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# Partition Functions and Invariant States in Two-Dimensional and Nearly Two-Dimensional Turbulence

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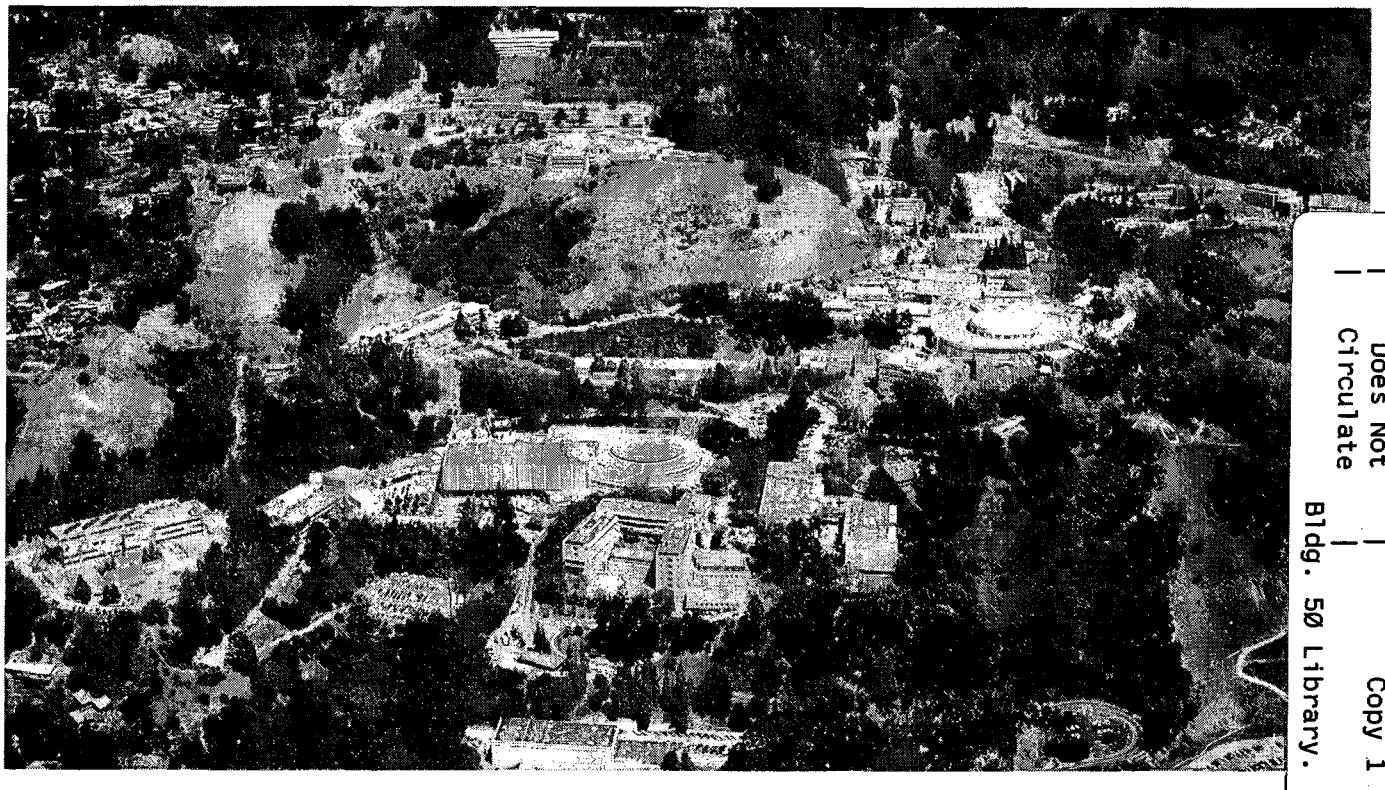

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# PARTITION FUNCTIONS AND INVARIANT STATES IN TWO-DIMENSIONAL AND NEARLY TWO-DIMENSIONAL TURBULENCE*

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## Abstract

The partition function is evaluated numerically for two-dimensional discretized incompressible fields with a hydrodynamical energy function and an infinite number of invariants; the behavior of the partition function is examined as the discretization is refined. The results are contrasted with those of recent mean-field theories, which are seen to be reasonable approximations only at moderate $|T|$, where $T$ is the temperature. At finite T the system has no phase transitions and no states invariant under refinement of the discretization. Invariant states may appear if a simple representation of vortex stretching is added to the system, in agreement with recent work on three-dimensional turbulence.

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**Introduction.** A large number of papers have examined the statistical mechanics of the two-dimensional incompressible Euler equations in a vortex representation[1--8]; it is well-known that negative temperature states play a major role in the analysis, and it is readily seen that at negative temperatures, $T < 0$, the partition function of the usual discrete vortex approximation diverges, as one can check by substituting $T < 0$ into its functional integral representation (see e.g. Ref. [9]). Most authors have therefore concentrated on the mean behavior of a vortex system, which is well-approximated by a mean-field theory, and have set aside questions regarding fluctuations in finite vortex systems.

The Euler equations in two dimensions admit as invariants the integrals $I_n = \int \xi^n dx$ where $\xi$ is the vorticity, $\xi = \text{curl } \mathbf{u}$, and $\mathbf{u}$ is the velocity, and $n$ is a positive integer. Miller et al.[10,11] noted that one can construct a family of discrete vorticity fields which share common values for a finite number of these invariants by simply permuting the vorticities in a finite number of boxes. The partition function then becomes a sum over the permutations, and remains well-defined for all $T$, negative as well as positive. Miller et al. then proceeded to approximate the free energy of their many-invariant vortex system with the help of an assumption of scale separation, and obtained a mean-field theory, identical to the mean-field theory with scale separation of Robert and Sommeria[12,13]. The assumption of separation of scales, discussed below, is essential in this analysis.

In the present paper the partition function , as formulated by Miller et al., will be evaluated numerically. Whether all the invariants are needed to describe the physics is an interesting question, and we shall discuss it briefly. The invariants are certainly needed at present to obtain a sensible partition function. We shall see that the Robert-Miller

mean-field theory is in general a poor approximation to the full theory, except at moderate and temperatures; this observation resolves certain paradoxes in the theory.

The main goal of the present paper is to set up a two-dimensional functional integral representation of random flow, to serve as an introduction and a contrast to a three-dimensional theory[6,14,15]. The observations that are important from this point of view are that, contrary to the conjecture in Ref. 11, the many-invariant system has no phase transition at a temperature $T \neq 0$, and no invariant measures in the limit of vanishing discretization.

The addition of vortex stretching to this two-dimensional system, though artificial, reverses these conclusions, in agreement with the conclusions in Refs. 6,14,15, according to which invariant states at non-zero temperature and vanishing discretization exist in three space dimensions and contain significant information about turbulence.

**The discrete many-invariant system.** Following Miller et al., we consider a vorticity field $\xi$ supported in a bounded domain $D$ (the unit square, for simplicity). The energy $U$ associated with this field is[16]:

$$U = \frac{1}{2}\int_D \mathbf{u}^2 d\mathbf{x}, \quad \mathbf{u} = (u_1(\mathbf{x}), u_2(\mathbf{x})), \quad \mathbf{x} = (x_1, x_2), \quad d\mathbf{x} = dx_1 dx_2,$$

where $\mathbf{u}$ is the velocity. Up to an immaterial additive constant $U$ can be rewritten as[5]:

$$U = -\frac{1}{4\pi}\int_D d\mathbf{x}\int_D d\mathbf{x}'\xi(\mathbf{x})G(\mathbf{x},\mathbf{x}')\xi(\mathbf{x}'), \tag{1}$$

where $G(\mathbf{x},\mathbf{x}')$ is the Green function appropriate to the boundary conditions on the boundary $\partial D$ of $D$; $G(\mathbf{x},\mathbf{x}') = -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}'| + \phi$, where $\phi$ is harmonic. In the calculations

below we set for convenience $\mathbf{u} \cdot \mathbf{n} = 0$ or $\partial D$, where $\mathbf{n}$ is the normal to $\partial D$; this choice is immaterial in the limit of vanishing discretization.

Divide $D$ into $N$ small squares of side $h$, $N = h^{-2}$, and approximate $U$ by $U_h$:

$$U_h = -\frac{1}{2} \sum_{ij} \sum_{i'j'} h^4 \xi_{ij} G_{ij,i'j'} \xi_{i'j'}, \tag{2}$$

where $G_{ij,i'j'}$ is the usual 5-point discretization of $G$, $G_{ij,i'j'} = G_h(ih, jh, i'h, j'h)$, and $\xi_{ij}$ is the constant value of $\xi$ on the $ij$ square.

It is shown in Ref. 11 that a family of discrete vorticity fields shares common values of $I_n = \int_D \xi^n dx$, $n = 1, \ldots, N$, if and only if they are obtained by permutations of the values of $\xi$ among the boxes. The partition function for a fixed $h$ is then

$$Z_h = \sum_{\text{permutations}} e^{-\beta U_h\{\xi_{ij}\}}, \tag{3}$$

where $U_h$ is given by (2). This sum is the appropriate discretization of the functional integral representation of the partition function. As $h \to 0$, $N$ and the number of invariants tend to infinity. The sum $Z_h$ can be evaluated by straightforward Metropolis sampling[11]. The only remark worth making is that it is more efficient to evaluate and store $G_h$ once and for all, rather than evaluate the energy by a Laplace inversion after each Metropolis move. The Metropolis algorithm converges very fast, and the statistical error in the calculations below is negligible.

Miller et al. as well as Robert and Sommeria assumed a separation of scales, according to which the energy $E = \langle U \rangle$ is determined by the large scales of $\xi$ and the entropy $S$ by the small scales (the brackets denote an ensemble average). All vorticity configurations are then equally likely in the small, and $S$ is readily found. In particular, $S$ is proportional

to $N$ for large $N$. Then, if one sets $T = h^2\bar{T}$, where $\bar{T}$ is independent of $h$, one finds $Z_h = \exp(-N\bar{\beta}F)$, where $\bar{\beta} = 1/\bar{T}$ and $F$ is the free energy $F = E - TS = E - \bar{T}(S/N)$; one finds the main contribution to $Z_h$ by maximizing $F$ for $T < 0$ and minimizing $F$ for $T > 0$. The curves $E =$ constant in the $(\beta, h^2)$ plane are the curves $\beta h^2 =$ constant.

As has been pointed out by several people, e.g. by Marcus[17], Turkington[18], this mean-field theory contradicts well-established facts about two-dimensional vortex systems, for example the existence of non-trivial small-scale vortex-vortex correlations in coherent structures[4]. Furthermore, in the presence of many invariants, $E$ has a well-defined maximum $E_{\max}$; for $\beta < 0$, $|\beta|$ large enough, that maximum is approached and $S \to 0$; energies larger than $E_{\max}$ cannot be accommodated by the system.

It may be worth noting that Miller et al. gave one derivation of their mean-field theory that seemed to be independent of separation of scales, but this independence was probably illusory; the derivation included an unjustified passage from a weak convergence of discrete Laplacians to a strong convergence, which needs something like separation of scales to be legitimate. The necessity of the separation of scales assumption has been stressed by Turkington and his coworkers[19,20]. It should also be noted that the main conclusions of the Miller-Robert theory: Validity of a mean-field approximation, constancy of $E$ or lines $\beta h^2 =$ constant, and the specific evaluation of $S$, are logically independent of each other; some may be true while others are false.

**Numerical results and discussion.** Numerical experiments were made with constants $I_n$ for which $\xi_{ij} = 1$ for half the cells and $\xi_{ij} = -1$ for the other half. The maximum value of $E$ is approximately 7.34, obtained when the positive vorticity fills half of the square and

negative vorticity the other half. The computational results are displayed in Figures 1–4/.

In Figure 1 we display the curves of constant mean energy $E$ in the $(\beta, h^2)$ plane for several values of $E$. As usual, $\beta < 0$ is to the right of $\beta > 0$ because negative temperatures are "hotter" than positive temperatures[6]. The system heats up as one goes from left to right in the figure. The main observation is that the constant $E$ curves never intersect the $h = 0$ line. More observations about Figure 1 will come in the next section.

The behavior of the $E$ = constant curves is better seen in Figure 2, where the horizontal axis corresponds to the variable $\beta \times h^2$. According to the mean-field theory the curves should be asymptotically (as $h \to 0$) vertical in these variables. They are indeed for moderate values of $E$, and hence asymptotically for low values of $E$, because the curves all move to the right as $h \to 0$ and do not intersect. As $E$ grows, a departure from the Miller-Robert mean-field scaling can be seen. For $E = E_{\max} \sim 7.34$ the curve is not defined and for $E > E_{\max}$ the curves do not exist.

Note that $E$ = constant on lines $\beta h^2$ = constant in the older few-constant theory, and that there is no bound on $E$ in the few-constant theory. It is thus plausible that the many-constant and the few-constant theories agree for smaller values of $E$ and not too far to the right on the $\beta$ axis; this has indeed been shown to be the case[21].

What is happening to the entropy $S$ and the mean energy $E$ is shown in Figures 3 and 4. The entropy $S_{2\times 2}$ in a $2 \times 2$ box can be readily calculated where $\xi$ takes on only two values[22], and this is why the present special case was chosen; if we identify two states that differ by an exchange of vorticity between boxes with the same vorticity, as was done in Ref. 11, then a $2 \times 2$ box has $2^4 = 16$ possible states; if they are all equally likely, the

entropy is $S_{2\times 2} = \log 16 \cong 2.77$. The real entropy is $\sum_1^{16} P_i \ln P_i$, where $P_i$ is the entropy of the $i$-th state which can be estimated from its frequency. In Figure 3 we display $S_{2\times 2}$ as a function of $h$ for various values of $E$; at the larger values of $E$ the equipartition assumed by Miller and Robert does not materialize, at least for the values of $h$ within numerical reach. For large enough values of $E$ the equipartition must fail. Note that $S_{2\times 2} = 0$ at $E \cong 7.34$.

In Figure 4 we display the ratio of the energy of the average configuration of the $\xi$'s to the average energy $E$, as a function of $h$. We have taken the precaution to rotate each individual solution so that its center of gravity (assimilating $\xi$ to mass) lies in one specific fourth of the square, so as not to produce a zero mean solution as a result of the invariance of the system under 90° rotations. If the Miller-Robert mean-field theory holds, the ratio should tend to 1 as $h \to 0$. The slow convergence of the ratio in Figure 4 is surprising; it is most likely the result of the slow convergence of the system at small $E$ to its asymptotic mean-field state, which can also be deduced from Figures 1 and 2. However, in the range of $N$ within numerical reach, the eventual convergence cannot be demonstrated or disproved. We cannot exclude the possibility that the stationary mean-field state predicted by the theory is not in fact reached for small values of $E$, maybe because there are too many different ways of reaching a given low energy.

**The connection with turbulence theory.** From Figures 1,2, one can conclude that nothing dramatic happens as one crosses the $\beta = 0$ line. Furthermore, one can check that the system has no phase transitions at any $T$, positive or negative, contrary to the conjecture in Miller et al. Indeed, such a transition would be a variant of the Kosterlitz-

Thouless transition, which requires the number of vortex pieces to depend on the chemical potential (here, roughly on $h$); this is not possible in the presence of all the invariants. Since the crossing of the line $h = 0$ by a curve $E =$ constant is connected with a phase transition[6], the picture is self-consistent.

The conjecture of Refs. 6,14,15 is that in three space dimensions, in the presence of vortex stretching, the lines $E =$ constant do cross the line $h = 0$, at a point determined by an analogue of the $\lambda$ vortex phase transition line, producing an invariant state for the vortex continuum that provides a rough description of turbulence. To exhibit this behavior, which contrasts with what is shown in Figure 1, consider the following crude representation of vortex stretching: Assume that vortex tubes are stretched in a direction perpendicular to the plane in which our calculation is performed, with the volume of the tubes conserved. Then, as $h \to 0$, a fraction $(1 - Q)$ of the cells will have $\xi_{ij} = 0$, with the other $Q$ having $\xi_{ij} = \pm 1/Q$ in equal numbers. Note that $\int \xi d\mathbf{x}$ is conserved while the other constants are not. In the calculation shown in Figure 5, $Q = \left(\frac{N}{144}\right)^{-\alpha}$ with $\alpha = 4/3$. The lines $E =$ constant then tilt towards the vertical (note the change of horizontal scale). One can obviously vary $\alpha$ to sharpen this effect, but there is no reason to do so here.

Note that as $Q$ increases one obtains a space of functions with ever more isolated and sharper vortices, which remain invariant for ever smaller $h$. One can then interpret the differential grinding away of background vorticity in two dimensions by viscosity and other non-equilibrium effect[21,23] as a relaxation towards the kind of equilibrium that prevails in three-dimensional turbulence. One can further speculate that similar mechanisms are responsible for the creation of isolated, coherent vortices in three dimensions as well.

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## Figure Captions

Figure 1: Constant energy lines in the $(\beta, h^2)$ plane.

Figure 2: Constant energy lines in the $(\beta h^2, h^2)$ plane.

Figure 3: Entropy as a function of $N = 1/h^2$ for various values of the mean energy $E$.

Figure 4: Ratio of energy of mean to mean energy as a function of $N$.

Figure 5: Constant energy lines in the $(\beta, h^2)$ plane in the presence of vortex stretching.

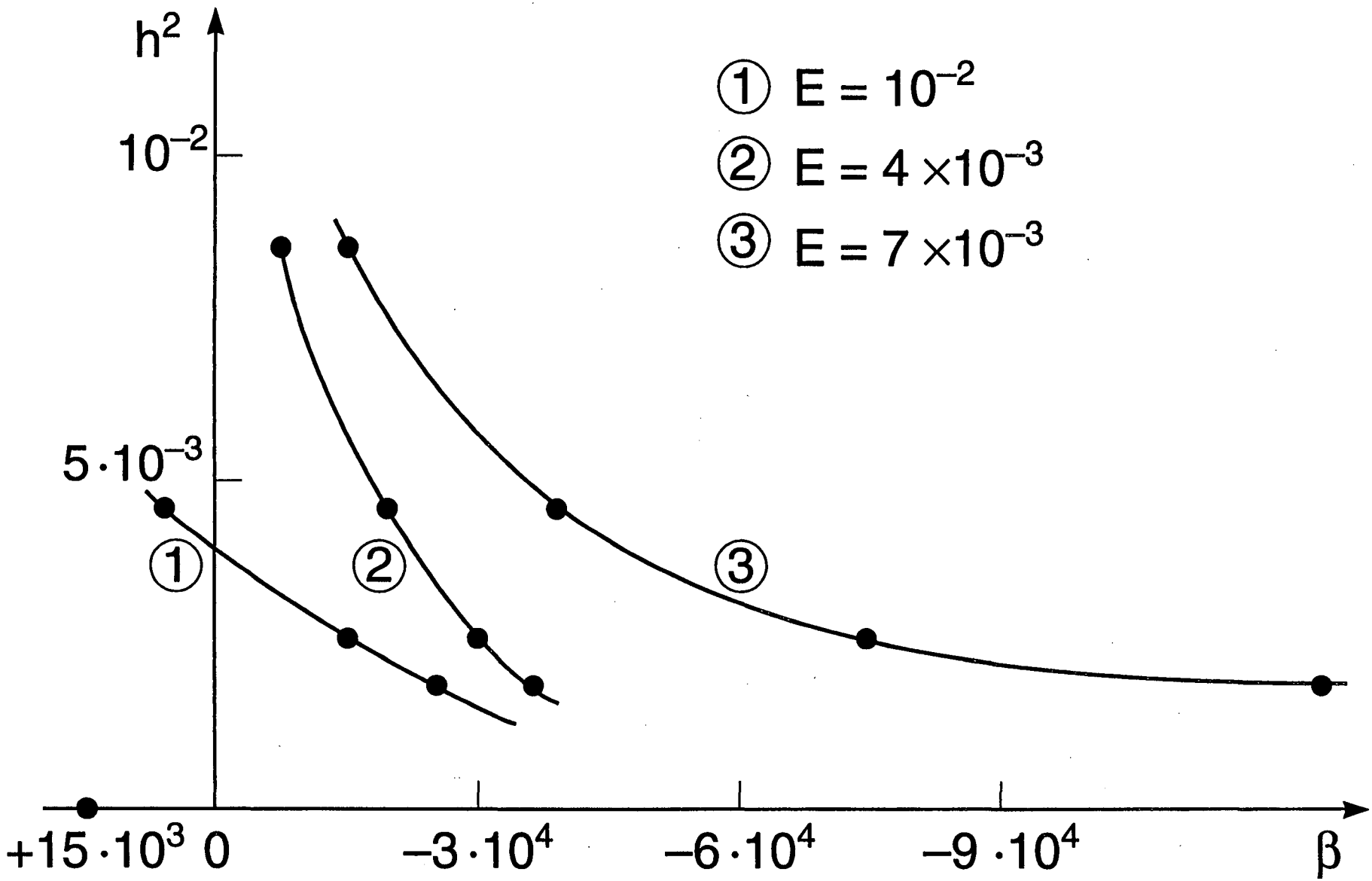


Figure 1

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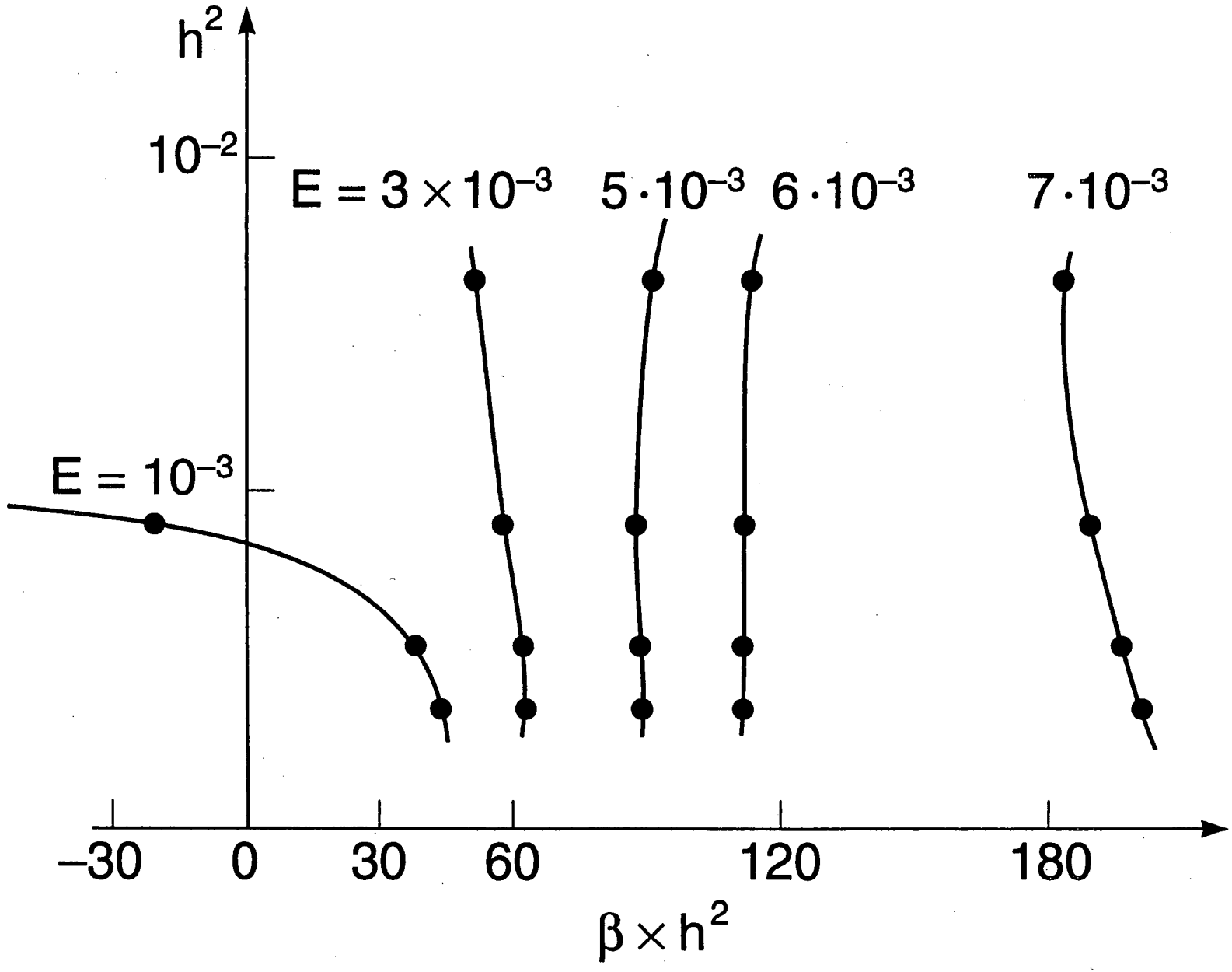


Figure 2

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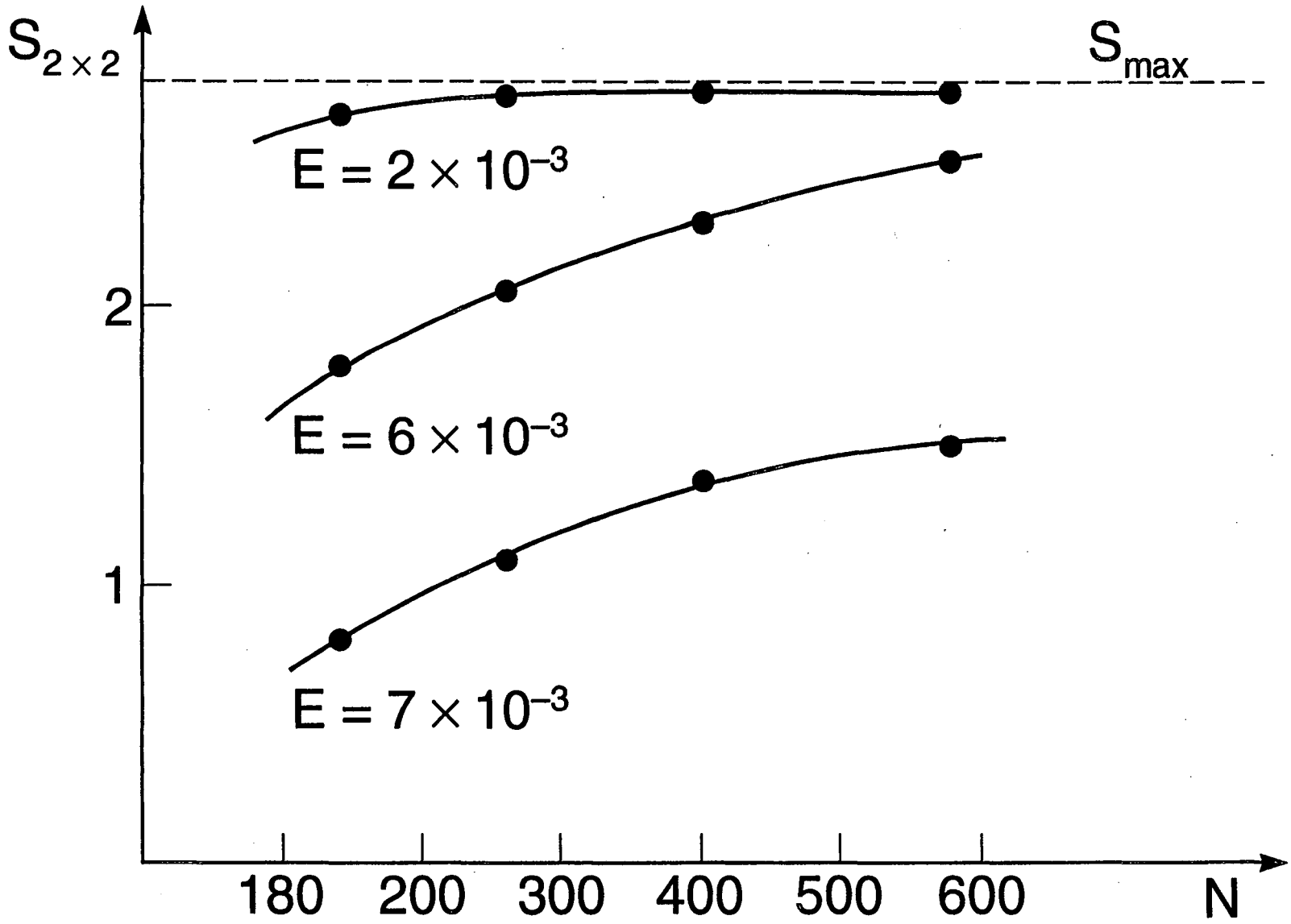


Figure 3

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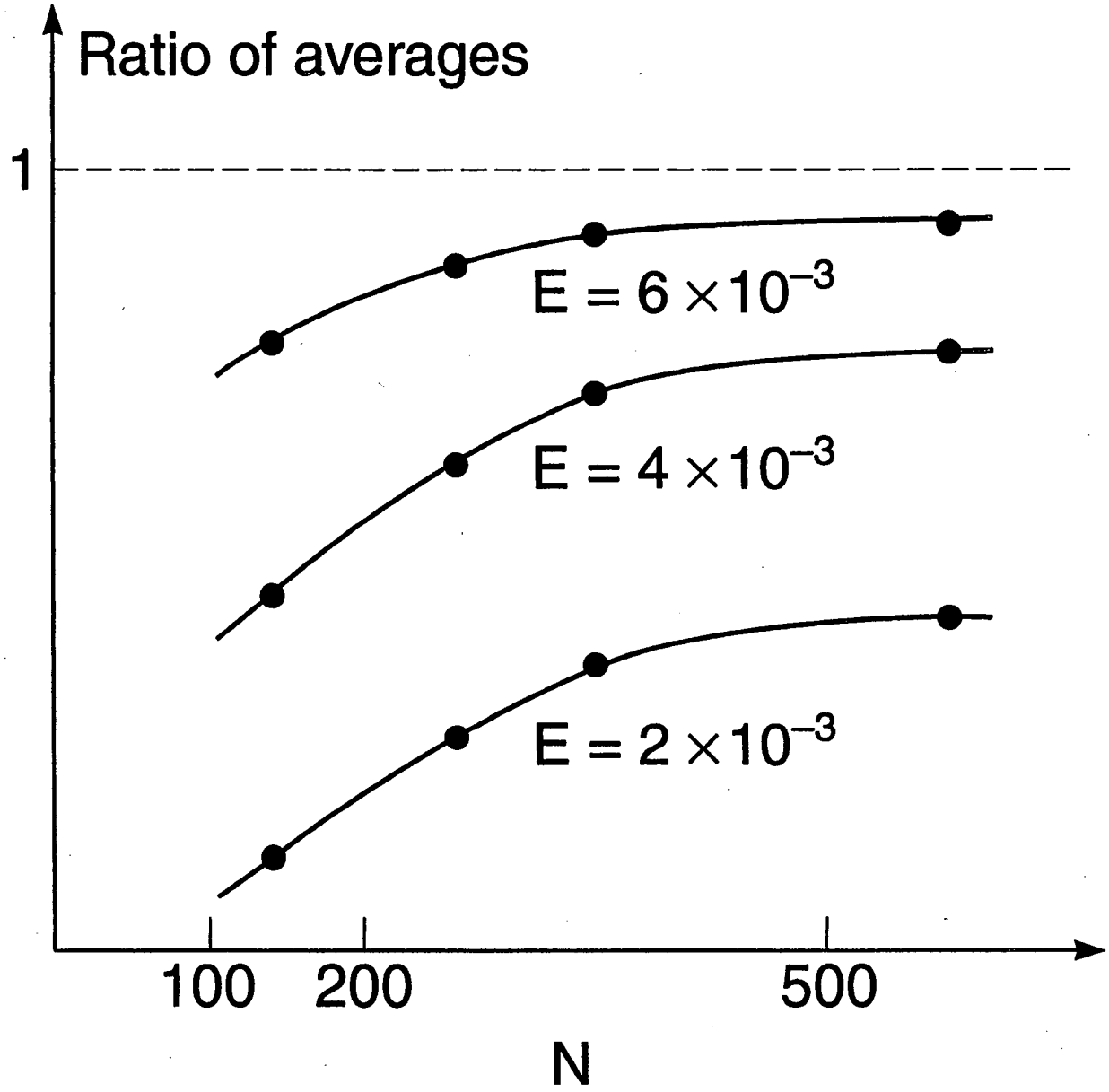


Figure 4

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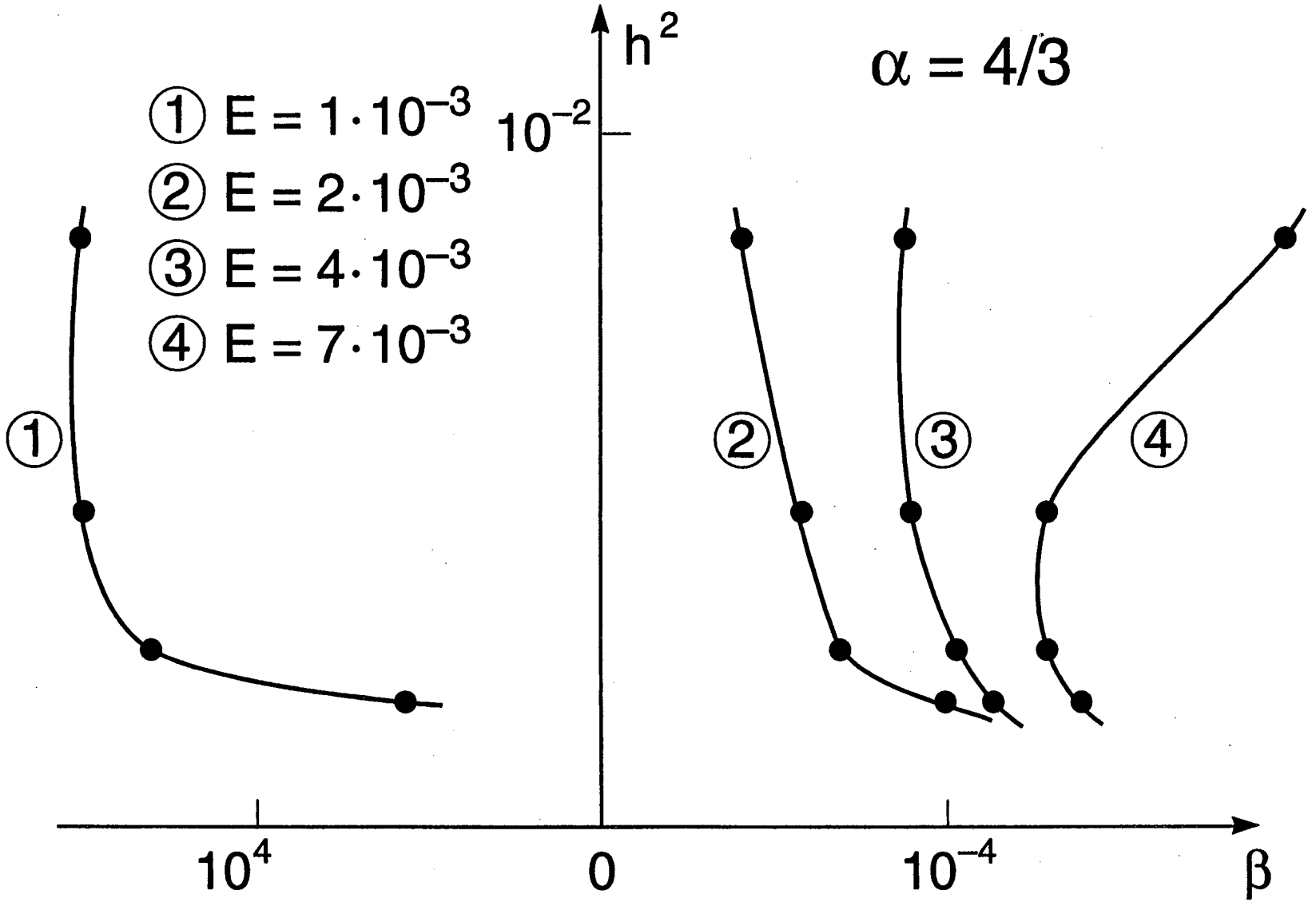


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Figure 5

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