thickening the vortex lines, one approaches the phase transition lines from any reasonable starting point (motion marked on Figure 1). Both phenomena are special cases of the general fact, already noted above, that only "critical" states survive a continuum limit at a finite energy. The expectation is that the points on the phase transition line approximate possible turbulence states. We are of course dealing with an approximation since the equations of motion have not yet been taken into account.

LINEAR RESPONSE THEORY

So far, the equations of motion have not been taken into account, and it is too much to expect that they will produce the equilibria we have constructed without some discrepancy. We wish to use linear response theory to evaluate the modifications needed to get a possible asymptotic state of the Euler or Navier-Stokes equations; this should yield at the same time a model of the effect of small scales on the large scales in a numerical "large-eddy" calculation because this effect is presumably contained in the equations of motion; in performing this modification we are correcting for a mismatch (hopefully small) between the equilibrium we have found and the set of solutions of the Navier-Stokes or Euler equations. That such a mismatch exists is of course obvious because turbulence is dissipative. Our analysis is closely related to earlier work on superfluids (see for example Ambageokar et al. (1978)). We sketch its simplest forms.

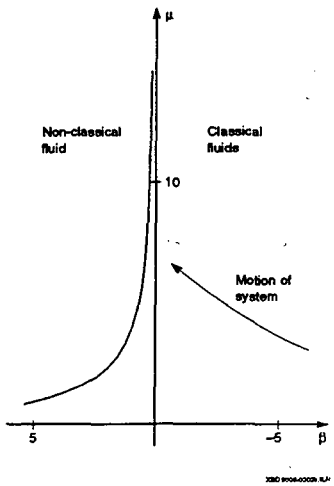

Figure 1: Phase diagram according to the approximate theory.

The zero-th order form of this correction is simple indeed: Assume the vortex system is very sparse, for example because a long time has passed since it was stirred and all small-scale structures have been lost. Then the parameter μ is large, and an easy calculation shows that the phase transition occurs at an infinite temperature T. At this temperature $\beta = 0$, there is no polarization, and small loops can be simply removed without harming the large scales. The removal modifies the equilibrium and creates an energy cascade from large to small scales, because the small scales are continuously replenished and must be continuously removed.

It would be desirable to do better. The difficulty lies in the fact that the equilibrium we are starting from is critical, and involves long range interactions and large fluctuations. Many of the standard perturbation schemes do not apply. One way of proceeding that may work in this situation is the following: Consider the distribution of vortex loops created by our equilibrium ensemble. Suppose we resolve all scales $\geq h$ in a flow, say by a finite-difference scheme. Consider the effect of the scales left unresolved on the scales that have been kept. these unresolved scales create a dielectric constant $e = e(h) = 1 + c_1 \int_\delta^h r^6 \exp(-K\mu r) dr$ (see equation (2)). If the computed velocity $\mathbf{u}$ in a cell of volume h^3 changes, equilibrium is not reached immediately; the time constant τ that characterizes the decay to equilibrium at scale r is $\tau = D/r^2$ (on dimensional grounds), where D is a diffusion coefficient that can be calculated from models of vortex interactions (Donnelly & Roberts (1971)). Thus the dielectric constant becomes time-dependent: $e = e(h,t) = 1 + C_1 \int_\delta^h r^6 \exp(-K\mu r)(1 - \exp(-t/\tau(r)) dr$. The simplest approximation is $D =$ constant. The complex part of the Fourier transform $\hat{e}(h,t)$ represents energy dissipation by the small scale equilibrium. As a result, the vorticity has two parts, one visible on the grid and one on subgrid scales; both affect the velocity and the second has a built-in delay that causes energy loss. An expansion of the delay function in a Taylor series produces an eddy viscosity model; the retention of the full delay function produces a rather simple integro-differential equation for the mean velocity. The whole machinery belongs to the class of approximation that go under the name of "linear response theory", and that are applicable in the immediate neighborhood of thermal equilibria. The obvious advantage of the approach is that it is based on an expansion in a parameter, the fugacity, which is typically quite small, rather than on an expansion in a Reynolds number R which is very large in all problems of interest. These ideas are now being tried on the computer (for example, in Kast & Chorin (1996)).

CONCLUSION

I have sketched a theory of turbulence, i.e., a set of a priori assumptions about the statistical solutions of the Navier-Stokes equations, of which the main one is that in three space dimensions one can find a Gibbsian ensemble of incompressible flows, with turbulence living in its vicinity. I think that this is a well-motivated theory, and likely to be true (in fact, a partial experimental verification has turned out to be possible, see Barenblatt & Chorin (1996)).

To make this theory into a practical computing tool one has to construct a practically implementable perturbation expansion around the equilibrium. While this is in principle a more tractable task than the more commonly attempted expansions that start from the Stokes equations, one must face the difficulties that arise from the critical nature of the proposed initial state, with its long range correlations. The tools proposed for performing this expansion come from condensed-matter physics, and there is good hope is that they will succeed. The obvious caveat is that many seemingly fool-proof theories of turbulence have failed in the past.

It is the criticality of the basic equilibrium that constitutes the major qualitative difference between turbulence theory as it has just been described and other statistical theories in fluid dynamics and kinetic theory. This criticality is closely connected to the fact that one must deal with collections of functions rather with ensembles of discrete particles.

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