Presented at the 4th International Conference
on Beam-Foil Spectroscopy and Heavy-Ion
Atomic Physics Symposium, Gatlinburg, TN,
September 15 - 19, 1975

LBL-3876 c.1

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November 11, 1975

Prepared for the U. S. Energy Research and
Development Administration under Contract W-7405-ENG-48

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I. INTRODUCTION

Recent progress in the measurement of the Lamb shift $S = E(2S_{1/2}) - E(2P_{1/2})$ in hydrogenic atoms and ions has been made in two directions. An experimental value with a substantial increase in precision over previous values has been obtained for atomic hydrogen,[1] and experimental values have been obtained for many higher-Z hydrogenic ions.[2,3] Quantum electrodynamics makes unambiguous predictions for the Lamb shift in these systems, so that a comparison between the theoretical and experimental values provides an important test of the theory. To aid in such a comparison, we compile here theoretical values for the various contributions to the Lamb shift for Z in the range 1 - 30. Compilations for small Z are given elsewhere,[4-7] but we include here some additional corrections which are significant at higher Z.

II. SELF ENERGY

Recently, the self-energy radiative correction of order α has been evaluated numerically to all orders in $Z\alpha$ for the $2S_{1/2}$ and $2P_{1/2}$ states for Z = 10, 20, 30, ..., 110.[7] The method of evaluation is based on the expansion of the bound electron propagation function in terms of the known Coulomb radial Green's functions.[8] In order to display the results for the self-energy contribution to the Lamb shift $S_{SE} = \Delta E(2S_{1/2}) - \Delta E(2P_{1/2})$, it is convenient to isolate the known low-order terms by writing[6,7]

TABLE I. Calculated Values of $G_{SE}(Z\alpha)$

Z	$G_{SE}(Z\alpha)$
10	- 20.13(34)
20	- 17.674(28)
30	- 15.776(11)
40	- 14.1376(62)
50	- 12.6650(28)

$$S_{SE} = \frac{\alpha}{\pi}\frac{(Z\alpha)^4}{6}mc^2\left[\ln(Z\alpha)^{-2} - \ln\frac{K_0(2,0)}{K_0(2,1)} + \frac{11}{24} + \frac{1}{2} + 3\pi\left(1 + \frac{11}{128} - \frac{1}{2}\ln 2\right)(Z\alpha) - \frac{3}{4}(Z\alpha)^2\ln^2(Z\alpha)^{-2} + \left(\frac{299}{240} + 4\ln 2\right)(Z\alpha)^2\ln(Z\alpha)^{-2} + (Z\alpha)^2 G_{SE}(Z\alpha)\right]. \qquad (1)$$

Values of the remainder $G_{SE}(Z\alpha)$, corresponding to the calculated values of S_{SE}, for Z in the range 10 - 50 appear in Table I. To obtain values for $G_{SE}(Z\alpha)$ for all Z in the range 1 - 30, we fit the interpolation function

$$a + b\ (Z\alpha)\ln(Z\alpha)^{-2} + c(Z\alpha) \qquad (2)$$

to the calculated values of $G_{SE}(Z\alpha)$ at Z = 10, 20, and 30.[7] The corresponding interpolated values of $G_{SE}(Z\alpha)$ appear in Table II. The numbers in parentheses with each entry in the table are error estimates which take into account the uncertainty in the calculated values of $G_{SE}(Z\alpha)$ and the uncertainty associated with the interpolation procedure.

III. VACUUM POLARIZATION

Evaluation of the energy level shift associated with the vacuum polarization of order α is facilitated by considering the expansion of the vacuum polarization potential in powers of the external Coulomb potential. Only odd powers of the external potential contribute as a consequence of Furry's theorem.[9]

The first term in the expansion gives rise to the modification of the external potential known as the Uehling potential[10,11]

$$U(r) = -\frac{\alpha}{\pi}\frac{Z\alpha}{r}\hbar c\int_1^\infty dt\,(t^2 - 1)^{\frac{1}{2}}\left(\frac{2}{3t^2} + \frac{1}{3t^4}\right)\exp(-2tr/\lambda) . \qquad (3)$$

The corresponding contribution to the Lamb shift is

$$\langle U \rangle_{2S_{1/2}} - \langle U \rangle_{2P_{1/2}} = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{6} mc^2 \left[-\frac{1}{5} + \frac{5}{64}\pi(Z\alpha) - \frac{1}{10}(Z\alpha)^2 \ln(Z\alpha)^{-2} + (Z\alpha)^2 G_U(Z\alpha) \right]. \tag{4}$$

The first three terms in the square brackets in (4) give the known contribution of the Uehling potential for small $Z\alpha$.[6] We have numerically evaluated the expectation values of $U(r)$ which appear in (4). The results are given in Table II in terms of the function $G_U(Z\alpha)$ which is defined by (4).

The second non-vanishing term in the expansion is third order in the external potential. Wichmann and Kroll have examined the vacuum polarization in detail and have obtained results from which one readily finds that the third order term contributes[12]

$$\frac{\alpha}{\pi} \frac{(Z\alpha)^6}{6} mc^2 \left[\frac{19}{60} - \frac{\pi^2}{36} + \left(\frac{3}{64} - \frac{31\pi^2}{3840} \right) \pi(Z\alpha) + \ldots \right] \tag{5}$$

to the Lamb shift. The terms omitted from (5) are higher order in $Z\alpha$.

The total vacuum polarization contribution to the Lamb shift is thus given by

$$S_{VP} = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{6} mc^2 \left[-\frac{1}{5} + \frac{5}{64}\pi(Z\alpha) - \frac{1}{10}(Z\alpha)^2 \ln(Z\alpha)^{-2} + (Z\alpha)^2 G_{VP}(Z\alpha) \right] \tag{6}$$

where

$$G_{VP}(Z\alpha) = G_U(Z\alpha) + .0425 - .1030(Z\alpha) \pm .1(Z\alpha). \tag{7}$$

The last term in (7) is an approximation for the uncertainty due to higher order omitted terms.

IV. FOURTH ORDER

The contribution to the Lamb shift from the fourth order radiative corrections is known exactly to lowest order in $Z\alpha$. It is given by

$$S_{FO} = \left(\frac{\alpha}{\pi}\right)^2 \frac{(Z\alpha)^4}{6} mc^2 \left[\pi^2 \ln 2 - \frac{37\pi^2}{144} - \frac{3767}{1728} - \frac{3}{2}\zeta(3) \pm \pi(Z\alpha) \right]. \tag{8}$$

Recent work on the evaluation of this term is discussed in Ref. 6. The last term in (8) is an approximation for the uncertainty due to higher order uncalculated terms.

V. REDUCED MASS AND RELATIVISTIC RECOIL

The lowest order reduced mass and relativistic recoil contributions to the Lamb shift are given by[6]

$$S_{RM} = \frac{\alpha}{\pi}\,\frac{(Z\alpha)^4}{6}\,mc^2\left(-\,3\frac{m}{M}\right)\left[\ln(Z\alpha)^{-2} - \ln\frac{K_0(2,0)}{K_0(2,1)} + \frac{23}{60}\right] \qquad (9)$$

and

$$S_{RR} = \frac{\alpha}{\pi}\,\frac{(Z\alpha)^4}{6}\,mc^2\left(\frac{Zm}{M}\right)\left[\frac{1}{4}\ln(Z\alpha)^{-2} - 2\ln\frac{K_0(2,0)}{K_0(2,1)} + \frac{97}{12}\right] \qquad (10)$$

where M is the nuclear mass. We employ the uncertainty estimate of Erickson[5] for these terms for Z = 1 and the Z-dependent estimate of Erickson and Yennie[13]

$$\Delta S = \frac{m}{M}(Z\alpha)^6 mc^2 \qquad (11)$$

for Z > 1. We also include Salpeter's estimate of $0.1 \times S_{RR}$ for the uncertainty due to uncalculated nuclear structure effects in deuterium.[14]

VI. NUCLEAR SIZE

The potential for an electron in a hydrogenlike ion is not a pure Coulomb potential due, in part, to the fact that the nuclear charge is distributed over a finite radius. We evaluate the energy shift due to the difference between the finite size potential and the Coulomb potential $\delta V(r)$ in first order perturbation theory with Dirac wavefunctions. Assuming that the nuclear charge is distributed uniformly inside a sphere, we obtain

$$S_{NS} = \left[1 + 1.70(Z\alpha)^2\right]\frac{(Z\alpha)^2}{12}\,mc^2\left(\frac{Z\alpha R}{\lambda\!\!\!^{-}}\right)^{2s} \qquad (12)$$

for the nuclear size contribution to the Lamb shift. In Eq. (12) $s = \sqrt{1-(Z\alpha)^2}$ and R is the r.m.s. charge radius of the nucleus. The expression in (12) neglects terms of relative order $(Z\alpha)^4$ or $Z\alpha R/\lambda\!\!\!^{-}$. This nuclear size correction S_{NS} agrees to lowest order in $(Z\alpha)^2$ with the nonrelativistic expression in Ref. 13.

It is known that first order perturbation theory gives inaccurate results for the finite size correction when Z is large.[15,16] We find that the fractional difference between the perturbation result and the result obtained by numerically solving the Dirac equation for a finite nucleus is approximately 3% at Z = 30. We therefore assign an uncertainty ΔS_{NS} to Eq. (12), where

$$\Delta S_{NS} = \left(0.7(Z\alpha)^2 + 2\,\frac{\Delta R}{R}\right)S_{NS}\,, \qquad (13)$$

TABLE II. Values for the functions $G_{SE}(Z\alpha)$, $G_U(Z\alpha)$, assumed values for the nuclear charge radii, and Lamb shift values

Z	A	$G_{SE}(Z\alpha)$	$G_U(Z\alpha)$	R[fm]	LAMB SHIFT[GHz]
1	1	-23.4(1.2)	-0.5587	.81(2)	1.057867(13)
1	2	"	"	2.10(2)	1.059241(27)
2	4	-22.9(1.0)	-0.5493	1.644(5)	14.04205(55)
3	6	-22.49(88)	-0.5411	2.56(5)	62.7375(66)
4	9	-22.10(77)	-0.5339	2.52(2)	179.791(25)
5	11	-21.72(68)	-0.5273	2.4(1)	404.57(10)
6	12	-21.37(60)	-0.5213	2.45(1)	781.99(21)
7	14	-21.04(52)	-0.5157	2.54(2)	1361.37(47)
8	16	-20.72(45)	-0.5106	2.72(3)	2196.21(92)
9	19	-20.42(39)	-0.5059	2.90(2)	3343.1(1.6)
10	20	-20.13(34)	-0.5015	3.02(4)	4861.1(2.7)
11	23	-19.85(29)	-0.4974	2.94(4)	6809.0(4.0)
12	24	-19.58(24)	-0.4936	3.01(3)	9256.0(5.8)
13	27	-19.31(20)	-0.4900	3.03(3)	12,264.7(8.0)
14	28	-19.06(17)	-0.4867	3.09(2)	15,907(11)
15	31	-18.81(14)	-0.4836	3.19(2)	20,254(13)
16	32	-18.57(11)	-0.4807	3.24(2)	25,373(17)
17	35	-18.338(84)	-0.4781	3.34(3)	31,347(20)
18	40	-18.111(63)	-0.4756	3.45(5)	38,250(25)
19	39	-17.890(44)	-0.4733	3.41(3)	46,133(29)
20	40	-17.674(38)	-0.4711	3.48(3)	55,116(37)
21	45	-17.464(34)	-0.4691	3.54(8)[a]	65,259(55)
22	48	-17.259(31)	-0.4673	3.60(1)	76,651(56)
23	51	-17.059(28)	-0.4656	3.60(5)	89,345(78)
24	52	-16.863(26)	-0.4641	3.66(5)	103,482(98)
25	55	-16.672(24)	-0.4627	3.72(7)	$1.1912(13) \times 10^5$
26	56	-16.485(22)	-0.4614	3.73(6)	$1.3632(15) \times 10^5$
27	59	-16.302(21)	-0.4603	3.80(5)	$1.5525(18) \times 10^5$
28	58	-16.123(19)	-0.4593	3.78(3)	$1.7585(21) \times 10^5$
29	63	-15.948(18)	-0.4584	3.93(3)	$1.9854(26) \times 10^5$
30	64	-15.776(17)	-0.4576	3.95(4)	$2.2303(32) \times 10^5$

[a]Interpolated value

to allow for this error, for model dependence error, and for the measured charge radius uncertainty ΔR.

Values we assume for the nuclear r.m.s. charge radii are given in Table II. Those numbers are representative values based on values in the compilations of nuclear charge radii deduced from electron scattering data,[17] and from muonic atom transition energies.[18]

VII. LAMB SHIFT VALUES

The sum of the contributions listed in the preceding sections gives the total Lamb shift S. The values are listed in Table II. We employ the recently recommended values[19] $R_\infty c = 3.28984200(25) \times 10^{15}$ Hz and $\alpha^{-1} = 137.03604(11)$. The uncertainty in α produces a relative uncertainty of $3\Delta\alpha/\alpha = 2.4 \times 10^{-6}$ in the total Lamb shift. The uncertainty listed with each Lamb shift value is the quadrature sum of the contributing uncertainties and is meant to be considered on a par with a one standard deviation uncertainty in the experimental value. The theoretical values listed here differ somewhat from the values compiled by Erickson.[5,6] The difference is due mainly to differences between the values used for the self-energy contribution.

Measured values for the n=2 hydrogenic Lamb shifts are listed

TABLE III. Comparison between theory and direct measurement of the Lamb shift $E(2S_{1/2}) - E(2P_{1/2})$

	THEORY (1σ)	EXPERIMENT (1σ)	REF.
H	1057.867(13) MHz	1057.893(20) MHz	1
		1057.90(6) "	20[a]
		1057.77(6) "	21[a]
D	1059.241(27) MHz	1059.24(6) MHz	22[a]
		1059.00(6) "	21[a]
$^4He^+$	14,042.05(55) MHz	14,046.2(1.2) MHz	23
		14,040.2(1.8) "	24[a]
$^6Li^{2+}$	62,737.5(6.6) MHz	62,765(21) MHz	25
		62,880(190) "	26
		63,031(327) "	27
$^{12}C^{5+}$	781.99(21) GHz	780.1(8.0) GHz	28
$^{16}O^{7+}$	2196.21(92) GHz	2215.6(7.5) GHz	29
		2202.7(11.0) "	30

[a]See Reference 31 for a discussion of the experimental value.

in Table III. In addition to the measurements listed there, we note the recent work of Kugel et al. who have measured the separation $E(2P_{3/2}) - E(2S_{1/2})$ in hydrogenlike fluorine.[32] They combine their result with the theoretical value of the fine structure splitting to obtain the Lamb shift value S = 3339(35) GHz.

ACKNOWLEDGMENT

I wish to thank Professor Glen Erickson for a helpful discussion of the nuclear recoil uncertainty estimate.

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