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Quantum Groups: Geometry and Applications

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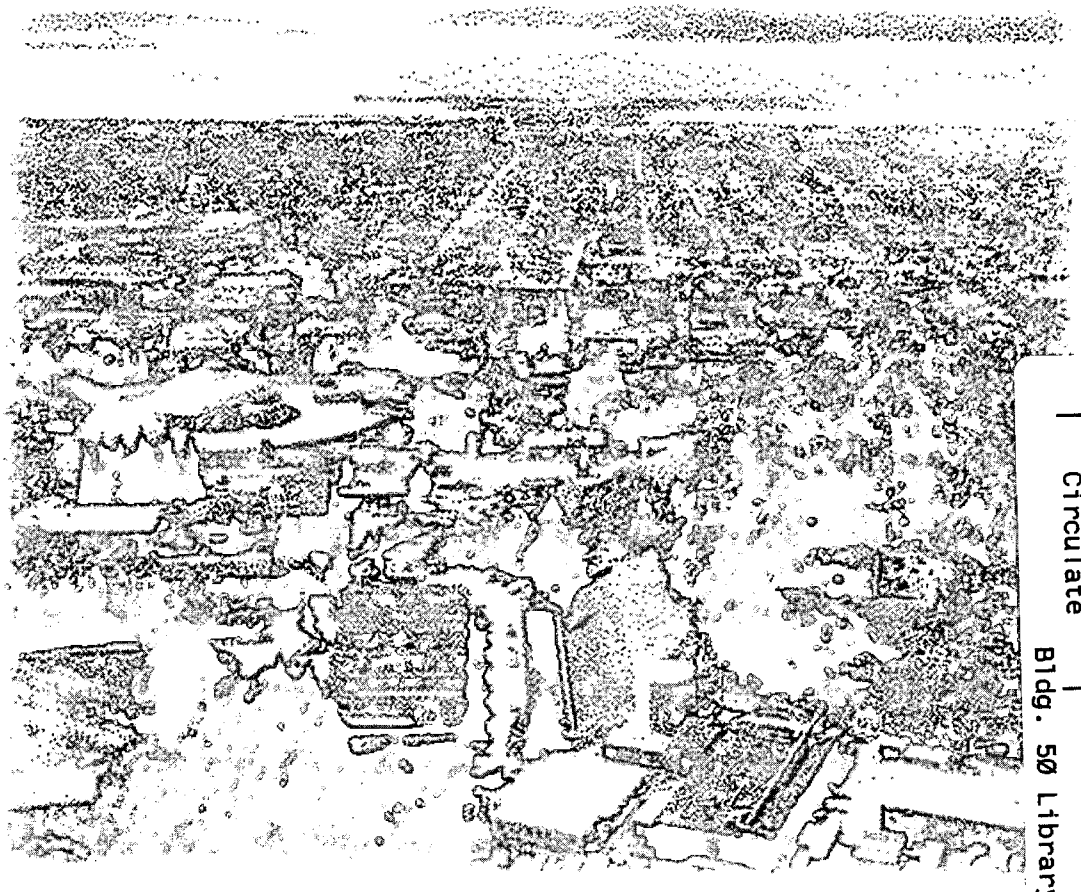

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Quantum Groups: Geometry and Applications [1]

(Ph.D. Thesis)

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Abstract

The main subject of this thesis is the geometry of quantum groups and quantum spaces. The main tool used is the Faddeev-Reshetikhin-Takhtajan description of quantum groups. A few content-rich examples of quantum complex spaces with quantum group symmetry are treated in details. In chapter 1, we review some of the basic concepts and notions for Hopf algebras and other background materials. In chapter 2, we study the vector fields of quantum groups. A compact realization of these vector fields as pseudodifferential operators acting on the linear quantum spaces is given. In chapter 3, we describe the quantum sphere as a complex quantum manifold by means of a quantum stereographic projection. A covariant calculus is introduced. An interesting property of this calculus is the existence of a one-form realization of the exterior differential operator. The concept of a braided comodule is introduced and a braided algebra of quantum spheres is constructed. In chapter 4, we consider the more general higher dimensional quantum complex projective spaces and the quantum Grassman manifolds. Differential calculus, integration and braiding can be introduced as in the one dimensional case. A sufficient condition for the existence of a one-form realization of the exterior differential operators $\delta, \bar{\delta}$ is given and is applied to the case of complex projective spaces and quantum Grassmannians. Finally, in chapter 5, we study the framework of quantum principal bundle and construct the q-deformed Dirac monopole as a quantum principal bundle with a quantum sphere as the base and a $U(1)$ with non-commutative calculus as the fiber. The first Chern class can be introduced and integrated to give the monopole charge.

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Introduction

Quantum groups were discovered in Leningrad by studying the quantum integrable 1+1 dimensional systems, using the method of quantum inverse scattering [1, 2]. In the Hamiltonian approach to the inverse scattering method [2, 3], one introduces the pair of first order differential operators,

$$L = \frac{d}{dx} - U, \quad M = \frac{d}{dt} - V, \tag{0.1}$$

where $U(x,\lambda), V(x,\lambda)$ are matrices with matrix elements being functions of the basic field variables (e.g. ψ, ψ^* for the nonlinear Schödinger system) and their spatial derivatives, and λ is a complex parameter. Given the Hamiltonian H and a Poisson bracket $\{,\}$, one requires that the equation of motion (EOM) admits the Lax representation

$$\text{EOM} \Leftrightarrow [L, M] = 0. \tag{0.2}$$

The Poisson bracket , written in terms of the matrix $U(x,\lambda)$ is of the form

$$\{U(x,\lambda) \overset{\otimes}{,} U(y,\mu)\} = [r(\lambda-\mu), U(x,\lambda)\otimes I + I \otimes U(y,\mu)]\delta(x-y), \tag{0.3}$$

where

$$\{A \overset{\otimes}{,} B\}^{ik}_{lm} \equiv \{A^i_l \overset{\otimes}{,} B^k_m\} \tag{0.4}$$

and $r(\lambda)$ is the classical r-matrix with spectral parameter λ satisfying the classical Yang-Baxter equation (CYBE),

$$[r_{12}(\lambda-\mu), r_{13}(\lambda) + r_{23}(\mu)] + [r_{13}(\lambda), r_{23}(\mu)] = 0. \tag{0.5}$$

In a quantum theory, one often uses a lattice to regularize the theory, therefore it is preferable to consider the above classical system on a one dimensional spatial lattice with lattice sites labelled by integer n. Define $L_n(\lambda)$ by

$$exp \int_{x_n}^{x_{n+1}} dx U(x,\lambda) \simeq L_n(\lambda) + [\delta x]^2, \tag{0.6}$$

then (0.3) becomes

$$\{L_n(\lambda) \overset{\otimes}{,} L_m(\mu)\} = [r(\lambda-\mu), L_n(\lambda)\otimes L_m(\mu)]\delta_{nm}. \tag{0.7}$$

This formula has the remarkable property that (0.7) still holds after replacing $L_n(\lambda)$ by $L_{n+1}(\lambda)L_n(\lambda)$. This suggest that we are dealing with a group comultiplication (loop group) that is compatible with the Poisson structure. It was this property that led Drinfeld to the notion of a Poisson-Lie group and thus revealed the mathematical structure of Poisson-Lie groups behind the theory of solvable lattice models. Upon quantization, Sklyanin, Takhtajan and Faddeev found that the operators $\hat{L}_n(\lambda)^i_j$ satisfy the commutation relations

$$R_{12}(\lambda-\mu)(\hat{L}_n(\lambda)\otimes I)(I\otimes\hat{L}_n(\mu)) = (I\otimes\hat{L}_n(\mu))(\hat{L}_n(\lambda)\otimes I)R_{12}(\lambda-\mu), \qquad (0.8)$$

where $R(\lambda)$ is the quantum R-matrix satisfying the quantum Yang-Baxter equation (QYBE),

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu) \qquad (0.9)$$

and the matrices $\hat{L}_n(\lambda)$ have commuting matrix elements for different lattice sites. (0.8) reduces to (0.7) in the classical limit of $h\to 0$:

$$\tfrac{1}{h}[\hat{L}_n(\lambda)^i_j, \hat{L}_n(\mu)^k_l] \to \{L_n(\lambda)^i_j, L_n(\mu)^k_l\}, \qquad (0.10)$$

$$\tfrac{1}{h}(R(\lambda)-I) \to r(\lambda). \qquad (0.11)$$

Remarkably, (0.8) also has the property that the matrix product $\hat{L}_{n+1}(\lambda)\hat{L}_n(\lambda)$ satisfies the same Poisson bracket (0.8) also. This "comultiplication structure" suggests an underlying structure of bialgebra compatible with the quantum algebra (0.8) specified by the matrix $R(\lambda)$ and eventually led Drinfeld and Jimbo [4, 5, 6, 7] to identify the relevant algebraic structure as a quasi-triangular Hopf algebra eventually. As such, quantum groups are deformations of their classical counterparts, the algebra of functions $Fun(G)$ over a Lie group G respectively its dual, the universal enveloping algebra $U(\mathbf{g})$ of a Lie algebra $\mathbf{g}$. The standard 1-parameter deformations of the classical groups and the universal enveloping algebras were given in [8]. The one-parameter deformation of the universal enveloping algebra was also constructed by Drinfeld and Jimbo from the point of view of quantization of Co-Poisson universal enveloping algebra (dual to the algebra of functions on the Poisson-Lie groups).

From a different point of view, Woronowicz initiated the study of quantum groups as nontrivial examples of non-commutative geometry. If M is a topological

space, then the algebra $C_\infty(M)$ of all complex-valued continuous functions over M which vanish at ∞ is commutative and is a C^*-algebra. The converse statement is also true: if A is a commutative C^*-algebra, then A is isomorphic to $C_\infty(M)$ for some locally compact topological space M. This is the Gelfand-Naimark theorem. When A is non-commutative, such a M does not exist. However, from the point of view of category theory, it is convenient to introduce non-commutative spaces as objects of the category which is dual to the category of C^*-algebras. The theory of group structures on non-commutative spaces is quite old [9, 10, 11]. It was Woronowicz who revived the interest in this subject. He proposed a useful set of axioms [12] for compact matrix quantum (pseudo) groups and which turn out to have a very rich structure and representation theory. In [13, 14], Woronowicz introduced covariant differential calculi on quantum groups. Differential calculi on linear quantum spaces were later constructed by Wess and Zumino [15]. See for example [16, 17, 18, 19, 20, 21, 22, 23] for many interesting constructions of the differential geometry on quantum groups and quantum spaces.

Quantum groups as "symmetries" of quantum spaces were first proposed by Manin [24, 25]. The suggestion of quantizing spacetime is not new, see for example [26, 27] for early attempts. On the other hand, the idea of a quantum symmetry with a quantized spacetime as carrier is new. It is very attractive and naturally attracted a lot of activities from the physics community: extensive work has been carried out on the q-deformation of the Lorentz [28, 29] and the Poincare [30, 31] group. The corresponding "Lie algebras" have also been constructed [32, 33]. In addition to the q-deformation, there exists also the κ-deformation of Poincare algebra [34]. Woronowicz and Zakrzewski made some natural and reasonable requirements on how a deformation of the Lorentz group $SL(2,\mathbf{C})$ should look like and gave a complete classification from a Hopf $*$-algebraic point of view [35]. The classification of deformed Poincare groups has been worked out recently [36]. Wave equations on Euclidean spaces [37] and on the quantum Minkowski spaces are studied [38]. The introduction of gauge symmetry was a very important step in physics. The QFT of strong, weak and electromagnetic interactions are modelled as gauge theories nowadays. However, it is likely that a new understanding of gauge symmetry is necessary for the ultimate goal of unification of all forces and a lot of attempts were made in constructing quantum group gauge theories [39, 40, 41].

It is commonly believed that our usual description of spacetime using the language of a smooth manifold is only a low energy classical approximation and will very likely be modified at short distances by quantum gravity effects. The use of a spacetime continuum is a basic and fundamental assumption in Einstein's geometric theory of gravity. This assumption, however, is tested experimentally only for distance much larger than the Planck length. It is natural to expect that one needs some new spacetime structures at small scales for a quantum theory of gravity. This belief is reinforced by the fact that quantum field theory of gravity is plagued by ultraviolet divergences which is not removable by the usual procedure of renormalization. Some alternative descriptions have been suggested, notably string theory [42] and non-commutative geometry [43, 44]. As an example of the framework of non-commutative geometry, the effects of electroweak unification on the nature of spacetime is studied in [45] by Lott and Connes who proposed a minimal two-point internal space and thus provided a geometric origin of the Higgs mechanism: the Higgs field appears as the component of the gauge field in the direction of the internal space. See [46] for a review of the prescription for particle physics model building in the non-commutative geometric framework, outstanding problems in this framework and many references. In particular, it is not clear how to incorporate the physics of quantum gravity using the language of non-commutative geometry.

Quantum groups and Hopf algebras make their appearance in many different branches of physics and mathematics. In physics, they occur as symmetries of lower dimensional theories such as spin chains and solvable lattice models (see for example [47]), and as hidden symmetry of WZNW models and conformal field theory [48, 49]. In some two dimensional CFT [50, 51], Mack and Schomerus found the more general quasi-Hopf algebra symmetry at certain values of the parameter space, generalizing the usual Hopf algebra symmetry. On the mathematical side, quantum group have stimulated a lot of interesting topics such as representation theory, knot theory and non-commutative geometry.

It will be very interesting to understand the role of quantum group symmetry in higher dimensional physics. The main theme of this thesis is a study of the geometry of quantum groups and quantum spaces, with the hope that they will be useful for the construction of quantum field theory with quantum group symmetry.

In this context, we study the realization of quantum group symmetries, we construct the q-deformed Dirac monopole as a quantum principal bundle and we study a few other content-rich examples of non-commutative geometry. It is hoped that these simple models of quantum spaces may capture some of the interesting features of the spacetime at the Planck scale.

Chapter 1

Hopf Algebras

We review in this chapter some of the most basic concepts and constructions of Hopf algebras and quantum groups.

1.1 Basic Definitions

Definition 1.1.1 (Bialgebra)
A bialgebra $\mathcal{A}$ over the field k is a unital algebra and a coalgebra. i.e. with also the two maps, comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit $\epsilon : \mathcal{A} \to k$ such that

$$(\Delta \otimes id) \circ \Delta(a) = (id \otimes \Delta) \circ \Delta(a), \tag{1.1}$$

$$(\epsilon \otimes id) \circ \Delta(a) = (id \otimes \epsilon) \circ \Delta(a) = a, \tag{1.2}$$

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(1) = 1 \otimes 1, \tag{1.3}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \quad \epsilon(1) = 1. \tag{1.4}$$

for any $a, b \in \mathcal{A}$.

Definition 1.1.2 (Hopf Algebra)
A Hopf algebra $\mathcal{A}$ is a bialgebra together with a map $S : \mathcal{A} \to \mathcal{A}$ such that [1]

$$a_1 S(a_2) = S(a_1) a_2 = 1\epsilon(a), \tag{1.5}$$

where 1 is the unit of $\mathcal{A}$.

[1] Here we use the Sweedler's notation [52] $\Delta(a) = \sum_i a_{i(1)} \otimes a_{i(2)} = a_{(1)} \otimes a_{(2)}$ for simplicity.

Definition 1.1.3 (Quasi-triangular Hopf Algebra)
A quasi-triangular Hopf algebra $\mathcal{A}$ is a Hopf algebra together with an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ satisfying the relations

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{1.6}$$

$$(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{23}, \tag{1.7}$$

$$(\tau \circ \Delta)(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \tag{1.8}$$

where

$$\begin{aligned} \tau : \mathcal{A} \otimes \mathcal{A} &\to \mathcal{A} \otimes \mathcal{A}, \\ a \otimes b &\mapsto b \otimes a \end{aligned} \tag{1.9}$$

is the permutation map and

$$\mathcal{R}_{12} = r_i \otimes r^i \otimes 1 \tag{1.10}$$

etc. if we denote $\mathcal{R} = r_i \otimes r^i$. $\mathcal{R}$ is called the universal R-matrix of $\mathcal{A}$ and satisfies the quantum Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{1.11}$$

as a consequence of the definition.

Definition 1.1.4 (*-Hopf Algebra)
*A *-Hopf algebra is a Hopf algebra together with an *-involution satisfying*

$$(\beta a)^* = \beta^* a^*, \quad \forall \beta \in k, \tag{1.12}$$

$$(ab)^* = b^* a^*, \tag{1.13}$$

$$\Delta(a^*) = (* \otimes *)\Delta(a), \tag{1.14}$$

$$\epsilon(a^*) = \epsilon(a)^*, \tag{1.15}$$

$$(S(a^*))^* = S^{-1}(a), \tag{1.16}$$

*where β^* is the *-conjugation of $\beta \in k$.*

Definition 1.1.5 (Dual Pairing of Hopf Algebras)
Two Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ are called dually paired if there exists a non-degenerate

inner product $< \cdot, \cdot >: \mathcal{U} \otimes \mathcal{A} \to k$ *such that*

$$\begin{aligned} < x, ab > &= < x_{(1)}, a >< x_{(2)}, b >, & (1.17)\\ < xy, a > &= < x, a_{(1)} >< y, a_{(2)} >, & (1.18)\\ < 1_{\mathcal{U}}, a > &= \epsilon(a), & (1.19)\\ < x, 1_{\mathcal{A}} > &= \epsilon(x), & (1.20)\\ < S(x), a > &= < x, S(a) >, & (1.21) \end{aligned}$$

where $x, y \in \mathcal{U}$ *and* $a, b \in \mathcal{A}$.

If $\mathcal{U}$ *and* $\mathcal{A}$ *are* $*$*-Hopf algebras, then we require also*

$$< x^*, a >=< x, (S(a))^* >^* . \tag{1.22}$$

Let $\mathcal{U}$ be a Hopf algebra and

$$\rho : \mathcal{U} \to M_N(k) \tag{1.23}$$

be an $N \times N$ faithful representation of $\mathcal{U}$. We can define a new Hopf algebra $\mathcal{A}$, dual to $\mathcal{U}$ in the following way: $\mathcal{A}$ is generated by the N^2 matrix elements A^i_j satisfying

$$\rho(x) =< x, A^i_j > \tag{1.24}$$

for $x \in \mathcal{U}$. Since ρ is faithful, A^i_j is well defined. The multiplication in $\mathcal{A}$ is determined by the coproduct in $\mathcal{U}$ through (1.24). As for the other Hopf structures, it is

$$\begin{aligned} \Delta(A^i_j) &= A^i_k \otimes A^k_j,\\ \epsilon(A^i_j) &= \delta^i_j,\\ S(A^i_j) &= (A^{-1})^i_j. & (1.25) \end{aligned}$$

1.2 Quasi-triangular $\mathcal{U}$ and the Dual Pair $(\mathcal{U}, \mathcal{A})$

When $\mathcal{U}$ is quasi-triangular, we have

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \forall x \in \mathcal{U}, \tag{1.26}$$

where $\mathcal{R}$ is the universal R-matrix and one can determine the multiplication in $\mathcal{A}$. It is convenient to introduce the $N^2 \times N^2$ numerical R-matrix for the representation ρ,

$$R^{ik}_{mn} = (\rho^i_m \otimes \rho^k_n)(\mathcal{R}) =< \mathcal{R}, A^i_m \otimes A^k_n >, \tag{1.27}$$

then

$$\begin{aligned} 0 &= < \mathcal{R}\Delta(x) - \Delta'(x)\mathcal{R}, A^i_j \otimes A^k_l >\\ &= < \mathcal{R}, A^i_m \otimes A^k_n >< \Delta(x), A^m_j \otimes A^n_l > - < \Delta'(x), A^i_m \otimes A^k_n >< \mathcal{R}, A^m_j \otimes A^n_l >\\ &= < x, R^{ik}_{mn} A^m_j A^n_l - A^k_n A^i_m R^{mn}_{jl} > . & (1.28) \end{aligned}$$

Since x is arbitrary, we have[8]

$$R^{ik}_{mn} A^m_j A^n_l = A^k_n A^i_m R^{mn}_{jl}, \tag{1.29}$$

which can be written compactly in tensor product notation as

$$R_{12} A_1 A_2 = A_2 A_1 R_{12}, \tag{1.30}$$

where

$$\begin{aligned} R_{12} &= (\rho_1 \otimes \rho_2)(R),\\ A_1 &= A \otimes 1, \quad A_2 = 1 \otimes A, & (1.31) \end{aligned}$$

i.e.

$$(R_{12})^{ij}_{kl} = R^{ij}_{kl}, \quad (A_1)^{ij}_{kl} = A^i_k \delta^j_l \tag{1.32}$$

etc.. The QYBE becomes

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{1.33}$$

It is convenient to introduce the numerical $\hat{R}$-matrix,

$$\hat{R} = P \circ R, \tag{1.34}$$

where $P^{ij}_{kl} = \delta^i_l \delta^j_k$ is the permutation matrix. Explicitly

$$\hat{R}^{ij}_{kl} = R^{ji}_{kl}. \tag{1.35}$$

And the QYBE becomes

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{1.36}$$

The numerical R-matrix for q-deformation of classical groups is given in [8], for example, it is

$$\hat{R}^{kl}_{mn} = \delta^l_m\delta^k_n(1+(q-1)\delta^k_l) + \lambda\delta^k_m\delta^l_n\theta(l>k), \quad \text{for } GL_q(N), \tag{1.37}$$

where $\theta(l>k)$ is equal to 1 if $l>k$ and is zero for $l \le k$.

$GL_q(N)$ R-matrix properties

For ease of reference, we list here also some other useful properties of the $GL_q(N)$ R-matrix. The numerical R-matrix for $GL_q(N)$ is

$$\hat{R}^{kl}_{mn} = \delta^l_m\delta^k_n(1+(q-1)\delta^k_l) + \lambda\delta^k_m\delta^l_n\theta(l>k), \tag{1.38}$$

where the indices run from 1 to N. The step function $\theta(l>k)$ is equal to 1 if $l>k$ and is zero for $l \le k$. q is any nonzero complex number. $\hat{R}_{12}$ satisfies

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}, \tag{1.39}$$

$$\hat{R}_{12}(q^{-1}) = \hat{R}^{-1}_{21}(q), \tag{1.40}$$

$$\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}, \quad \hat{R}^{i'j'}_{k'l'} = \hat{R}^{ji}_{lk}, \tag{1.41}$$

where

$$i' = N - i + 1. \tag{1.42}$$

The quantum $\epsilon^{i_1 i_2 \ldots i_N}$ is given by

$$\epsilon^{i_1 i_2 \ldots i_N} = (-q)^{l(\sigma)} \tag{1.43}$$

when $(i_1 i_2 \ldots i_N)$ is a permutation of $(12\ldots N)$: $(i_1 i_2 \ldots i_N) = \sigma(12\ldots N)$ and is zero otherwise. Here $l(\sigma)$ is the length of the permutation σ. It is

$$R_{0N}\ldots R_{02}R_{01}\epsilon^{12\ldots N} = q\epsilon^{12\ldots N}I_0 = R_{10}R_{20}\ldots R_{N0}\epsilon^{12\ldots N}. \tag{1.44}$$

The characteristic equation is

$$\hat{R} - \hat{R}^{-1} = \lambda I \tag{1.45}$$

and the spectral representation is

$$\hat{R} = qP_+ - q^{-1}P_-, \tag{1.46}$$

where

$$P_+ = \frac{1}{q+q^{-1}}(\hat{R}+q^{-1}I), \quad P_- = \frac{1}{q+q^{-1}}(-\hat{R}+qI). \tag{1.47}$$

One can introduce the inverse of $(\hat{R}^{-1})^{jk}_{il}$ with respect to the indices i,j,

$$\Phi^{ij}_{kl} = \hat{R}^{ji}_{lk}q^{2(i-l)} = \hat{R}^{ji}_{lk}q^{2(k-j)}, \tag{1.48}$$

which satisfies

$$\Phi^{ri}_{sj}(\hat{R}^{-1})^{jk}_{il} = (\hat{R}^{-1})^{ri}_{sj}\Phi^{jk}_{il} = \delta^r_l\delta^k_s. \tag{1.49}$$

It satisfies the interesting "trace" properties (sum over the index k),

$$\Phi^{ik}_{jk} = \delta^i_j q^{2i-1}, \tag{1.50}$$

$$\Phi^{ki}_{kj} = \delta^i_j q^{2(N-i)+1}. \tag{1.51}$$

Similarly, one can introduce the inverse of $\hat{R}^{jk}_{il}$ with respect to the indices i,j,

$$\Psi^{ij}_{kl} = (\hat{R}^{-1})^{ji}_{lk}q^{2(i-l)} = (\hat{R}^{-1})^{ji}_{lk}q^{2(k-j)} \tag{1.52}$$

which satisfies

$$\Psi^{ri}_{sj}\hat{R}^{jk}_{il} = \hat{R}^{ri}_{sj}\Psi^{jk}_{il} = \delta^r_l\delta^k_s \tag{1.53}$$

and the "trace" properties (sum over the index k),

$$\Psi^{ik}_{jk} = \delta^i_j q^{-2(N-i)-1}, \tag{1.54}$$

$$\Psi^{ki}_{kj} = \delta^i_j q^{-2i+1}. \tag{1.55}$$

Generators for $\mathcal{U}$

It is convenient to introduce the $N \times N$ matrices $L^\pm$ with entries in $\mathcal{U}$ by

$$\begin{aligned} L^+ &= \langle \mathcal{R}, id \otimes A \rangle, \\ L^- &= \langle \mathcal{R}^{-1}, A \otimes id \rangle. \end{aligned} \tag{1.56}$$

The commutation relations[8] follow immediately from the QYBE,

$$\begin{aligned} R_{12}L_2^+L_1^+ &= L_1^+L_2^+R_{12}, \\ R_{12}L_2^-L_1^- &= L_1^-L_2^-R_{12}, \\ R_{12}L_2^+L_1^- &= L_1^-L_2^+R_{12}. \end{aligned} \tag{1.57}$$

For example,

$$\begin{aligned} 0 &= < \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} - \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, id \otimes A_1 \otimes A_2 > \\ &= R_{12}L_2^+L_1^+ - L_1^+L_2^+R_{12}. \end{aligned}$$

The Hopf structures follow from the properties of $\mathcal{R}$ also,

$$\begin{aligned} \Delta(L^\pm) &= L^\pm \dot{\otimes} L^\pm, \\ \epsilon(L^\pm) &= I, \\ S(L^\pm) &= (L^\pm)^{-1}, \end{aligned} \tag{1.58}$$

where the notation $(M \dot{\otimes} N)^i_j = M^i_k \otimes N^k_j$ is used.

1.2.1 Example: $U_q(su(2))$ and $SU_q(2)$

A typical example of a nontrivial $*$-dual pair is $\mathcal{U} = U_q(su(2))$ and $\mathcal{A} = SU_q(2)$.

The Hopf Algebra $U_q(su(2))$

The quantum enveloping algebra $U_q(su(2))$ of $su(2)$ is the associative unital algebra generated by the Drinfeld-Jimbo generators H, X_+, and X_- modulo the commutation relations

$$[H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \tag{1.59}$$

where q is a nonzero real number. The Hopf structures are given by

$$\begin{gathered} \Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X_\pm) = X_\pm \otimes q^H + q^{-H} \otimes X_\pm, \\ \epsilon(H) = 0, \quad \epsilon(X_\pm) = 0, \\ S(H) = -H, \quad S(X_\pm) = -q^{\pm 1}X_\pm, \end{gathered} \tag{1.60}$$

and the $*$-structure is

$$H^* = H, \quad X_\pm^* = X_\mp. \tag{1.61}$$

The universal R-matrix for $\mathcal{U}$ is given by

$$\mathcal{R} = q^{H \otimes H/2} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_{1/q}!} (q^{H/2}X_+ \otimes q^{-H/2}X_-)^n \tag{1.62}$$

with

$$[x]_q := \frac{q^{2x} - 1}{q^2 - 1}, \ x \text{ any complex number} \tag{1.63}$$

and

$$[n]_q! := \prod_{m=1}^{n} [m]_q \tag{1.64}$$

for integer n.

The Dual Hopf algebra $SU_q(2)$

A faithful 2×2 representation for $U_q(su(2))$ is given by

$$H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{1.65}$$

The numerical R-matrix in this representation is a 4×4 matrix,

$$R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{1.66}$$

where $\lambda := q - q^{-1}$. $GL_q(2)$ is the algebra generated by the matrix elements $\alpha, \beta, \gamma, \delta$ of the quantum matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{1.67}$$

Eqn.(1.30) then give

$$\begin{gathered} \alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\delta = \delta\alpha + (q - q^{-1})\beta\gamma, \\ \beta\gamma = \gamma\beta, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma. \end{gathered} \tag{1.68}$$

The comultiplication is given by

$$\Delta\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\dot{\otimes}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha\otimes\alpha+\beta\otimes\gamma & \alpha\otimes\beta+\beta\otimes\delta\\ \gamma\otimes\alpha+\delta\otimes\gamma & \gamma\otimes\beta+\delta\otimes\delta\end{pmatrix} \tag{1.69}$$

and antipode by

$$S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = (det_qT)^{-1}\begin{pmatrix}\delta & -q^{-1}\beta\\ -q\gamma & \alpha\end{pmatrix}, \tag{1.70}$$

where $det_qT := \alpha\delta - q\beta\gamma$ is central in the algebra and is group-like

$$\Delta(det_qT) = det_qT\otimes det_qT. \tag{1.71}$$

It is called the quantum determinant.

It is consistent to set $det_qT = 1$ and the antipode becomes

$$S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\delta & -q^{-1}\beta\\ -q\gamma & \alpha\end{pmatrix} \tag{1.72}$$

and we get $SL_q(2)$.

One can introduce an involution $*$ to $SL_q(2)$,

$$\alpha^* = \delta, \quad \beta^* = -q\gamma, \quad \gamma^* = -q^{-1}\beta, \quad \delta^* = \alpha \tag{1.73}$$

and get $SU_q(2)$. Note that as $q \to 1, R$ becomes the identity matrix and everything commute. Thus the "classical limit" is given by $q = 1$. For $q \neq 1$, we get $Fun(SU_q(2))$, or $SU_q(2)$ for short. For $q = 1, *$ is nothing but complex conjugation and these relations is the familiar condition $T^\dagger = T^{-1}$ of $SU(2)$.

The $L^\pm$ Generators

The 2×2 matrices $L^\pm$ are related to the Drinfeld-Jimbo generators as

$$L^+ = \begin{pmatrix} q^{-H/2} & -q^{-1/2}\lambda X_+\\ 0 & q^{H/2}\end{pmatrix}, \quad L^- = \begin{pmatrix} q^{H/2} & 0\\ q^{1/2}\lambda X_- & q^{-H/2}\end{pmatrix}. \tag{1.74}$$

And the eqns. (1.89), (1.58) reproduce (1.59), (1.60). The $*$-structure (1.61) is

$$(L_q^\pm)^\dagger = L_{1/q}^\mp, \tag{1.75}$$

where the matrix $L_q^\pm$ is given by (1.74) and $L_{1/q}^\pm$ is given by the same matrix but with q changed to $1/q$.

1.3 The Smash Product

1.3.1 Action and Coaction

Definition 1.3.1 (Left $\mathcal{A}$-Module and Module Algebra)
Let $\mathcal{A}$ be an algebra, we say that a vector space V is a left $\mathcal{A}$-module if there is a map

$$\triangleright : \mathcal{A}\otimes V \to V : \; x\otimes v \mapsto x\triangleright v, \tag{1.76}$$

such that

$$(xy)\triangleright v = x\triangleright(y\triangleright v), \quad 1\triangleright v = v. \tag{1.77}$$

The map $\triangleright$ is called the left action of $\mathcal{A}$ on V.

If V is an algebra, then we can require a Hopf algebra $\mathcal{A}$ to act on it so as to respect its algebraic structure. Thus, if in addition to (1.77), also

$$x\triangleright(uv) = (x_{(1)}\triangleright u)(x_{(2)}\triangleright v), \quad x\triangleright 1 = 1\epsilon(x), \tag{1.78}$$

then we say that V is a left $\mathcal{A}$-module algebra.

Right $\mathcal{A}$-module and right $\mathcal{A}$-module algebra can be defined similarly.

Definition 1.3.2 (Left $\mathcal{A}$-Comodule and Comodule Algebra)
Let $\mathcal{A}$ be a coalgebra, we say that a vector space V is a left $\mathcal{A}$-comodule if there is a map $\Delta_L : V \to \mathcal{A}\otimes V$ such that

$$(\Delta\otimes id)\Delta_L = (id\otimes\Delta_L)\Delta_L, \quad (\epsilon\otimes id)\Delta_L = id. \tag{1.79}$$

The map $\Delta_L$is called the left coaction of $\mathcal{A}$ on V.

If V is an algebra, then we can require a Hopf algebra $\mathcal{A}$ to coact on it so as to respect its algebraic structure. Thus, if in addition to (1.79), also

$$\Delta_L(uv) = \Delta_L(u)\Delta_L(v), \quad \Delta_L(1_V) = 1_{\mathcal{A}}\otimes 1_V, \tag{1.80}$$

i.e. Δ_L is an algebra map also, then we say that V is a left $\mathcal{A}$-comodule algebra.

Right $\mathcal{A}$-comodule and right $\mathcal{A}$-comodule algebra can be defined similarly.

1.3.2 Smash Product

Let $\mathcal{U}$ and $\mathcal{A}$ be two dually paired Hopf algebras. We want to construct a new associative algebra, their smash product $\mathcal{A}\rtimes\mathcal{U}$. The smash product $\mathcal{A}\rtimes\mathcal{U}$ is isomorphic to the tensor product $\mathcal{A}\otimes\mathcal{U}$ as a vector space, and is equipped with the "twisted" multiplication. First, instead of the usual multiplication in tensor product space

$$(b\otimes x)(a\otimes y) = ba\otimes xy, \tag{1.81}$$

where $b, a \in \mathcal{A}, x, y \in \mathcal{U}$, one can also introduced a twisted multiplication on $\mathcal{A}\otimes\mathcal{U}$ as

$$(1\otimes x)(a\otimes 1) =< x_{(1)}, a_{(2)} > a_{(1)}\otimes x_{(2)} \tag{1.82}$$

for $x \in \mathcal{U}, a \in \mathcal{A}$, where

$$\Delta_{\mathcal{U}} x = x_{(1)}\otimes x_{(2)}, \quad \Delta_{\mathcal{A}} a = a_{(1)}\otimes a_{(2)} \tag{1.83}$$

are the coproducts on $\mathcal{A}$ and $\mathcal{U}$ respectively. This multiplication is associative and is induced from the action of $\mathcal{U}$ on $\mathcal{A}$

$$x \triangleright a = a_{(1)} < x, a_{(2)} > . \tag{1.84}$$

For simplicity, we will drop the tensor product sign $\otimes$ and write

$$a\otimes x = ax, \tag{1.85}$$

the above relation (1.82) becomes

$$xa = a_{(1)}x_{(2)} < x_{(1)}, a_{(2)} > . \tag{1.86}$$

1.3.3 Example: Quantum Group

Now consider the case that $\mathcal{U}$ is quasi-triangular and $(\mathcal{A},\mathcal{U})$ be a dual pair defined by a faithful representation

$$\rho : \mathcal{U} \to M_N(k). \tag{1.87}$$

The relations

$$R_{12}A_1A_2 = A_2A_1R_{12} \tag{1.88}$$

and

$$\begin{aligned} R_{12}L_2^+L_1^+ &= L_1^+L_2^+R_{12}, \\ R_{12}L_2^-L_1^- &= L_1^-L_2^-R_{12}, \\ R_{12}L_2^+L_1^- &= L_1^-L_2^+R_{12} \end{aligned} \tag{1.89}$$

have been obtained.

Using (1.86), one obtains, for example

$$\begin{aligned} L^{+i}{}_jA_l^k &= A_n^k < L^{+i}{}_m, A_l^n > L^{+m}{}_j \\ &= A_n^kR_{lm}^{ni}L^{+m}{}_j . \end{aligned}$$

In tensor notation, the full set of mixed relations are

$$\begin{aligned} L_1^+A_2 &= A_2R_{21}L_1^+, \\ L_1^-A_2 &= A_2R_{12}^{-1}L_1^-. \end{aligned} \tag{1.90}$$

Bicovariant Generators

It will be convenient to introduce the matrix Y_j^i

$$Y = L^+(L^-)^{-1} \tag{1.91}$$

and instead of (1.89),(1.90), we have[53, 54, 55, 56]

$$\hat{R}_{12}Y_2\hat{R}_{12}Y_2 = Y_2\hat{R}_{12}Y_2\hat{R}_{12}, \tag{1.92}$$

$$Y_1A_2 = A_2\hat{R}_{12}Y_2\hat{R}_{12}. \tag{1.93}$$

One reason for introducing these generators is that the relations (1.92),(1.93) have the remarkable property that they are covariant under the transformation[53]

$$A \to A'A, \quad Y \to Y, \tag{1.94}$$

or

$$A \to AA', \quad Y \to A'^{-1}YA', \tag{1.95}$$

where A' is a second copy of the quantum group satisfying (1.30) but is inert to A and Y, i.e. the matrix elements of A' commute with those of A as well as those of Y. The commutation relations (1.92),(1.93) are said to be "bicovariant" and the matrix elements of Y are bicovariant vector fields on the quantum group $\mathcal{A}$.

1.4 Differential Calculus on an Algebra

In this section, we want to introduce the notion of a differential calculus on an algebra.

1.4.1 First Order Differential Calculus on an Algebra

Classically, to introduce a differential calculus, one first has to know what is one-form. Then one can proceed to construct completely antisymmetrized tensor product of one-form and get the exterior algebra, with the wedge product. So the first thing we need to know is the properties of first order differential form in the classical case. Remember that we have the Leibniz rule

$$d(ab) = adb + (da)b, \quad a, b \in Fun(M) \tag{1.96}$$

and any one-form ρ can be written in the form $\rho = \sum_k a_k db_k, a_k, b_k \in Fun(M)$. Classically, one also knows that forms and functions commute i.e. $b(da) = (da)b$. In the deformed case, even the "functions" themselves don't commute, so one shouldn't expect this latter condition to be true. (Still, to be a useful calculus, it is required to give some sort of commutations. We will see how to do that later.) Therefore we will keep the first two properties in our definition of first order differential calculus on an algebra.

Definition 1.4.1 (First Order Differential Calculus)
Let $\mathcal{A}$ be an albegra with unity, Γ be a bimodule [2] *over $\mathcal{A}$ and $d : \mathcal{A} \to \Gamma$ be a linear map. We say that (Γ, d) is a first order differential calculus over $\mathcal{A}$ if*

1. $\forall a, b \in \mathcal{A}, d(ab) = adb + (da)b$,
2. *Any element $\rho \in \Gamma$ can be written in the form $\rho = \sum_k a_k db_k, a_k, b_k \in \mathcal{A}$.*

1.4.2 Universal Calculus and General Calculus

We have the definition. But it is not good for anything until we can actually construct "useful" calculus. This can be done in two steps. First, we construct

[2] A bimodule Γ of $\mathcal{A}$ is a vector space in which the algebra acts on it (or is represented) from both left and right.

a universal calculus over $\mathcal{A}$, and then we get a non-universal one called general calculus (the useful one).

Universal Calculus

The universal calculus $(\mathcal{A}^2, d)$ on $\mathcal{A}$ can be constructed as follows. Introduce

$$\mathcal{A}^2 = \ker\cdot \subset A \otimes A, \tag{1.97}$$

where $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map in $\mathcal{A}$ and $d : \mathcal{A} \to \mathcal{A}^2$ defined by

$$da = 1 \otimes a - a \otimes 1. \tag{1.98}$$

$\mathcal{A}^2$ can be given an $\mathcal{A}$ bimodule structure given by

$$c(\sum_k a_k \otimes b_k) = \sum_k (ca_k) \otimes b_k \tag{1.99}$$

$$(\sum_k a_k \otimes b_k)c = \sum_k a_k \otimes (b_k c) \tag{1.100}$$

for any $\sum_k a_k \otimes b_k \in \mathcal{A}^2$, $c \in \mathcal{A}$. Then d clearly obeys the Leibniz rule since,

$$d(ab) = 1 \otimes ab - ab \otimes 1, \tag{1.101}$$

$$adb = a \otimes b - ab \otimes 1, \quad (da)b = 1 \otimes ab - a \otimes b. \tag{1.102}$$

Furthermore, it is easy to see that every element of $\mathcal{A}^2$ can be represented in the form $\sum_k a_k db_k$. In this way $(\mathcal{A}^2, d)$ is indeed a first order differential calculus over $\mathcal{A}$ as stated. Note that the construction of universal calculus is canonical and it is constructed precisely to observe the Leibniz rule.

General Calculus

We still don't know the relations between adb and $(db)a$. Why then do we construct such an object? It is because every first order differential calculus on $\mathcal{A}$ can be obtained as a quotient of it. The way is to mode out a submodule in $\mathcal{A}^2$ to restrict the universal calculus to a general calculus. Taking the quotient is equivalent to imposing the commutation relations. More precisely,

Proposition 1.4.1 *Let N be a submodule of $\mathcal{A}^2$, define $\Gamma = \mathcal{A}^2/N, \pi$ be the canonical epimorphism $\mathcal{A}^2 \to \Gamma$ and $d' = \pi \circ d$. Then (Γ, d') is a first order differential calculus on $\mathcal{A}$.*

For example, if we take $N =< adb - (db)a, a, b \in Fun(M) >$, then we obtain the classical calculus.

Covariant Properties of Calculus

When $\mathcal{A}$ is a Hopf algebra, the notion of calculus is more fruitful. Introduce the coaction on $\mathcal{A}^2$ by

$$\Delta_L : \mathcal{A}^2 \to \mathcal{A} \otimes \mathcal{A}^2 : adb \mapsto a_{(1)}b_{(2)} \otimes a_{(2)}db_{(2)}, \tag{1.103}$$

$$\Delta_R : \mathcal{A}^2 \to \mathcal{A}^2 \otimes \mathcal{A} : adb \mapsto a_{(1)}db_{(1)} \otimes a_{(2)}b_{(2)}. \tag{1.104}$$

The first order differential calculus is said to be

1. left covariant if $\sum_k a_k db_k = 0 \Rightarrow \Delta_L(\sum_k a_k db_k) = 0$,
2. right covariant if $\sum_k a_k db_k = 0 \Rightarrow \Delta_R(\sum_k a_k db_k) = 0$,
3. bi-covariant if both.

Classically, the calculus on a Lie group is bicovariant in the sense that a zero differential form remains zero under a left or right group action.

For Hopf algebras, Woronowicz [14] gave a way to choose N so that the resulting general calculus enjoy some desired properties that must be there in the classical limit:

1. The prescription is to pick a right ideal M of $\mathcal{A}$ contained in $ker\epsilon$, then $N = \kappa(\mathcal{A} \otimes M)$ give a left covariant calculus.
2. If M is Ad_R invariant also, i.e. $Ad_R M \subset M \otimes \mathcal{A}$, then the calculus is bicovariant.

Here the maps κ and Ad_R are,

$$\kappa : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad \kappa(a \otimes a') = \sum aSa'_{(1)} \otimes a'_{(2)}, \tag{1.105}$$

$$Ad_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad Ad_R(a) = \sum a_{(2)} \otimes Sa_{(1)}a_{(3)}. \tag{1.106}$$

Note that Ad_R is just $T \to T'^{-1}TT'$ in the classical.

1.4.3 3-D Calculus on $SU_q(2)$

For $SU_q(2)$, the 3-D [3] calculus are defined [13] by picking the right ideal $M \in SU_q(2)$ which is generated by six elements,

$$\begin{aligned} &\delta + q^2\alpha - (1+q^2), \quad \gamma^2, \quad \beta\gamma, \\ &\beta^2, \quad (\alpha - 1)\gamma, \quad (\alpha - 1)\beta. \end{aligned}$$

We choose the basis of the space of the left-invariant 1-forms on P as

$$\begin{aligned} \omega^0 &= \delta d\beta - q^{-1}\beta d\delta, \\ \omega^1 &= \delta d\alpha - q^{-1}\beta d\gamma, \\ \omega^2 &= \gamma d\alpha - q^{-1}\alpha d\gamma. \end{aligned} \tag{1.107}$$

We have the following commutation relations between ω^i, $i = 0, 1, 2$ and the generators of $SU_q(2)$

$$\begin{aligned} \omega^0\alpha &= q^{-1}\alpha\omega^0, \quad \omega^0\beta = q\beta\omega^0, \\ \omega^1\alpha &= q^{-2}\alpha\omega^1, \quad \omega^1\beta = q^2\beta\omega^1, \\ \omega^2\alpha &= q^{-1}\alpha\omega^2, \quad \omega^2\beta = q\beta\omega^2. \end{aligned} \tag{1.108}$$

The remaining relations can be obtained by the replacement $\alpha \to \gamma$, $\beta \to \delta$. The relation between exterior differential d and basic one-forms ω^i is given by

$$d\alpha = \alpha\omega^1 - q\beta\omega^2, \quad d\beta = \alpha\omega^0 - q^2\beta\omega^1 \tag{1.109}$$

and similarly with α replaced by γ and β replaced by δ.

The higher order calculus is also determined

$$\begin{aligned} \omega^0\omega^0 &= 0 \quad , \quad \omega^2\omega^0 = -q^2\omega^0\omega^2, \\ \omega^1\omega^1 &= 0 \quad , \quad \omega^1\omega^0 = -q^4\omega^0\omega^1, \\ \omega^2\omega^2 &= 0 \quad , \quad \omega^2\omega^1 = -q^4\omega^1\omega^2. \end{aligned} \tag{1.110}$$

[3] It is called 3-D calculus because the dimension of the space of one-forms is 3 in this calculus. One can construct another calculus called the 4-D calculus on $SU(2)$ which is bicovariant. That one has a dimension of four for the space of one-forms,one too many!

We also have the Maurer-Cartan equations

$$\begin{aligned} d\omega^0 &= q^2(1+q^2)\omega^0\omega^1, \\ d\omega^1 &= q\omega^0\omega^2, \\ d\omega^2 &= q^2(1+q^2)\omega^1\omega^2. \end{aligned} \tag{1.111}$$

Chapter 2

Pseudodifferential Operators Realization of Vector Fields

In this chapter, we consider the realization of the vector fields of the quantum Lie algebra of the quantum groups $GL_q(N), SL_q(N)$ and $SO_q(N)$ as pseudodifferential operators on the linear quantum spaces covariant under the corresponding quantum groups. Their expressions are simple and compact. These vector fields satisfy certain characteristic polynomial identities. The real forms $SU_q(N)$ and $SO_q(N,R)$ are also discussed.

2.1 Vector Fields for Quantum Groups

As seen in Chapter 1, a quantum group can be described in terms of $N \times N$ matrices A with noncommuting elements satisfying the equation

$$\hat{R}_{12}A_1A_2 = A_1A_2\hat{R}_{12}, \tag{2.1}$$

with the $\hat{R}$ matrix appropriate to the particular quantum group. The matrix elements generate the algebra of functions on the group. We also derived the vector fields on the quantum group. It can be described by the matrix elements of a matrix Y satisfying the commutation relation

$$\hat{R}_{12}Y_2\hat{R}_{12}Y_2 = Y_2\hat{R}_{12}Y_2\hat{R}_{12}, \tag{2.2}$$

which corresponds to the Lie algebra relations in the classical case. The action of the vector fields on the group is then given by the commutation relation

$$Y_1 A_2 = A_2 \hat{R}_{12} Y_2 \hat{R}_{12}. \tag{2.3}$$

The quantum group matrices can coact on a quantum space, for instance by right multiplication. A point of coordinates x_{0i} is transformed into $x_i = x_{0j} A^j_i$ or, more compactly,

$$x = x_0 A. \tag{2.4}$$

Keeping the original point x_0 fixed, the action of a vector field on the quantum group induces an action on the quantum space

$$Y_1 x_2 = x_2 \hat{R}_{12} Y_2 \hat{R}_{12}, \tag{2.5}$$

i.e.

$$Y^i_j x_k = x_m \hat{R}^{im}_{ln} Y^n_r \hat{R}^{lr}_{jk}. \tag{2.6}$$

We shall consider the case when a differential calculus covariant with respect to the coaction of the quantum group exists on the quantum space. In this case it is natural to ask whether it is possible to realize the vector fields Y as pseudodifferential operators satisfying (2.2) and (2.5). We shall show that this can be done for the quantum groups $GL_q(N), SL_q(N)$ and $SO_q(N)$. Their real forms are also considered.

2.2 q-Determinant for FRT Type Algebra

For the sake of completeness, we construct and discuss the properties of the q-determinant for FRT type algebras. The discussion is self-contained. By assuming the top form of the associated exterior quantum plane to be unique, we construct the completely antisymmetric tensor ϵ and the q-determinant for FRT type algebras. Contrary to [57], we don't need to assume the existence of a metric.

First we recall the definition of the FRT algebra $\mathcal{A}$ and the associated quantum exterior algebra. FRT algebra is an associative algebra generated by 1 and $t^i{}_j$; $i, j = 1, \ldots, N$ which obey the RTT equation (2.1). Projectors can be introduced by

looking at the spectral decomposition of the R matrix. In particular, the q-exterior algebra is defined by the q-symmetrizer P_S:

$$dx_1 dx_2 (P_S)^{12} = 0. \tag{2.7}$$

Consider the top forms $dx_{i_1} dx_{i_2} \ldots dx_{i_N}$. Assuming that it is unique, then one can define an $N-$dimensional tensor ϵ by,[1]

$$dx_{i_1} dx_{i_2} \ldots dx_{i_N} = \epsilon_{i_1 i_2 \ldots i_N} dv \tag{2.8}$$

Like in the classical, our ϵ is completely antisymmetric:

$$\epsilon_{i_1 i_2 \ldots j_r j_{r+1} \ldots i_N} P_S{}^{j_r j_{r+1}}_{i_r i_{r+1}} = 0$$

from the definition of the exterior algebra.

Now, we can prove an important property for ϵ,

Proposition 2.2.1

$$\epsilon_{i_1 i_2 \ldots i_N} t^{i_1}_{j_1} t^{i_2}_{j_2} \ldots t^{i_N}_{j_N} = \epsilon_{j_1 j_2 \ldots j_N} D \tag{2.9}$$

for some $D \in \mathcal{A}$.

Proof Consider $dx_{i_1} dx_{i_2} \ldots dx_{i_N} = \epsilon_{i_1 i_2 \ldots i_N} dv$, multiply both side by $t^{i_1}_{j_1} t^{i_2}_{j_2} \ldots t^{i_N}_{j_N}$, we get

$$dx_{i_1} dx_{i_2} \ldots dx_{i_N} \otimes t^{i_1}_{j_1} t^{i_2}_{j_2} \ldots t^{i_N}_{j_N} = dv \otimes \epsilon_{i_1 i_2 \ldots i_N} t^{i_1}_{j_1} t^{i_2}_{j_2} \ldots t^{i_N}_{j_N}. \tag{2.10}$$

But the left side is equal to

$$\Delta_{\mathcal{A}} (dx_{j_1} dx_{j_2} \ldots dx_{j_N}), \tag{2.11}$$

where $\Delta_{\mathcal{A}}$ is the left coaction of $\mathcal{A}$ on the quantum plane

$$\Delta_{\mathcal{A}} x_i = x_j \otimes t^j_i, \quad \Delta_{\mathcal{A}} dx_i = dx_j \otimes t^j_i. \tag{2.12}$$

Since

$$\Delta_{\mathcal{A}} (dx_{j_1} dx_{j_2} \ldots dx_{j_N}) = \epsilon_{j_1 j_2 \ldots j_N} \Delta_{\mathcal{A}} (dv) = \epsilon_{j_1 j_2 \ldots j_N} dv \otimes D, \tag{2.13}$$

[1] The ϵ tensor introduced here has lower indices while the one in Chapter 1 and Chapter 4 has upper indices. The reason is because we have used coordinates x_i with lower indices here. The two are however identical.

where we have used the fact that the top form is unique and so $\Delta_{\mathcal{A}}(dv) = dv \otimes D$ for some $D \in \mathcal{A}$. Comparing the coefficient of dv in (2.10) and (2.13), we obtain $\epsilon_{i_1 i_2 \ldots i_N} t^{i_1}_{j_1} t^{i_2}_{j_2} \ldots t^{i_N}_{j_N} = \epsilon_{j_1 j_2 \ldots j_N} D$. □

We call this D the q-determinant. Notice that equation (2.9) can be written compactly as

$$\epsilon_{12\ldots N} T_1 T_2 \ldots T_N = D \epsilon_{12\ldots N}. \tag{2.14}$$

D has the following properties,

Proposition 2.2.2 *$\epsilon(D) = 1$ and $\Delta(D) = D \otimes D$.*

Proof The first equation is trivial. For the second one, apply Δ to (2.9)

$$\begin{aligned} \epsilon_{j_1 j_2 \ldots j_N} \Delta(D) &= \epsilon_{i_1 i_2 \ldots i_N} t^{i_1}_{k_1} t^{i_2}_{k_2} \ldots t^{i_N}_{k_N} \otimes t^{k_1}_{j_1} t^{k_2}_{j_2} \ldots t^{k_N}_{j_N} \\ &= D \epsilon_{k_1 k_2 \ldots k_N} \otimes t^{k_1}_{j_1} t^{k_2}_{j_2} \ldots t^{k_N}_{j_N} \\ &= D \otimes D \epsilon_{j_1 j_2 \ldots j_N}. \end{aligned}$$

□

Proposition 2.2.3 *D is central in the algebra $\mathcal{A}$.*

Proof First, notice that the FRT algebra admits the representations

$$\rho_+(T_1)_0 = R_{10}, \quad \rho_-(T_1)_0 = R^{-1}_{01} \tag{2.15}$$

Thus we have[2]

$$\epsilon_{12\ldots N} R^{-1}_{01} R^{-1}_{02} \ldots R^{-1}_{0N} = \rho_-(D) \epsilon_{12\ldots N} I_0, \tag{2.18}$$

[2]For example, it is

$$\epsilon_{12\ldots N} R_{0N} \ldots R_{01} = c \epsilon_{12\ldots N} I_0 = \epsilon_{12\ldots N} R_{10} R_{20} \ldots R_{N0}, \tag{2.16}$$

where $c = q$ and 1 for $GL_q(N)$ and $SO_q(N)$, respectively. In terms of the compact notation to be introduced in the next subsection, this can be written as

$$\epsilon_{12\ldots N} R_{0,(12\ldots N)} = c \epsilon_{12\ldots N} I_0 = \epsilon_{12\ldots N} R_{(12\ldots N),0}. \tag{2.17}$$

where I_0 denotes the identity matrix in the zeroth vector space. We also have the inverse relation

$$\epsilon_{12\ldots N} R_{0N} \ldots R_{01} = \rho_-(D)^{-1} \epsilon_{12\ldots N} I_0. \tag{2.19}$$

Now, consider

$$\begin{aligned} \epsilon_{12\ldots N} T_0 D &= \epsilon_{12\ldots N} T_0 T_1 T_2 \ldots T_N \\ &= \epsilon_{12\ldots N} R^{-1}_{01} R^{-1}_{02} \ldots R^{-1}_{0N} T_1 T_2 \ldots T_N T_0 R_{0N} \ldots R_{01} \\ &= \rho_-(D) \epsilon_{12\ldots N} T_1 T_2 \ldots T_N T_0 R_{0N} \ldots R_{01} \\ &= \rho_-(D) D T_0 \epsilon_{12\ldots N} R_{0N} \ldots R_{01} \\ &= \rho_-(D) \rho_-(D)^{-1} D T_0 \epsilon_{12\ldots N} \\ &= D T_0 \epsilon_{12\ldots N}. \end{aligned} \tag{2.20}$$

Hence $t^i_j D = D t^i_j$. □

2.2.1 q-Determinant for Bicovariant Vector Fields Y

Recall that the bicovariant vector field matrix Y^i_j is defined by

$$Y^i_j = < \mathcal{R}_{21} \mathcal{R}_{12}, A^i_j \otimes id > \tag{2.21}$$

and satisfy

$$R_{21} Y_1 R_{12} Y_2 = Y_2 R_{21} Y_1 R_{12}, \tag{2.22}$$

or equivalently,

$$\hat{R}_{12} Y_2 \hat{R}_{12} Y_2 = Y_2 \hat{R}_{12} Y_2 \hat{R}_{12}. \tag{2.23}$$

Following [56], it will be convenient to introduce an associative $\bullet$-product by

$$\begin{aligned} Y_1 \bullet Y_2 \bullet \cdots \bullet Y_k &:= < \mathcal{R}_{21} \mathcal{R}_{12}, A_1 A_2 \cdots A_k \otimes id > \\ &= (R^{-1}_{12} R^{-1}_{13} \cdots R^{-1}_{1k} Y_1 R_{1k} \cdots R_{12}) \cdot \\ &\quad \cdot (R^{-1}_{23} R^{-1}_{24} \cdots R^{-1}_{2k} Y_2 R_{2k} \cdots R_{23}) \cdots \\ &\quad \cdot (R^{-1}_{k-1\,k} Y_{k-1} R_{k-1\,k}) Y_k, \end{aligned} \tag{2.24}$$

then (2.22) can be rewritten in the FRT form

$$R_{12} Y_1 \bullet Y_2 = Y_2 \bullet Y_1 R_{12}. \tag{2.25}$$

And so one can define the q-determinant

$$\epsilon_{12\ldots N} Y_1 \bullet Y_2 \bullet \cdots \bullet Y_N = Det_q Y \epsilon_{12\ldots N}, \tag{2.26}$$

which is central and group-like. Remember that in chapter 1, we introduced the numerical R-matrix that gives the commutation relations between two quantum matrices in the fundamental representation

$$R^{ik}_{mn} = < \mathcal{R}, A^i_m \otimes A^k_n > . \tag{2.27}$$

For higher tensor product representations: A_1A_2, $A_1A_2A_3$, ...,$A_1A_2A_3\cdots A_m$ one can introduce the corresponding numerical R-matrix [56],

$$\begin{aligned} R_{I,II} &\equiv < \mathcal{R}, A_I \otimes A_{II} > \\ &= \begin{array}{ccccc} R_{1'm} & \cdot R_{1'(m-1)} & \cdot \ldots \cdot & R_{1'1} \\ \cdot R_{2'm} & \cdot R_{2'(m-1)} & \cdot \ldots \cdot & R_{2'1} \\ \vdots & \vdots & & \vdots \\ \cdot R_{n'm} & \cdot R_{n'(m-1)} & \cdot \ldots \cdot & R_{n'1} \end{array} \end{aligned}$$

where

$$A_I \equiv A_{(1'2'\cdots n')} \equiv A_{1'}A_{2'}\cdots A_{n'}, \quad A_{II} \equiv A_{(12\cdots m)} \equiv A_1A_2\cdots A_m \tag{2.28}$$

satisfy

$$R_{I,II}A_IA_{II} = A_{II}A_IR_{I,II} \tag{2.29}$$

and $R_{I,II}$ satisfies the QYBE

$$R_{I,II}R_{I,III}R_{II,III} = R_{II,III}R_{I,III}R_{I,II}. \tag{2.30}$$

Eq. (2.26) can be rewritten compactly as

$$\epsilon_{12\cdots N} Y_{(12\cdots N)} = Det_q Y \epsilon_{12\cdots N}, \tag{2.31}$$

where we have introduced the notation

$$Y_{(12\cdots N)} = Y_1 \bullet Y_2 \bullet \cdots \bullet Y_N \tag{2.32}$$

in the same way as in (2.28). For simplicity, we will sometimes write Det_qY as $DetY$. For arbitrary $I = (1'2'\cdots n')$, $II = (12\cdots m)$, we have

$$Y_I \bullet Y_{II} = R^{-1}_{I,II} Y_I R_{I,II} Y_{II}. \tag{2.33}$$

2.3 $GL_q(N), SL_q(N)$ and $SU_q(N)$

2.3.1 $GL_q(N)$

The calculus for the quantum plane covariant under $GL_q(N)$ is well known[58]. The coordinates x_i in the plane satisfy the commutation relations

$$x_1x_2 = q^{-1}x_1x_2\hat{R}_{12} \tag{2.34}$$

and the derivatives ∂^i satisfy

$$\partial^i x_j = \delta^i_j + q\hat{R}^{ik}_{jl}x_k\partial^l \tag{2.35}$$

and

$$\partial_2\partial_1 = q^{-1}\hat{R}_{12}\partial_2\partial_1. \tag{2.36}$$

All indices run from 1 to N and $\hat{R}$ is the $GL_q(N)$ matrix, which satisfies the characteristic equation

$$\hat{R}^2 = 1 + \lambda\hat{R}, \quad \lambda = q - q^{-1}. \tag{2.37}$$

Using (2.37) and the above commutation relations, it is easy to verify that

Construction 2.3.1 (Realization for $GL_q(N)$)
The differential operator

$$Y^i_j = q^{-2}\delta^i_j + q^{-1}\lambda\partial^i x_j \tag{2.38}$$

satisfies (2.5),

$$\partial_2 Y_1 = \hat{R}_{12}Y_2\hat{R}_{12}\partial_2 \tag{2.39}$$

and (2.2).

Characteristic Equation

Let μ be the rescaling operator in the plane

$$\mu = 1 + q\lambda x_i\partial^i, \tag{2.40}$$

which satisfies

$$\mu x_i = q^2 x_i\mu, \quad \partial^i\mu = q^2\mu\partial^i. \tag{2.41}$$

It is obvious that μ commutes with the elements of Y.

It is very easy to verify that the matrix given by (2.38) satisfies the identity

$$(Y-\mu)(Y-q^{-2})=0, \tag{2.42}$$

where matrix multiplication is implied. This is a special example of polynomial characteristic equations satisfied by quantum vector fields[59]. In general these equations are of higher order but for the realization (2.38) we see that the polynomial is quadratic in Y.

The invariant quantum trace of the k-th power of the matrix Y is defined as

$$t_k = TrD^{-1}Y^k, \tag{2.43}$$

where D is the diagonal matrix $(1, q^2, ..., q^{2(N-1)})$. The t_k's commute with the matrix elements of Y. In general only the first r of them ($k = 1, 2, ..., r$) are independent and they generate the center of the Y algebra (2.2), where r is the rank of the group[8]. For Y given by (2.38) all t_k are simply functions of μ. For instance,

$$\begin{aligned} t_1 &= [N]-1+\mu = q^{-2}t_0 - q^{-2N}+\mu, \\ t_2 &= q^{-2}t_1 - \mu q^{-2N}+\mu^2, \\ t_3 &= q^{-2}t_2 - \mu^2 q^{-2N}+\mu^3, \end{aligned} \tag{2.44}$$

etc., where

$$[N] = 1+q^{-2}+q^{-4}+...+q^{-2(N-1)}. \tag{2.45}$$

$GL_q(2)$ characteristic equation

For $GL_q(2)$, denote

$$Y = \begin{pmatrix} y_1 & y_+ \\ y_- & y_2 \end{pmatrix}, \tag{2.46}$$

then (2.2) gives explicitly,

$$\begin{aligned} y_1y_+ - y_+y_1 + \lambda q^{-1}y_+y_2 &= 0, \\ y_1y_- - y_-y_1 - \lambda q^{-1}y_2y_- &= 0, \\ y_2y_+ &= q^2y_+y_2, \end{aligned}$$

$$\begin{aligned} y_2y_- &= q^{-2}y_-y_2, \\ y_1y_2 &= y_2y_1, \\ y_+y_- - y_-y_+\lambda q^{-1}y_2(y_1-y_2) &= 0. \end{aligned} \tag{2.47}$$

Using these, it was discovered by Professor B. Zumino [54] that a characteristic identity exists for this algebra,

$$Y^2 - p_1Y - p_2 = 0, \tag{2.48}$$

where

$$\begin{aligned} p_1 &= TrD^{-1}Y = y_1 + q^{-2}y_2, \\ p_2 &= \tfrac{1}{[2]}(t_2 - t_1^2) = y_+y_- - q^{-2}y_1y_2 = -q^{-2}DetY. \end{aligned} \tag{2.49}$$

Using (2.44), (2.48) reduces to (2.42) if

$$DetY = \mu \tag{2.50}$$

holds for our realization (2.38). This is in fact true, first we have the lemma

Lemma 2.3.1

$$DetYx = q^2xDetY, \quad DetY\partial = q^{-2}\partial DetY. \tag{2.51}$$

Proof First we need

$$Y_Ix_0 = x_0R_{0I}Y_IR_{I0} \tag{2.52}$$

for any $I = (12\cdots m)$. This is true for $I = (1)$. Assume it is true for I and II, then

$$\begin{aligned} Y_{III}x_0 &= Y_I \bullet Y_{II}x_0 \\ &= R^{-1}_{I,II}Y_IR_{I,II}Y_{II}x_0 \\ &= R^{-1}_{I,II}Y_IR_{I,II}x_0R_{0,II}Y_{II}R_{II,0} \\ &= x_0R^{-1}_{I,II}R_{0I}Y_IR_{I0}R_{I,II}R_{0,II}Y_{II}R_{II,0} \\ &= x_0R^{-1}_{I,II}R_{0I}R_{0,II}Y_IR_{I,II}R_{I0}Y_{II}R_{II,0} \\ &= x_0R_{0,II}R_{0I}R^{-1}_{I,II}Y_IR_{I,II}Y_{II}R_{I0}R_{II,0} \\ &= x_0R_{0,(III)}Y_{III}R_{(III),0}. \end{aligned} \tag{2.53}$$

Hence

$$\begin{aligned} DetY x_0 \epsilon_{12\cdots N} &= \epsilon_{12\cdots N} Y_{(12\cdots N)} x_0 \\ &= \epsilon_{12\cdots N} x_0 R_{0,(12\cdots N)} Y_{(12\cdots N)} R_{(12\cdots N),0} \\ &= q^2 x_0 DetY \epsilon_{12\cdots N}. \end{aligned} \tag{2.54}$$

We have used the fact (2.16)

$$\epsilon_{12\cdots N} R_{(12\cdots N),0} = q I_0 \epsilon_{12\cdots N} = \epsilon_{12\cdots N} R_{0,(12\cdots N)}. \tag{2.55}$$

The proof for $DetY\partial = q^{-2}\partial DetY$ is similar. □

Proposition 2.3.1 *For the particular representation (2.38) of the Y matrix, it is*

$$DetY = \mu. \tag{2.56}$$

Proof Remember that $DetY$ is central in the Y algebra. Now, since the quantum traces t_k, $k = 1, 2, \cdots, r$ generate the center of the Y algebra and t_k is a polynomial of μ of degree k, we have

$$DetY = \sum_{n=0}^{N} a_n \mu^n, \tag{2.57}$$

where the numerical coefficients a_n are functions of q and N. Due to lemma 2.3.1, $DetY/\mu$ commutes with x and ∂, we have

$$DetY = a\mu = a + \lambda q a x \cdot \partial, \tag{2.58}$$

where a is a constant. We claim that $a = 1$. First recall the definition of $DetY$

$$\epsilon_{j_1 j_2 \cdots j_N} DetY = \epsilon_{i_1 i_2 \cdots i_N} (Y_{(12\cdots N)})^{i_1 i_2 \cdots i_N}_{j_1 j_2 \cdots j_N}, \tag{2.59}$$

where

$$\begin{aligned} Y_{(12\cdots N)} = &(R_{12}^{-1} R_{13}^{-1} \cdots R_{1N}^{-1} Y_1 R_{1N} \cdots R_{12}) \cdot \\ &\cdot (R_{23}^{-1} R_{24}^{-1} \cdots R_{2N}^{-1} Y_2 R_{2N} \cdots R_{23}) \cdots (R_{N-1\,N}^{-1} Y_{N-1} R_{N-1\,N}) Y_N. \end{aligned} \tag{2.60}$$

Using

$$Y^i_j = q^{-2}\delta^i_j + q^{-1}\lambda \partial^i x_j = \delta^i_j + \lambda \hat{R}^{ik}_{jl} x_k \partial^l, \tag{2.61}$$

one can easily extract the constant term in (2.60)

$$(Y_{(12\cdots N)})^{i_1 i_2 \cdots i_N}_{j_1 j_2 \cdots j_N} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_N}_{j_N} + \cdots, \tag{2.62}$$

where $\cdots$ is a polynomial of x and ∂ with at least one x and one ∂ in each term. Substituting into (2.59),

$$\epsilon_{i_1 i_2 \cdots i_N} DetY = 1 \epsilon_{i_1 i_2 \cdots i_N} + \cdots \tag{2.63}$$

Comparing with (2.58) shows that $a = 1$. □

2.3.2 $SL_q(N)$

The quantum subgroup $SL_q(N)$ can be obtained from $GL_q(N)$ as follows[55]. For the quantum matrices one uses the standard quantum determinant $det_q A$ and defines a new matrix

$$T = (det_q A)^{-1/N} A \tag{2.64}$$

having quantum determinant equal to one. For the vector fields, one uses the determinant $DetY$ and defines a new matrix of vector fields[55, 56]

$$Z = (DetY)^{-1/N} Y \tag{2.65}$$

having determinant one. The number of independent elements of the matrix Z is $N^2 - 1$, as in the classical case.

Construction 2.3.2 (Realization for $SL_q(N)$)

$$Z = \mu^{-1/N} Y \tag{2.66}$$

realizes the $SL_q(N)$ vector fields as pseudodifferential operators in the quantum plane.

2.3.3 Real Forms $U_q(N)$ and $SU_q(N)$

If $|q| = 1$ the calculus given by (2.34-2.36) for the quantum plane can be given a reality structure[58, 60] by requiring x_i to be real

$$x_i^* = x_i \tag{2.67}$$

and by defining conjugate derivatives as

$$\partial^{i*} = -q^{2i'}\partial^i, \tag{2.68}$$

where we have introduced the notation

$$i' = N + 1 - i, \quad i = 1, 2, ..., N. \tag{2.69}$$

Here we consider instead the case when q is real and the complex conjugates of x_i and of ∂^i are new independent variables. The complex conjugation $*$ is an involution which inverts the order of factors in a product. It will be convenient to give them new names, i.e. we set

$$x_i^* = \hat{x}^i \tag{2.70}$$

and

$$\partial^{i*} = -\hat{\partial}_i. \tag{2.71}$$

The commutation relations of these new variables can be obtained immediately from (2.34-2.36) by complex conjugation. Using the symmetry property

$$\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}, \tag{2.72}$$

we see that

$$\hat{x}_2\hat{x}_1 = q^{-1}\hat{R}_{12}\hat{x}_2\hat{x}_1, \tag{2.73}$$

$$\hat{x}^j\hat{\partial}_i = -\delta^j_i + q\hat{R}^{jl}_{ik}\hat{\partial}_l\hat{x}^k \tag{2.74}$$

and

$$\hat{\partial}_1\hat{\partial}_2 = q^{-1}\hat{\partial}_1\hat{\partial}_2\hat{R}_{12}. \tag{2.75}$$

Eq.(2.74) can be written in a form more analogous to (2.35) if one introduces the matrix

$$\Psi^{ir}_{js} = (\hat{R}^{-1})^{ri}_{sj}q^{2(j-r)} = (\hat{R}^{-1})^{ri}_{sj}q^{2(i-s)}, \tag{2.76}$$

which satisfies

$$\hat{R}^{kj}_{li}\Psi^{ir}_{js} = \Psi^{kj}_{li}\hat{R}^{ir}_{js} = \delta^k_s\delta^r_l, \tag{2.77}$$

$$\Psi^{ri}_{si} = \delta^r_s q^{-2(N-r)-1} \tag{2.78}$$

and

$$\Psi^{ir}_{is} = \delta^r_s q^{-2(r-1)-1}. \tag{2.79}$$

It takes the form

$$\hat{\partial}_i\hat{x}^j = \delta^j_i q^{-2i'} + q^{-1}\Psi^{jl}_{ik}\hat{x}^k\hat{\partial}_l, \tag{2.80}$$

where i' is given by (2.69).

To complete the algebra of the complex calculus, we must now give commutation relations between the variables x_i, ∂^i and their conjugates $\hat{x}^i, \hat{\partial}_i$. A consistent set is given by

$$\hat{x}^i x_j = q(\hat{R}^{-1})^{ik}_{jl}x_k\hat{x}^l, \tag{2.81}$$

$$\partial^i\hat{x}^j = q(\hat{R}^{-1})^{ji}_{lk}\hat{x}^k\partial^l, \tag{2.82}$$

$$\hat{\partial}_i x_j = q^{-1}\hat{R}^{kl}_{ij}x_k\hat{\partial}_l \tag{2.83}$$

and

$$\partial^i\hat{\partial}_j = q^{-1}\hat{R}^{ik}_{jl}\hat{\partial}_k\partial^l. \tag{2.84}$$

Consistency can be checked by verifying that all these relations braid correctly with each other.

Having the complex calculus we can now ask how the vector field realization of (2.38) acts on the conjugate variables. It is not hard to verify that

$$\hat{x}_2 Y_1 = \hat{R}_{12}Y_2\hat{R}^{-1}_{12}\hat{x}_2 \tag{2.85}$$

and

$$Y_1\hat{\partial}_2 = \hat{\partial}_2\hat{R}_{12}Y_2\hat{R}^{-1}_{12}. \tag{2.86}$$

On the other hand, by complex conjugation, (2.5),(2.39), (2.85) and (2.86) give

$$\hat{x}_2 Y_1^\dagger = \hat{R}_{12}Y_2^\dagger\hat{R}_{12}\hat{x}_2, \tag{2.87}$$

$$Y_1^\dagger\hat{\partial}_2 = \hat{\partial}_2\hat{R}_{12}Y_2^\dagger\hat{R}_{12}, \tag{2.88}$$

$$Y_1^\dagger x_2 = x_2\hat{R}^{-1}_{12}Y_2^\dagger\hat{R}_{12} \tag{2.89}$$

and

$$\partial_2 Y_1^\dagger = \hat{R}^{-1}_{12}Y_2^\dagger\hat{R}_{12}\partial_2, \tag{2.90}$$

where $Y^\dagger$ is the hermitian conjugate of the matrix Y

$$(Y^\dagger)^i_j = Y_i^{j*} = q^{-2}\delta^i_j - q^{-1}\lambda\hat{x}^i\hat{\partial}_j, \tag{2.91}$$

which satisfies the equation conjugate to (2.2)

$$\hat{R}_{12}Y_2^\dagger\hat{R}_{12}Y_2^\dagger = Y_2^\dagger\hat{R}_{12}Y_2^\dagger\hat{R}_{12}, \tag{2.92}$$

as well as the commutation relation with Y

$$\hat{R}_{12}Y_2\hat{R}_{12}^{-1}Y_2^\dagger = Y_2^\dagger\hat{R}_{12}Y_2\hat{R}_{12}^{-1}. \tag{2.93}$$

Until now, we have considered two $GL_q(N)$ groups complex conjugate of each other, i.e. a truly complex $GL_q(N)$[61, 62, 63]. The quantum group can be restricted to $U_q(N)$ by imposing on its matrices the unitarity condition

$$A^\dagger = A^{-1} \tag{2.94}$$

and to $SU_q(N)$ by further normalizing the matrices as in (2.64) so that they have quantum determinant equal to one.

The vector fields of the $U_q(N)$ subgroup can be defined as the elements of the Hermitian matrix

$$U = YY^\dagger. \tag{2.95}$$

Indeed, it is very easy to check that U commutes with the Hermitian length

$$\mathcal{L} = x_i\hat{x}^i = x_ix_i^* \tag{2.96}$$

(Y and $Y^\dagger$ separately do not), i.e. the U vector fields leave $\mathcal{L}$ invariant. U is a perfectly good matrix of vector fields and satisfies equations similar to (2.2) and (2.5)

$$\hat{R}_{12}U_2\hat{R}_{12}U_2 = U_2\hat{R}_{12}U_2\hat{R}_{12}, \tag{2.97}$$

$$U_1x_2 = x_2\hat{R}_{12}U_2\hat{R}_{12} \tag{2.98}$$

and

$$\hat{x}_2U_1 = \hat{R}_{12}U_2\hat{R}_{12}\hat{x}_2, \tag{2.99}$$

as a consequence of the equations for Y and $Y^\dagger$ given above. Notice that

$$q^2U^i_j = q^{-2}\delta^i_j + q^{-1}\lambda\partial^ix_j - q^{-1}\lambda\hat{x}^i\hat{\partial}_j - \lambda^2\partial^i\mathcal{L}\hat{\partial}_j, \tag{2.100}$$

which will be useful to us later.

Finally we observe that, if we want to reduce the vector fields to the number appropriate to $SU_q(N)$, we must normalize U, i.e., take the matrix

$$ZZ^\dagger = U/(DetU)^{1/N}. \tag{2.101}$$

Lemma 2.3.2 *It is*

$$DetU = \mu\mu^*. \tag{2.102}$$

Proof By use of the same techniques as in lemma 2.3.1, one can verify that with

$$U \equiv YY^\dagger \tag{2.103}$$

then

$$U_I = Y_IY_I^\dagger \tag{2.104}$$

for any $I = (12\cdots m)$. Hence

$$\begin{aligned}\epsilon_{12\cdots N}DetU &= \epsilon_{12\cdots N}U_{(12\cdots N)}\\ &= \epsilon_{12\cdots N}Y_{(12\cdots N)}Y^\dagger_{(12\cdots N)}\\ &= \epsilon_{12\cdots N}DetY\,Det(Y^\dagger)\\ &= \mu\mu^*\epsilon_{12\cdots N}.\end{aligned} \tag{2.105}$$

□

In addition to commuting with Y^i_j, the rescaling operator μ in (2.40) commutes with $\hat{x}^i, \hat{\partial}_i$ and therefore with $(Y^\dagger)^i_j$ and

$$\mu^* = 1 - q\lambda\hat{\partial}_i\hat{x}^i. \tag{2.106}$$

On the other hand μ^* commutes with $(Y^\dagger)^i_j, x_i, \partial^i, Y^i_j$ and satisfies

$$\mu^*\hat{x}^i = q^{-2}\hat{x}^i\mu^*, \quad \hat{\partial}_i\mu^* = q^{-2}\mu^*\hat{\partial}_i. \tag{2.107}$$

Clearly $\mu\mu^*$ commutes with $\mathcal{L}$, therefore so does $ZZ^\dagger$. Z and $Z^\dagger$ satisfy equations analogous to (2.2),(2.92),(2.93). Using this fact one can show that

$$DetZZ^\dagger = (DetZ)(DetZ^\dagger) = 1. \tag{2.108}$$

Notice that the vector field matrix $ZZ^\dagger$ is Hermitian, which is the natural reality condition for $SU_q(N)$.

Construction 2.3.3 (Realization for $SU_q(N)$)

$$ZZ^\dagger = U/(\mu\mu^*)^{1/N} \tag{2.109}$$

provides the pseudodifferential operator realization for the $SU_q(N)$ vector fields.

2.4 $SO_q(N)$ and $SO_q(N,R)$

We shall call T the quantum matrix of $SO_q(N)$, instead of A. In addition to

$$\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}, \tag{2.110}$$

they satisfy the orthogonality relations [8]

$$T^t g T = g, \quad T g^{-1} T^t = g^{-1}, \tag{2.111}$$

where the numerical quantum metric matrices $g = g_{ij}$ and $g^{-1} = g^{ij}$ can be chosen to be equal $g_{ij} = g^{ij}$. The $SO_q(N)$ $\hat{R}$ matrix satisfies also orthogonality conditions

$$(\hat{R}^{-1})^{ij}_{kl} = g^{im}\hat{R}^{jn}_{mk}g_{nl} = g_{km}\hat{R}^{mi}_{ln}g^{nj}, \tag{2.112}$$

as well as the usual symmetry relations

$$\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}. \tag{2.113}$$

The $SO_q(N)$ vector field matrix, which we shall call Z, satisfies

$$\hat{R}_{12}Z_2\hat{R}_{12}Z_2 = Z_2\hat{R}_{12}Z_2\hat{R}_{12}, \tag{2.114}$$

$$Z_1T_2 = T_2\hat{R}_{12}Z_2\hat{R}_{12}, \tag{2.115}$$

as well as an orthogonality constraint in one of the two equivalent forms[54, 56]

$$g_{ij}(Z_2\hat{R}_{12}Z_2)^{ij}_{kl} = q^{1-N}g_{kl}, \tag{2.116}$$

$$(Z_2\hat{R}_{12}Z_2)^{ij}_{kl}g^{kl} = q^{1-N}g^{ij}. \tag{2.117}$$

Eq.(2.116) or (2.117) reduces the number of independent vector fields from N^2 to $N(N-1)/2$ as in the classical case.

The projector decomposition of the $\hat{R}$ matrix for $SO_q(N)$ is

$$\hat{R} = qP^+ - q^{-1}P^- + q^{1-N}P^0. \tag{2.118}$$

Here P^+ is the traceless part of the symmetrizer, P^- is the antisymmetrizer and P^0 is the trace operator. It is related to the metric by

$$(P^0)^{ij}_{kl} = \nu g^{ij}g_{kl}, \quad \nu = \frac{\lambda}{(q^N-1)(q^{1-N}+q^{-1})}. \tag{2.119}$$

The coordinates x_i of the quantum Euclidean space satisfy the commutation relations

$$x_kx_l(P^-)^{kl}_{ij} = 0, \tag{2.120}$$

or in one of the two equivalent forms[3]

$$x_kx_l\hat{R}^{kl}_{ij} = qx_ix_j - \lambda Lg_{ij}, \tag{2.121}$$

$$x_kx_l(\hat{R}^{-1})^{kl}_{ij} = q^{-1}x_ix_j + \lambda q^{N-2}Lg_{ij}, \tag{2.122}$$

where the length L is defined as

$$L = \alpha x \cdot x, \tag{2.123}$$

$x \cdot x \equiv x_kx_lg^{kl} = x_kx^k$ and

$$\alpha = \frac{1}{1+q^{N-2}}. \tag{2.124}$$

As a consequence of (2.121), the length L commutes with all the coordinates,

$$Lx_i = x_iL. \tag{2.125}$$

Since $g_{ij} \neq g_{ji}$, we need a convention to raise and lower the indices, we adopt

$$m_i = g_{ij}m^j, \quad m^i = g^{ij}m_j. \tag{2.126}$$

[3]Due to the $SO_q(N)$ characteristic equation for the numerical R matrix,

$$\hat{R}^{ij}_{kl} - (\hat{R}^{-1})^{ij}_{kl} = \lambda(\delta^i_k\delta^j_l - g^{ij}g_{kl})$$

and

$$\hat{R}^{ij}_{kl}g^{kl} = q^{1-N}g^{ij}.$$

A calculus on quantum Euclidean space can be obtained by introducing derivatives ∂^i which satisfy

$$\partial^i x_j = \delta^i_j + q\hat{R}^{ik}_{jl} x_k \partial^l \tag{2.127}$$

and

$$(P^-)^{ij}_{kl}\partial^l\partial^k = 0. \tag{2.128}$$

Explicitly, the latter is

$$\partial_k\partial_l \hat{R}^{kl}_{ij} = q\partial_i\partial_j - \lambda\Delta g_{ij}, \tag{2.129}$$

or equivalently

$$\partial_k\partial_l(\hat{R}^{-1})^{kl}_{ij} = q^{-1}\partial_i\partial_j - \lambda q^{N-2}\Delta g_{ij}, \tag{2.130}$$

where the Laplacian

$$\Delta \equiv \alpha\partial\cdot\partial = \alpha\partial_i\partial^i \tag{2.131}$$

commutes with all derivatives,

$$\Delta\partial^i = \partial^i\Delta. \tag{2.132}$$

One can define a rescaling operator

$$\Lambda = 1 + q\lambda x_i\partial^i + q^N\lambda^2 L\Delta, \tag{2.133}$$

which satisfies

$$\Lambda x_i = q^2 x_i\Lambda, \quad \partial^i\Lambda = q^2\Lambda\partial^i. \tag{2.134}$$

A useful relation is

$$\partial^i L = q^2 L\partial^i + q^{2-N}x^i. \tag{2.135}$$

The action of the vector fields Z on $SO_q(N)$ induces in the standard way an action on Euclidean space analogous to (2.5)

$$Z^i_j x_k = x_m \hat{R}^{im}_{ln} Z^n_r \hat{R}^{lr}_{jk}. \tag{2.136}$$

For q real, the quantum Euclidean space can be endowed with a reality structure as follows. For the coordinates one imposes the reality condition

$$x_i^* = g^{ij}x_j = x^i. \tag{2.137}$$

Let us now define derivatives $\hat{\partial}_i$ in terms of the conjugate derivatives by

$$\hat{\partial}_i = g_{ij}\hat{\partial}^j = -q^N\partial^{i*}. \tag{2.138}$$

The complex conjugate of (2.127) can be transformed to the form

$$\hat{\partial}^i x_j = \delta^i_j + q^{-1}(\hat{R}^{-1})^{ik}_{jl} x_k \hat{\partial}^l \tag{2.139}$$

and that of (2.129) to

$$\hat{\partial}_k\hat{\partial}_l(\hat{R}^{-1})^{kl}_{ij} = q^{-1}\hat{\partial}_i\hat{\partial}_j + \lambda\hat{\Delta}g_{ij}, \tag{2.140}$$

where

$$\hat{\Delta} \equiv \alpha q^{N-2}\hat{\partial}\cdot\hat{\partial}. \tag{2.141}$$

The relation between the derivatives ∂^i and their complex conjugates or the $\hat{\partial}_i$ can be written[64] in the nonlinear form

$$\hat{\partial}^i = \Lambda^{-1}(\delta^i_j + q^{N-1}\lambda\alpha x^i\partial_j)\partial^j, \tag{2.142}$$

which can be shown to satisfy (2.139). Using (2.142), one can show that

$$\hat{\partial}^i\partial^j = q\hat{R}^{ji}_{lk}\partial^k\hat{\partial}^l. \tag{2.143}$$

We wish to find a realization for the vector fields Z of $SO_q(N,R)$ as pseudodifferential operators on Euclidean space. It must satisfy (2.136), (2.114), the orthogonality relations (2.116, 2.117) and the reality condition for $SO_q(N,R)$

$$Z^\dagger = Z. \tag{2.144}$$

One way to find the appropriate expression is to proceed in analogy with (2.100) by writing similar terms but adjusting the coefficients so that all relations required of Z are satisfied. It turns out that the correct formula is

$$Z^i_j = q^{-2}\delta^i_j + q^{-1}\lambda\partial^i x_j - q^{1-N}\lambda x^i\hat{\partial}_j - \lambda^2 L\partial^i\hat{\partial}_j. \tag{2.145}$$

In fact, using the relations given above for the calculus on Euclidean space, one can verify that Z^i_j satisfies (2.136) as well as

$$\partial^i Z^j_k = \hat{R}^{ji}_{lm} Z^m_n \hat{R}^{ln}_{kr}\partial^r \tag{2.146}$$

and

$$\hat{\partial}^i Z^j_k = \hat{R}^{ji}_{lm} Z^m_n \hat{R}^{ln}_{kr}\hat{\partial}^r. \tag{2.147}$$

Combining (2.136), (2.146) and (2.147), one finds that Z satisfies also (2.114).

It is very easy to see that Z, as given by (2.145) satisfies the reality condition (2.144) if one observes that (2.145) can be written in the more symmetric form

$$q^2 Z^i_j = \delta^i_j + q\lambda\partial^i x_j + q\lambda x_i{}^* \partial^{j*} + \alpha q^N \lambda^2 \partial^i x_k x_k{}^* \partial^{j*} \tag{2.148}$$

by using (2.135).

Finally, the orthogonality condition is

$$g_{ij} Z^j_m \hat{R}^{im}_{kn} Z^n_l = q^{1-N} g_{kl}. \tag{2.149}$$

Using (2.145), it is easy to get

$$g_{ij} Z^j_m \hat{R}^{im}_{kn} = q^{1-N} g_{kr} \hat{Z}^r_n, \tag{2.150}$$

where the quantity $\hat{Z}^r_n$

$$\hat{Z}^r_n = q^2 \delta^r_n - q\lambda \hat{\partial}^r x_n + q^{N-1}\lambda x^r \partial_n - \lambda^2 \hat{L} \hat{\partial}^r \partial_n \tag{2.151}$$

is obtained from Z^r_n by exchanging

$$x \leftrightarrow \hat{x} \equiv x, \quad \partial \leftrightarrow \hat{\partial}, \tag{2.152}$$

$$q \leftrightarrow q^{-1}, \quad \hat{R} \leftrightarrow \hat{R}^{-1}, \quad g \leftrightarrow g. \tag{2.153}$$

Note that this operation exchanges (2.127) with (2.139), (2.129) with (2.140) and leaves (2.143) invariant and is therefore a symmetry of the quantum Euclidean space. We claim

Lemma 2.4.1

$$Z^l_m \hat{Z}^m_n = \delta^l_n. \tag{2.154}$$

Proof It can be easily checked by direct computations and making use of the following useful equalities

$$\lambda \hat{\partial}_m x^m = (q^N - 1)(q^{1-N} + q^{-1}) + \lambda q^{-N} x_m \hat{\partial}^m, \tag{2.155}$$

$$\hat{\partial}_n = q^{-2}\Lambda^{-1}\partial_n + \lambda q^{N-3}\Lambda^{-1}\Delta x_n, \tag{2.156}$$

$$\hat{L} = q^{N-2} L, \tag{2.157}$$

$$\hat{\Delta} = q^{2N-4}\Lambda^{-1}\Delta, \tag{2.158}$$

$$\hat{\Lambda} = \Lambda^{-1}. \tag{2.159}$$

□

Combining all these, we have

Construction 2.4.1 (Realization for $SO_q(N,R)$)

$$Z^i_j = q^{-2}\delta^i_j + q^{-1}\lambda\partial^i x_j - q^{1-N}\lambda x^i \hat{\partial}_j - \lambda^2 L \partial^i \hat{\partial}_j \tag{2.160}$$

realizes the $SO_q(N,R)$ vector fields as pseudodifferential operators in the quantum Euclidean space.

It is remarkable that Z as given by (2.160) satisfies even the orthogonality relations (2.116) and (2.117), without need for any further normalization as was necessary in (2.66) and (2.101). This is due, apparently, to the fact that the $SO_q(N)$ $\hat{R}$ matrix already satisfies orthogonality relations.

On the other hand, if one does not impose (2.137) and doesn't identify $\hat{\partial}_i$, as given in (2.142), with the complex conjugate derivative ∂^{i*} by (2.138), then (2.144) will not be true. However, (2.160) would still give a realization of vector fields for the complex quantum group $SO_q(N)$ on Euclidean space.

2.5 A Few Remarks

1. In the differential calculus on a quantum space, one naturally introduces the differentials of the coordinates

$$\xi_i = dx_i. \tag{2.161}$$

For quantum Euclidean space, they satisfy the commutation relations

$$\xi_k \xi_l (P^+)^{kl}_{ij} = 0, \quad \xi_k \xi_l (P^0)^{kl}_{ij} = 0, \tag{2.162}$$

$$x_i \xi_j = q \xi_k x_l \hat{R}^{kl}_{ij}, \tag{2.163}$$

$$\partial^i \xi_j = q^{-1} (\hat{R}^{-1})^{ik}_{jl} \xi_k \partial^l. \tag{2.164}$$

According to (2.137) it is natural to introduce variables $\hat{\xi}_i$ related to $\xi_i{}^*$ by

$$\xi_i{}^* = g^{ij}\hat{\xi}_j = \hat{\xi}^i. \tag{2.165}$$

The complex conjugate of (2.163) can be written as

$$\hat{\xi}_i x_j = q x_k \hat{\xi}_l \hat{R}^{kl}_{ij}. \tag{2.166}$$

It was shown[64] that the $\hat{\xi}_i$ can be related to ξ_i by a (nonlinear) transformation which was given explicitly there. It turns out that that transformation can be written very compactly as

$$\hat{\xi}_i = \sigma q^N \Lambda \xi_k Z^k_i, \tag{2.167}$$

where Λ is given by (2.133). In this form one can easily verify that $\hat{\xi}$ satisfies all desired relations. For instance (2.166) follows immediately from (2.134), (2.136) and (2.163). The requirement that complex conjugation be an involution restricts σ to be a phase, $|\sigma| = 1$. Vice versa, if one knows the correct expression for $\hat{\xi}_i$, one can infer from it the formula for Z^k_i.

2. All of the above equations are "covariant". This means that they go into themselves by coaction transformations. For instance, for all equations for $GL_q(N)$ from Eq.(2.1) to (2.39), it is easy to see that the transformation

$$A \to AB, \quad x \to xB, \quad \partial \to B^{-1}\partial, \tag{2.168}$$

$$Y \to B^{-1}YB, \quad x_i\partial^i \to x_i\partial^i \tag{2.169}$$

leaves them invariant. Here the matrix elements of B are taken to commute with everything (which is the reason for using the word coaction) but B is itself a quantum matrix, satisfying the analogue of Eq.(2.1). It holds similarly for the complex conjugate sector of $GL_q(N)$,

$$A^\dagger \to B^\dagger A^\dagger, \quad \hat{x} \to B^\dagger \hat{x}, \quad \hat{\partial} \to \hat{\partial}(B^\dagger)^{-1}, \tag{2.170}$$

$$Y^\dagger \to B^\dagger Y^\dagger (B^\dagger)^{-1}, \quad \hat{\partial}_i \hat{x}^i \to \hat{\partial}_i \hat{x}^i \tag{2.171}$$

(the relation $(B^\dagger)^{-1} = (B^{-1})^\dagger$ is used). Analogous transformation laws leave invariant the $SL_q(N), SO_q(N)$ equations as well as their respective real forms.

3. The realization of vector fields for $GL_q(N)$ and $SL_q(N)$ given above is equivalent to that given earlier[65]. The formulas given here are simpler because

of a more convenient choice of notations and definitions. For instances, we use a right coaction and a corresponding more convenient lower index for the coordinates x_i and upper index for the derivatives ∂^i. The same applies to a comparison between the formulas written above for $SO_q(N)$ and earlier ones[64]. The reader should have no difficulty in establishing the correspondence between the conventions of these different references.

4. A realization of vector fields for the orthogonal group in terms of pseudodifferential operators on quantum Euclidean space has been given by Gaetano Fiore[66]. He uses the explicit description of the quantum Lie algebra by Drinfeld and Jimbo, instead of (2.114), (2.116) and (2.117) and gives explicit realizations for the vector fields in that basis. Ours is an alternative solution of the same problem which has perhaps the advantage of being more symmetric and also covariant, as explained above.

Chapter 3

Quantum Complex Sphere S_q^2

In this chapter, we introduce complex coordinate to the two dimensional quantum sphere of Podleś [67, 68, 69, 70] by means a stereographic projection. The coaction of $SU_q(2)$ on the sphere is given by fractional transformations on the complex variable in the plane, analogous to the classical ones. Left covariant differential calculus is introduced. To cover the whole sphere, we need at least one more coordinate patch. The quantum sphere appears then as the quantum deformation of the classical two-sphere described as a complex manifold. We also discuss a very interesting property of the calculus: the admission of a one-form realization of the exterior differential operator.

3.1 The Algebra S_q^2 and the Patch C^+

A family of quantum 2-spheres was introduced in [67]. There, the algebra of functions over the sphere is generated by 3 coordinates, subjected to a condition that reduces the number of independent generators to 2. The case of $c = 0$ is of special interest[71]. In this case, the algebra over the sphere S_q^2 is the C^*-algebra generated by the elements $b_+ = \gamma\delta, b_- = \alpha\beta, b_3 = \alpha\delta$, (where $\alpha, \beta, \gamma, \delta \in SU_q(2)$) with commutations

$$\begin{aligned}(1-b_3)b_- &= q^{-2}b_-(1-b_3),\\ (1-b_3)b_+ &= q^2 b_+(1-b_3),\\ q^{-2}b_-b_+ &= q^2 b_+ b_- - \lambda(b_3-1),\end{aligned} \tag{3.1}$$

where $\lambda \equiv q - q^{-1}$ and constraint

$$b_3^2 = b_3 + q^{-1}b_-b_+. \tag{3.2}$$

The $*$-algebra structure is $b_\pm^* = -q^{\mp 1}b_\mp$, $b_3^* = b_3$. q is a real number.

One can construct a stereographic projection to rewrite this algebra in terms of coordinates of the complex plane $z, \bar{z}$. Define

$$\begin{aligned}z &= -qb_-(1-b_3)^{-1} = \alpha\gamma^{-1},\\ \bar{z} &= b_+(1-b_3)^{-1} = -\delta\beta^{-1},\end{aligned} \tag{3.3}$$

which is the projection from the north pole of the sphere to the plane with coordinates $z, \bar{z}$. It is easy to derive the commutation relation

$$z\bar{z} = q^{-2}\bar{z}z + q^{-2} - 1 \tag{3.4}$$

and the $*$-structure

$$z^* = \bar{z}. \tag{3.5}$$

We will denote the $*$-algebra generated by z and $\bar{z}$ subjected to the commutation relation (3.4) and $*$-structure (3.5) by C^+. Note that the relation (3.4) for the patch C^+ differs from the usual quantum plane (see for example [58]) by an additional inhomogeneous constant term.

The inverse relations to (3.3) can also be obtained easily. It is

$$\begin{aligned}b_+ &= \rho^{-1}\bar{z},\\ b_- &= -qz\rho^{-1},\\ 1-b_3 &= q^2\rho^{-1},\end{aligned} \tag{3.6}$$

where ρ is defined as

$$\rho = 1 + \bar{z}z \tag{3.7}$$

and satisfies

$$\begin{aligned}\rho z &= q^2 z\rho,\\ \rho\bar{z} &= q^{-2}\bar{z}\rho.\end{aligned} \tag{3.8}$$

One can check directly that Eq.(3.4) is covariant under the fractional transformation, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_q(2)$,

$$z \to (az+b)(cz+d)^{-1}, \quad \bar{z} \to -(c-d\bar{z})(a-b\bar{z})^{-1}, \tag{3.9}$$

which is induced from the $SU_q(2)$ coproduct, interpreted as a left transformation. Here a, b, c and d commute with z and $\bar{z}$.

3.2 Differential Calculus

In Refs.[68, 69, 70], differential structures on S_q^2 are studied and classified. In this section, we give a differential calculus on the sphere in terms of the complex coordinates z and $\bar{z}$ of the patch. Just as the algebras of functions on C^+ can be inferred from those of $SU_q(2)$, so can be the differential calculus.

For $SU_q(2)$ there are several well-known calculi[13, 14]: the 3D left- and right-covariant differential calculi, and the $4D_+, 4D_-$ bi-covariant calculi. The 4D bi-covariant calculi have one extra dimension in their space of one-forms compared with the classical case. The right-covariant calculus will not give a projection on C^+ in a closed form in terms of z, $\bar{z}$, which are defined to transform from the left. Therefore we shall choose the left-covariant differential calculus.

It is straightforward to obtain the following relations from those for $SU_q(2)$:

$$zdz = q^{-2}dzz, \qquad \bar{z}dz = q^2 dz\bar{z}, \tag{3.10}$$

$$zd\bar{z} = q^{-2}d\bar{z}z, \qquad \bar{z}d\bar{z} = q^2 d\bar{z}\bar{z}, \tag{3.11}$$

$$(dz)^2 = (d\bar{z})^2 = 0, \tag{3.12}$$

and

$$dzd\bar{z} = -q^{-2}d\bar{z}dz. \tag{3.13}$$

We can also define derivatives ∂, $\bar{\partial}$ such that on functions,

$$d = dz\partial + d\bar{z}\bar{\partial}. \tag{3.14}$$

From the requirement $d^2 = 0$ and the undeformed Leibniz rule for d together with Eqs. (3.10) to (3.12) it follows that:

$$\partial z = 1 + q^{-2}z\partial, \quad \partial\bar{z} = q^2\bar{z}\partial, \tag{3.15}$$

$$\bar{\partial}z = q^{-2}z\bar{\partial}, \quad \bar{\partial}\bar{z} = 1 + q^2\bar{z}\bar{\partial}, \tag{3.16}$$

and

$$\partial\bar{\partial} = q^{-2}\bar{\partial}\partial. \tag{3.17}$$

It can be checked explicitly that these commutation relations are covariant under the transformation (3.9) and

$$dz \to dz(q^{-1}cz+d)^{-1}(cz+d)^{-1}, \tag{3.18}$$

$$\partial \to (cz+d)(q^{-1}cz+d)\partial, \tag{3.19}$$

which follow from (3.9) and the fact that d is invariant.

The $*$-structure also follows from that of $SU_q(2)$:

$$(dz)^* = d\bar{z}, \tag{3.20}$$

$$\partial^* = -q^{-2}\bar{\partial} + (1+q^{-2})z\rho^{-1}, \tag{3.21}$$

$$\bar{\partial}^* = -q^2\partial + (1+q^2)\rho^{-1}\bar{z}., \tag{3.22}$$

where the $*$-involution inverts the order of factors in a product.

The inhomogeneous pieces on the RHS of the Eqs.(3.21) and (3.22) reflect the fact that the sphere has curvature. Incidentally all the commutation relations in this section admit another possible involution:

$$(dz)^* = d\bar{z}, \tag{3.23}$$

$$\partial^* = -q^2\bar{\partial}, \tag{3.24}$$

$$\bar{\partial}^* = -q^{-2}\partial. \tag{3.25}$$

This involution is not covariant under the fractional transformations and cannot be used for the sphere. However, it can be used when we have a quantum plane defined by the same algebra of functions and calculus. We shall take Eqs. (3.10) to (3.22) as the definition of the differential calculus on the patch C^+.

Symmetries

It is interesting to note that there exist two different types of symmetries in the calculus. The first symmetry is that if we put a bar on all unbarred variables (z, dz, ∂), take away the bar from any barred ones and at the same time replace q by $1/q$ in any statement about the calculus, the statement is still true.

The second symmetry is the consecutive operation of the two $*$-involutions above, so that

$$\partial \to -q^2\bar{\partial}^* = q^4\partial - q^2(1+q^2)\rho^{-1}\bar{z}, \tag{3.26}$$

$$\bar{\partial} \to -q^{-2}\partial^* = q^{-4}\bar{\partial} - q^{-2}(1+q^{-2})z\rho^{-1}, \tag{3.27}$$

with $z, \bar{z}, dz, d\bar{z}$ unchanged. This replacement can be iterated n times and gives a symmetry which resembles that of a gauge transformation on a line bundle:

$$\partial \to \partial^{(n)} \equiv q^{4n}\partial - q^2[2n]_q\rho^{-1}\bar{z} \tag{3.28}$$

$$= q^{4n}\rho^{2n}\partial\rho^{-2n}, \tag{3.29}$$

$$\bar{\partial} \to \bar{\partial}^{(n)} \equiv q^{-4n}\bar{\partial} - q^{-2}[2n]_{1/q}z\rho^{-1} \tag{3.30}$$

$$= q^{-4n}\rho^{2n}\bar{\partial}\rho^{-2n}, \tag{3.31}$$

where $[n]_q = \frac{q^{2n}-1}{q^2-1}$. For example, we have

$$\partial^{(n)}z = 1 + q^{-2}z\partial^{(n)}. \tag{3.32}$$

Making a particular choice of $\partial, \bar{\partial}$ is like fixing a gauge.

Many of the features of a calculus on a classical complex manifold are preserved. Define $\delta = dz\partial$ and $\bar{\delta} = d\bar{z}\bar{\partial}$ as the exterior derivatives on the holomorphic and antiholomorphic functions on C^+ respectively. We have:

$$[\delta, z] = dz, \quad [\delta, \bar{z}] = 0, \tag{3.33}$$

$$[\bar{\delta}, z] = 0, \quad [\bar{\delta}, \bar{z}] = d\bar{z}, \tag{3.34}$$

$$d = \delta + \bar{\delta}. \tag{3.35}$$

The action of δ and $\bar{\delta}$ can be extended consistently on forms as follows

$$\delta dz = dz\delta = 0, \quad \bar{\delta}d\bar{z} = d\bar{z}\bar{\delta} = 0, \tag{3.36}$$

$$\{\delta, d\bar{z}\} = 0, \quad \{\bar{\delta}, dz\} = 0, \tag{3.37}$$

$$\delta^2 = \bar{\delta}^2 = 0, \tag{3.38}$$

$$\{\delta, \bar{\delta}\} = 0, \tag{3.39}$$

where $\{\cdot,\cdot\}$, $[\cdot,\cdot]$ are the anticommutator and commutator respectively.

3.2.1 One-form Realization of the Exterior Differential Operator d

The calculus described in the previous section has a very interesting property. There exists a one-form Ξ having the property that

$$\Xi f \mp f\Xi = \lambda df, \tag{3.40}$$

where, as usual, the minus sign applies for functions or even forms and the plus sign for odd forms. Indeed, it is very easy to check that

$$\Xi = \xi - \xi^* \tag{3.41}$$

$$\xi = q dz\rho^{-1}\bar{z} \tag{3.42}$$

satisfies Eq.(3.40) and

$$\Xi^* = -\Xi. \tag{3.43}$$

It is also easy to check that

$$d\Xi = 2q d\bar{z}\rho^{-2}dz \tag{3.44}$$

and

$$\Xi^2 = q\lambda d\bar{z}\rho^{-2}dz. \tag{3.45}$$

Suitably normalized, $d\Xi$ is the natural area element on the quantum sphere. Notice that Ξ^2 commutes with all functions and forms, as required for consistency with the relation

$$d^2 = 0. \tag{3.46}$$

The existence of the form Ξ within the algebra of $z, \bar{z}, dz, d\bar{z}$ is especially interesting because no such form exists for the 3-D calculus on $SU_q(2)$, from which we have derived the calculus on the quantum sphere (a one-form analogous to Ξ does exist for the two bicovariant calculi on $SU_q(2)$, but we have explained before why we didn't choose either of them). It is also interesting that $d\Xi$ and Ξ^2 do not vanish (as the corresponding expressions do in the bicovariant calculi on the quantum groups or in the calculus on quantum Euclidean space).

The one-form Ξ is regular everywhere on the sphere, except at the point $z = \bar{z} = \infty$, which classically corresponds to the north pole. This is discussed in [72] where we argue that the pole singularity at that point can be included by allowing forms with distribution valued coefficients. The area element $d\Xi$ is regular everywhere on the sphere.

3.3 Patching Two Quantum Planes

The variables z and $\bar{z}$ cover the sphere with the exception of the north pole. In analogy with the classical case, we can introduce new variables $w = z^{-1}$ and $\bar{w} = \bar{z}^{-1}$ which describe the sphere without the south pole. These variables satisfy the commutation relation

$$w\bar{w} = q^{-2}\bar{w}w + (q^{-2} - 1)w\bar{w}^2 w \tag{3.47}$$

and the $*$-structure

$$w^* = \bar{w}. \tag{3.48}$$

It is clear that (3.47) is covariant under the transformation

$$w \to (dw + c)(bw + a)^{-1}, \quad \bar{w} \to -(a\bar{w} - b)(c\bar{w} - d)^{-1}. \tag{3.49}$$

Notice that the commutation relation (3.47) is different from that satisfied by z and $\bar{z}$; our way of quantizing the sphere is inherently asymmetric between the north and the south pole.

The calculus in z and $\bar{z}$ induces a calculus in w and $\bar{w}$. It is not hard to derive the commutation relations for this $w, \bar{w}$ calculus as well as the mixed commutation relations. For example, we have

$$wdw = q^2 dww, \tag{3.50}$$

$$\partial_w w = 1 + q^2 w\partial_w \tag{3.51}$$

and

$$dzw = q^{-2}wdz. \tag{3.52}$$

Denote by $\mathcal{C}^-$ the $*$-algebra generated by the variables w and $\bar{w}$, subjecting to (3.47) and (3.48). The coordinates $w, \bar{w}$ are related to the coordination $b_\pm, b_3$ by

$$\begin{aligned} b_+ &= w\rho_w^{-1}, \\ b_- &= -q\rho_w^{-1}\bar{w}, \\ b_3 &= \rho_w^{-1}, \end{aligned} \tag{3.53}$$

where ρ_w is defined as

$$\rho_w = 1 + \bar{w}w. \tag{3.54}$$

The relations (3.6) (respectively (3.53)) defines a $*$-algebra homomorphism between the algebra S_q^2 and $\mathcal{C}^+$ (respectively $\mathcal{C}^-$) and the sphere S_q^2 is covered by the two patches $\mathcal{C}^\pm$ with the transition relation

$$wz = zw = 1. \tag{3.55}$$

Singularity

Since w and $\bar{w}$ are functions of z and $\bar{z}$, Eq.(3.40) is valid for functions and forms in w and $\bar{w}$, with the same Ξ. In terms of w and $\bar{w}$ the one-forms ξ and ξ^* are given by

$$\xi = -w^{-1}dw(1 + \bar{w}w)^{-1}, \quad \xi^* = -(1 + \bar{w}w)^{-1}d\bar{w}\bar{w}^{-1}. \tag{3.56}$$

Clearly they are singular at the north pole $w = \bar{w} = 0$. This polar singularity is an intrinsic feature of our asymmetric quantization and of our calculus. This asymmetry is also apparent when we go to the classical limit of the Poisson sphere [73] and it seems to be unavoidable in our approach [72]. A different description of Podleś spheres was given in an interesting paper by Šťovíček[74]. He shows that the sphere can be understood as the patching of two complex quantum planes. His choice of variable is symmetric between the two planes, but the coaction of $SU_q(2)$ is very complicated in terms of his variable. Also, Šťovíček does not consider the non-commutative calculus on the sphere.

We believe that the singularity can be controlled by allowing distributions, rather than just functions as the elements of our algebra and as coefficients of differential forms. This point of view is explained in [72] for the limit of the Poisson sphere so as to avoid the need to develop the concept of distribution in the framework of noncommutative algebra.

3.4 Right Invariant Vector Fields on S_q^2

In this section we want to define vector fields on S_q^2 which generate the fractional transformation mentioned above. We will see that these vector fields can be inferred from those on $SU_q(2)$.

First let us recall some well-known facts about the vector fields on $SU_q(2)$ (see for example Ref.[53]). The enveloping algebra $\mathcal{U}$ of $SU_q(2)$ is usually said to be generated by the left-invariant vector fields $H_L, X_{L\pm}$ which are arranged in two matrices L^+ and L^-. The action of these vector fields corresponds to infinitesimal right transformation: $T \to TT'$. What we want now is the infinitesimal version of the left transformation given by Eq.(3.9), hence we shall use the right-invariant vector fields $H_R, X_{R\pm}$. Since only the right-invariant ones will be used, we will drop the subscript R hereafter.

The properties of the right-invariant vector fields are similar to those of the left-invariant ones. Note that if an $SU_q(2)$ matrix T is transformed from the right by another $SU_q(2)$ matrix T', then it is equivalent to say that the $SU_{1/q}(2)$ matrix T^{-1} is transformed from the left by another $SU_{1/q}(2)$ matrix T'^{-1}. Therefore one can simply write down all properties of the left-invariant vector fields and then make the replacements: $q \to 1/q$, $T \to T^{-1}$ and left-invariant fields$\to$right-invariant fields.

FRT Basis

Using the matrices:

$$M^+ = \begin{pmatrix} q^{-H/2} & q^{-1/2}\lambda X_+ \\ 0 & q^{H/2} \end{pmatrix}, \quad M^- = \begin{pmatrix} q^{H/2} & 0 \\ -q^{1/2}\lambda X_- & q^{-H/2} \end{pmatrix}, \tag{3.57}$$

the commutation relations between the vector fields are given by,

$$R_{12}M_2^+M_1^+ = M_1^+M_2^+R_{12}, \tag{3.58}$$

$$R_{12}M_2^-M_1^- = M_1^-M_2^-R_{12}, \tag{3.59}$$

$$R_{12}M_2^+M_1^- = M_1^-M_2^+R_{12}, \tag{3.60}$$

while the commutation relations between the vector fields and the elements of the quantum matrix in the smash product of $\mathcal{U}$ and $SU_q(2)$ are,

$$T_1M_2^+ = M_2^+\mathcal{R}_{12}T_1, \tag{3.61}$$

$$T_1M_2^- = M_2^-\mathcal{R}_{21}^{-1}T_1, \tag{3.62}$$

where T is a $SU_q(2)$ matrix, $\mathcal{R} = q^{-1/2}R$ and R is the $GL_q(2)$ R-matrix. Clearly M^+, and M^- are the right-invariant counterparts of L^+ and L^-. The commutation relations between the M's and the T's tell us how the functions on $SU_q(2)$ are transformed by the vector fields H, X_+, X_-.

Basis suitable for the Patch

It is convenient to define a different basis for the vector fields,

$$\mathcal{Z}_+ = X_+q^{H/2}, \tag{3.63}$$

$$\mathcal{Z}_- = q^{H/2}X_- \tag{3.64}$$

and

$$\mathcal{H} = [H]_q = \tfrac{q^{2H}-1}{q^2-1}. \tag{3.65}$$

They satisfy the commutation relations

$$\mathcal{H}\mathcal{Z}_+ - q^4\mathcal{Z}_+\mathcal{H} = (1+q^2)\mathcal{Z}_+, \tag{3.66}$$

$$\mathcal{Z}_-\mathcal{H} - q^4\mathcal{H}\mathcal{Z}_- = (1+q^2)\mathcal{Z}_- \tag{3.67}$$

and

$$q\mathcal{Z}_+\mathcal{Z}_- - q^{-1}\mathcal{Z}_-\mathcal{Z}_+ = \mathcal{H}. \tag{3.68}$$

Using the expressions of $z, \bar{z}$ in terms of $\alpha, \beta, \gamma, \delta$, one can easily find the action of these vector fields on the variables $z, \bar{z}$,

$$\mathcal{Z}_+z = q^2z\mathcal{Z}_+ + q^{1/2}z^2, \tag{3.69}$$

$$\mathcal{Z}_+\bar{z} = q^{-2}\bar{z}\mathcal{Z}_+ + q^{-3/2}, \tag{3.70}$$

$$\mathcal{H}z = q^4z\mathcal{H} + (1+q^2)z, \tag{3.71}$$

$$\mathcal{H}\bar{z} = q^{-4}\bar{z}\mathcal{H} - q^{-4}(1+q^2)\bar{z}, \tag{3.72}$$

$$\mathcal{Z}_-z = q^2z\mathcal{Z}_- - q^{1/2} \tag{3.73}$$

and

$$\mathcal{Z}_-\bar{z} = q^{-2}\bar{z}\mathcal{Z}_- - q^{-3/2}\bar{z}^2. \tag{3.74}$$

It is clear that a $*$-involution can be given:

$$\mathcal{Z}_+{}^* = \mathcal{Z}_-, \quad \mathcal{H}^* = \mathcal{H}. \tag{3.75}$$

Since all the relations listed above are closed in the vector fields and z, $\bar{z}$ (this would not be the case if we had used the left-invariant fields), we can now take these equations as the definition of the vector fields that generate the fractional transformation on S_q^2. We shall take our vector fields to commute with the exterior differentiation d. This is consistent for right-invariant vector fields in a left-covariant calculus and allows us to obtain the action of our vector fields on the differentials dz and $d\bar{z}$, as well as on the derivatives ∂ and $\bar{\partial}$. For instance (3.69) gives

$$\mathcal{Z}_+ dz = q^2 dz \mathcal{Z}_+ + q^{1/2}(dzz + zdz) \tag{3.76}$$

and

$$\partial \mathcal{Z}_+ = q^2 \mathcal{Z}_+ \partial + q^{-3/2}(1+q^2) z\partial. \tag{3.77}$$

It is interesting to see how Ξ and $d\Xi$ transform under the action of the right invariant vector fields or under the coaction of the fractional transformations (3.9). Using (3.69) to (3.74) one finds

$$\mathcal{Z}_+ \Xi = \Xi \mathcal{Z}_+ + q^{-1/2} dz \tag{3.78}$$

and

$$\mathcal{H}\Xi = \Xi\mathcal{H}. \tag{3.79}$$

These equations are consistent with (3.40). For instance,

$$\mathcal{Z}_+(\lambda dz - \Xi z + z\Xi) = q^2(\lambda dz - \Xi z + z\Xi)\mathcal{Z}_+ \tag{3.80}$$

$$+q^{1/2}(\lambda dz^2 - \Xi z^2 + z^2\Xi) - q^{-1/2}(dzz - q^2 zdz). \tag{3.81}$$

Eqs. (3.78) and (3.79) imply that $d\Xi$ commutes with $Z_\pm$ and $\mathcal{H}$, as expected for the invariant area element.

For the fractional transformation (3.9) one finds $\xi \to \xi'$ where

$$\xi' - \xi = -q(dz)cd^{-1}(1 + cd^{-1}z)^{-1} \tag{3.82}$$

and a similar formula for ξ^*. The right hand side of (3.82) is a closed one-form, since $(dz)^2 = 0$, so one could write

$$\xi' - \xi = -qd[\log_q(1 + cd^{-1}z)] \tag{3.83}$$

with a suitably defined quantum function $\log_q$. At any rate

$$d\xi' = d\xi \tag{3.84}$$

so that the area element two-form is invariant under finite transformations as well.

Pseudo-differential Operators Realization

It is natural to ask whether one can realize the vector fields $\mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H}$ as pseudo-differential operators acting on $\mathcal{C}^+$. The answer is yes. Introduce the differential operators,

$$C = 1 - \lambda q^{-1} z\partial, \tag{3.85}$$

$$D = 1 + \lambda q \bar{z}\bar{\partial} \tag{3.86}$$

and

$$B = 1 - \lambda q^{-1} z\partial + \lambda q \bar{z}\bar{\partial} - \lambda^2 q^{-2} \rho \bar{\partial}\partial. \tag{3.87}$$

One finds the following realizations of $\mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H}$ as pseudo-differential operators, which satisfy Eqs. (3.66) to (3.75):

$$q^{3/2}\mathcal{Z}_+ = (z^2\partial + q^2\bar{\partial}B^{-1})C^{-1}, \tag{3.88}$$

$$-q^{3/2}\mathcal{Z}_- = (q^2\bar{z}^2\bar{\partial} + \partial B^{-1})D^{-1} \tag{3.89}$$

and

$$\mathcal{H} = \frac{1 - B^{-2}}{1 - q^2}. \tag{3.90}$$

Equivalently,

$$q^{-1}\rho^2\bar{\partial} = (\mathcal{Z}_- z \mathcal{Z}_+ - q^4 \mathcal{Z}_+ z \mathcal{Z}_- + q^{1/2}(1+q^2)\mathcal{Z}_+)B \tag{3.91}$$

and

$$q^{-1}\rho^2\partial = (q^4 \mathcal{Z}_+ \bar{z} \mathcal{Z}_- - \mathcal{Z}_- \bar{z} \mathcal{Z}_+ - q^{1/2}(1+q^2)\mathcal{Z}_-)B. \tag{3.92}$$

3.5 Braided Quantum Sphere

We first review the general formulation [75] for obtaining the braiding of quantum spaces in terms of the universal R-matrix of the quantum group which coacts on the quantum space.

3.5.1 Braiding for Quantum Group Comodules

Let $\mathcal{A}$ be the algebra of functions on a quantum group and $\mathcal{V}$ an algebra on which $\mathcal{A}$ coacts on the left:

$$\begin{aligned} \Delta_L : \mathcal{V} &\rightarrow \mathcal{A} \otimes \mathcal{V} \\ v &\mapsto v^{(1')} \otimes v^{(2)}, \end{aligned}$$

where we have used the Sweedler-like notation for $\Delta_L(v)$.

Let $\mathcal{W}$ be another left $\mathcal{A}$-comodule algebra,

$$\begin{aligned} \Delta_L : \mathcal{W} &\rightarrow \mathcal{A} \otimes \mathcal{W} \\ w &\mapsto w^{(1')} \otimes w^{(2)}. \end{aligned}$$

It is known[75] that one can put $\mathcal{V}$ and $\mathcal{W}$ into a single left $\mathcal{A}$-comodule algebra with the multiplication between elements of $\mathcal{V}$ and $\mathcal{W}$ given by

$$vw = \mathcal{R}(w^{(1')}, v^{(1')}) w^{(2)} v^{(2)}. \tag{3.93}$$

Here $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ is the universal R-matrix for the quantum enveloping algebra $\mathcal{U}$ dual to $\mathcal{A}$ (with respect to the pairing $< \cdot, \cdot >$) and

$$\mathcal{R}(a, b) = < \mathcal{R}, a \otimes b > .$$

It satisfies:

$$\mathcal{R}(f_{(1)}, g_{(1)}) f_{(2)} g_{(2)} = g_{(1)} f_{(1)} \mathcal{R}(f_{(2)}, g_{(2)}), \tag{3.94}$$

$$\mathcal{R}(fg, h) = \mathcal{R}(f, h_{(1)}) \mathcal{R}(g, h_{(2)}), \tag{3.95}$$

$$\mathcal{R}(f, gh) = \mathcal{R}(f_{(1)}, h) \mathcal{R}(f_{(2)}, g), \tag{3.96}$$

$$\mathcal{R}(1, f) = \mathcal{R}(f, 1) = \epsilon(f). \tag{3.97}$$

Take 3 elements $v, \tilde{v} \in \mathcal{V}$, $w \in \mathcal{W}$, notice that if the multiplication on $\mathcal{V}$ is given by

$$v\tilde{v} = c\mathcal{R}(\tilde{v}^{(1')}, v^{(1')}) \tilde{v}^{(2)} v^{(2)}, \tag{3.98}$$

where c is a proportional constant, then

$$\begin{aligned} (v\tilde{v})w &= c\mathcal{R}(\tilde{v}^{(1')}, v^{(1')}) \tilde{v}^{(2)} v^{(2)} w \\ &= c\mathcal{R}(\tilde{v}^{(1')}, v^{(1')}) \mathcal{R}(w^{(1')}, v^{(2')}) \tilde{v}^{(2)} w^{(2)} v^{(3)} \\ &= c\mathcal{R}(\tilde{v}^{(1')}, v^{(1')}) \mathcal{R}(w^{(1')}, v^{(2')}) \mathcal{R}(w^{(2')}, \tilde{v}^{(2')}) w^{(3)} \tilde{v}^{(3)} v^{(3)} \\ &= c\mathcal{R}(w^{(1')}, \tilde{v}^{(1')}) \mathcal{R}(w^{(2')}, v^{(1')}) \mathcal{R}(\tilde{v}^{(2')}, v^{(2')}) w^{(3)} \tilde{v}^{(3)} v^{(3)} \\ &\qquad \text{(due to Yang-Baxter equation)} \\ &= \mathcal{R}(w^{(1')}, \tilde{v}^{(1')}) \mathcal{R}(w^{(2')}, v^{(1')}) w^{(3)} v^{(2)} \tilde{v}^{(2)} \\ &= \mathcal{R}(w^{(1')}, \tilde{v}^{(1')}) v w^{(2)} \tilde{v}^{(2)} \\ &= v(\tilde{v}w). \end{aligned}$$

Similarly, for elements $v \in \mathcal{V}$, $w, \tilde{w} \in \mathcal{W}$, if

$$w\tilde{w} = c'\mathcal{R}(\tilde{w}^{(1')}, w^{(1')}) \tilde{w}^{(2)} w^{(2)}, \tag{3.99}$$

then $v(w\tilde{w}) = (vw)\tilde{w}$. In many cases of interest, one consider algebras with multiplication given by (3.98) and in these cases, the multiplication (3.93) is associative.

One can check that (3.93) is left covariant

$$\Delta_L(vw) = \Delta_L(v)\Delta_L(w). \tag{3.100}$$

[1] It is because

$$\begin{aligned} \Delta_L(vw) &= \mathcal{R}(w^{(1')}, v^{(1')}) w^{(2)(1')} v^{(2)(1')} \otimes w^{(2)(2)} v^{(2)(2)} \\ &= \mathcal{R}(w^{(1')}{}_{(1)}, v^{(1')}{}_{(1)}) w^{(1')}{}_{(2)} v^{(1')}{}_{(2)} \otimes w^{(2)} v^{(2)} \\ &\qquad \text{(since } (id \otimes \Delta_L) \circ \Delta_L = (\Delta \otimes id) \circ \Delta_L \text{)} \\ &= v^{(1')}{}_{(1)} w^{(1')}{}_{(1)} \mathcal{R}(w^{(1')}{}_{(2)}, v^{(1')}{}_{(2)}) \otimes w^{(2)} v^{(2)} \end{aligned}$$

[1] If on the other hand, one start with two right $\mathcal{A}$-comodule algebras,

$$\begin{aligned} \Delta_R : \mathcal{V} &\rightarrow \mathcal{V} \otimes \mathcal{A} \\ v &\mapsto v^{(1)} \otimes v^{(2')}, \\ \Delta_R : \mathcal{W} &\rightarrow \mathcal{W} \otimes \mathcal{A} \\ w &\mapsto w^{(1)} \otimes w^{(2')}, \end{aligned} \tag{3.101}$$

Then the multiplication

$$vw = w^{(1)} v^{(1)} \mathcal{R}(v^{(2')}, w^{(2')}) \tag{3.102}$$

is associative (under the corresponding assumption), right covariant under Δ_R and makes $\mathcal{V}$ and $\mathcal{W}$ together as a right $\mathcal{A}$-comodule algebra.

$$
\begin{aligned}
&\text{(due to property (3.94) of } \mathcal{R}) \\
&= v^{(1')}w^{(1')}\mathcal{R}(w^{(2)(1')}, v^{(2)(1')}) \otimes w^{(2)(2)}v^{(2)(2)} \\
&= v^{(1')}w^{(1')} \otimes v^{(2)}w^{(2)} \\
&= \Delta_L(v)\Delta_L(w).
\end{aligned}
$$

For $\mathcal{A} = SU_q(2)$, it is

$$\mathcal{R}(T^i_j, T^k_l) = q^{-1/2}\hat{R}^{ki}_{jl},$$

where $\hat{R}$ is the $GL_q(2)$ R-matrix. For example,

$$
\mathcal{R}(a,T) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad \mathcal{R}(b,T) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$
$$
\mathcal{R}(d,T) = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}, \quad \mathcal{R}(c,T) = \begin{pmatrix} 0 & \lambda q^{-1/2} \\ 0 & 0 \end{pmatrix},
$$

where $\lambda = q - q^{-1}$.

The braiding formula (3.93) can be used for any number of ordered $\mathcal{A}$-comodules $\{\mathcal{V}_n\}_{n=1}^N$ so that it holds for $v \in \mathcal{V}_m$ and $w \in \mathcal{V}_n$ if $m < n$.

3.5.2 The Braided Sphere

Since we know how z, z' and $\bar{z}'$ transform, we can use (3.93) to derive the braided commutation relations[76]. We will not repeat the derivation here but will only give the results

$$z\bar{z} = q^{-2}\bar{z}z - \lambda q^{-1}, \tag{3.103}$$

$$zz' = q^2 z'z - \lambda q z'^2, \tag{3.104}$$

$$z\bar{z}' = q^{-2}\bar{z}'z - \lambda q^{-1}. \tag{3.105}$$

For consistency with the $*$-involution of the braided algebra the braiding order of $z, \bar{z}, z'$ and $\bar{z}'$ has to be $z < z' < \bar{z}' < \bar{z}$ after we have fixed $z < z'$ and $z < \bar{z}$ as assumed in [72]. It is crucial that we braid separately $\mathcal{A} = \langle\{1, z\}\rangle$ with $\mathcal{A}'$ and $\bar{\mathcal{A}}'$, and $\bar{\mathcal{A}} = \langle\{1, \bar{z}\}\rangle$ with $\mathcal{A}'$ and $\bar{\mathcal{A}}'$ instead of simply braiding the whole algebra $\langle\{1, z, \bar{z}\}\rangle$ with $\langle\{1, z', \bar{z}'\}\rangle$. Otherwise we will not be able to have the usual properties of the $*$-involution (e.g. $(f(z)g(z'))^* = g(z')^* f(z)^*$) for the braiding relations.

Anharmonic Ratio

Let z_i, $i = 1, 2, 3, 4$ be four braided spheres with commutation relations

$$z_i z_j = q^2 z_j z_i - \lambda q z_j^2, \quad i \leq j, \tag{3.106}$$

one can verify that the anharmonic ratio $[12][24]^{-1}[34][13]^{-1}$, $[12][23]^{-1}[34][14]^{-1}$, as well as a number of others are invariant under the projective transformation

$$z_i \to (az_i + b)(cz_i + d)^{-1}. \tag{3.107}$$

It is natural to ask whether these invariants are independent of each other. A detailed analysis has been carried out in [72] and the answer is that all the invariants are related and can be written as a function of any one of them. There is only one independent anharmonic ratios, exactly the same as in the undeformed case.

This interesting quantum projective invariant in the algebra of four braided spheres was first discovered by Pei-Ming Ho. Its existence was later explained by Professor Zumino. They have also worked out the similar projective invariants for the higher projective space $CP_q(N)$ [77, 78].

Extending to the Differential Algebra

The differential calculus can also be defined on the braided spheres by imposing the Leibniz rule on the exterior derivatives d and d' so that d' acts on z' and $\bar{z}'$ in the same way d acts on z and $\bar{z}$, and

$$
\begin{aligned}
dz' &= z'd, \quad d\bar{z}' = \bar{z}'d, \\
d'z &= zd', \quad d'\bar{z} = \bar{z}d'
\end{aligned}
$$

together with

$$dd' = -d'd.$$

Then (3.104),(3.105) and their $*$-involution will imply commutation relations between functions and forms of different copies of the sphere. As a consequence, the area element of the second copy $K' = dz'd\bar{z}'(1 + \bar{z}'z')^{-2}$ is central in the whole braided algebra, while $K = dzd\bar{z}(1 + \bar{z}z)^{-2}$ is only central in the original copy $(z, \bar{z})$.

3.6 Integration

We want to determine the invariant integral $< f >$ of a function $f(z,\bar{z})$ over the sphere.

Using the Definition

A left-invariant integral can be defined, up to a normalization constant, by requiring invariance under the action of the right-invariant vector fields

$$< \chi f(z,\bar{z}) >= 0, \tag{3.108}$$

for $\chi = \mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H}$.

Using $\mathcal{H}$ and Eqs.(3.71) and (3.72) one finds that

$$< z^k \bar{z}^l g(\bar{z}z) >= 0, \quad \text{unless } k = l. \tag{3.109}$$

(Here g is a convergence function such that $z^k\bar{z}^l g(\bar{z}z)$ belongs to the sphere.) Therefore we can restrict ourselves to integrals of the form $< f(\bar{z}z) >$.

Eqs.(3.69) and (3.70) imply

$$\mathcal{Z}_+\rho = \rho\mathcal{Z}_+ + q^{1/2}z\rho \tag{3.110}$$

and

$$\mathcal{Z}_+\rho^{-l} = \rho^{-l}\mathcal{Z}_+ - q^{-3/2}[l]_{1/q}z\rho^{-l}. \tag{3.111}$$

From $< \mathcal{Z}_+(\bar{z}\rho^{-l}) >= 0$, $l \geq 1$, one finds easily the recursion formula

$$[l+1]_q < \rho^{-l} >= [l]_q < \rho^{-l+1} >, \quad l \geq 1, \tag{3.112}$$

which gives

$$< \rho^{-l} >= \frac{1}{[l+1]_q} < 1 >, \quad l \geq 0. \tag{3.113}$$

Similarly

$$< \frac{\bar{z}z}{(1+\bar{z}z)^l} >= (\frac{1}{[l]_q} - \frac{1}{[l+1]_q}) < 1 >, \quad l \geq 1. \tag{3.114}$$

We leave it to the reader to find the expression for

$$< \frac{(\bar{z}z)^k}{(1+\bar{z}z)^l} >, \quad l \geq k. \tag{3.115}$$

As an application of the stereographic projection, we can define an integration on the complex quantum plane C_q by inserting an appropriate measure factor ρ^2. C_q has the same algebra (3.4) and differential calculus (3.10) to (3.17), but a different $*$-structure (3.23) to (3.25). Classically, it holds $\int_C dzd\bar{z}/2\pi i f(z,\bar{z}) = \int_{S^2} \rho^2 f(z,\bar{z})$ Motivated by this, we define an integration over the quantum plane as,

$$\int_{C_q} f(z,\bar{z}) \equiv < \rho^2 f(z,\bar{z}) > . \tag{3.116}$$

We need to check that this integration is translationally invariant, namely, $\int \partial f = \int \bar{\partial} f = 0$. It follows immediately from (3.91), (3.92) and the definition (3.116). Formally, we have

$$\int_{C_q} \partial f =< \rho^2 \partial f >=< \mathcal{Z}_+ \cdots > - < \mathcal{Z}_- \cdots > \tag{3.117}$$

and

$$\int_{C_q} \bar{\partial} f =< \rho^2 \partial f >=< \mathcal{Z}_- \cdots > - < \mathcal{Z}_+ \cdots >, \tag{3.118}$$

which are both zero since the integral on the sphere is defined by $< \mathcal{O} f >= 0$ for $\mathcal{O} = \mathcal{Z}_\pm, \mathcal{H}$. So the integral defined by (3.116) is translational invariant.

Using the Braiding

We can also compute the left-invariant integral by requiring its consistency with the braiding relations.

Since z' and $\bar{z}'$ are always on the same side of the variables of their braided copy, z and $\bar{z}$, in the braiding order ($z < z' < \bar{z}' < \bar{z}$), the integration on $z', \bar{z}'$, has the following property:

if

$$f(z',\bar{z}')g(z,\bar{z}) = \sum_i g_i(z,\bar{z}) f_i(z',\bar{z}'),$$

then

$$< f(z',\bar{z}') > g(z,\bar{z}) = \sum_i g_i(z,\bar{z}) < f_i(z',\bar{z}') >, \tag{3.119}$$

where $< \cdot >$ is the invariant integral on S^2_q. However,

$$f(z',\bar{z}') < g(z,\bar{z}) >\neq \sum_i < g_i(z,\bar{z}) > f_i(z',\bar{z}').$$

The above property (3.119) can be used to derive explicit integral rules. For example, consider the case of $f(z',\bar{z}') = \bar{z}'\rho'^{-n}$, where $\rho' = 1 + \bar{z}'z'$ and $g(z,\bar{z}) = z$. Since

$$\bar{z}'\rho'^{-n}z = q^2 z\bar{z}'\rho'^{-n} + q^{1-2n}\lambda([n+1]_q - [n]_q\rho')\rho'^{-n}, \quad n \geq 0,$$

where $[n]_q = \frac{q^{2n}-1}{q^2-1}$, using (3.119) and $< \bar{z}'\rho'^{-n} >= 0$ we get the recursion relation:

$$[n+1]_q < \rho'^{-n} >= [n]_q < \rho'^{-(n-1)} >, \quad n \geq 1. \tag{3.120}$$

This agree with the first method.

Notice that one can also compute the same integral by using the "cyclic property" of the quantum integral [2]

$$< f(z,\bar{z})g(z,\bar{z}) >=< g(z,\bar{z})f(q^{-2}z, q^2\bar{z}) > . \tag{3.121}$$

[2]Similar cyclic properties have been found by H. Steinacker[79] for integrals over higher dimensional quantum spheres in quantum Euclidean space.

Chapter 4

Quantum Complex Projective Space

In this chapter, we define the quantum projective space $CP_q(N)$ in terms of both homogeneous and inhomogeneous complex coordinates and we study the differential calculus on it. $CP_q(N)$ is shown to be the quantum deformation of a Kähler manifold with the Fubini-Study metric.

4.1 $CP_q(N)$ as a Complex Manifold

4.1.1 $SU_q(N-1)$ Covariant Quantum Space C_q^{N+1}

First, let us define the complex quantum space C_q^{N+1} from which the projective space can be obtained. C_q^{N+1} is the algebra spanned by the coordinates x_i and its complex conjugate $\bar{x}^i, i = 0, 1, ..., N$ which satisfy the relations

$$x_i x_j = q^{-1}\tilde{R}^{kl}_{ij}x_k x_l, \tag{4.1}$$

$$\bar{x}^i x_j = q(\tilde{R}^{-1})^{ik}_{jl}x_k\bar{x}^l, \tag{4.2}$$

$$\bar{x}^i\bar{x}^j = q^{-1}\tilde{R}^{ji}_{lk}\bar{x}^k\bar{x}^l. \tag{4.3}$$

The indices run from 0 to N, instead of from 1 to $N+1$ because in the next section, we will introduce the inhomogeneous coordinates z_a for $CP_q(N)$ and we want the

indices a of z_a to run from 1 to N, instead of from 2 to $N+1$. The later is not suitable for a compact R-matrix description.

q is a real number and $\tilde{R}^{kl}_{ij}$ is the $GL_q(N+1)$ $\hat{R}$-matrix [8] with indices running from 0 to N. The complex conjugate coordinate $\bar{x}^i$ is related to the coordinate x_i by a $*$-involution

$$x_i^* = \bar{x}^i. \tag{4.4}$$

It is trivial to check that the Hermitian length

$$L \equiv x_i \bar{x}^i \tag{4.5}$$

is real and central

$$L x_i = x_i L. \tag{4.6}$$

Differential Calculus

As usual, the differential calculus can be introduced by imposing commutation relations between the functions and forms. We propose the differentials $\xi_i = dx_i, \bar{\xi}^i = (\xi_i)^*$ to satisfy

$$x_i \xi_j = q \tilde{R}^{kl}_{ij} \xi_k x_l, \tag{4.7}$$

$$\bar{x}^i \xi_j = q (\tilde{R}^{-1})^{ik}_{jl} \xi_k \bar{x}^l \tag{4.8}$$

and

$$\xi_i \xi_j = -q \tilde{R}^{kl}_{ij} \xi_k \xi_l, \tag{4.9}$$

$$\bar{\xi}^i \xi_j = -q (\tilde{R}^{-1})^{ik}_{jl} \xi_k \bar{\xi}^l. \tag{4.10}$$

We will discuss the possibilities of other choices later.

To introduce derivatives $D^i, \bar{D}_i$ acting on the "functions", [1] we require the exterior derivatives $\delta \equiv \xi^i D_i, \bar{\delta} \equiv \bar{\xi}^i \bar{D}_i$ on the holomorphic and antiholomorphic functions satisfy the undeformed Leibniz rule, $\delta^2 = \bar{\delta}^2 = 0$ and $\bar{\delta} x_j = x_j \bar{\delta}$ etc. These imply

$$D^i x_j = \delta^i_j + q \tilde{R}^{ik}_{jl} x_k D^l, \qquad D^i \bar{x}^j = q (\tilde{R}^{-1})^{ji}_{lk} \bar{x}^k D^l, \tag{4.11}$$

$$\bar{D}_i \bar{x}^j = \delta^i_j + q^{-1} (\tilde{R}^{-1})^{lj}_{ki} \bar{x}^k \bar{D}_l, \quad \bar{D}_i x_j = q^{-1} \tilde{\Phi}^{lk}_{ji} x_k \bar{D}_l \tag{4.12}$$

[1] The usual symbols $\partial^a, \bar{\partial}_b$ are reserved below for the derivatives on $CP_q(N)$.

and

$$D^i D^j = q^{-1} \tilde{R}^{ji}_{lk} D^k D^l, \tag{4.13}$$

$$D^i \bar{D}_j = q^{-1} \tilde{\Phi}^{ki}_{lj} \bar{D}_k D^l, \tag{4.14}$$

$$\bar{D}_i \bar{D}_j = q^{-1} \tilde{R}^{kl}_{ij} \bar{D}_k \bar{D}_l. \tag{4.15}$$

The matrix $\tilde{\Phi}^{ij}_{kl}$ is defined as

$$\tilde{\Phi}^{ij}_{kl} = \tilde{R}^{ji}_{lk} q^{2(i-l)} = \tilde{R}^{ji}_{lk} q^{2(k-j)}, \tag{4.16}$$

which satisfies

$$\tilde{\Phi}^{ri}_{sj} (\tilde{R}^{-1})^{jk}_{il} = (\tilde{R}^{-1})^{ri}_{sj} \tilde{\Phi}^{jk}_{il} = \delta^r_l \delta^k_s \tag{4.17}$$

and (sum over the index k)

$$\bar{\Phi}^{ik}_{jk} = \delta^i_j q^{2i+1}, \quad \bar{\Phi}^{ki}_{kj} = \delta^i_j q^{2(N-i)+1}. \tag{4.18}$$

Because we have chosen to run the indices from 0 to $N+1$, (4.18) is slightly different from the formulas in chapter 1.

Symmetry and $*$-involution

Using

$$\tilde{R}^{ij}_{kl}(q^{-1}) = (\tilde{R}^{-1})^{ji}_{lk}(q) \tag{4.19}$$

and

$$\tilde{R}^{ij}_{kl} = \tilde{R}^{kl}_{ij}, \tag{4.20}$$

one can show that if we do the following replacement

$$q \to q^{-1}, \tag{4.21}$$

$$x_i \to k q^{-2i} \bar{x}^i, \quad \bar{x}^i \to l x_i, \tag{4.22}$$

$$\xi_i \to k q^{-2i} \bar{\xi}^i, \quad \bar{\xi}^i \to l \xi_i \tag{4.23}$$

and

$$D^i \to k^{-1} q^{2i} \bar{D}_i, \quad \bar{D}_i \to l^{-1} D^i, \tag{4.24}$$

where k and l are arbitrary constants, then all the commutation relations just go back to themselves and the replacement (4.21) - (4.24) is hence a symmetry of the

algebra. Exchanging the barred and unbarred quantities in (4.21) - (4.24), we get another symmetry which is related to the inverse of this one.

Since L commutes with $x_i, \bar{x}^i$, a $*$-involution can be defined for D^i

$$(D^i)^* = -q^{-2i'} L^n \bar{D}_i L^{-n}, \tag{4.25}$$

where

$$i' = N - i + 1 \tag{4.26}$$

for any real number n. The $*$-involutions corresponding to different n's are related to one another by the symmetry of conjugation by L

$$a \to L^m a L^{-m}, \tag{4.27}$$

where a can be any function or derivative and m is the difference in the n's.

Finally, all the above relations are covariant under the transformation

$$x_i \to x_j T^j_i, \qquad \bar{x}^i \to (T^{-1})^i_j \bar{x}^j, \tag{4.28}$$

$$D^i \to (T^{-1})^i_j D^j, \quad \bar{D}_i \to \bar{D}_j q^{2i'} T^j_i q^{-2j'}, \tag{4.29}$$

$$\xi_i \to \xi_j T^j_i, \qquad \bar{\xi}^i \to (T^{-1})^i_j \bar{\xi}^j, \tag{4.30}$$

where $T^i_j \in SU_q(N+1)$.

One can check that

$$L\xi_i = q^2 \xi_i L, \tag{4.31}$$

which will be useful to us later.

4.1.2 Complex Projective Space $CP_q(N)$

Define for $a = 1, ..., N$, the inhomogeneous coordinates [2]

$$z_a = x_0^{-1} x_a, \quad \bar{z}^a = \bar{x}^a (\bar{x}^0)^{-1}. \tag{4.32}$$

As a consequence of (4.1, 4.2), we have

$$z_a z_b = q^{-1} \hat{R}^{ce}_{ab} z_c z_e, \tag{4.33}$$

[2]The letters a, b, c, e etc. run from 1 to N, while i, j, k, l run from 0 to N.

$$\bar{z}^a z_b = q^{-1} (\hat{R}^{-1})^{ac}_{be} z_c \bar{z}^e - \lambda q^{-1} \delta^a_b, \tag{4.34}$$

where $\hat{R}^{ac}_{be}$ is the $GL_q(N)$ $\hat{R}$-matrix with indices running from 1 to N and $\lambda = q - 1/q$. The formulas

$$x_0 x_a = q x_a x_0, \quad x_0 \bar{x}^0 = \bar{x}^0 x_0 \tag{4.35}$$

and

$$x_0 \bar{x}^a = q^{-1} \bar{x}^a x_0 \tag{4.36}$$

are very useful to obtaining (4.33) and (4.34).

Differential Calculus

The commutation relations for the calculus can be induced from that of C_q^{N+1}. It is obvious that

$$dz_a = x_0^{-1}(\xi_a - \xi_0 z_a), \quad d\bar{z}^a = (\bar{\xi}^a - \bar{z}^a \bar{\xi}^0)(\bar{x}^0)^{-1} \tag{4.37}$$

and

$$x_0 \xi_0 = q^2 \xi_0 x_0, \quad x_0 \bar{\xi}^0 = \bar{\xi}^0 x_0. \tag{4.38}$$

Using these and (4.7) and (4.8), we have

$$z_a dz_b = q \hat{R}^{ce}_{ab} dz_c z_e, \tag{4.39}$$

$$\bar{z}^a dz_b = q^{-1} (\hat{R}^{-1})^{ac}_{be} dz_c \bar{z}^e, \tag{4.40}$$

$$dz_a dz_b = -q \hat{R}^{ce}_{ab} dz_c dz_e \tag{4.41}$$

and

$$d\bar{z}^a dz_b = -q^{-1} (\hat{R}^{-1})^{ac}_{be} dz_c d\bar{z}^e. \tag{4.42}$$

One can introduce derivatives $\partial^a, \bar{\partial}_a$ by requiring $\delta \equiv dz_a \partial^a$ and $\bar{\delta} \equiv d\bar{z}^a \bar{\partial}_a$ to be exterior differentiations, i.e. $\delta^2 = \bar{\delta}^2 = 0$ and satisfy the Leibniz rule. It follows from (4.39) and (4.40) that

$$\partial^a z_b = \delta^a_b + q \hat{R}^{ac}_{be} z_c \partial^e, \tag{4.43}$$

$$\partial^a \bar{z}^b = q^{-1} (\hat{R}^{-1})^{ba}_{ec} \bar{z}^c \partial^e, \tag{4.44}$$

$$\bar{\partial}_a z_b = q\Phi^{ec}_{ba} z_c \bar{\partial}_e, \tag{4.45}$$

$$\bar{\partial}_a \bar{z}^b = \delta^b_a + q^{-1}(\hat{R}^{-1})^{eb}_{ca}\bar{z}^c\bar{\partial}_e, \tag{4.46}$$

$$\partial^b\partial^a = q^{-1}\hat{R}^{ab}_{ce}\partial^e\partial^c \tag{4.47}$$

and

$$\partial^a\bar{\partial}_b = q\Phi^{ca}_{eb}\bar{\partial}_c\partial^e, \tag{4.48}$$

where the Φ matrix is defined by

$$\Phi^{ca}_{db} = \hat{R}^{ac}_{bd}q^{2(c-b)} = \hat{R}^{ac}_{bd}q^{2(d-a)}. \tag{4.49}$$

Symmetry and $*$-involution

The algebra of the differential calculus on $CP_q(N)$ has the symmetry:

$$q \to q^{-1}, \tag{4.50}$$

$$z_a \to rq^{-2a}\bar{z}^a, \quad \bar{z}^a \to sz_a, \tag{4.51}$$

$$dz_a \to rq^{-2a}d\bar{z}^a, \quad d\bar{z}^a \to sdz_a \tag{4.52}$$

and

$$\partial^a \to r^{-1}q^{2a}\bar{\partial}_a, \quad \bar{\partial}_a \to s^{-1}\partial^a, \tag{4.53}$$

where $rs = q^2$. And also another symmetry by exchanging the barred and unbarred quantities in the above.

One can define a $*$-involution

$$z^*_a = \bar{z}^a, \tag{4.54}$$

$$dz^*_a = d\bar{z}^a \tag{4.55}$$

and

$$\partial^{a*} = -q^{2n-2a'}\rho^n\bar{\partial}_a\rho^{-n}, \tag{4.56}$$

where

$$a' = N - a + 1 \tag{4.57}$$

and

$$\rho = 1 + \sum_{a=1}^{N} z_a\bar{z}^a \tag{4.58}$$

for any n. The different choice of involution for different n's are related to one another by the symmetry of conjugation by ρ to some powers followed by a rescaling by appropriate powers of q. In particular, the correct classical limit of Hermitian conjugation with the standard measure $\rho^{-(N+1)}$ of $CP(N)$ is reproduced by $*$-involution with the choice of $n = N + 1$.

Covariance

The transformation for z_a is induced from (4.28) on C_q^{N+1}, it is

$$z_a \to (T^0_0 + z_bT^b_0)^{-1}(T^0_a + z_cT^c_a). \tag{4.59}$$

The differentials transform as

$$dz_a \to dz_bM^b_a, \quad d\bar{z}^a \to (M^\dagger)^a_b d\bar{z}^b, \tag{4.60}$$

where M^b_a is a matrix of functions in z_a with coefficients in $SU_q(N+1)$. It is computable from (4.59) but we don't need to know their explicit from. It is $(M^\dagger)^a_b \equiv (M^b_a)^*$. The transformation on the derivatives are

$$\partial^a \to (M^{-1})^a_b\partial^b, \quad (\partial^a)^* \to (\partial^b)^*((M^\dagger)^{-1})^b_a. \tag{4.61}$$

It follows from the fact that $\delta, \bar{\delta}$ are invariant. The covariance of the $CP_q(N)$ relations under the transformation (4.59), (4.60) and (4.61) follows directly from the covariance in C_q^{N+1}.

4.1.3 Other Choices for the Defining Relations for C_q^{N+1}

One may ask why do we choose to define the C_q^{N+1} algebra and calculus as in section 4.1.1. Keeping the other relations the same, one can use the alternative

$$\bar{x}^ix_j = q^{-1}\tilde{R}^{ik}_{jl}x_k\bar{x}^l, \tag{4.62}$$

and

$$x_i\xi_j = q^{-1}(\tilde{R}^{-1})^{kl}_{ij}\xi_kx_l \tag{4.63}$$

instead of (4.2) and (4.7). We have 4 different choices:

Type:	Defining relations	Braiding order
Type IA:	(4.2) and (4.7)	$d\bar{x} < \bar{x} < x < dx$
Type IB:	(4.62) and (4.7)	$\bar{x} < d\bar{x} < dx < x$
Type IIA:	(4.62) and (4.63)	$dx < x < \bar{x} < d\bar{x}$
Type IIB:	(4.2) and (4.63)	$x < dx < d\bar{x} < \bar{x}$

We have also listed the corresponding order that will yield the commutation relations. Since Type II is essentially the same as Type I (with a replacement $x \to \bar{x}, dx \to d\bar{x}$, one goes from IA to IIA, IB to IIB), we will only discuss the other choice IB here. It is easy to check that in IB, we still have $x_i L = L x_i$. But instead of (4.31), we have

$$\xi_i L = L\xi_i + \lambda q x_i \delta L \tag{4.64}$$

or equivalently

$$x_i \delta L = q^{-2} \delta L x_i. \tag{4.65}$$

This doesn't fit with the construction for the one-form realization that we are going to introduce in the next section. Interestingly enough, with the same definitions (4.32) and (4.37) for $z_i, \bar{z}^i, dz_i, d\bar{z}^i$, we get exactly the same relations for the $CP_q(N)$ as in section 4.1.2. In particular we have the same one-form realization (see next section)

$$\rho z_a = q^{-2} z_a \rho, \quad \rho dz_a = dz_a \rho \tag{4.66}$$

and

$$\eta = -q^{-1}\delta\rho\rho^{-1}, \quad \bar{\eta} = q\bar{\delta}\rho\rho^{-1}. \tag{4.67}$$

We choose to work with Type I because the corresponding relations for C_q^{N+1} have nicer property. But there is really no difference at the level of projective space $CP_q(N)$.

4.2 One-Form Realization of Exterior Differential Operators

Let us first recall that in Connes' non-commutative geometry[44], the calculus is quantized using the following operator representation for the differentials,

$$d\omega = F\omega - (-1)^k \omega F \tag{4.68}$$

where ω is a k-form and F is an operator such that $F^* = F$ and $F^2 = 1$. [3] In the bicovariant calculus of Quantum Groups[14], there exists a one-form η with the properties $\eta^* = -\eta$, $\eta^2 = 0$ and

$$df = [\eta, f]_\pm \tag{4.69}$$

where $[a, b]_\pm = ab \pm ba$ is the graded commutator with plus sign only when both a and b are odd. [4] It is interesting to ask when will such a realization of differentials exist? And what will be the properties of this special one-form? Instead of studying the operator aspect, we will first consider these questions in the simpler algebraic sense.

4.2.1 A Special One-Form

Let us first look at an example.

Example 4.2.1 ($SO_q(N)$ quantum space) *[8, 64]*
The quantum matrices T of $SO_q(N)$ satisfy in addition to

$$\hat{R}_{12} T_1 T_2 = T_1 T_2 \hat{R}_{12}, \tag{4.70}$$

also the orthogonality relations[8]

$$T^t g T = g, \quad T g^{-1} T^t = g^{-1}, \tag{4.71}$$

[3] The appropriate setting is a Fredholm module $(\mathcal{H}, F)$ where all these relations take place in the Hilbert space $\mathcal{H}$.

[4] For the 3D left or right covariant calculus of $SU_q(2)$[13], such a one-form doesn't exist and has to be introduced as an additional one-form.

where the numerical quantum metric matrices $g = g_{ij}$ and $g^{-1} = g^{ij}$ can be chosen to be equal $g_{ij} = g^{ij}$. The coordinates x_i of the quantum Euclidean space satisfy the commutation relations

$$x_k x_l \hat{R}^{kl}_{ij} = q x_i x_j - \lambda\alpha(x \cdot x) g_{ij}, \tag{4.72}$$

where $L \equiv x \cdot x = x_k x_l g^{kl} = x_k x^k$ and $\alpha = \frac{1}{1+q^{N-2}}$. The differentials of the coordinates $\xi_i = dx_i$ satisfy the commutation relations

$$x_i \xi_j = q \xi_k x_l \hat{R}^{kl}_{ij}, \tag{4.73}$$

It can be verified that

$$L x_i = x_i L, \quad L dx_i = q^2 dx_i L. \tag{4.74}$$

Hence $\eta = -q^{-1} dL L^{-1}$ satisfies [5]

$$\lambda df = [\eta, f]_{\pm}. \tag{4.75}$$

Generalizing the idea, we have the following construction:

Construction 4.2.1 (Sufficient condition)

Let A be an algebra generated by coordinates x_i. $(\Omega(A), d)$ be a differential calculus [6] *over A. If there exists an element $a \in A$, constants r, s such that*

$$a x_i = r x_i a, \quad a dx_i = s dx_i a, \quad \forall i, \tag{4.76}$$

then

$$\lambda df = [\eta, f]_{\pm} \tag{4.77}$$

with

$$\eta = \frac{\lambda}{1 - s/r} da a^{-1}. \tag{4.78}$$

The normalization constant λ is introduced such that $\lambda/(1 - s/r)$ is well defined as $r, s \to 1$.

[5] The existence and construction of this one-form is first discovered and pointed out to the author by Professor B. Zumino.

[6] By this we mean an A bimodule $\Omega(A)$ generated by x_i, dx_i with commutation relations specified such that $(d1) = 0$, graded Leibniz rule and $d^2 = 0$.

Proof

$$\begin{aligned}
a dx_i = s dx_i a &\Rightarrow da dx_i = -s dx_i da \\
&\Rightarrow [da a^{-1}, dx_i]_+ = 0. \\
a x_i = r x_i a &\Rightarrow da x_i + a dx_i = r dx_i a + r x_i da \\
&\Rightarrow r da a^{-1} x_i + s dx_i = r dx_i + r x_i da a^{-1} \\
&\Rightarrow [(1 - s/r)^{-1} da a^{-1}, x_i] = dx_i.
\end{aligned}$$

□

It is not hard to prove that $\eta^2 = d\eta = 0$. We give a few examples.

Example 4.2.2 ($SU_q(N+1)$ covariant quantum spaces C_q^{N+1}) *[8]*

The $SU_q(N+1)$ symmetry is represented in the complex quantum space C_q^{N+1} with coordinates $x_i, \bar{x}^i$ and differentials $\xi_i, \bar{\xi}^i, i = 0, 1, ..., N$. They satisfy the relations:

$$x_i x_j = q^{-1} x_k x_l \tilde{R}^{kl}_{ij}, \tag{4.79}$$

$$\bar{x}^i x_j = q (\tilde{R}^{-1})^{ik}_{jl} x_k \bar{x}^l, \tag{4.80}$$

$$x_i \xi_j = q \tilde{R}^{kl}_{ij} \xi_k x_l, \tag{4.81}$$

$$\bar{x}^i \xi_j = q (\tilde{R}^{-1})^{ik}_{jl} \xi_k \bar{x}^l \tag{4.82}$$

where $\tilde{R}$ is the $GL_q(N+1)\hat{R}$ matrix. The Hermitian length

$$L = x_i \bar{x}^i \tag{4.83}$$

satisfies

$$L x_i = x_i L, \quad L dx_i = q^2 dx_i L. \tag{4.84}$$

Hence

$$\eta = -q^{-1} dL L^{-1}. \tag{4.85}$$

Example 4.2.3 ($GL_q(N)$ quantum group) *[8, 55]*

The algebra is generated by the elements of the quantum matrix $T = (T^i_j)_{i,j=1,...N}$ and the differentials dT^i_j. The quantum determinant $L = det_q T$ satisfies

$$L T^i_j = T^i_j L, \quad L dT^i_j = q^2 dT^i_j L \tag{4.86}$$

and

$$\eta = -q^{-1} dL L^{-1}. \tag{4.87}$$

4.2.2 One-form Realization of the Exterior Differential for a $*$-Algebra

In the same manner as in the construction in section 4.2.1, we have the following:

Construction 4.2.2 ($*$-Algebra)
Let A be a $*$-involutive algebra with coordinates $z_i, \bar{z}_i$ and differentials $dz_i = \delta z_i, d\bar{z}_i = \bar{\delta}\bar{z}_i$ such that $\bar{z}_i = z_i^*, d\bar{z}_i = (dz_i)^*$. If there exists a real element $a \in A$ and real unequal nonvanishing constants r, s such that

$$az_i = rz_i a, \quad adz_i = sdz_i a, \quad \forall i, \tag{4.88}$$

then, as easily seen,

$$\lambda\delta f = [\eta, f]_\pm, \quad \eta = \frac{\lambda}{1 - s/r}\delta a a^{-1}, \tag{4.89}$$

$$\lambda\bar{\delta} f = [\bar{\eta}, f]_\pm, \quad \bar{\eta} = \frac{\lambda}{1 - r/s}\bar{\delta} a a^{-1} \tag{4.90}$$

and

$$\lambda df = [\Xi, f]_\pm, \quad \Xi = \eta + \bar{\eta}, \tag{4.91}$$

where $\pm$ applies for odd/even forms f.

Notice that (4.89) and (4.90), and therefore (4.88), imply that

$$ra\delta a = s\delta a a, \quad r\bar{\delta} a a = sa\bar{\delta} a. \tag{4.92}$$

Kähler Form

It can be proved that $\eta^* = -\bar{\eta}$ and so $\Xi^* = -\Xi$. It holds that $\eta^2 = \bar{\eta}^2 = 0$. However $\Xi^2 = \eta\bar{\eta} + \bar{\eta}\eta = \lambda\delta\bar{\eta} = \lambda\bar{\delta}\eta$ will generally be nonzero. Note that

$$\lambda d\Xi = [\Xi, \Xi]_+ = 2\Xi^2. \tag{4.93}$$

Define

$$K = \delta\bar{\eta} = \bar{\delta}\eta \tag{4.94}$$

then

$$K = \frac{1}{2}d\Xi. \tag{4.95}$$

It follows that $dK = 0$ and $K^* = K$. Thus in the case $K \neq 0$, we will call it a Kähler form and K^n [7] will be non-zero and define a real volume element for an integral (invariant integral if K^n is invariant). K also has the very nice property of commuting with everything

$$Kz_a = z_a K, \quad Kdz_a = dz_a K. \tag{4.96}$$

We see here an example of Connes' calculus [44] of the type $F^2 \neq 0$ rather than $F^2 = 0$.

Quantum Sphere S_q^2

In the case of S_q^2, The element $\rho = 1 + \bar{z}z$ is introduced which satisfies

$$\rho z = q^2 z\rho, \quad \rho dz = dz\rho. \tag{4.97}$$

Therefore, we get

$$\eta = qdz\rho^{-1}\bar{z}, \quad \bar{\eta} = -qd\bar{z}\rho^{-1}z \tag{4.98}$$

and K is just the area element

$$\begin{aligned} K &= \bar{\delta}\eta \\ &= -q^3 dz d\bar{z}\rho^{-2}. \end{aligned} \tag{4.99}$$

One can introduce the Kähler potential V defined by

$$K = \delta\bar{\delta}V. \tag{4.100}$$

It is

$$V = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^{2k-1}}{[k]_q}\bar{z}^k z^k \tag{4.101}$$

and where the quantum number $[x]_q$ is defined as

$$[x]_q = \frac{q^{2x} - 1}{q^2 - 1}. \tag{4.102}$$

[7] n = complex dimension of the algebra. We consider only deformations such that the Poincaré series of the deformed algebra and its classical counterpart match.

Complex Projective Space

Such a one-form representation for the calculus exists on both C_q^{N+1} and $CP_q(N)$. For C_q^{N+1}, we saw in the above that

$$Lx_i = x_i L, \quad L\xi_i = q^2 \xi_i L \tag{4.103}$$

and

$$\eta_0 = -q^{-1}\delta L L^{-1}, \quad \bar{\eta}_0 = q\bar{\delta} L L^{-1}. \tag{4.104}$$

In this case, K is not the Kähler form one usually assigns to C_q^{N+1}. Rather, it gives C_q^{N+1} the geometry of $CP_q(N)$ written in homogeneous coordinates.

Similar relations hold for $CP_q(N)$ in inhomogeneous coordinates. It is

$$\rho z_a = q^{-2} z_a \rho, \quad \rho dz_a = dz_a \rho \tag{4.105}$$

and therefore

$$\eta = -q^{-1}\delta\rho\rho^{-1}, \quad \bar{\eta} = q\bar{\delta}\rho\rho^{-1}. \tag{4.106}$$

One can then compute

$$\begin{aligned} K &= \bar{\delta}\eta && (4.107)\\ &= dz_a g^{a\bar{b}} d\bar{z}^b, && (4.108) \end{aligned}$$

where the metric $g^{a\bar{b}}$ is

$$g^{a\bar{b}} = q^{-1}\rho^{-2}(\rho\delta_{ab} - q^2 \bar{z}^a z_b) \tag{4.109}$$

with inverse $g_{\bar{b}c}$

$$g_{\bar{b}c} g^{c\bar{a}} = g^{a\bar{c}} g_{\bar{c}b} = \delta_{ab} \tag{4.110}$$

given by

$$g_{\bar{b}c} = q\rho(\delta_{bc} + \bar{z}^b z_c). \tag{4.111}$$

This metric is the quantum deformation of the standard Fubini-Study metric for $CP(N)$. It is $K = \delta\bar{\delta}V$, where the Kähler potential V is

$$V = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^{2k-1}}{[k]_q} \sum_{1 \le a_1, a_2, \cdots, a_k \le N} z_{a_k} z_{a_{k-1}} \cdots z_{a_1} \bar{z}^{a_1} \cdots \bar{z}^{a_{k-1}} \bar{z}^{a_k}. \tag{4.112}$$

Notice that under the transformation (4.59)

$$\eta \to \eta + q f^{-1}\delta f, \quad f = T_0^0 + z_b T_0^b \tag{4.113}$$

and so K is invariant. From (4.60) and (4.108), it follows that

$$g^{a\bar{b}} \to (M^{-1})^a_c g^{c\bar{d}} ((M^\dagger)^{-1})^{\bar{d}}_{\bar{b}}, \tag{4.114}$$

$$g_{\bar{b}a} \to (M^\dagger)^{\bar{b}}_{\bar{d}} g_{\bar{d}c} M^c_a. \tag{4.115}$$

One can show that the following form dv_x in C_q^{N+1}

$$\begin{aligned} dv_x &\equiv \Pi_{j=0}^N (\bar{\xi}^j L^{-1/2}) \Pi_{i=0}^N (L^{-1/2}\xi_i) && (4.116)\\ &= \rho^{-(N+1)} d\bar{z}^N \cdots d\bar{z}^1 dz_1 \cdots dz_N \cdot \bar{\xi}^0 (\bar{x}^0)^{-1} (x_0)^{-1} \xi_0 && (4.117) \end{aligned}$$

is invariant. Using this, one can prove that

$$dv_x \equiv \rho^{-(N+1)} d\bar{z}^N \cdots d\bar{z}^1 dz_1 \cdots dz_N \tag{4.118}$$

is invariant also and is in fact equal to K^N (up to a numerical factor). The factor $\rho^{-(N+1)}$ justifies the choice $n = N+1$ for the involution (4.56).

Having a quantum Kähler metric one can define connections, curvature, a Ricci tensor and a Hodge star operation. We shall not do it here because there seems to be no unique way to define these constructs. Still, once certian choices are made, the full differential geometry can be developed. See [80] for a very nice discussion of the quantum Riemannian case.

4.3 Integration

We now turn to the discussion of integration on $CP_q(N)$. We shall use the notation $< f(z,\bar{z}) >$ for the right-invariant integral of a function $f(z,\bar{z})$ over $CP_q(N)$. It is defined, up to a normalization factor, by requiring

$$< \mathcal{O} f(z,\bar{z}) > = 0 \tag{4.119}$$

for any left-invariant vector field $\mathcal{O}$ of $SU_q(N+1)$. We can work out the integral by looking at the explicit action of the vector fields on functions. This approach

has been worked out for the case of the sphere but it get quite complicated for the higher dimensional projective spaces. We shall follow a different and simpler approach here. First we notice that the identification

$$x_i/L^{1/2} = T_i^N, \quad \bar{x}^i/L^{1/2} = (T^{-1})_N^i, \quad i = 0, 1, ..., N \tag{4.120}$$

reproduces (4.1)-(4.5). Thus if we define

$$< f(z,\bar{z}) > \equiv < f(z,\bar{z})|_{z_a=(T_0^N)^{-1}T_a^N, \bar{z}^a=(T^{-1})_N^a/(T^{-1})_N^0} >_{SU_q(N+1)}, \tag{4.121}$$

where $< \cdot >_{SU_q(N+1)}$ is the Haar measure [12] on $SU_q(N+1)$, then it follows immediately that (4.119) is satisfied. [8] Next we claim that

$$< (z_1)^{i_1}(\bar{z}^1)^{j_1} \cdots (z_N)^{i_N}(\bar{z}^N)^{j_N} >= 0 \text{ unless } i_1 = j_1, ..., i_N = j_N. \tag{4.122}$$

This is because the integral is invariant under the finite transformation (4.59). For the particular choice $T_j^i = \delta_j^i \alpha_i$, with $|\alpha_i| = 1, \Pi_{i=0}^N \alpha_i = 1$, this gives

$$z_a \to (\alpha_a/\alpha_0) z_a \tag{4.123}$$

and so (4.122) follows.

In [12], Woronowicz proved the following interesting property for the Haar measure

$$< f(T)g(T) >_{SU_q(N+1)} = < g(T)f(DTD) >_{SU_q(N+1)}, \tag{4.124}$$

where

$$(DTD)_j^i = D_k^i T_m^k D_j^m \tag{4.125}$$

and

$$D_j^i = q^{-N+2i}\delta_j^i \tag{4.126}$$

is the D matrix for $SU_q(N+1)$. It follows from (4.124) that

$$< f(z,\bar{z})g(z,\bar{z}) > = < g(z,\bar{z})f(\mathcal{D}z, \mathcal{D}^{-1}\bar{z}) >, \tag{4.127}$$

where

$$\mathcal{D}_b^a = \delta_b^a q^{2a}, \quad a, b = 1, 2, ..., N. \tag{4.128}$$

[8] A similar strategy of using the "angular" measure to define an integration has been employed by H. Steinacker [81] in constructing integration over quantum Euclidean space.

Introducing

$$\rho_r = 1 + \textstyle\sum_{a=1}^r z_a \bar{z}^a, \tag{4.129}$$

one finds from (4.33) and (4.34) that

$$\rho_r z_a = \begin{cases} z_a \rho_r & r < a \\ q^{-2} z_a \rho_r & r \geq a \end{cases}, \tag{4.130}$$

$$\rho_r \rho_s = \rho_s \rho_r \tag{4.131}$$

and

$$\bar{z}^a z_a = q^{-2}\rho_a - \rho_{a-1} \quad \text{(no sum)}. \tag{4.132}$$

Because of (4.122), it is sufficient to determine integrals of the form

$$< \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} > . \tag{4.133}$$

The values of the integers i_a for (4.133) to make sense will be determined later.

Consider

$$\begin{aligned} < \bar{z}_a \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} z_a > &= < \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} z_a (q^{-2a}\bar{z}^a) > \\ &= q^{-2a} < \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} (\rho_a - \rho_{a-1}) >, \end{aligned} \tag{4.134}$$

where (4.127) is used. Applying (4.130)

$$\begin{aligned} \text{L.S.} &= q^{2(i_a + \cdots + i_N)} < \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} \bar{z}^a z_a > \\ &= q^{2I_a} < \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} \bar{z}^a z_a >, \end{aligned} \tag{4.135}$$

where we have denoted

$$I_a = i_a + \cdots + i_N. \tag{4.136}$$

Using (4.132) we get the recursion formula

$$\begin{aligned} &< \rho_1{}^{-i_1} \cdots \rho_{a-1}{}^{-i_{a-1}+1} \rho_a{}^{-i_a} \cdots \rho_N{}^{-i_N} > [I_a + a]_q \\ &= < \rho_1{}^{-i_1} \cdots \rho_{a-1}{}^{-i_{a-1}} \rho_a{}^{-i_a+1} \cdots \rho_N{}^{-i_N} > [I_a + a - 1]_q. \end{aligned} \tag{4.137}$$

It is obvious then that

$$< \rho_1{}^{-i_1} \cdots \rho_a{}^{-i_a} > = < \rho_1{}^{-i_1} \cdots \rho_{a-1}{}^{-i_{a-1}-i_a} > \frac{[a]_q}{[I_a + a]_q}. \tag{4.138}$$

By repeated use of the recursion formula, $< \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} >$ reduces finally to $< \rho_1{}^{-i_1-i_2\cdots-i_N} >$ and

$$< \rho_1{}^{-I_1} >= \frac{1}{[I_1+1]_q} < 1 > . \tag{4.139}$$

Therefore

$$< \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} >=< 1 > \Pi_{a=1}^{N} \frac{[a]_q}{[I_a+a]_q}. \tag{4.140}$$

For this to be positive definite, i_a should be restricted such that $I_a + a > 0$ for $a = 1, \cdots, N$.

4.4 Braided $CP_q(N)$

As described in [76] and also in section 3.5.1, it is sufficient to know the transformation property of the algebra to derive the braiding . But as demonstrated in [76], it is already quite complicated in the case of one dimensional algebra. Therefore although we can derive the braiding for the $CP_q(N)$ using the general framework of 3.5.1, we will follow a different easier path: first introduce the braiding for C_q^{N+1} quantum planes and then use it to induce a braiding on the $CP_q(N)$'s.

Braided C_q^{N+1}

Let the first copy of quantum plane be denoted by $x_i, \bar{x}^i$ and the second by $x'_i, \bar{x}'^i$ and let their commutation relations be:

$$x_i x'_j = \tau \tilde{R}^{kl}_{ij} x'_k x_l, \tag{4.141}$$

$$\bar{x}^i x'_j = \nu (\tilde{R}^{-1})^{ik}_{jl} x'_k \bar{x}^l \tag{4.142}$$

and their $*$-involutions for arbitrary numbers τ, ν. These are consistent and covariant, as one can easily check. One can choose $\tau = \nu^{-1}$ and the Hermitian length L will be central, $Lf' = f'L$, for any function f' of $x', \bar{x}'$. However, L' does not commute with $x, \bar{x}$. In the following, we don't need to assume that $\tau = \nu^{-1}$.

By assuming that the exterior derivatives of the two copies satisfy the Leibniz rule

$$\delta' f = \pm f \delta', \quad \bar{\delta}' f = \pm f \bar{\delta}', \tag{4.143}$$

$$\delta f' = \pm f' \delta, \quad \bar{\delta} f' = \pm f' \bar{\delta}, \tag{4.144}$$

where the plus (minus) signs apply for even (odd) f and f', and

$$\delta\delta' = -\delta'\delta, \quad \delta\bar{\delta}' = -\bar{\delta}'\delta, \tag{4.145}$$

$$\bar{\delta}\delta' = -\delta'\bar{\delta}, \quad \bar{\delta}\bar{\delta}' = -\bar{\delta}'\bar{\delta}, \tag{4.146}$$

One can derive the commutation relations between functions and forms. Identify $\delta = dx_i D^i, \bar{\delta} = d\bar{x}^i \bar{D}_i$ for both copies, one can derive also the commutation relations between derivatives and functions of different copies. We will not bother to write them down here.

Braided $CP_q(N)$

Using (4.141, 4.142), one can derive the braiding relations of two braided copies of $CP_q(N)$ in terms of the inhomogeneous coordinates

$$z_a z'_b = q \hat{R}^{ce}_{ab} (z'_c - q^{-1} \lambda z_c) z_e, \tag{4.147}$$

$$\bar{z}'^a z_b = q^{-1} (\hat{R}^{-1})^{ac}_{be} z_c \bar{z}'^e - q^{-1} \lambda \delta^a_b \tag{4.148}$$

and their $*$-involutions. Notice that these are independent of the particular choice of τ and ν. Similarly, one can work out the commutation relations between functions and forms of different copies following the assumption that their exterior derivatives anticommute. We will not list them here.

See [77, 78] for a detailed discussion on the anharmonic ratios for braided copies of $CP_q(N)$.

4.5 Quantum Grassmannians $G_q^{M,N}$

4.5.1 The Algebra

Let C^i_a, $i = 1, 2, \cdots, M$, $a = 1, 2, \cdots, M+N$ be an $M \times (M+N)$ rectangular matrix satisfying the commutation relations

$$\hat{R}'^{ij}_{kl} C^k_c C^l_d = C^i_a C^j_b \hat{R}^{ab}_{cd}, \tag{4.149}$$

where $\hat{R}'^{ij}_{kl}$ is a $GL_q(M)$ $\hat{R}$-matrix, with indices i, j, k, l etc. going from 1 to M and $\hat{R}^{ab}_{cd}$ is a $GL_q(M+N)$ $\hat{R}$-matrix, with indices a, b, c, d etc. going from 1 to $M+N$.

In compact notation, it is

$$\hat{R}'_{12}C_1C_2 = C_1C_2\hat{R}_{12} \tag{4.150}$$

and (4.150) is covariant under the transformation

$$C \to CT, \tag{4.151}$$

where T^a_b is a $GL_q(M+N)$ quantum matrix and also under the transformation

$$C \to SC, \tag{4.152}$$

where S^i_j is a $GL_q(M)$ quantum matrix. Writing

$$C^i_a = (A^i_j, B^i_\alpha) \tag{4.153}$$

with $\alpha = 1, 2, \cdots, N$, we have

$$\begin{aligned} \hat{R}'_{12}A_1A_2 &= A_1A_2\hat{R}'_{12}, \\ \hat{R}'_{12}B_1B_2 &= B_1B_2\hat{R}''_{12}, \\ A_1B_2 &= R'_{21}B_2A_1, \end{aligned} \tag{4.154}$$

where $\hat{R}''^{\alpha\beta}_{\gamma\delta}$ is a $GL_q(N)$ $\hat{R}$-matrix, with indices $\alpha, \beta, \gamma, \delta$ etc. going from 1 to N.

Define the coordinates Z^i_α for the quantum Grassmannians

$$Z = A^{-1}B. \tag{4.155}$$

Z is invariant under the transformation (4.152), while under (4.151), it transforms as

$$Z \to (\alpha + Z\gamma)^{-1}(\beta + Z\delta), \tag{4.156}$$

where $\alpha, \beta, \gamma, \delta$ are the sub-matrices of T

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{4.157}$$

It follows from (4.154) that Z satisfies

$$\hat{R}'_{21}Z_1Z_2 = Z_1Z_2\hat{R}''_{12}. \tag{4.158}$$

$*$−structure

We consider q to be a real number. One can introduce the $*$−conjugate variables $(C^i_a)^*$ and impose the commutation relation

$$C^\dagger_1\hat{R}'^{-1}_{12}C_1 = C_2\hat{R}^{-1}_{12}C^\dagger_2, \tag{4.159}$$

i.e.

$$\begin{aligned} (A^{-1})^\dagger_2\hat{R}'_{12}(A^{-1})_2 &= (A^{-1})_1\hat{R}'_{12}(A^{-1})^\dagger_1, \\ B^\dagger_1A^{-1}_2 &= A^{-1}_2B^\dagger_1R'^{-1}_{12}, \\ B^\dagger_1\hat{R}'^{-1}_{12}B_1 &= B_2\hat{R}''^{-1}_{12}B^\dagger_2 - \lambda I_1(AA^\dagger)_2, \end{aligned} \tag{4.160}$$

where $(I_1(AA^\dagger)_2)^{\alpha i}_{\beta j} = \delta^\alpha_\beta(AA^\dagger)^i_j$. These implies

$$Z^\dagger_1\hat{R}'_{21}Z_1 = Z_2\hat{R}''^{-1}_{12}Z^\dagger_2 - \lambda I_{12}. \tag{4.161}$$

Explicitly,

$$(Z^\dagger)^\alpha_i\hat{R}'^{si}_{tj}Z^j_\beta = Z^s_\gamma(\hat{R}''^{-1})^{\alpha\gamma}_{\beta\delta}(Z^\dagger)^\delta_t - \lambda\delta^\alpha_\beta\delta^s_t. \tag{4.162}$$

4.5.2 Calculus

One can introduce the following commutation relation for functions and one-forms

$$\hat{R}'^{-1}_{12}C_1dC_2 = dC_1C_2\hat{R}_{12}, \tag{4.163}$$

i.e.

$$\begin{aligned} \hat{R}'^{-1}_{12}A_1dA_2 &= dA_1A_2\hat{R}'_{12}, \\ \hat{R}'^{-1}_{12}B_1dB_2 &= dB_1B_2\hat{R}''_{12}, \\ dA_1B_2 &= R'^{-1}_{12}B_2dA_1, \\ A_1dB_2 &= R'_{21}(dB_2A_1 + \lambda dA_2B_1P_{12}), \end{aligned} \tag{4.164}$$

where $(P_{12})^{ij}_{kl} = \delta^i_l\delta^j_k$. Since $Z^i_\alpha = (A^{-1})^i_kB^k_\alpha$, it is easy to derive

$$dZ = A^{-1}(dB - dAZ) \tag{4.165}$$

and

$$\begin{aligned} Z_1 dA_2 &= dA_2 R'^{-1}_{12} Z_1, \\ dZ_1 A_2 &= A_2 R'^{-1}_{12} dZ_1, \\ Z_1 dB_2 &= (dB_2 Z_1 - \lambda Z_1 (dAZ)_2 P_{12}) R''_{12}. \end{aligned} \tag{4.166}$$

It follows

$$\hat{R}'^{-1}_{21} Z_1 dZ_2 = dZ_1 Z_2 \hat{R}''_{12}. \tag{4.167}$$

To introduce a $*-$structure for the calculus, it is consistent to take $(dZ^i_\alpha)^* = d(Z^{i*}_\alpha)$, this implies

$$Z^\dagger_1 \hat{R}'_{21} dZ_1 = dZ_2 \hat{R}''^{-1}_{12} Z^\dagger_2. \tag{4.168}$$

4.5.3 One-Form Realization

Introduce the matrix

$$E^i_j = C^i_a (C^\dagger)^a_j. \tag{4.169}$$

It is

$$E^\dagger = E \tag{4.170}$$

and

$$\hat{R}'^{-1}_{12} E_1 \hat{R}'^{-1}_{12} E_1 = E_1 \hat{R}'^{-1}_{12} E_1 \hat{R}'^{-1}_{12}. \tag{4.171}$$

Since

$$\hat{R}'^{-1}_{12}(q) = \hat{R}'_{21}(q^{-1}), \tag{4.172}$$

one can rewrite (4.171) as

$$\hat{R}'_{12}(q^{-1}) E_2 \hat{R}'_{12}(q^{-1}) E_2 = E_2 \hat{R}'_{12}(q^{-1}) E_2 \hat{R}'_{12}(q^{-1}). \tag{4.173}$$

This shows that the commutation for the E matrix is like the bi-covariant vector fields Y (c.f. (2.23)), but with q^{-1} as its parameter. Similarly one also has

$$\begin{aligned} C_2 E_1 &= \hat{R}'^{-1}_{12}(q^{-1}) E_2 \hat{R}'_{12}(q^{-1}) C_2, \\ dC_2 E_1 &= \hat{R}'_{12}(q^{-1}) E_2 \hat{R}'_{12}(q^{-1}) dC_2. \end{aligned} \tag{4.174}$$

One can show that

$$R'^{-1}_{I,J} E_J R'^{-1}_{J,I} E_I = E_I R'^{-1}_{I,J} E_J R'^{-1}_{J,I} \tag{4.175}$$

or equivalently,

$$R'_{I,J} E_I \bullet E_J = E_J \bullet E_I R'_{I,J}, \tag{4.176}$$

where the bullet product is defined inductively by

$$E_I \bullet E_J \equiv E_I R'^{-1}_{I,J} E_J R'_{I,J} \tag{4.177}$$

for any $I = (12 \cdots m)$, $J = (12 \cdots n)$. Hence one can introduce the quantum determinant [56] for the generators E,

$$Det E \epsilon^{12\cdots M} = E_{(12\cdots M)} \epsilon^{12\cdots M}, \tag{4.178}$$

where

$$E_{(12\cdots M)} \equiv E_1 \bullet E_2 \bullet \cdots \bullet E_M \tag{4.179}$$

and $\epsilon^{12\cdots M}$ is the same ϵ tensor (1.43) for $GL_q(M)$.

Using

$$R'_{(12\cdots M),0} \epsilon^{12\cdots M} = q \epsilon^{12\cdots M} I_0, \tag{4.180}$$

one can show that $Det E$ commutes with E^i_j:

$$\begin{aligned} \epsilon^{12\cdots M} Det E E_0 &= E_{(12\cdots M)} E_0 \epsilon^{12\cdots M} \\ &= E_{(12\cdots M)} R'^{-1}_{(12\cdots M),0} E_0 \epsilon^{12\cdots M} \\ &= q R'^{-1}_{(12\cdots M),0} E_0 R'^{-1}_{0,(12\cdots M)} E_{(12\cdots M)} R'_{0,(12\cdots M)} \epsilon^{12\cdots M} \\ &= E_0 Det E \epsilon^{12\cdots M}. \end{aligned}$$

Denote

$$L = Det E. \tag{4.181}$$

It follows from (4.174) that

$$\begin{aligned} CL &= LC, \\ dCL &= q^{-2} L dC. \end{aligned} \tag{4.182}$$

Using the general procedure stated in section 4.2 , we obtain the one-form realization on the algebra generated by C^i_a, dC^i_a and their $*-$conjugates,

$$\eta = -q L^{-1} \delta L. \tag{4.183}$$

To find the one-form realization for the exterior differential operating on the complex Grassmannians Z, dZ, we introduce

$$\begin{aligned} X_l^k &= (A^{-1})_i^k E_j^i (A^{\dagger -1})_l^j \\ &= \delta_l^k + Z_\alpha^k (Z^\dagger)_l^\alpha. \end{aligned} \tag{4.184}$$

It is not hard to check that

$$\begin{aligned} A_1^{-1} R'_{21} E_2 R'^{-1}_{21} &= E_2 A_1^{-1}, \\ X_1 R'_{12} A_2^{-1} &= R'_{12} A_2^{-1} X_1 \end{aligned} \tag{4.185}$$

and hence

$$\hat{R}'_{12} X_2 \hat{R}'_{12} X_2 = X_2 \hat{R}'_{12} X_2 \hat{R}'_{12}. \tag{4.186}$$

Since X commutes like the vector field Y, the quantum determinant

$$\begin{aligned} \rho &\equiv DetX \\ &= X_{(12\cdots M)} \epsilon^{12\cdots M} \end{aligned} \tag{4.187}$$

is central in the algebra of X. Here, the $\bullet$-product for X is

$$X_I \bullet X_J = R'^{-1}_{IJ} X_I R'_{IJ} X_J, \tag{4.188}$$

as for the vector field Y. Using the techniques as in ([56]), one can show that for any $I = (12\cdots m)$,

$$X_I = A_I^{-1} E_I (A^{\dagger -1})_I, \tag{4.189}$$

where

$$A_I = A_1 A_2 \cdots A_m, \quad A_I^{-1} = A_m^{-1} \cdots A_2^{-1} A_1^{-1}. \tag{4.190}$$

Introducing the quantum determinant $det(A^{-1})$, $det(A^{\dagger -1})$

$$\begin{aligned} det(A^{-1}) \epsilon^{12\cdots M} &= A_M^{-1} \cdots A_2^{-1} A_1^{-1} \epsilon^{12\cdots M}, \\ det(A^{\dagger -1}) \epsilon^{12\cdots M} &= (A^{\dagger -1})_1 (A^{\dagger -1})_2 \cdots (A^{\dagger -1})_M \epsilon^{12\cdots M} \end{aligned} \tag{4.191}$$

for the "RTT"-like A^{-1} matrix and $A^{\dagger -1}$ matrix

$$\begin{aligned} \hat{R}'_{12}(q^{-1}) A_1^{-1} A_2^{-1} &= A_1^{-1} A_2^{-1} \hat{R}'_{12}(q^{-1}), \\ \hat{R}'_{12} (A^{\dagger -1})_1 (A^{\dagger -1})_2 &= (A^{\dagger -1})_1 (A^{\dagger -1})_2 \hat{R}'_{12}, \end{aligned} \tag{4.192}$$

it is straightforward to obtain

$$DetX = det(A^{-1})\, DetE\, det(A^{\dagger -1}). \tag{4.193}$$

It is trivial to check that

$$\begin{aligned} Bdet(A^{-1}) &= qdet(A^{-1})B, \\ Bdet(A^{\dagger -1}) &= qdet(A^{\dagger -1})B \end{aligned} \tag{4.194}$$

and noticing that

$$\rho = det(A^{-1})\, L\, det(A^{\dagger -1}), \tag{4.195}$$

we have

$$Z\rho = q^2 \rho Z. \tag{4.196}$$

Using (4.166), one can check that

$$\begin{aligned} dZdet(A^{-1}) &= qdet(A^{-1})dZ, \\ dZdet(A^{\dagger -1}) &= qdet(A^{\dagger -1})dZ. \end{aligned} \tag{4.197}$$

Together with

$$dZL = q^{-2} LdZ, \tag{4.198}$$

we have

$$dZ\rho = \rho dZ. \tag{4.199}$$

As a result, we have the one-form realization

$$\eta = -q^{-1} \rho^{-1} \delta\rho \tag{4.200}$$

for the exterior operator acting on the algebra generated by Z_α^i, dZ_α^i and their $*$-conjugate. The Kähler form

$$K = \bar{\delta}\eta \tag{4.201}$$

is central as usual.

4.5.4 Braided $G_q^{M,N}$

Let Z, Z' be two copies of the quantum Grassmannians $G_q^{M,N}$ defined by

$$Z = A^{-1}B, \quad Z' = A'^{-1}B', \tag{4.202}$$

where $C_a^i = (A_j^i, B_\alpha^i)$, $C_a'^i = (A_j'^i, B_\alpha'^i)$ both satisfy the relations (4.150). Let the mixed commutation relations be

$$Q_{12}C_1C_2' = C_1'C_2\hat{R}_{12}, \tag{4.203}$$

where Q is a numerical matrix. For (4.203) to be consistent with (4.150), we can take Q to be $\hat{R}'^{\pm 1}$. For either of these two choice, (4.203) is covariant under

$$C \to CT, \quad C' \to C'T, \tag{4.204}$$

where T_b^a is a $GL_q(M+N)$ quantum matrix and also under the transformation

$$C \to SC, \quad C' \to SC', \tag{4.205}$$

where S_j^i is a $GL_q(M)$ quantum matrix. We will pick $Q = \hat{R}'$ in the following

$$\hat{R}_{12}'C_1C_2' = C_1'C_2\hat{R}_{12}. \tag{4.206}$$

Explicitly, it is

$$\begin{aligned} R_{12}'A_1A_2' &= A_2'A_1R_{12}', \\ R_{12}'B_1B_2' &= B_2'B_1R_{12}'', \\ B_1A_2' &= R_{12}'^{-1}A_2'B_1, \\ A_1B_2' &= R_{12}'^{-1}B_2'A_1 + \lambda B_1A_2'P_{12}. \end{aligned} \tag{4.207}$$

Since

$$Z = A^{-1}B, \quad Z' = A'^{-1}B', \tag{4.208}$$

it follows that[9]

$$Z_1Z_2' = R_{12}'Z_2'Z_1R_{12}'' - \lambda Z_1Z_2\hat{R}_{12}''. \tag{4.209}$$

[9]If we had chosen the other choice $Q = \hat{R}'^{-1}$ in the above, the relations (4.207) would be different, but (4.209) would remain the same.

One can introduce a $*$-structure to this braided algebra, the relation

$$C_1^\dagger\hat{R}_{12}'^{-1}C_1' = C_2'\hat{R}_{12}^{-1}C_2^\dagger \tag{4.210}$$

is consistent and is covariant under

$$C \to CT, \quad C' \to C'T, \tag{4.211}$$

and also under the transformation

$$C \to SC, \quad C' \to SC' \tag{4.212}$$

with the same T, S quantum matrix as explained above. It follows immediately

$$Z_1^\dagger\hat{R}_{21}'Z_1' = Z_2'\hat{R}_{12}''^{-1}Z_2^\dagger - \lambda I_{12}. \tag{4.213}$$

One can also show that the Kähler form K of the original copy (4.201) commutes also with the $Z', Z'^\dagger, dZ', dZ'^\dagger$.

This concludes our discussion for the quantum Grassmannians, with the case of complex projective spaces $CP_q(N) = G_q^{1,N}$ as a special case.[10]

Strictly speaking, we have given in this chapter only a local description of the complex projective spaces and the quantum Grassmannians, i.e. in a certain coordinate patch. Presumably, there is no difficulty to introduce other patches in the picture and this has been illustrated in details in Chapter 3 for the simplest case of the sphere.

[10]Notice that for $M = 1$, the numerical R-matrix becomes a number: $\hat{R}_{12}' = q$.

Chapter 5

q-Deformed Dirac Monopole

A major step towards a q-deformed gauge theory is to find a suitable concept of "quantum" fiber bundles. Recently some versions of quantum bundles have been proposed [71, 82, 83, 84], where both the base space and the fiber are quantum spaces. In [71], a detailed formulation for quantum principal fiber bundle is proposed and as examples, the q-deformed Dirac monopole for charge 1 and 2 are constructed. Essentially, their construction is based on the isomorphisms

$$S_q^2 = SU_q(2)/U(1) \tag{5.1}$$

and

$$S_q^2 = SO_q(3)/U(1). \tag{5.2}$$

However, in the case of charge 1, their trivalizations involve square root of algebra elements and are formal. In this chapter, we will construct the deformed Dirac monopole on the quantum sphere S_q^2 for arbitrary charge n and show that it is a quantum principal bundle in the sense of [71]. We also get the monopole charge by integrating the curvature over the base S_q^2.

5.1 Quantum Principal Bundle

The definition of a quantum principal bundle follows the motto of non-commutative geometry, dualize everything and then introduce deformation. We first review the definitions of [71].

5.1.1 Universal Calculus

Definition 5.1.1 *[71, Def 4.1] $P = P(B, A)$ is a quantum principal bundle (short: QPB) with universal differential calculus, structure quantum group A and base B if*

1. *A is a Hopf algebra*
2. *(P, Δ_R) is a right A -comodule algebra; write $\Delta_R(p) = p^1 \otimes p^2 \in P \otimes A$*
3. *$B = P^A = \{u \in P : \Delta_R u = u \otimes 1\}$*
4. *$(\cdot \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_R)$: $P \otimes P \to P \otimes A$ is a surjection(freeness condition)*
5. *$\ker \tilde{} = \Gamma_{hor}$ (exactness condition for the differential envelope)*

where horizontal forms Γ_{hor} are defined by

$$\Gamma_{hor} = P(\Gamma_B)P \subseteq \Gamma_P \tag{5.3}$$

and satisfy $\tilde{}(\Gamma_{hor}) = 0$ identically. The left P- module map $\tilde{}$ is defined as

$$\tilde{} = (\cdot \otimes id) \circ (id \otimes \Delta_R)|_{P^2} : \Gamma_P \to P \otimes \ker \epsilon. \tag{5.4}$$

In the dual picture, it generates the fundamental (vertical) vector fields on the bundle. We will use the same symbol $\tilde{}$ for the extended map in condition 4.

Trivialization

Classically, the trivialization

$$\phi : P \to G, \tag{5.5}$$

is given by

$$\phi(u)a = \phi(u \cdot a). \tag{5.6}$$

In the deformed case, let A be a Hopf algebra and P an A-comodule algebra with invariant subalgebra B. Suppose that there exists a convolution invertible map $\Phi : A \hookrightarrow P$ such that

$$\Delta_R \circ \Phi = (\Phi \otimes id) \circ \Delta, \qquad \Phi(1_A) = 1_P \tag{5.7}$$

(so Φ is an intertwiner for the right coaction). It is proved in [71] that P is a quantum principal bundle. We call $P(B, A, \Phi)$ a trivial bundle with *trivialization* Φ.

Connection

The topological aspects of bundles is embodied in the set of transition functions. The notion of a connection plays the same essential role in the differential geometry of bundles. A connection defines a covariant derivatives which contains a gauge field and specifies how to parallel transport a vector in P along a curve lying in the base B. Therefore we have to first be able to define what is a horizontal vector in the principal bundle P. A connection Π on P is just a prescription to separate the tangent space T_uP into the vertical subspace V_uP and the horizontal subspace H_uP such that

1. $T_uP = V_uP \oplus H_uP$
2. $H_{u\cdot g} = R_{g*}H_uP, \forall u \in P, g \in G$

This definition is geometric but not practical in computation. To acheive this, it is common to introduce a Lie-algebra valued 1-form $\omega \in g \otimes T^*P$ called the connection 1-form which satisfies

1. $\omega(\tilde{\xi}) = \xi$ for any $\xi \in g$
2. $(R_a)^*\omega = ad(a^{-1})\omega$, i.e. $\omega((R_a)_*X) = ad(a^{-1})\omega(X)$ for any $a \in g$ and any vector field X.

The relation between the 2 definitions is given by

$$H_uP = \{X \in T_uP : \omega(X) = 0\} \tag{5.8}$$

In the dual formulation, one define a *connection* Π on a quantum principal bundle P as an assignment of a left P-submodule $\Gamma_{ver} \subseteq \Gamma_P$ such that:

1. $\Gamma_P = \Gamma_{hor} \oplus \Gamma_{ver}$,
2. projection $\Pi : \Gamma_P \to \Gamma_{ver}$ is right invariant i.e.

$$\Delta_R\Pi = (\Pi \otimes id)\Delta_R. \tag{5.9}$$

A connection in P is characterized by a right-invariant left P-module map $\sigma : P \otimes \ker\epsilon \to \Gamma_P$ splitting the exact sequence

$$0 \to \Gamma_{hor} \to \Gamma_P \xrightarrow{\sim} P \otimes \ker\epsilon \to 0, \tag{5.10}$$

i.e.

$$\tilde{}\circ\sigma = id. \tag{5.11}$$

The connection form $\omega : A \to \Gamma_P$ is given by

$$\omega(a) = \sigma(1 \otimes (a - \epsilon(a))). \tag{5.12}$$

Conversely, $\sigma(p \otimes a) = p\omega(a)$ for $p \otimes a \in P \otimes \ker\epsilon$.

Gauge Field

Classically, a gauge field is a Lie algebra valued 1-form living on the base. It is the pull back of the connection one form by some section σ_i. Conversely, given a gauge field A over an open set U of the base B, let $\phi : \pi^{-1}(U) \to G$ be a trivialization, then

$$\omega = \phi^{-1}\pi^*A\phi + \phi^{-1}d\phi \tag{5.13}$$

give a connection 1-form.

In the dual we have Let $\beta : A \to \Gamma_B$ be a linear map such that $\beta(1) = 0$. Then the map [1]

$$\omega = \Phi^{-1} * j \circ \beta * \Phi + \Phi^{-1} * d\Phi \tag{5.18}$$

[1] Here $f_1 * f_2$ is the convolution product of two maps. Given

$$f_i : A \to B, \quad i = 1, 2, \tag{5.14}$$

where A is a coalgebra and B is a algebra, we define $f_1 * f_2$ as a map from A to B by

$$(f_1 * f_2)(a) = f_1(a_{(1)})f_2(a_{(2)}). \tag{5.15}$$

Similarly, if V is a left A-comodule and $f_1 : A \to B$, $f_2 : V \to B$, then $f_1 * f_2$ is defined as the map from V to B by

$$(f_1 * f_2)(v) = f_1(v^{(-1)})f_2(v^{(0)}), \tag{5.16}$$

where

$$\Delta_L(v) = v^{(-1)} \otimes v^{(0)} \in A \otimes V \tag{5.17}$$

is the left coaction of A on V.

is a connection 1-form in the trivial principal bundle $P(B, A, \Phi)$ with trivialization Φ. β is called a gauge field.

5.1.2 General Calculus

Let M_A be a right ideal of A^2 and N_P be a sub-bimodule of P^2. Introduce a first-order differential calculus on A and P by

$$\Gamma_A = A^2/N_A, \tag{5.19}$$

where

$$N_A = \kappa(A \otimes M_A) \tag{5.20}$$

and

$$\Gamma_P = P^2/N_P, \tag{5.21}$$

where the map $\kappa : A \otimes A \to A \otimes A$ is given by

$$\kappa(a \otimes a') = \sum aSa'_{(1)} \otimes a'_{(2)}. \tag{5.22}$$

Definition 5.1.2 *[71, Def 4.1] We say that $P = P(B, A, N_P, M_A)$ is a* quantum principal bundle *with structure quantum group A and base B and quantum differential calculi defined by N_P, M_A if:*

1. *A is a Hopf algebra.*
2. *(P, Δ_R) is a right A-comodule algebra.*
3. *$B = P^A = \{u \in P : \Delta_R u = u \otimes 1\}$.*
4. *$(\cdot \otimes id)(id \otimes \Delta_R) : P \otimes P \to P \otimes A$ is a surjection (freeness condition).*
5. *$\Delta_R N_P \subset N_P \otimes A$ (right covariance of differential structure).*
6. *$\tilde{}(N_P) \subset P \otimes M_A$ (fundamental vector fields compatibility condition)*
7. *$\ker \tilde{}_{N_P} = \Gamma_{hor}$ (exactness condition).*

The map $\tilde{}_{N_P}$ is induced by $\tilde{}$

$$\tilde{}_{N_P}(\rho) = (id \otimes \pi_A) \circ \tilde{}(\rho_U) \tag{5.23}$$

where

$$\pi_{N_P} : P^2 \to \Gamma_P$$

and

$$\pi_A : ker\epsilon \to ker\epsilon/M_A$$

are the canonical epimorphisms and $\rho_U \in \pi_{N_P}^{-1}(\rho)$ is any representative of ρ. (5.23) is well defined because of condition 6. of the definition.

Connection

A connection on a QPB with general calculus is again determined by a splitting σ of the sequence

$$0 \to \Gamma_{hor} \to \Gamma_P \overset{\tilde{}_{N_P}}{\to} P \otimes \ker\epsilon/_{M_A} \to 0. \tag{5.24}$$

The connection form is given by

$$\omega(a) = \sigma(1 \otimes \pi_A(a - \epsilon(a))). \tag{5.25}$$

An element $\alpha \in \Gamma_{ver}$ is called a *vertical form*. If there exists a connection in P then any one-form $\alpha \in \Gamma_P$ can be uniquely written as a sum of a horizontal and a vertical forms.

Notice that one can replace the Point (6) of the definition of QPB by the slightly stronger condition [71]

6.' $\tilde{}(N_P) = P \otimes M_A$, which we will adopt in our construction.

5.2 Monopole Bundle: Global

The monopole bundle that we are going to construct is characterized by an integer n and we will refer to it as the charge of the bundle. Let the algebras P, A, B be

$$P = SU_q(2), \tag{5.26}$$

$$A = k < Z^{1/2}, Z^{-1/2} >= U(1), \tag{5.27}$$

$$B = < 1, b_-, b_+, b_3 >, \tag{5.28}$$

where $b_- = \alpha\beta, b_+ = \gamma\delta, b_3 = \alpha\delta$. For charge n, define the following subalgebras,

$$P^{(n)} = < \{p \in SU_q(2), deg(p) = nk, k \in \mathbf{Z}\} >, \tag{5.29}$$
$$A^{(n)} = k < Z^{n/2}, Z^{-n/2} >, \tag{5.30}$$

where the degree for a monomial in $SU_q(2)$ is defined as

$$deg(\alpha^a\beta^b\gamma^c\delta^d) = a + c - b - d \tag{5.31}$$

irrespective of ordering. Introduce the right coaction $\Delta_R : P \to P \otimes A$ defined by

$$\Delta_R \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha \otimes Z^{1/2} & \beta \otimes Z^{-1/2} \\ \gamma \otimes Z^{1/2} & \delta \otimes Z^{-1/2} \end{pmatrix}. \tag{5.32}$$

It is easy to see that

$$B = (P^{(n)})^{A^{(n)}}. \tag{5.33}$$

As for the calculus on $P^{(n)}$ and $A^{(n)}$, take the right ideal $M_{P^{(1)}}$ is generated by the six elements

$$\delta + q^2\alpha - (1 + q^2), \gamma^2, \beta\gamma, \beta^2, (\alpha - 1)\gamma, (\alpha - 1)\beta \tag{5.34}$$

and

$$M_{A^{(1)}} = \pi(M_{P^{(1)}}) =< \{Z^{-1/2} + q^2 Z^{1/2} - (1 + q^2)\} > . \tag{5.35}$$

The projection $\pi : P \to A$ (dual of $U(1) \subset SU(2)$) is an algebra map

$$\pi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} Z^{1/2} & 0 \\ 0 & Z^{-1/2} \end{pmatrix}. \tag{5.36}$$

For charge n, take $N_{P^{(n)}} = N_{P^{(1)}} \cap (P^{(n)})^2$, $M_{A^{(n)}}$ generated by $Z^{-n/2} + q^{2n}Z^{n/2} - (1 + q^{2n})$, i.e. $Z^{n/2}dZ^{n/2} = q^{2n}dZ^{n/2}Z^{n/2}$ or equivalently $N_{A^{(n)}} = N_{A^{(1)}} \cap (A^{(n)})^2$.

We have

Proposition 5.2.1 *$P^{(n)}(B, A^{(n)}, N_{P^{(n)}}, M_{A^{(n)}})$ is a QPB.*

Proof Condition 1-3 of the definition 5.1.2 are obviously satisified. To show condition 4, consider the monomial $p \otimes Z^{nk/2} \in P^{(n)} \otimes A^{(n)} \subset P^{(1)} \otimes A^{(1)}$ for $k \in \mathbf{Z}$. Since $P^{(1)}$ is a QPB, there exists $\sum p_{i_1} \otimes p_{i_2} \in P^{(1)} \otimes P^{(1)}$ such that $\tilde{}(\sum p_{i_1} \otimes p_{i_2}) = p \otimes Z^{nk/2}$. Now, $\tilde{}(\sum p_{i_1} \otimes p_{i_2}) = \sum p_{i_1}p_{i_2} \otimes Z^{deg(p_{i_2})/2}$. Therefore, $deg(p_{i_2}) = nk$, and $p_{i_2} \in P^{(n)}$ for all i_2. Also, $deg(p_{i_1}p_{i_2}) = deg(p)$ and $p \in P^{(n)}$, so $deg(p_{i_1}) \in n\mathbf{Z}$. Hence, $p_{i_1} \in P^{(n)}$ for all i_1. Surjectivity is proved. Condition 5 is obvious because of our simple definition of Δ_R. For condition 6, notice that $P^{(1)}(B, A^{(1)}, N_{P^{(1)}}, M_{A^{(1)}})$ is a QPB, so

$$\ker\tilde{}_{N_{P^{(n)}}} = (\ker\tilde{}_{N_{P^{(1)}}}) \cap \Gamma_{P^{(n)}} = \Gamma^{(1)}_{hor} \cap \Gamma^{(n)}_P \cap \Gamma^{(n)}_P) = \Gamma^{(n)}_{hor} \tag{5.37}$$

since deg(B)=0. For the same reason, we know that $\tilde{}N_{P^{(1)}} \subset P^{(1)} \otimes M_{A^{(1)}}$. Therefore, $\tilde{}N_{P^{(n)}} \subset P^{(1)} \otimes M_{A^{(1)}}$. But $\tilde{}(p_1dp_2) = p_1p_2^{(1)} \otimes p_2^{(2)} - p_1p_2 \otimes 1 \in P^{(n)} \otimes A^{(n)}$, for all $p_1, p_2 \in P^{(n)}$, so

$$\tilde{}N_{P^{(n)}} \subset P^{(n)} \otimes M_{A^{(n)}}. \tag{5.38}$$

Hence we see that $P^{(n)}(B, A^{(n)}, N_{P^{(n)}}, M_{A^{(n)}})$ is a QPB. Also, note that the 3D - calculus respects the $*$-structure [14]. □

A possible connection one-form on $P^{(n)}$ is given by

$$\omega(Z^{kn/2}) = S((\alpha^{kn})_{(1)})d(\alpha^{kn})_{(2)} = \kappa(1 \otimes (\alpha^{kn} - 1)), \tag{5.39}$$
$$\omega(Z^{-kn/2}) = S((\delta^{kn})_{(1)})d(\delta^{kn})_{(2)} = \kappa(1 \otimes (\delta^{kn} - 1)) \tag{5.40}$$

for $k > 0$, where κ is defined in [71]. This ω is well defined, since $S((\alpha^{kn})_{(1)})$, $S((\delta^{kn})_{(1)})$, $(\alpha^{kn})_{(2)}$, $(\delta^{kn})_{(2)} \in P^{(n)}$. This connection was found observing that this is the trivial connection [71] obtained form the trivialization $\Phi(Z^{n/2}) = \alpha^n$, which is a gauge - transformation of the trivialization (5.54) that we will introduce later. Note that the above trivialization does not respect the $*$-structure even for $q = 1$, nevertheless it is useful to e.g. find a connection; in the 3D - calculus, it simplifies to (5.43), and for even n, we would have obtained the same ω using (5.54). Quite generally, gauge transformations tend to spoil the $*$-structure (and algebra structure, as pointed out in [71]) of a trivialization.

To prove that ω defines a connection, we use Proposition 4.10 in [71]. We have to show

1. $\omega(1) = 0$ and $\omega(M_{A^{(n)}}) = 0$.

2. ${}^{\sim}{}_{N_{P^{(n)}}}\omega(a) = 1 \otimes \pi_{A^{(n)}}(a - \epsilon(a))$ for all $a \in A$

3. $\Delta_R \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{Ad_R}$

2. holds since for $k > 0$,

$$\begin{aligned} {}^{\sim}\omega(Z^{kn/2}) &= S((\alpha^{kn/2})_{(1)})(\alpha^{kn/2})^{(1)}_{(2)} \otimes (\alpha^{kn/2})^{(2)}_{(2)} - 1 \otimes 1 \\ &= S((\alpha^{kn/2})_{(1)})(\alpha^{kn/2})_{(2)} \otimes Z^{kn/2} - 1 \otimes 1 \\ &= 1 \otimes (Z^{kn/2} - 1) \end{aligned} \tag{5.41}$$

and similarly for $k < 0$ as is easily seen from our coaction. For 3.,

$$\begin{aligned} \Delta_R \omega(Z^{kn/2}) &= S((\alpha^{kn/2})_{(1)}) d(\alpha^{kn/2})_{(2)} \otimes Z^{-kn/2} Z^{kn/2} \\ &= \omega(Z^{kn/2}) \otimes 1 = (\omega \otimes id) Ad_R(Z^{kn/2}) \end{aligned} \tag{5.42}$$

and similarly for $k < 0$. As for 1., this is clear since $\omega(Z^{-n/2} + q^{2n} Z^{n/2} - (1 + q^{2n})) = \kappa(1 \otimes (\delta^n + q^{2n}\alpha^n - (1 + q^{2n}))) \in \kappa(1 \otimes M_{P^{(n)}})$.

Thus, ω is a connection form on the bundle $P^{(n)}$ and is given by

$$\begin{aligned} \omega(Z^{kn/2}) &= [kn]_{q^{-2}} \omega^1, \\ \omega(Z^{-kn/2}) &= -[-kn]_{q^{-2}} \omega^1 = -q^{2kn} \omega(Z^{kn/2}) \end{aligned} \tag{5.43}$$

if viewed in $SU_q(2)$, where $[n]_q = \frac{q^n - 1}{q - 1}$. This generalizes the result of [71] for $n = 1$ and 2. Since $(\omega^1)^* = -\omega^1$, ω is a $*$-map for $q = 1$ only. We have used $\omega^1 \alpha = q^{-2} \alpha \omega^1, \omega^1 \gamma = q^{-2} \gamma \omega^1$, where $\omega^1 = \delta d\alpha - q^{-1} \beta d\gamma$ is a left invariant form in $SU_q(2)$.

To our knowledge, (5.29) is also a new description of the classical Dirac monopole.

5.3 General Statements on Patching of Trivial QPBs

Let us first show how in general nontrivial QPB's can be obtained by "glueing" together "local" bundles. To avoid repeating ourselves too much, we will give the following statements for the case of a general calculus only; the universal calculus is recovered by putting $M_A = N_A = N_P = 0$. We first observe that the conditions 4. and 7. in definition 5.1.2 are equivalent to the exactness of the sequence (5.24).

Lemma 5.3.1 *$P(B, A, N_P, M_A)$ satisfying conditions 1. to 3.,5. and 6.' of the definition 5.1.2 is a QPB with general calculus if and only if the sequence (5.24) is exact.*

Proof Exactness of (5.24) at Γ_P is just the condition 7. above.

Assume first P is a QPB. Then by condition 4. , for any $p \otimes a \in P \otimes \ker \epsilon / M_A$ there exists $p_1 \otimes p_2 \in P \otimes P$ with ${}^{\sim}(p_1 \otimes p_2) = p_1 p_2^1 \otimes p_2^2 = p \otimes a$. Applying $\mathrm{id} \otimes \epsilon$ to this equation we get $0 = p_1 p_2^1 \epsilon(p_2^2) = p_1 p_2$, i.e. $p_1 \otimes p_2 \in \Gamma_P$, which shows that ${}^{\sim}$ in (5.24) is surjective, so it is exact.

Conversely, suppose (5.24) is exact. Take any $p \otimes a = p \otimes (a - \epsilon(a)) + p \otimes \epsilon(a) \in P \otimes A$. Since ${}^{\sim}{}_{N_P}$ is surjective, there exists $p_1 dp_2 \in \Gamma_P$ with ${}^{\sim}{}_{N_P}(p_1 dp_2) = p \otimes (a - \epsilon(a)) + P \otimes M_A$. Now ${}^{\sim}(p \otimes \epsilon(a)) = p \otimes \epsilon(a)$ and from 6'. ${}^{\sim}(N_P) = P \otimes M_A$, so condition 4. is satisfied. □

Assume now we have 3 quantum principal bundles

$$P_0(B_0, A, N_0, M_A), P_1(B_1, A, N_1, M_A) \subset P_{01}(B_{01}, A, N_{01}, M_A) \tag{5.44}$$

(P_{01} corresponds to the bundle on the "overlap" B_{01} of B_0 and B_1) and we would like to know if P_0 and P_1 can be understood as two patches of a "global" quantum bundle $P(B, A, N, M_A) \subset P_0, P_1 \subset P_{01}$. A natural guess is that $P = P_0 \cap P_1$. In this case the coactions $\Delta_{R_i} : P_i \to P_i \otimes A$ certainly must agree in P. If we want a connection on P, then we should also have connection forms $\omega_i : A \to \Gamma_{P_i}$ which agree on the overlap, i.e. $\omega_0(a) = \omega_1(a)$ in $\Gamma_{P_{01}}$.

However, some care must be taken if we want to compare differential forms on different patches. First of all, the differential structures on P_i must be compatible, i.e. we should have $N_0 = N_{01} \cap P_0^2$, $N_1 = N_{01} \cap P_1^2$ and $N_P \equiv N = N_{01} \cap P^2 = N_0 \cap N_1$. But this is not enough: Suppose we have any 2 differential forms - not necessarily connections - $\omega_0 \in \Gamma_{P_0}$ and $\omega_1 \in \Gamma_{P_1}$ and find by doing calculations in $\Gamma_{P_{01}}$ that they are equal. One would certainly like to conclude, as in the classical case, that they determine a "global" form ω in Γ_P. This is not evident, it is a condition on the calculus. It motivates the following definition: The above calculi on P_0, P_1, P_{01} are called *admissible* if

$$\omega_0 = \omega_1 + n_{01} \quad \text{for} \quad \omega_i \in \Gamma_{P_i} \tag{5.45}$$

implies that there exists a $\omega \in \Gamma_P$ such that

$$\omega = \omega_0 + n_0 = \omega_1 + n_1, \quad n_i \in N_i. \tag{5.46}$$

In other words, $\omega_0 = \omega_1$ determines a $\omega \in \Gamma_{P_0} \cap \Gamma_{P_1} = \Gamma_P$, where the intersection is defined as intersection of the cosets. A calculus which does not satisfy this condition would be highly unpracticable for global statements. The universal calculus is certainly admissible since $(P_0 \otimes P_0) \cap (P_1 \otimes P_1) = P \otimes P$ implies $\Gamma_{P_0} \cap \Gamma_{P_1} = \Gamma_P$. The calculus we will consider on the monopole - bundle will be shown to be admissible too, using a fairly general line of reasoning.

Theorem 5.3.1 *In the above situation, $P = P_0 \cap P_1 = P(B, A, N, M_A)$ is a quantum principal bundle with base $B = B_0 \cap B_1$ and connection if we have admissible differential structures which satisfy $\tilde{}(N) = P \otimes M_A$, connection forms $\omega_0 = \omega_1$ on P_0 resp. P_1, and $\Gamma_{0hor} \cap \Gamma_{1hor} = \Gamma_{hor}$. Conversely, if $P = P_0 \cap P_1$ is a quantum principal bundle, then $\Gamma_{0hor} \cap \Gamma_{1hor} = \Gamma_{hor}$.*

Proof First, $\Delta_R(p) \in P_0 \otimes A \cap P_1 \otimes A = P \otimes A$ for $p \in P$ implies 2. in Def. 5.1.2. Further, $B = P^A = (P_0 \cap P_1)^A = P_0^A \cap P_1^A = B_0 \cap B_1$. By the above definition of the differential structures condition 5. is satisfied, since $\Delta_{R_i} : P_i \to P_i \otimes A$ do not "leave" the bundles.

Assume $\Gamma_{0hor} \cap \Gamma_{1hor} = \Gamma_{hor}$. Since $\omega_0(a) = \omega_1(a)$ and the calculus is admissible, this defines $\omega(a) \in \Gamma_P$ and $\sigma(p \otimes a) = p\omega(a) \in \Gamma_P$ for $(p \otimes a) \in P \otimes \ker \epsilon$. From proposition 4.10 in [71] it follows that ω is a connection 1 - form. Now $\tilde{}\sigma(p \otimes a) = p \otimes a$ shows that the map $\tilde{}$ in (5.24) is surjective.

It remains to show $\ker\tilde{} = \Gamma_{hor}$. Let $p_1 dp_2 \in \Gamma_P$. Since P_0 and P_1 are quantum bundles, $\tilde{}(p_1 dp_2) = 0$ implies $p_1 dp_2 \in \Gamma_{0hor} \cap \Gamma_{1hor} = \Gamma_{hor}$ by assumption. Now Lemma 5.3.1 tells us that $P(B, A, N, M_A)$ is a quantum principal bundle.

Conversely, assume $P = P_0 \cap P_1$ is a quantum principal bundle. Let $p_1 dp_2 \in \Gamma_{0hor} \cap \Gamma_{1hor}$. Then $\tilde{}(p_1 dp_2) = 0$. Since $\Gamma_{0hor} \cap \Gamma_{1hor} \subset \Gamma_{P_0} \cap \Gamma_{P_1} = \Gamma_P$ and P is a QPB, this implies $p_1 dp_2 \in \Gamma_{hor}$. The other inclusion $\Gamma_{0hor} \cap \Gamma_{1hor} \supset \Gamma_{hor}$ is trivial. □

If there are several "patches" P_i, then the above theorem generalizes inductively in an obvious way. One can show that if $\tilde{}(N_P) = P \otimes M_A$, then $\Gamma_{0hor} \cap \Gamma_{1hor} = \Gamma_{hor}$ follows from $\Gamma^u_{0hor} \cap \Gamma^u_{1hor} = \Gamma^u_{hor}$ (universal calculus). More generally, we have

Lemma 5.3.2 *If $P(B, A)$ is a QPB with universal calculus and we have N_P and M_A satisfying conditions 5. and 6'. of definition 5.1.2, then $P(B, A, N_P, M_A)$ is a QPB with general calculus. Conversely, if $P(B, A, N_P, M_A)$ is a QPB and $\tilde{}(n) = 0$ for $n \in N_P$ implies $n \in \Gamma^u_{hor}$, then $P(B, A)$ is a QPB with universal calculus.*

Proof First suppose $P(B, A)$ is a QPB; we have to show that $\ker\tilde{}_{N_P} \subset \Gamma_{hor}$. Let $\tilde{}_{N_P}(\gamma) = 0$. This means $\tilde{}(\gamma) \in P \otimes M_A = \tilde{}(N_P)$ by 6'. . So there is a $n \in N_P$ with $\tilde{}(n - \gamma) = 0$. But P is a QPB with universal calculus, so it follows $\gamma \in \Gamma^u_{hor} + n$, i.e. $\gamma \in \Gamma_{hor}$.

The converse statement can be proved similarly. □

It is possible to give a trivalization of our monopole bundle. This will be done for even "charge" only; for odd charge, the trivializations etc. would only be formal. We define two trivial QPBs $P_0^{(2n)}$ and $P_1^{(2n)}$, and then show that $P^{(2n)} = P_0^{(2n)} \cap P_1^{(2n)}$ is the monopole of charge $2n$.

5.4 Monopole Bundle: Patching

We now present the second construction of the Dirac monopoles for general calculus as an illustration of the general method above. This will be done for even "charge" only; for odd charge, the trivializations etc. would only be formal.

5.4.1 Universal Calculus

We define two trivial QPBs $P_0^{(2n)}$ and $P_1^{(2n)}$, and then show that $P^{(2n)} = P_0^{(2n)} \cap P_1^{(2n)}$ is the monopole of charge $2n$.

Patches

For $P_0^{(2n)}$, as motivated by the charge 2 case in [71], we now try to define the base B_0, fiber $A^{(2n)}$ and trivial bundle $P_0^{(2n)}$ be specified by their generators as:

$$B_0 = < \{1, b_-, b_+, b_3, (b_3 + q^{2m} - 1)^{-1}; \quad m \in \mathbf{Z}\} >, \tag{5.47}$$

$$A^{(2n)} = < \{Z^n, Z^{-n}\} >, \tag{5.48}$$

$$P_0^{(2n)} = < B_0 \cup \{(\delta^{-1}\alpha)^n, (\alpha^{-1}\delta)^n\} > . \tag{5.49}$$

The commutation relations between the generators of P_0 are induced by $SU_q(2)$ through the following expressions [67]:

$$b_- = \alpha\beta, \quad b_+ = \gamma\delta, \quad b_3 = \alpha\delta, \tag{5.50}$$

where $\alpha, \beta, \gamma, \delta$ are generators of $SU_q(2)$ with the well-known relations stated before. The commutation relations involving inverses are obtained by multiplying them from both sides by inverses of generators. In the classical limit $q = 1$, B_0 becomes the algebra of the functions on $S^2\backslash\{\text{south pole}\}$, and $b_\pm = \pm(x \pm iy)$, $b_3 = z + 1/2$, where x, y, z are the Cartesian coordinates. Note that $\alpha\delta - \beta\gamma = 1$ is equivalent to $x^2 + y^2 + z^2 = (1/2)^2$. The somewhat complicated definition here (see [85]) will become clear below. $P_0^{(2n)}$ as a trivial bundle is generated by the base B_0 and the fibers, cp. (5.54).

Define a coaction Δ_R on $P_0^{(2n)}$ such that $B_0 = (P_0^{(2n)})^{A^{(2n)}}$:

$$\begin{aligned} \Delta_R(1) &= 1 \otimes 1, && (5.51)\\ \Delta_R(b_i) &= b_i \otimes 1, \quad i = -, +, 3, && (5.52)\\ \Delta_R((\delta^{-1}\alpha)^{\pm n}) &= (\delta^{-1}\alpha)^{\pm n} \otimes Z^{\pm n}. && (5.53) \end{aligned}$$

The trivialization Φ_0 is defined as

$$\Phi_0(1) = 1, \qquad \Phi_0(Z^{\pm n}) = (\delta^{-1}\alpha)^{\pm n} \tag{5.54}$$

which generalizes the trivialization in [71].

Proposition 5.4.1 *$P_0^{(2n)}(B, A^{(2n)}, \Phi_0)$ is a trivial QPB.*

Proof To see that we have a trivial QPB, we first have to show that B_0 is the invariant subalgebra of $P_0^{(2n)}$ under the above coaction. This is clear if any $p_0 \in P_0^{(2n)}$ can be written as a sum of terms $B_0(\delta^{-1}\alpha)^{kn}$. Thus we must be able to commute B_0 through $(\delta^{-1}\alpha)$. Writing down the commutation relations explicitely, one can always obtain relations like $\alpha B_0 \alpha^{-1} \in B_0$. Note that for $m \in \mathbb{Z}$, $q^{-2m}\alpha^{-m} b_3^{-1} \alpha^m = q^{-2m}\delta^m b_3^{-1}\delta^{-m} = (b_3 + q^{2m} - 1)^{-1}$ (cp. [85]) and so in general,

$$q^{-4nk}(\alpha^{-1}\delta)^{nk}(b_3 + q^{2m} - 1)^{-1}(\delta^{-1}\alpha)^{nk} = (b_3 + q^{4nk+2m} - 1)^{-1}, \quad k \in \mathbb{Z}. \tag{5.55}$$

This shows that B_0, as defined in (5.47) is the invariant subalgebra, and one can also see the necessity to include all the generators of B_0.

Φ_0 is convolution-invertible with $\Phi_0^{-1}(Z^{\pm n}) = (\delta^{-1}\alpha)^{\mp n}$, and is also an intertwiner: $\Delta_R \circ \Phi_0 = (\Phi_0 \otimes id) \circ \Delta_A$, where $\Delta_A(Z^n) = Z^n \otimes Z^n$ is the coproduct on $A^{(2n)}$. So $P_0^{(2n)}$ is a trivial QPB. □

The discussion on $P_1^{(2n)}$ is parallel to that on $P_0^{(2n)}$, but much easier. Therefore we just give the relevant equations:

$$\begin{aligned} B_1 &= < \{1, b_-, b_+, b_3, (b_3 - 1)^{-1}\} >, && (5.56)\\ P_1^{(2n)} &= < B_1 \cup \{(\gamma\beta^{-1})^n, (\beta\gamma^{-1})^n\} >, && (5.57)\\ A^{(2n)} &= < \{Z^n, Z^{-n}\} >, && (5.58)\\ \Delta_R(b_i) &= b_i \otimes 1, \quad i = -, +, 3, && (5.59)\\ \Delta_R((\gamma\beta^{-1})^{\pm n}) &= (\gamma\beta^{-1})^{\pm n} \otimes Z^{\pm n}, && (5.60)\\ \Phi_1(Z^{\pm n}) &= (-\gamma\beta^{-1})^{\pm n} && (5.61) \end{aligned}$$

and $P_1^{(2n)}$ is also a trivial QPB. Note again that $deg(B_1) = 0$ and $deg(\Phi_i(Z^n)) = 2n$.

Overlap

The "overlap" $P_{01}^{(2n)}$ of $P_0^{(2n)}$ and $P_1^{(2n)}$ is similarly defined by

$$\begin{aligned} B_{01} &= < B_0 \cup \{(b_3 - 1)^{-1}\} >, && (5.62)\\ P_{01}^{(2n)} &= < B_{01} \cup \{(\gamma\beta^{-1})^{\pm n}, (\delta^{-1}\alpha)^{\pm n}\} > && (5.63) \end{aligned}$$

and so on as above. On $P_{01}^{(2n)}$, both trivializations can be used, with the transition function

$$\gamma_{01}(Z^n) = \Phi_0(Z^n)\Phi_1^{-1}(Z^n) = (-q^2 b_3^{-1} b_-^2 (b_3 - 1)^{-1})^n \in B_{01}. \tag{5.64}$$

It should be noted that while these trivial bundles are closed under the $*$-operation, the maps Φ_i respect this $*$-structure only for $q = 1$. This appears to be very hard to avoid in this framework, and we accept it here.

Global Bundle by Patching

Now define the Dirac - monopole bundle with charge $2n$ by

$$P^{(2n)} = P_0^{(2n)} \cap P_1^{(2n)}. \tag{5.65}$$

We will now show that for even charges this construction agrees with the one in section 2. First, we need the following useful equivalent representation for $P_0^{(2n)}$,

Lemma 5.4.1

$$P_0^{(2n)} = \{p \in < SU_q(2) \cup \{(\delta\alpha)^{-1}, (\alpha\delta)^{-1}\} >: deg(p) = 2kn, k \in \mathbf{Z}\} \equiv \tilde{P}_0^{(2n)}, \quad (5.66)$$

i.e. the algebra generated by $SU_q(2)$ and $(\delta\alpha)^{-1}, (\alpha\delta)^{-1}$, with degrees being multiples of $2n$.

Proof To see this, note that $P_0^{(2n)} \subset \tilde{P}_0^{(2n)}$ because $\delta^{-1}\alpha = (\alpha\delta)^{-1}\alpha^2$ etc. and $b_3^{-1} = (\alpha\delta)^{-1}$, so $\alpha^{-n}b_3^{-1}\alpha^n \in \tilde{P}_0^{(2n)}$ also. To see the other inclusion, we first show that B_0 is also the invariant subalgebra (under the coaction of $A^{(2n)}$) of $\tilde{P}_0^{(2n)}$: we have just seen $B_0 \subset \tilde{P}_0^{(2n)}$, and the same commutation relations as above show that indeed $B_0 = (\tilde{P}_0^{(2n)})^{A^{(2n)}}$. But this means that $\tilde{P}_0^{(2n)}$ is a QPB with the same trivialization Φ_0 as above. Thus we know (from [71] Example 4.2) that $\tilde{P}_0^{(2n)} = B_0\Phi_0(A^{(2n)}) = P_0^{(2n)}$. □

With this, it is easy to see that the $P^{(2n)}$ defined in (5.65) agrees with the one constructed in section 2

Proposition 5.4.2

$$P^{(2n)} =< p \in SU_q(2) : deg(p) = 2nk, k \in \mathbf{Z} > . \quad (5.67)$$

Proof Let $p_0, p_1 \in P_0^{(2n)}$ resp. $P_1^{(2n)}$ and $p_0 = p_1$. Note that $\beta, \beta^{-1}, \gamma, \gamma^{-1}$ can be commuted through any terms by just picking up powers of q. Multiplying α^{-1}, δ^{-1} to the relation $\alpha\delta = \delta\alpha + (q-1)(\frac{q+1}{q})\beta\gamma$ appropriately from both sides, one gets relations like $\delta\alpha^{-1} = \alpha^{-1}\delta + (q-1)(...)$ and $\alpha^{-1}\delta^{-1} = \delta^{-1}\alpha^{-1} + (q-1)(...)$, i.e. *one can order thing in any way up to terms proportional to* $(q-1)$.

Let us define a *normal form for* p_1 as follows: bring all β, γ to the right of all α, δ and order α to the left of δ, picking up terms proportional to $(q-1)$. Then replace all terms $\alpha\delta$ by $(1 + q\beta\gamma)$. Putting γ to the right of β, p_1 finally has the form either $\alpha^n\beta^x\gamma^y + (q-1)(...)$; $\delta^n\beta^x\gamma^y + (q-1)(...)$ or $\beta^x\gamma^y + (q-1)(...)$ with $x, y \in \mathbf{Z}, n \in \mathbb{N}$.

Similarly, define a *normal form for* p_0 as follows: bring all β, γ to the right of all α, δ, order β to the left of γ and replace all terms $\beta\gamma$ by $(\alpha\delta - 1)/q$. Now order

α to the left of δ picking up terms prop. to $(q-1)$. p_0 finally has the form either $\alpha^x\delta^y\beta^n + (q-1)(...)$, $\alpha^x\delta^y\gamma^n + (q-1)(...)$ or $\alpha^x\delta^y + (q-1)(...)$ with $x, y \in \mathbf{Z}, n \in \mathbb{N}$. Now consider the equation

$$p_0 = p_1. \quad (5.68)$$

and put terms in p_1 which do not contain inverses to the left side, in normal form for p_0 (only for monomials which are not proportional to $(q-1)$, say). Then let $q = 1$ and consider both sides as classical functions on $SU(2)$. All terms proportional to $(q-1)$ vanish, and all remaining monomials are in normal form on both sides and are easily seen to be independent as functions on $SU(2)$. This implies that all coefficients are actually zero, i.e. all terms on both sides are proportional to $(q-1)$. (or simply: classical functions defined on both patches are defined globally on $SU(2)$). We can now cancel the greatest common power of $(q-1)$, put regular terms to the left and apply the same argument. This cannot go on forever since the right side can be ordered completely, so both sides must be zero eventually, proving that $p_0 = p_1 \in SU_q(2)$. Using (5.66), this immediately shows that

$$P^{(2n)} =< p \in SU_q(2) : deg(p) = 2nk, k \in \mathbf{Z} >, \quad (5.69)$$

as claimed. □

The essence of the proof is to write things in the form ("class")+$(q-1)$ ("quantum") and to apply classical reasoning to ("class"), which should be a fairly general strategy. Proposition 5.4.2 and (5.66) generalize the result of [71] for $n = 1$.

5.4.2 General Calculus

We can now introduce the same induced 3-D calculus on the bundles as in section 2, i.e. the calculus on the patches $P_i^{(2n)}$ is defined by

$$N_{P_i^{(2n)}} = P_i^{(2n)} N_{P^{(2n)}} P_i^{(2n)}, \quad (5.70)$$

with the same ideals as in section 2. Using $\tilde{} = (id \otimes \pi)\kappa^{-1}$ in a Hopf algebra one can easily see $\tilde{}(N_P^{(1)}) = P^{(1)} \otimes M_{A^{(1)}}$, and $\tilde{}(N_{P^{(2n)}}) = P^{(2n)} \otimes M_{A^{(2n)}}$ with a similar argument as in section 2. So $P_i^{(2n)}$ are trivial QPB with this calculus by example (4.11) in [71].

It was already shown in section 2 that

$$\omega(Z^{kn}) = S((\alpha^{2kn})_{(1)})d(\alpha^{2kn})_{(2)} = \kappa(1 \otimes (\alpha^{2kn} - 1)), \tag{5.71}$$

$$\omega(Z^{-kn}) = S((\delta^{2kn})_{(1)})d(\delta^{2kn})_{(2)} = \kappa(1 \otimes (\delta^{2kn} - 1)) \tag{5.72}$$

for $k \in \mathbb{N}$ defines a connection one-form. Any monomials of degree $2nk$ in $P_i^{(2n)}$ can be written in the form $\Phi_i(Z^{2nk})B$ or $B\Phi_i(Z^{2nk})$, and so one can put ω in the standard form of a connection one-form in $P_0^{(2n)}$ and $P_1^{(2n)}$:

$$\omega(a) = \Phi_i^{-1}(a)\beta_i(a)\Phi_i(a) + \Phi_i^{-1}(a)d\Phi_i(a); i = 0, 1; a \in A^{(2n)}, \tag{5.73}$$

where $\beta_i \in \Gamma_{ihor}^{(2n)}$ and $\beta_i(1) = 0$.

Now let us show the following

Proposition 5.4.3 *The calculus on $P^{(2n)}, P_0^{(2n)}, P_1^{(2n)}$ is admissible.*

Proof The reasoning is as in the previous proposition. Assume we have ω_0, ω_1 in $\Gamma_{P_0^{(2n)}}$ resp. $\Gamma_{P_1^{(2n)}}$ with $\omega_0 = \omega_1$ in $\Gamma_{P_{01}^{(2n)}}$. Since in the 3D - calculus all one - forms on $SU_q(2)$ and thus on $P_i^{(2n)}$ can be written in terms of three left - invariant Maurer - Cartan forms $\omega^0, \omega^1, \omega^2$ which have simple commutation relations

$$\begin{aligned} \omega^0\alpha &= q^{-1}\alpha\omega^0, \quad \omega^0\beta = q\beta\omega^0, \\ \omega^1\alpha &= q^{-2}\alpha\omega^1, \quad \omega^1\beta = q^2\beta\omega^1, \\ \omega^2\alpha &= q^{-1}\alpha\omega^2, \quad \omega^2\beta = q\beta\omega^2. \end{aligned} \tag{5.74}$$

and similarly with the inverses α^{-1} etc., we can commute the forms to the right and have $\omega_0 = f_k\omega^k, \quad \omega_1 = g_k\omega^k$ (summation implied), so

$$f_k\omega^k = g_k\omega^k. \tag{5.75}$$

As in proposition 5.4.2 put both f_k and g_k in their respective normal form ("class") + $(q-1)$("quant") and bring all regular terms of g_k to the left side. Then putting $q = 1$, the "classical" parts are all independent as one - forms since the ω_i are and therefore vanish. Cancelling $(q-1)$ and repeating the argument, it follows that ω_0 and ω_1 are elements of $\Gamma_{P^{(1)}}$ and in fact in $\Gamma_{P^{(2n)}}$, since the degree is conserved. □

Now we can use theorem 5.3.1 Suppose $\rho \in \Gamma_{0hor}^{(2n)} \cap \Gamma_{1hor}^{(2n)}$, so $\rho \in \Gamma_{P^{(2n)}}$. We can expand it as above

$$\rho = f_0\omega^0 + f_1\omega^1 + f_2\omega^2, \tag{5.76}$$

with $f_i \in P^{(2n)}$. But ω^0 and ω^2 are horizontal (explicitely: $\omega^0 = \delta^2 db_- + q^{-2}\beta^2 db_+ - q^{-1}(1+q^{-2})\beta\delta db_3$ and $\omega^2 = -\gamma^2 db_- - q^{-2}\alpha^2 db_+ + q^{-1}(1+q^{-2})\alpha\gamma db_3$), while ω^1 is not. Therefore $f_1 = 0$, and $\rho \in \Gamma_{hor}^{(2n)}$, since all coefficients of dB must have degree $2n$. So $P^{(2n)}$ is a QPB with a general differential calculus, with the same connection form ω restricted to elements $a \in A^{(2n)}$.

Finally we would like to mention that since the trivializations are not "real" for $q \neq 1$, one might just go ahead and use trivializations such as $\Phi(Z^{1/2}) = \alpha$ which do not respect the $*$-structure even for $q = 1$, at least as computational tools. Since we know that the "global" bundle with the $*$-structure does have the correct classical limit, this may be an acceptable and useful strategy, and deserves further consideration.

5.5 Two Remarks

5.5.1 A Note on the Gauge Transformations

A gauge transformation is a convolution invertible map $\gamma : A \to B$:

$$\gamma * \gamma^{-1} = \gamma^{-1} * \gamma = 1. \tag{5.77}$$

Let us define the "primitive charge" of a monomial in B as $(n_- - n_+)$, where $n_\pm$ are the total powers of $b_\pm$ appearing in the monomial or equivalently (power of α− power of δ). This is preserved by the commutation relations, as our previous degree. Suppose that $\gamma = \sum_{k=i}^{j} \gamma^{(k)}$, where each $\gamma^{(k)}$ contains only monomials that have primitive charge k. Hence i and j are the minimum and maximum of the primitive charges of all monomials in γ. Let the convolution inverse of γ be denoted in the same way: $\gamma^{-1} = \sum_{k=i'}^{j'} \gamma^{(k)\prime}$. So

$$1 = \epsilon(Z^{n/2}) \cdot 1 = \gamma * \gamma^{-1} = \sum_{k=i+i'}^{j+j'} \gamma^{(k)\prime\prime} \tag{5.78}$$

which implies that $i + i' = j + j' = 0$. The only possibility that this can be true is that $i = j = -i' = -j'$, which means that all monomials in γ have the same

primitive charge n. However, this means in theclassical limit that γ is proportional to $e^{in\phi}$. That is, by admitting only finite sums in a convolution invertible γ one is restricting oneself to a very special, rigid class of gauge transformations. Thus infinite series cannot be avoided in general.

5.5.2 A Note on the Chern Class

Classically, the monopole charge n is given by an integration over the base of the first Chern class

$$\frac{1}{2\pi i}\int_{S^2} F = n, \tag{5.79}$$

where $F = dA_+ = dA_-$. Here A_+, A_- are the connection form on the northern and southern hemisphere respectively and the global connection form is given in terms of trivalizations as

$$\omega = \begin{cases} A_+ + id\varphi_+, on H_+ \\ A_- + id\varphi_-, on H_- \end{cases} \tag{5.80}$$

with $e^{i\varphi_{+,-}}$ being the local trivalization.

In the deformed case, we have the global connection form ω. Suppose it is written in terms of trivialization as [71]

$$\omega = \phi_i^{-1}\beta_i\phi_i + \phi_i^{-1}d\phi_i, \tag{5.81}$$

then it is not hard to check that

$$d\omega = \phi_i^{-1}(d\beta_i + \beta_i\beta_i)\phi_i = d\omega + \omega\omega, \tag{5.82}$$

which is in fact the curvature 2-form on $P^{(n)}$ ([86], cp. [71]). Carrying the ϕ_i through the $d\omega$, we get

$$d\beta_i + \beta_i\beta_i = q^{2n}d\omega = q^{2n}[n]_{q^{-2}}d\omega^1 \tag{5.83}$$

which is again equal for the two patches and explicitely horizontal. It leads us to define the deformed Chern class as

$$\frac{1}{2\pi i}F = \frac{1}{2\pi i}(d\beta_i + \beta_i\beta_i) = \frac{q^{2n}}{2\pi i}[n]_{q^{-2}}d\omega^1. \tag{5.84}$$

Consider the base $B = S_q^2 =< b_+, b_-, b_3 >\subset SU_q(2)$, with the calculus inherited from the 3-D calculus on $SU_q(2)$. Denote $\Gamma_B = BdB$ and introduce the set $\Gamma_B^{\wedge^2}$ of 2

forms on B. Notice that $\Gamma_B^{\wedge^2}$ contains elements of the form $Bdb_idb_j, i, j = -, +, 3$. Since

$$\begin{aligned} db_+ &= \gamma^2\omega^0 - q^2\delta^2\omega^2, \\ db_- &= \alpha^2\omega^0 - q^2\beta^2\omega^2, \\ db_3 &= \alpha\gamma\omega^0 - q^2\beta\delta\omega^2. \end{aligned} \tag{5.85}$$

So, $\Gamma_B^{\wedge^2} = B\omega^0\omega^2 = Bd\omega^1$. Because $d\omega^1$ is a central element in B_{01}, under a gauge transformation $U \in B_{01}$ we have $F \to U^{-1}FU = F$.

Notice that $\omega^0\omega^2$ is manifestly left invariant under the coaction of $SU_q(2)$, and is the unique top 2-form on B. This allow us to introduce a linear functional

$$\int_B : \Gamma_B^{\wedge^2} \to C,$$

$$\int_{S_q^2} ad\omega^1 = 2\pi i < a >_{SU_q(2)}, \forall a \in B, \tag{5.86}$$

where $<>_{SU_q(2)}$ is the invariant "Haar" measure on $SU_q(2)$[12]. This integral is obviously left- and right- invariant under the coaction of $SO_q(3)$ and unique as such. The normalization is choosen to give the correct classical limit. Classically, $d\omega^1 = i/2d\Omega$.

Therefore the deformed monople charge is obtained as in the classical case

$$\frac{1}{2\pi i}\int_{S_q^2} F = \frac{q^{2n}}{2\pi i}\int_{S_q^2}[n]_{q^{-2}}d\omega^1 = q^{2n}[n]_{q^{-2}}. \tag{5.87}$$

This is actually gauge - invariant in the sense that it does not depend on the trivialization chosen, but this appears to be the case only for our particular connection.

Bibliography

[1] E. K. Sklyanin, L. A. Takhtajan, and L. D. Faddeev, *Quantum Inverse Problem Method I*, Theoret. Math. Phys. **40**, 194 (1979).

[2] L. D. Faddeev, *Integrable Models in (1+1)-Dimensional Quantum Field Theory*, Les Houches Session XXXIX (Elsevier Amsterdam, 1982) pp. 563–608.

[3] L.A. Takhtajan and L.D. Faddeev, *Hamiltonian Methods in the Theory of Solitons*, Berlin-Heidelberg-New York, Springer, 1987.

[4] M. Jimbo, *A q-Difference Analogue of $U(g)$ and the Yang-Baxter Equation*, Lett. Math. Phys. **10**, 63 (1985).

[5] M. Jimbo, *A q-Analog of $U(GL(N+1))$, Hecke Algebra and the Yang-Baxter Equation*, Lett. Math. Phys. **11**, 247 (1986).

[6] V. G. Drinfeld, *Hopf Algebras and the Quantum Yang-Baxter Equation*, Sov. Math. Dokl. **32**, 254 (1985).

[7] V. G. Drinfeld, *Quantum Groups*, Proceedings of the Int. Cong. of Math. 1986 (Berkeley) (A. Gleason, ed.), A.M.S., Providence, Rhode Island, 1987, pp. 798–820.

[8] N. Yu. Reshetikhin, L.A. Takhtajan and L.D. Faddeev, *Quantization of Lie Groups and Lie Algebras*, Alg. i Anal. **1**, 178 (1989). (Leningrad Math. J. **1**, 193 (1990)).

[9] G. Kac, *Ring-Groups and the Principle of Duality I and II*, Trudy Moskov. Mat. Obsc. **12**, 259 (1963); **13**, 84 (1965).

[10] M. Takesaki, *Duality and von Neumann Algebras*, Tulane University lecture notes (1970).

[11] M. Enock, J. Schwarz, *Une dualité dans les algèbres de von Neumann*, Bull. Soc. Math. France, Suplément mémoire **44** 1 (1979).

[12] S.L. Woronowicz, *Compact Matrix Pseudogroups*, Commun. Math.Phys. **111**, 613 (1987).

[13] S.L. Woronowicz, *Twisted $SU(2)$ Group. An Example of a Non-Commutative Differential Calculus*, Publ.RIMS, Kyoto Univ. **23**, 117 (1987).

[14] S.L. Woronowicz, *Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups)*, Commun. Math. Phys. **122**, 125 (1989).

[15] J. Wess and B. Zumino, *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl. Phys. B (Proc. Suppl.) **18**, 302 (1990).

[16] D. Bernard, *Quantum Lie Algebras and Differential Calculus on Quantum Groups*, Prog. of Theoretical Phys. Supp. **102**, 49 (1990).

[17] U. Carow-Watamura, M. Schlieker, S. Watamura, and W. Weich, *Bicovariant Differential Calculus on Quantum Groups $SU_q(N)$ and $SO_q(N)$* Commun.Math. Phys. **142** 605 (1991).

[18] P. Schupp, P. Watts, and B. Zumino, *Differential Geometry on Linear Quantum Groups*, Lett. Math. Phys. **25**, 139 (1992).

[19] P. Aschieri, L. Castellani, *An Introduction to Non-Commutative Differential Geometry on Quantum Groups*, Int. J. Mod. Phys. **A 8**, 1667 (1993).

[20] A. Schüler, *Covariant Bimodules and Differential Calculi on Quantum Groups of Type B, C, D*, Seminar Sophus Lie **2**, 115 (1992).

[21] K. Schmüdgen, A. Schüler, *Classification of Bicovariant Differential Calculi on Quantum Groups of type A, B, C, D*, University of Leipzig preprint-Nr. 1/1994.

[22] L.D.Faddeev and P.N.Pyatov, *The Differential Calculus on Quantum Linear Groups*, hep-th/9402070.

[23] P. Aschieri, P. Schupp, *Vector Fields on Quantum Groups*, IFUP-TH 15/95, LMU-TPW 94-14, q-alg/9505023.

[24] Yu. I. Manin, *Quantum Groups and Non-Commutative Geometry*, preprint CRM-1561, C.R.M., Montreal, 1988.

[25] Yu. I. Manin, *Multiparameter Quantum Deformations of the General Linear Supergroup*, Commun. Math. Phys. **123**, 163 (1989).

[26] H. S. Synder, *Quantized Space-Time*, Phys. Rev. **71**, 38 (1947); *The Electromagnetic Field in Quantized Space-Time*, Phys. Rev. **72**, 68 (1947).

[27] C. N. Yang, *On Quantized Space-Time*, Phys. Rev. **72**, 874 (1947).

[28] U. Carow-Watamura, M. Schlieker, M. Scholl, S. Watamura, *Tensor Representation of the Quantum Group $SL_q(2,C)$ and Quantum Minkowski Space*, Z. phys. C **48** 159 (1990).

[29] P. Podles , S. L. Woronowicz, *Quantum Deformation of Lorentz Group*, Commun. Math. Phys. **130**, 381 (1990).

[30] M. Schlieker, W. Weich, and R. Weixler, *Inhomogeneous quantum groups*, Lett. Math. Phys. **27**, 217 (1993).

[31] P. Podles , S. L. Woronowicz, *On the structure of inhomogeneous quantum groups*, hep-th/9412058.

[32] O. Ogievetsky, W. B. Schmidke, J. Wess, B. Zumino, *6 Generator q-Deformed Lorentz Algebra*, Lett. Math. Phys. **23**, 233 (1991).

[33] O. Ogievetsky, W. B. Schmidke, J. Wess, B. Zumino, *q-Deformed Poincare Algebra*, Commun. Math. Phys. **150**, 495 (1992).

[34] J. Lukierski, H. Ruegg. *q-Deformation of Poincare Algebra*, Phys. Lett. **B 264** (3,4) 331 (1991) J. Lukierski, A. Nowicki, H. Ruegg, *New Quantum Poincare Algebra and κ-Deformed Field Theory* Phys. Lett. **B 293**, 344 (1992).

[35] S.L. Woronowicz, S. Zakrzewski, *Quantum Deformations of the Lorentz Group. The Hopf ∗-Algebra Level*, Compositio Mathematica 90, No.2, 211 (1994).

[36] P. Podles , S. L. Woronowicz, *On the classification of quantum Poincare groups*, hep-th/9412059.

[37] U. Carow-Watamura, M. Schlieker, S. Watamura, *$SO_q(N)$ Covariant Differential Calculus on Quantum Space and Quantum Deformation of Schodinger Equation*, Z. Phys. C, **49**, 439 (1991).

[38] P. Podles , *Solutions of Klein–Gordon and Dirac equations on quantum Minkowski spaces*, hep-th/9510019.

[39] T. Brzezinski, S. Majid, *Quantum Group Gauge Theory on Classical Spaces*, Phys. Lett.**B 298**, 339 (1993).

[40] I.Ya. Aref'eva, I.V. Volovich, *Quantum Group Gauge Fields*, Mod. Phys. Lett. A 6(10): 893 (1991).
A.P. Isaev, Z. Popowicz, *q-Trace for Quantum Groups and q-Deformed Yang-Mills Theory*, Phys. Lett. **B 281**, 271 (1992); *Quantum Group Gauge Theories and Covariant Quantum Algebras*, Phys. Lett. **B 307**, 353 (1993).
K. Wu, R.-J. Zhang, *Algebraic Approach to Gauge Theory in Non-Commutative Extension*, Commun. Theor. Phys.**17**, 175 (1992).

[41] P. Watts, *Towards a q-Deformed Standard Model*, UMTG-189, hep-th/9603143.

[42] M. Green, J. Schwarz, E. Witten, *Superstring Theory*, Cambridge University Press, 1986.

[43] A. Connes, *The Action Funcctional in Non-Commutative Geometry*, Commun. Math. Phys. **117**, 673 (1988).

[44] A. Connes, *Non-Commutative Geometry*, Academic Press (1994).

[45] A. Connes, J. Lott, *Particle Models and Non-Commutative Geometry*, Nucl. Phys. B (Proc. Suppl.) **18B**, 29 (1990).

[46] A. H. Chamseddine, *Review of Non-Commutative Geometric Methods Applied to Particle Physics*, ETH/TH/94-33 and references therein.

[47] M. Jimbo, T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, CBMS Regional Conference Series in Mathematics.

[48] A. Alekseev, S. Shatashvili, *Quantum Groups and WZNW Models*, Commun. Math. Phys. **133**, 353 (1990).

[49] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge Univ. Press, Cambridge, 1992.

[50] G. Mack, V. Schomerus, *Quasi Quantum Group Symmetry and Local Braid Relations in the Conformal Ising Model*, Phys. Lett. **B 267**, 207 (1991).

[51] G. Mack, V. Schomerus, *Quasi Hopf Quantum Symmetry in Quantum Theory*, Nucl. Phys. **370**, 185 (1992).

[52] M.E. Sweedler, *Hopf algebras*, Benjamin, (1969).

[53] B. Zumino, *Introduction to the Differential Geometry of Quantum Groups*, In: K. Schmüdgen (ed.), Math. Phys. X, Proc. X-th IAMP Conf. Leipzig (1991), Springer-Verlag (1992).

[54] B. Zumino, *Differential Calculus on Quantum Spaces and Quantum Groups*, In: M.O., M.S., J.M.G. (eds.), Proc. XIX-th ICGTMP Salamanca (1992), CIEMAT/RSEF Madrid (1993).

[55] P. Schupp, P. Watts, and B. Zumino, *Differential Geometry on Linear Quantum Groups*, Lett. Math. Phys. **25**, 139 (1992).

[56] P. Schupp, P. Watts, and B. Zumino, *Bicovariant Quantum Algebras and Quantum Lie Algebras*, Commun. Math. Phys. **157**, 305 (1993).

[57] U. Meyer, *Quantum Determinants*, DAMTP/94-54, hep-th/9406172.

[58] J. Wess and B. Zumino, *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl. Phys. B (Proc. Suppl.) **18B**, 302 (1990).

[59] M.D.Gould, R.B. Zhang and A.J. Bracken, *Generalized Gel'fand Invariants and Characteristic Identities for Quantum Groups*, J. Math. Phys. **32**, 2298 (1991).

[60] K. SchmHudgen, *Operator Representation of the Real Twisted Commutation Relations*, J. Math. Phys. **33**, 3211 (1994).

[61] B. Drabant, M. Schlieker, W. Weich and B. Zumino, *Complex Quantum Groups and their Quantum Enveloping Algebras*, Commun. Math. Phys. **147**, 625 (1992).

[62] C. Chryssomalakos, B. Drabant, M. Schlieker, W. Weich and B. Zumino, *Vector Fields on Complex Quantum Groups*, Commun. Math. Phys. **147**, 635 (1992).

[63] P. Podlés, *Complex Quantum Groups and their Real Representations*, Publ. R.I.M.S., Kyoto University **28**, 709 (1992).

[64] O.Ogievetsky and B. Zumino, *Reality in the Differential Calculus on q-Euclidean Spaces*, Lett. Math. Phys. **25**, 121 (1992).

[65] C. Chryssomalakos, P. Schupp and B. Zumino, *Induced Extended Calculus on the Quantum Plane* , Alg. i Anal. **6**, 252 (1994).

[66] G.Fiore, *Realization of $U_q(so(N))$ within the Differential Algebra on $\mathbf{R}_q^N$* , preprint SISSA 90/93/EP, hep-th/9403033.

[67] P. Podleś, *Quantum Spheres*, Lett.Math.Phys. **14**, 193 (1987).

[68] P. Podleś, *Differential Calculus on Quantum Spheres*, Lett.Math.Phys. **18**, 107 (1989).

[69] P. Podleś, *Differential Calculus on Quantum Spheres*, RIMS Kokyuroku Series, No.751, May (1991).

[70] P. Podleś, *The Classification of Differential Structures on Quantum 2-Spheres*, Commun. Math. Phys. **150**, 167 (1992).

[71] T. Brzezinski and S. Majid, *Quantum group gauge theory on quantum spaces*, Commun. Math. Phys. 157, 591 (1993).

[72] C.-S. Chu, P.-M. Ho, B. Zumino, *The Quantum 2-Sphere as a Complex Quantum Manifold*, q-alg/9504003, to appear in Z. Phys. C.

[73] A.J.-L. Sheu with J.-H. Lu and A. Weinstein, *Quantization of Poisson $SU(2)$ and the Two-Sphere*, Commun. Math. Phys. **135**, 217 (1991).

[74] P.Šťovíček, *Quantum Line Bundles on S^2 and Method of Orbits for $SU_q(2)$*, J. Math. Phys., **34**, 1606 (1993).

[75] S. Majid, *Beyond Supersymmetry and Quantum Symmetry (An Introduction to Braided Groups and Braided Matrices)* , in M-L. Ge and H.J. de Vega, editors, *Quantum Groups, Integrable Statistical Models and Knot Theory*, p. 231-282, World Sci. (1993).

[76] C.-S. Chu, P.-M. Ho, B. Zumino, *The Braided Quantum 2-Sphere*, q-alg/9507013, to appear in Mod. Phys. Lett.

[77] C.-S. Chu, P.-M. Ho, B. Zumino, *Geometry of the Quantum Complex Projective Space $CP_q(N)$*, q-alg/9510021, to apppear in Z. Phys. C.

[78] P.-M. Ho, *Geometrical Aspects of Quantum Spaces, Ph.D. Thesis*, Berkeley (1996).

[79] H. Steinacker, *Integration on Quantum Euclidean Space and Sphere in N Dimensions*, q-alg/9506020, to appear in J. Math. Phys.

[80] P.-M. Ho, *Riemannian Geometry on Quantum Spaces*, LBL-37902, UCB-PTH-95/37, q-alg/9505021.

[81] H. Steinacker, *Integration on Quantum Euclidean Space and Sphere in N Dimensions*, LBL-37431, UCB-PTH-95/21, q-alg/9506020, to appear in J. Math. Phys.

[82] M. Pflaum, *Quantum groups on Fibre Bundles*, preprint-Nr. gk-mp-9403/1, (1994).

[83] R. Budzynski, W. Kondracki, *Quantum Principal Fiber Bundles: Topological Aspects*, preprint hep-th/9401019.

[84] M. Durdevic, *Quantum Gauge Transformations and Braided Structure on Quantum Principal Bundles*, q-alg/9605011.

[85] T. Brzezinski and S. Majid, Erratum for *Quantum group gauge theory on quantum spaces*, Commun. Math. Phys. 157, 591 (1993).

[86] P.M. Hajac, *Strong connections and $U_q(2)$ - Yang - Mills Theory on Quantum Principal Bundles, Ph.D. Thesis*, Berkeley (1994).

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