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UNIVERSALITY, TWISTED FANS, AND THE ISING MODEL*

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ABSTRACT

We calculate critical exponents for the Ising model using universality in the form of "twisted fans" previously introduced in Reggeon field theory. The universality is with respect to scales induced through renormalization. Exact twists are obtained at $\beta = 0$ in one loop for $D = 2,3$ with $\nu = 0.75$ and 0.60 respectively. In two loops we obtain $\nu \approx 1.32$ and 0.68. No twists are obtained for η, however. The results for the standard two loop calculations are also presented as functions of a scale.

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The possibility of applying the n-loop expansion to the calculation of critical indices in solid state physics directly in the dimension D of interest was proposed and carried out by Parisi to two loops, with optimistic results[1]. The method requires the introduction of a massive theory to avoid severe infrared massless problems. The same approach followed in the high energy scattering theory of Gribov's Reggeon calculus [2,3] led to a proposal that universality be employed as a working tool to determine approximations to critical exponents using the concept of "twisted fans" [4]. The idea is simple. Consider a critical exponent function $\gamma(g)$ defined with the aid of a wave function renormalization constant Z which depends on the renormalized dimensionless coupling g. At the critical coupling g_c defined by setting the Gell-Mann - Low function $\beta(g_c) = 0$, $\gamma_c \equiv \gamma(g_c)$ must be independent of a change of the momentum renormalization points of Green's functions, generically denoted by p_N^2. This is so because at the phase transition correlations are of infinite range and universality demands that γ_c be independent of any finite scale [5]. The only effect is a finite renormalization of Green's functions which does not affect the scaling laws. This ceases to be true, however, if $g \neq g_c$ since correlations are then not of infinite range and γ can depend on p_N^2. Defining $\xi = p_N^2/m^2$, m being a mass scale, and expanding γ and β in powers of g we have

$$\gamma(g,\xi) = \sum_{\ell=\ell_0}^{\infty} h_\ell(\xi)\, g^\ell \tag{1}$$

$$\beta(g,\xi) = \sum_{k=k_0}^{\infty} b_k(\xi)\, g^k \tag{2}$$

In the exact theory, $\beta = 0$ at $g = g_c(\xi)$ and $\gamma_c = \gamma[g_c(\xi),\xi]$ then becomes independent of ξ. If either $g \neq g_c(\xi)$ or the sums are truncated, both γ and β will depend on ξ.

Now consider plotting β vs. γ for fixed ξ using g as a parameter along the curve. In the full theory the curve will start at the origin and pass through $\beta = 0$ at $\gamma = \gamma_c$ where $\beta'(g_c) > 0$. As ξ is varied a series of curves with the same property results, resembling a fan which twists at the point $\beta = 0$, $\gamma = \gamma_c$. The idea is then to arrange approximations which possess this universality property of the exact theory as far as possible. In so doing, one is essentially maximizing the information at one's disposal at a given order. The utility of such an approach can be seen in cases where β does not have a zero in a given order. This in fact happens in two loops, both in solid state physics [1] and in the Reggeon calculus [3]. Nonetheless twisted fans are obtained in the latter case to high accuracy [4].

Several points should be kept in mind. First, several scales ξ_j can in general be introduced into the renormalization procedure; the demand of universality with respect to all these parameters provides maximal information. Although the exact theory should respect invariance with respect to any simultaneous variation of the ξ_j, in finite order one may have to settle for an approximate invariance, e.g. along a ray

in the $\{\xi_j\}$ space. Presumably one could try to maximize the degree of invariance, perhaps obtaining reliable bounds, by varying the direction $\hat{n}$ of the ray. A sophisticated use of the method would involve minimizing a suitably defined velocity $\vec{v}$ of the twist as a function of $\hat{n}$. It should therefore be clear that the absence of a fan in a given approximation with a fixed assumption about the direction of the ray does not a-priori mean that the method has failed.

Next, it may or may not be profitable in a given order to estimate higher order terms with Pade approximants; indeed the presence of approximate universality could be employed as justification for using them. Third, twist positions in successive orders may oscillate, probably depending on the existence or non-existence of a zero of β in those orders. Fourth, in a given order a twist may only be approximate. At least in the Reggeon calculus infrared divergences of the massless theory prevent the limit $\xi \to \infty$ from being valid in perturbation theory [4], though it must be valid in the complete theory [1,6,7]. Use of Pade approximates can also restrict the available range in ξ. Fifth, if more than one critical exponent γ_j exists the fan will be a multi-dimensional construct in $\{\beta, \gamma_j\}$.

Finally, our use of universality seems similar in philosophy to that of Ref. 8, although the details are quite different.

We now describe the calculation, carried out in Euclidean ϕ^4 theory [1,6,7]. Details will be presented elsewhere[9]. The unrenormalized proper vertex function $\Gamma^{(N,L)}$ for N ϕ fields and L ϕ^2 fields are used to define parameters in terms of which we will re-express the theory.

These are the renormalized mass m, the dimensionless renormalized coupling g, and the ϕ and ϕ^2 wave function renormalization constants Z_1 and Z_2, defined by

$$\Gamma^{(2,0)}(p^2)\Big|_{p^2=m^2} = 0 \tag{3}$$

$$\frac{d}{dp^2}\Gamma^{(2,0)}(p^2)\Big|_{p^2=\xi_2 m^2} \equiv Z_1^{-1} \tag{4}$$

$$\Gamma^{(2,1)}(p_i)\Big|_{NP(\xi_3)} \equiv (Z_1 Z_2)^{-1} \tag{5}$$

$$\Gamma^{(4,0)}(p_i)\Big|_{NP(\xi_4)} \equiv Z_1^{-2}\, m^{4-D}\, g \tag{6}$$

The $\Gamma^{(4,0)}$ renormalization point is defined by the symmetric "decay" condition $p_1 = 3p_N = -3p_j$ $(j = 2,3,4)$ where $p_N^2 = \xi_4 m^2$. To obtain the renormalization point for $\Gamma^{(2,1)}$, p_3 and p_4 are joined to form the ϕ^2 line with momentum $-2p_N$; we define $p_N^2 = \xi_3 m^2$ for this function. Other normalizations are of course possible and should be investigated. We have in fact found that a more symmetric normalization of $\Gamma^{(2,1)}$ coupled with the above "decay" normalization of $\Gamma^{(4,0)}$ produces less satisfactory results.

We define for $i = 1,2$

$$c_i(g, \xi_j) = m^2 \frac{\partial}{\partial m^2} \ln Z_i \Big|_{\xi_j, \lambda_0} \tag{7}$$

and

$$\beta(g, \xi_j) = m^2 \frac{\partial g}{\partial m^2} \Big|_{\xi_j, \lambda_0} \tag{8}$$

The derivatives are to be evaluated at fixed ξ_j and fixed bare coupling λ_0 (λ_0 has dimension m^{4-D}). The critical exponent functions ν and η are then

$$\nu(g, \xi_j) = \left[2 + 2c_2(g, \xi_j)\right]^{-1} \tag{9}$$

$$\eta(g, \xi_j) = 2c_1(g, \xi_j) \tag{10}$$

The corresponding Callan-Symanzik operator for $\Gamma^{(N,L)}$ is [6]

$$m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - \frac{N\eta}{4} - \frac{L}{2}\left(\frac{1}{\nu} - 2\right) \tag{11}$$

We set

$$I_1(\xi) = \frac{\Gamma(3 - D/2)}{(4\pi)^{D/2}} \int dx\, dy\, \delta(1-x-y) \left[1 + 4\xi xy\right]^{D/2 - 2} \tag{12}$$

$$I_2(\xi) = \frac{\Gamma(5-D)}{(4\pi)^D} \int \frac{dx\,dy\,dz\,\delta(1-x-y-z)\,xyz}{(xy+yz+xz)^{1+D/2}} \left[1 + \frac{\xi xyz}{(xy+yz+xz)}\right]^{D-4} \tag{13}$$

$$I^{ice}_{A,B}(\xi) = \frac{\Gamma(5-D)}{(4\pi)^D} \int \prod_{i=1}^{4} dx_i\, \delta\left(1 - \sum^4 x_i\right) \frac{F_{A,B}(x_i, \xi)}{\left[(x_1+x_2+x_4)(x_3+x_4) - x_4^2\right]^{D/2}} \tag{14}$$

where

$$F_{A,B}(x_i,\xi) = \left\{ 1 + \frac{\xi\left[4x_1x_2(x_3+x_4) + x_3x_4(x_1+\alpha x_2)\right]}{\left[(x_1+x_2+x_4)(x_3+x_4) - x_4^2\right]} \right\}^{D-4} \tag{15}$$

and $\alpha = 1,9$ for F_A, F_B respectively. I_1 corresponds to the one loop graphs in $\Gamma^{(2,1)}$ and $\Gamma^{(4,0)}$; I_2 corresponds to the two loop graph in $\Gamma^{(2,0)}$; I_A^{ice} corresponds to the "ice-cream" two loop graphs in $\Gamma^{(2,1)}$ and $\Gamma^{(4,0)}$ where the $3p_N$ leg in $\Gamma^{(4,0)}$ is adjacent to the ice-cream, and Γ_B^{ice} corresponds to the ice-cream graph in $\Gamma^{(4,0)}$ with the $3p_N$ leg not adjacent to the ice-cream. We obtain

$$c_1(g,\xi_2) = \frac{1}{6} g^2 I_2(\xi_2) \tag{16}$$

$$c_2(g,\xi_j) = -\frac{1}{2} g I_1(\xi_3) - c_1(g,\xi_2) + g^2 \left\{ \frac{1}{2} I_A^{ice}(\xi_3) + \frac{I_1(\xi_3)}{2(4-D)}\left[I_1(\xi_3) - 3I_1(\xi_4)\right] \right\} \tag{17}$$

$$\beta(g;\xi_2,\xi_4) = (D/2-2)g + \frac{3}{2} g^2 I_1(\xi_4) + 2g c_1(g,\xi_2) + g^3\left\{ -\frac{3}{2}\left[I_A^{ice}(\xi_4) + I_B^{ice}(\xi_4)\right] + \frac{6}{(4-D)}\left[I_1(\xi_4)\right]^2 \right\} \tag{18}$$

The apparent poles at $D = 4$ cancel; all expressions are finite for $D \leq 4$.

The results in one loop are easily examined since $c_1 = \eta = 0$ to that order. Setting $\beta[g_c(\xi_4), \xi_4] = 0$ yields

$$g_c(\xi_4) = \frac{1}{3}(4-D) \Big/ I_1(\xi_4) \tag{19}$$

so that

$$c_2[g_c(\xi_4), \xi_3] = \frac{1}{6}(D-4)\, I_1(\xi_3) \Big/ I_1(\xi_4) \tag{20}$$

We see that we need only set $\xi_3 = \xi_4$ to obtain a perfect twist at $\beta = 0$ with critical exponent

$$\nu = 3(2+D)^{-1} \tag{21}$$

independent of ξ_4. This result is identical to Parisi's,[1] and yields $\nu = 0.75$ for $D = 2$ and $\nu = 0.60$ for $D = 3$. The best values[1] are $\nu = 1$ and 0.64, respectively.

We learn a lesson at this point by requiring $\xi_3 \neq \xi_4$, spoiling the perfect twist. The point is, of course, that we are at liberty to express the theory in terms of those boundary values of the $\Gamma^{(N,L)}$ which are the best choices from the point of view of universality. We do not need to do so, we want to do so; in this order $\xi_3 = \xi_4$ reproduces the universality present in the exact theory as well as possible.

The results for the two loop approximation to the $D = 3$ Ising model with all $\xi_j = \xi$ are presented in Figs. 1,2. Since our momenta are Euclidean the perturbative singularities are at $\xi < 0$; hence we restrict our attention to $\xi > 0$. Pade approximates have been used for β and ν, following Parisi. We call these functions β_p and ν_p.

Pade approximates are necessary for the results which follow for ν. The region in ζ which we can examine is limited by their poles to $\zeta \lesssim 4$. An approximate twist with a spread of about 0.02 is observed for ν_p yielding

$$\nu_p \approx 0.68 \tag{22}$$

However no twist is found for η in this approximation.

The graphs also directly yield results for the standard two loop calculation[1] obtained by setting $\beta_p = 0$, but demonstrating the variation of the exponents on the scale ζ. We see that ν_p is relatively stable; we obtain

$$0.64 \lesssim \nu_p \lesssim 0.67 \tag{23}$$

However η is not stable; we obtain only $\eta > 0.03$ for $\zeta > 0$. The lower bounds at $\zeta = 0$ were obtained by Parisi[1].

The results for ν_p at D = 2 in two loops are qualitatively similar with a twist at $\nu_p \approx 1.32$ and a variation at $\beta_p = 0$ of $1.1 \lesssim \nu_p \lesssim 1.29$ over the range in ζ allowed by the Pade approximates, which in this case is $\zeta \lesssim 0.4$. Again η has no twist and is quite unstable; we obtain only $\eta > 0.04$.

We have not investigated the cause of the instability of the exponent η in any detail, although it is interesting that it is also more unstable than ν in the treatment of Ref. 8. After this work was nearly completed we received a preprint[10] in which a coupling constant

rescaling $g_f = f(\xi)g$, $\beta_f(g_f) = f(\xi)\beta(g)$ was shown to have substantial effects on η . (The condition for a twist $\partial\beta/\partial\xi = 0$ becomes $f'\beta + f\partial\beta/\partial\xi = 0$ which depends on f if $\beta \neq 0$). Further examination along these lines coupled with a less restrictive assumption on the ray in $\{\xi_j\}$ space than used here would be interesting.

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FIGURE CAPTIONS

1. Results for the Pade-approximated β_p vs ν_p in the $D = 3$ Ising model in the two loop approximation.
2. Results for β_p vs η at $D = 3$ in two loops.

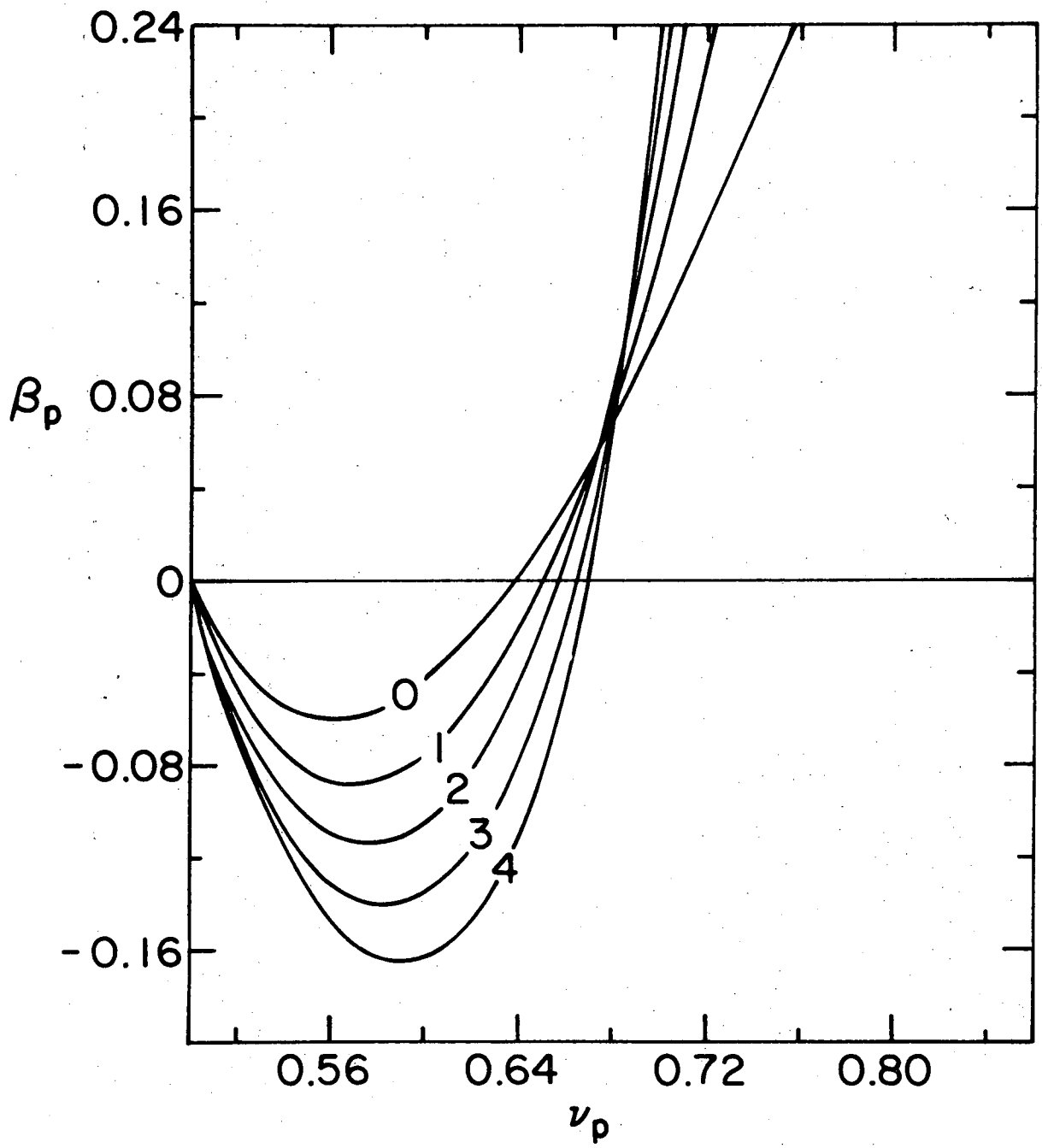

Fig. 1

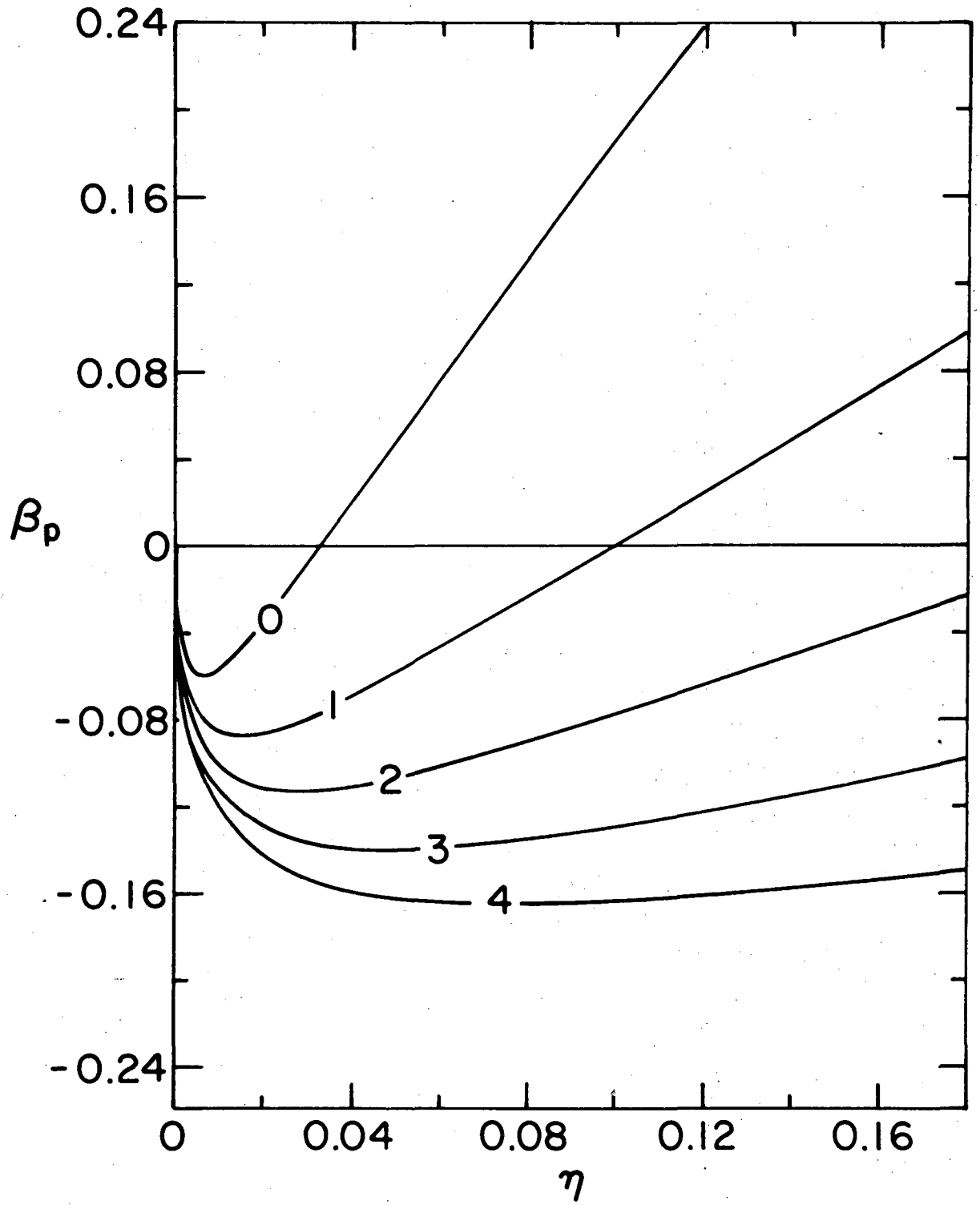

Fig. 2

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