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ABSTRACT

In a toroidal plasma with axial symmetry, the three adiabatically invariant actions of a particle are the magnetic moment, the canonical angular momentum, and the toroidal flux enclosed by the drift surface. Resonant interactions between particles and the normal modes of collective oscillation produce mode growth or decay and random changes in the actions. This random walk is represented by a diffusion equation in action space. Both the diffusion tensor and the growth rate depend upon a coupling coefficient which represents the work done by a normal-mode field eigenfunction on the current density of an unperturbed particle orbit. The diffusion of the plasma causes adiabatic changes in the electric and magnetic self-consistent fields. Accordingly, energy is not conserved, but is exchanged with external currents.

I. INTRODUCTION

Among the processes which contribute to the diffusion of plasma in a confined configuration is the resonant interaction of particles with collective modes of oscillation. Since the confinement may be interpreted as being due to the existence of adiabatic invariants for each particle, the loss of confinement is attributed to the breaking of one or more invariants by the perturbing fields of the oscillations. Thus the magnetic moment is broken by perturbing fields varying at the gyrofrequency or at harmonics thereof, while the other invariants are broken by resonances between the perturbation frequencies and the guiding-center drift frequencies.

In this paper, we formulate the quasilinear diffusion theory for an axisymmetric system, with special regard to a toroidal system, or tokomak. The unperturbed magnetic field may have both toroidal and poloidal components, generating magnetic surfaces. However, the special case of a purely poloidal field, either closed (multipole) or open (mirror machine), is included in the theory with slight modifications.

The present work may be considered as an extension of two previous studies of quasilinear diffusion in axisymmetric tori. Horton[1] investigated diffusion due to electrostatic low-frequency normal modes in a toroidal plasma with only poloidal field. Liu, Baxter, and Thompson[2] studied diffusion in tokomak geometry, again for electrostatic low-frequency modes, in the local approximation. Here we generalize by using the normal modes of the full Maxwell equations, and by including high frequency modes, so that gyro-resonance may occur. The unperturbed motion is treated in the guiding-center approximation, while the perturbing fields modify both the guiding-center drift and the gyration. Recent studies of the linear problem along these lines are those of Horton, Callen, and Rosenbluth,[3] who treated the mirror machine geometry, and Callen,[4] who treated the tokomak. Here we extend those studies (which include solutions of the linear integral equations) to the quasilinear regime.

As in the papers cited above, we remain within the Vlasov approximation, thus excluding collisional phenomena. Further, we include no strictly nonlinear effects,[5] as these would considerably complicate the treatment. Our work is thus the analog, for a finite system, of the classical quasilinear theory of an infinite plasma. The chief differences are three: (1) the conductivity tensor (28) is an integral operator; (2) the normal modes have a discrete spectrum; (3) the unperturbed electric and magnetic fields are themselves time-dependent, since the diffusing plasma contributes charge- and current-densities as sources for these self-consistent fields.

Because these quasistatic fields are adiabatically changing (slowly compared to the characteristic particle frequencies), the particle unperturbed energies have an explicit time-dependence. The particle action-variables, on the other hand, are adiabatic invariants, and vary only from resonances with the perturbing fields. Hence, in contrast to previous studies,[1,2] we use the three actions, and not the energy, as the independent variables for the diffusing zero-order distribution function.

The three actions for an axisymmetric system, when the guiding-center approximation is valid, are (1) μ, the magnetic moment; (2) p_φ, the canonical angular momentum; and (3) J_P, the toroidal flux enclosed by the drift surface. For a trapped particle, with a banana-shaped poloidal orbit, p_φ characterizes the magnetic surface of the banana-center, and its conjugate angle-variable $\bar{\phi}$ denotes its azimuth, which drifts at the constant frequency $\omega_\varphi \equiv \dot{\bar{\phi}}$; the banana encloses the toroidal flux J_P, whose conjugate angle θ_P increases at the bounce-frequency $\omega_P = \dot{\theta}_P$. (For a purely poloidal field, J_P is instead the bounce-action.) For circulating particles, p_φ represents (mainly) $v_\parallel$, while now J_P characterizes the magnetic surface, which approximates the drift surface. The canonical formalism of these variables is developed in Section II.

In terms of the action- and angle-variables, the solution of the linearized Vlasov equation is easily found, and from it the conductivity kernel is obtained, in Section III. The normal mode problem is then formulated in Section IV, as the solution of an integro-differential equation, with an unhermitian kernel. Treating the antihermitian part as a perturbation, we obtain an expression (37) for the growth rate γ_a of a normal mode, in terms of (i) a coupling coefficient (38), representing the work done by the normalized electric-field eigenfunction, of a zero-order normal mode, on the current density of an unperturbed orbit; (ii) the derivatives of the unperturbed distribution function; and (iii) a resonance condition

$$\omega_a = \ell_g \omega_g + \ell_\varphi \omega_\varphi + \ell_P \omega_P \qquad (1)$$

between the zero-order (real) normal-mode eigenfrequency ω_a and the action-dependent unperturbed angle-frequencies. The latter are ω_g, the poloidal-average gyrofrequency; and $(\omega_\varphi, \omega_P)$, defined above; while $(\ell_g, \ell_\varphi, \ell_P)$ are integers, possibly negative or zero. (The azimuthal mode number is ℓ_φ.)

The case $\ell_g = 0$ represents a low-frequency resonance, wherein the perturbing field, with the Doppler-shifted frequency $\omega_a - \ell_\varphi \omega_\varphi$ (as seen by the drifting guiding center), resonates with a multiple (ℓ_P) of the poloidal (bounce) frequency ω_P. This case thus includes the trapped-particle instabilities.

The sub-case $(\ell_g = 0, \omega_a = 0)$ characterizes orbits whose unperturbed drift curves are closed, and thus do not generate surfaces, since ω_P/ω_φ is rational. (In other words, the rotational transform of the guiding center is a rational multiple of 2π.) The guiding-center diffusion resulting from an axially-unsymmetric $(\ell_\varphi \neq 0)$ quasistatic perturbation is completely analogous to the corresponding diffusion of field lines, i.e., the destruction of magnetic surfaces in the neighborhood of rational rotational transforms. The latter problem has been studied by Rosenbluth, Sagdeev, Taylor, and Zaslavski.[6]

The case $\ell_g \neq 0$ represents gyroresonance (let $\ell_g = 1$, say); here the local (Doppler-shifted) resonance condition $\omega_a - k_\parallel v_\parallel = \Omega_g$ is replaced by $\omega_a - \ell_\varphi\omega_\varphi - \ell_P\omega_P = \omega_g$, which takes account of the poloidal variation of the local gyrofrequency Ω_g, and of gyrophase-shifts from one bounce to the next. This linear resonance condition is replaced by a nonlinear one in the work of Lichtenberg and Jaeger.[7]

The quasilinear diffusion in the three-dimensional action-space is derived in Section V. The diffusion tensor depends on (i) the normal-mode wave energies, (ii) the coupling coefficients described above, and (iii) the resonance condition (1). Since the latter condition determines a set of surfaces in action-space, on which the diffusion tensor is nonvanishing, one must appeal to a slight nonlinearity to argue a continuous tensor field for the diffusivity. With this proviso, an entropy theorem is obtained, indicating a continuing diffusion in the three actions, or equivalently, in minor radius (of magnetic surface), kinetic energy, and pitch-angle.

Since energy conservation serves as a helpful check on the theory, especially with the unperturbed fields being time-dependent, this question is discussed in Section VI. It is found that the resonant mode-particle interaction conserves the sum of unperturbed particle energy and wave energy, as expected; while the adiabatic field variation corresponds to an energy transfer between the plasma and the external currents.

II. INVARIANTS OF AN AXISYMMETRIC TOROID

We consider an axisymmetric field configuration, with slowly varying magnetic and electric fields. If the particle gyroradius is small compared to the spatial scale of these fields, we may use the guiding-center (g.c.) Lagrangian[8] for the g.c. motion:

$$L(\underset{\sim}{R},\underset{\sim}{V};\ \mu;\ \underset{\sim}{A}_0,\phi_0) = \frac{1}{2} V_\parallel^{\,2} - \mu\, B_0(\underset{\sim}{R};\ t) - e\,\phi_0(\underset{\sim}{R};\ t) + e\,\underset{\sim}{V}\cdot\underset{\sim}{A}_0(\underset{\sim}{R};\ t)\ . \tag{2}$$

Here $(\underset{\sim}{R},\underset{\sim}{V})$ are the g.c. position and velocity; $(\phi_0, \underset{\sim}{A}_0, B_0)$ are the quasistatic scalar potential, vector potential, and magnetic field magnitude evaluated at $\underset{\sim}{R}$; μ is the magnetic moment; $V_\parallel \equiv \underset{\sim}{V}\cdot\hat{B}$; and units are used with $m = c = 1$.

The magnetic field has toroidal and poloidal components:

$$\underset{\sim}{B}_0 = B_\varphi(R,Z)\hat{\varphi} + \underset{\sim}{B}^P(R,Z), \tag{3}$$

using standard cylindrical coordinates (R,φ,Z). The poloidal field is the curl of the azimuthal vector potential: $\underset{\sim}{B}^P = \nabla \times (A_\varphi\,\hat{\varphi})$, which in turn is expressed in terms of the poloidal flux function ψ :

$$A_\varphi = -R^{-1}\,\psi(R,Z), \qquad B_Z = -R^{-1}\,\partial\psi/\partial R, \qquad B_R = R^{-1}\,\partial\psi/\partial Z. \tag{4}$$

The curves $\psi(R,Z) =$ constant are the poloidal field lines and the projections of the total field lines:

$$\frac{dR}{B_R} = \frac{dZ}{B_Z} = \frac{R\,d\varphi}{B_\varphi} = \frac{ds}{B_0}\ . \tag{5}$$

(The magnetic axis is the point (R_0,Z_0) at which $\psi = 0$.) The parallel velocity is thus

$$V_\parallel \equiv \dot{s} = \frac{B_0 R}{B_\varphi}\,\dot{\varphi} = \dot{\varphi}\,\mathcal{R}(R,Z), \tag{6a}$$

where

$$\mathcal{R}(R,Z) \equiv B_0 R/B_\varphi\ . \tag{6b}$$

The Lagrangian (2) now reads

$$L(R,Z,\dot{\varphi},\underset{\sim}{V}^P) = \frac{1}{2}\mathcal{R}^2\,\dot{\varphi}^2 - \mu B_0 - e\phi_0 - e\dot{\varphi}\psi + e\underset{\sim}{V}^P\cdot\underset{\sim}{A}_0^{\ P}\ . \tag{7}$$

The poloidal vector potential can be expressed in terms of Euler potentials:[9] $\underset{\sim}{A}_0^{\ P} = \alpha\,\nabla\beta$, with $\alpha(R,Z)$ a function only of ψ , and $\beta(R,Z)$ increasing clockwise around the magnetic surface $\psi =$ constant, with period 2π. For a given ψ , the toroidal flux enclosed is

$$\iint dR\,dZ\,B_\varphi(R,Z) = \oint d\underset{\sim}{s}\cdot\underset{\sim}{A}_0^{\ P} = \oint \alpha\,d\beta = 2\pi\alpha;$$

thus $\alpha(\psi)$ is the toroidal flux function. We note that the rotational transform is $\iota/2\pi = d\psi/d\alpha$. Using (α,β) as the poloidal coordinates, we see that the last term of (7) is[10]

$$e\,\alpha\,\underset{\sim}{V}^P\cdot\nabla\beta \equiv e\,\alpha(\dot{\beta} - \frac{\partial\beta}{\partial t}),$$

where $\partial\beta/\partial t \equiv (\partial/\partial t)\,\beta(R,Z;\ t)$ is to be expressed in terms of $(\alpha,\ \beta,\ t)$. The Lagrangian (7) is now

$$L(\alpha,\beta,\dot{\varphi},\dot{\beta}) = \frac{1}{2}\mathcal{R}^2(\alpha,\beta)\dot{\varphi}^2 - \mu B_0(\alpha,\beta) - e\phi_0{}'(\alpha,\beta) - e\dot{\varphi}\psi(\alpha) + e\alpha\,\dot{\beta}\ , \tag{8}$$

where $\phi_0' \equiv \phi_0 + \alpha\,(\partial\beta/\partial t)$. (The time-dependence, e.g. $\mathcal{R}^2(\alpha,\beta;\ t)$, is here suppressed.)

The canonical angular momentum

$$p_\varphi \equiv \partial L/\partial\dot{\varphi} = \mathcal{R}^2(\alpha,\beta)\dot{\varphi} - e\,\psi(\alpha) \tag{9}$$

is invariant, since L is independent of φ. The momentum conjugate to β is, as usual,[10]

$$p_\beta \equiv \partial L/\partial\dot\beta = e\,\alpha . \tag{10}$$

The g.c. Hamiltonian is thus $[H_0 \equiv p_\beta\,\dot\beta + p_\varphi\,\dot\varphi - L]$

$$H_0(\alpha,\beta;\, p_\varphi;\, \mu;\, t) = \tfrac{1}{2}\,\mathcal{R}^{-2}(\alpha,\beta;\, t)[p_\varphi + e\psi(\alpha)]^2 + \mu\, B_0(\alpha,\beta;\, t) + e\,\phi_0'(\alpha,\beta;\, t) . \tag{11}$$

The magnetic moment μ is the action of the gyration (with the factor e); its conjugate angle-variable θ_g is the gyrophase, whose frequency $e\,\partial H_0/\partial\mu$ is the local gyrofrequency $e\,B_0(\alpha,\beta;\, t)/(mc)$. Thus for two degrees of freedom (gyration and azimuthal), the coordinates are ignorable. For the poloidal variables, the equations of motion are

$$\dot\beta = e^{-1}\,\partial H_0/\partial\alpha = (\iota/2\pi)\dot\varphi - e^{-1}\,\dot\varphi^2\,\mathcal{R}\,\partial\mathcal{R}/\partial\alpha + \mu e^{-1}\,\partial B_0/\partial\alpha + \partial\phi_0'/\partial\alpha , \tag{12}$$

$$\dot\alpha = -e^{-1}\,\partial H_0/\partial\beta = e^{-1}\,\dot\varphi^2\,\mathcal{R}\,\partial\mathcal{R}/\partial\alpha - \mu e^{-1}\,\partial B_0/\partial\beta - \partial\phi_0'/\partial\beta .$$

The first term of $\dot\beta$ represents the projection onto the poloidal plane of the $V_\parallel$-motion; if the other terms are negligible, $d\beta/d\varphi = \iota/2\pi$, so that in one azimuthal period $(\Delta\varphi = 2\pi)$, $\Delta\beta = \iota$, i.e., the g.c. follows the field line. The other terms represent the g.c. drifts, due to curvature, magnetic gradient, and electric field, respectively.

We now transform to action-angle variables $(J_P,\, \theta_P)$ for the poloidal motion (12) with the frozen Hamiltonian (11). That is, we solve (11) implicitly for $\alpha(\beta;\, H_0, p_\varphi, \mu;\, t)$, and define the poloidal action

$$2\pi\, J_P(H_0, p_\varphi, \mu;\, t) = \oint d\beta\, \alpha(\beta;\, H_0, p_\varphi, \mu;\, t). \tag{13}$$

This is thus the toroidal flux enclosed by the poloidal orbit or by the drift surface. For a circulating particle with negligible drift, it is just α. The angle variable θ_P satisfies $\dot\theta_P = \omega_P$, the poloidal frequency (which is the bounce frequency for a trapped particle). This frequency is given by

$$2\pi\, \omega_P^{-1} = \oint \frac{d\beta}{\dot\beta(\beta;\, H_0, p_\varphi, \mu;\, t)} , \tag{14}$$

where $\dot\beta$ is given by (12). [Note that in (12), $\dot\varphi$ is to be eliminated in favor of p_φ by (9), and α in favor of H_0 by (11).] The angle variable is thus defined by

$$\theta_P(\beta;\, H_0, p_\varphi, \mu;\, t) \equiv \omega_P(H_0, p_\varphi, \mu;\, t)\int_0^\beta \frac{d\beta'}{\dot\beta(\beta';\, H_0, p_\varphi, \mu;\, t)} .$$

This canonical transformation is accomplished by the generating function[11]

$$G(\theta_g, \varphi, \beta;\, \mu', p_\varphi', J_P;\, t) \equiv \theta_g\, \mu' + \varphi\, p_\varphi' + \int_0^\beta d\beta'\, \alpha(\beta';\, \mu', p_\varphi', J_P;\, t). \tag{15}$$

The time-dependence of G, caused by the quasistatic changes in the potentials, makes the transformed Hamiltonian slightly different from the Hamiltonian $H_0(\mu, p_\varphi, J_P;\, t)$ obtained from (13) by solving (implicitly) for H_0. However, this difference vanishes upon coarse-graining in time in the quasistatic limit, and will henceforth be neglected.

The new actions (μ', p_φ') are the same as the old (e.g., $p_\varphi = \partial G/\partial\varphi = p_\varphi'$), so we drop the primes. The new angle variables $(\theta_g \equiv \partial G/\partial\mu', \quad \Phi \equiv \partial G/\partial p_\varphi')$ are however different from the old, because of the final term of (15). Their meaning is easily seen from their equations of motion:

$$\dot{\theta}_g = e\partial H_0(\mu, p_\varphi, J_P; t)/\partial\mu \equiv \omega_g(\mu, p_\varphi, J_P; t),$$
$$\dot{\Phi} = \partial H_0(\mu, p_\varphi, J_P; t)/\partial p_\varphi \equiv \omega_\varphi(\mu, p_\varphi, J_P; t). \tag{16}$$

These frequencies vary only quasistatically; the β-dependence of $\dot{\theta}_g = eB_0$ and of $\dot{\varphi}$ [see (9)] is averaged over in (16). Thus ω_g is the mean gyrofrequency, and ω_φ the mean azimuthal frequency (i.e., the drift frequency for a banana). To be precise, e.g.,

$$\theta_g = \theta_g - \int_0^\beta \frac{d\beta'}{\dot{\beta}} \left[eB_0(\alpha', \beta') - e\bar{B}_0\right],$$

with

$$\bar{B}_0 \equiv \left(\oint \frac{d\beta}{\dot{\beta}} B_0\right)\left(\oint \frac{d\beta}{\dot{\beta}}\right)^{-1}.$$

In the following sections, we shall condense the notation as follows: $\underset{\sim}{J} \equiv (\mu/e, p_\varphi, J_P)$; $\underset{\sim}{\theta} \equiv (\theta_g, \Phi, \theta_P)$; $\underset{\sim}{\omega} \equiv (\omega_g, \omega_\varphi, \omega_P)$. Thus from $H_0(\underset{\sim}{J}; t)$, we have $\dot{\underset{\sim}{J}} = -\partial H_0/\partial\underset{\sim}{\theta} = 0$, indicating adiabatic invariance, while $\dot{\underset{\sim}{\theta}} = \partial H_0/\partial\underset{\sim}{J} = \underset{\sim}{\omega}(\underset{\sim}{J}; t)$.

III. CONDUCTIVITY KERNEL

Since the three actions $\underset{\sim}{J}$ are invariants of the quasistatic Hamiltonian H_0, the general quasistatic solution for the phase-space density is $f_0(\underset{\sim}{J})$. However, the quasilinear diffusion caused by the perturbing fields induces a quasistatic time-dependence $f_0(\underset{\sim}{J}; t)$, to be studied in Sec. V. The quasistatic variation will be suppressed in this section, which deals with the rapid variations of the perturbations.

For a particle at phase-point $(\underset{\sim}{J}, \underset{\sim}{\theta})$, the contribution to the current density at position $\underset{\sim}{x}$ is

$$\underset{\sim}{j}(\underset{\sim}{x} \mid \underset{\sim}{J}, \underset{\sim}{\theta}) \equiv e\, \underset{\sim}{v}(\underset{\sim}{J}, \underset{\sim}{\theta})\delta[\underset{\sim}{x} - \underset{\sim}{r}(\underset{\sim}{J}, \underset{\sim}{\theta})], \tag{17}$$

where $(\underset{\sim}{r}, \underset{\sim}{v})$ are the particle (not g.c) position and velocity corresponding to $(\underset{\sim}{J}, \underset{\sim}{\theta})$. Summing over all particles (with species summation implicit), we have the unperturbed current density

$$\langle \underset{\sim}{j} \rangle_0 (\underset{\sim}{x}) \equiv \int d^3J \int d^3\theta\, f_0(\underset{\sim}{J})\, \underset{\sim}{j}(\underset{\sim}{x} \mid \underset{\sim}{J}, \underset{\sim}{\theta}) \tag{18a}$$

and unperturbed charge density

$$\langle \rho \rangle_0 (\underset{\sim}{x}) \equiv \int d^3J \int d^3\theta\, f_0(\underset{\sim}{J})\, \rho(\underset{\sim}{x} \mid \underset{\sim}{J}, \underset{\sim}{\theta}). \tag{18b}$$

These are to be used in the Maxwell equations for the self-consistent unperturbed fields.

For the perturbed fields of the normal modes (Sec. IV), we need the perturbed current density

$$\delta\langle \underset{\sim}{j} \rangle (\underset{\sim}{x}, t) \equiv \int d^3J \int d^3\theta\, \delta f(\underset{\sim}{J}, \underset{\sim}{\theta}; t)\, \underset{\sim}{j}(\underset{\sim}{x} \mid \underset{\sim}{J}, \underset{\sim}{\theta}), \tag{19a}$$

or its Fourier transform

$$\delta\langle \mathbf{j} \rangle(\mathbf{x},\omega) \equiv \int d^3J \int d^3\theta \; \delta f(\mathbf{J},\boldsymbol{\theta};\omega)\, \mathbf{j}(\mathbf{x} \mid \mathbf{J},\theta). \qquad (19b)$$

Since the variables $\boldsymbol{\theta}$ are cyclic, any function of them can be expanded in a Fourier series:

$$g(\boldsymbol{\theta}) \equiv \sum_{\boldsymbol{\ell}}{}' g_{\boldsymbol{\ell}} \exp(i\, \boldsymbol{\ell} \cdot \boldsymbol{\theta}), \qquad (20)$$

$$g_{\boldsymbol{\ell}} \equiv \oint \frac{d^3\theta}{(2\pi)^3}\, g(\boldsymbol{\theta}) \exp(-i\, \boldsymbol{\ell} \cdot \boldsymbol{\theta}) .$$

Applying (20) to $\mathbf{j}$ and δf, we obtain for (19b)

$$\delta\langle \mathbf{j} \rangle(\mathbf{x},\omega) = (2\pi)^3 \int d^3J \sum_{\boldsymbol{\ell}} \mathbf{j}^*_{\boldsymbol{\ell}} (\mathbf{x}|\mathbf{J})\, \delta f_{\boldsymbol{\ell}}(\mathbf{J};\omega). \qquad (21)$$

The linearized Vlasov equation for δf is, in the $(\mathbf{J},\boldsymbol{\theta})$ variables,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\omega} \cdot \frac{\partial}{\partial \boldsymbol{\theta}}\right) \delta f(\mathbf{J},\boldsymbol{\theta};t) = -\,\delta\dot{\mathbf{J}} \cdot \frac{\partial f_0}{\partial \mathbf{J}} . \qquad (22)$$

To find $\delta\dot{\mathbf{J}}$, we need the perturbed Hamiltonian. As the exact particle (not g.c.) Hamiltonian is

$$H(\mathbf{p},\mathbf{r};t) = \frac{1}{2}[\mathbf{p} - e\,\mathbf{A}(\mathbf{r},t)]^2 + e\,\phi(\mathbf{r},t), \qquad (23)$$

its perturbation is

$$\begin{aligned}\delta H(\mathbf{p},\mathbf{r};t) &= -\,e\,\mathbf{v} \cdot \delta\mathbf{A}(\mathbf{r},t) + e\,\delta\,\phi(\mathbf{r},t)\\ &= -\int d^3x\; \mathbf{j}(\mathbf{x} \mid \mathbf{p},\mathbf{r}) \cdot \delta\mathbf{A}(\mathbf{x},t)\\ &\qquad + \int d^3x\; \rho(\mathbf{x} \mid \mathbf{p},\mathbf{r})\, \delta\phi(\mathbf{x},t).\end{aligned}$$

Transforming to $(\mathbf{J},\boldsymbol{\theta})$, this is

$$\begin{aligned}\delta H(\mathbf{J},\boldsymbol{\theta};t) &= -\int d^3x\; \mathbf{j}(\mathbf{x} \mid \mathbf{J},\boldsymbol{\theta}) \cdot \delta\mathbf{A}(\mathbf{x},t)\\ &\qquad + \int d^3x\; \rho(\mathbf{x} \mid \mathbf{J},\boldsymbol{\theta})\, \delta\phi(\mathbf{x},t).\end{aligned} \qquad (24a)$$

For time-dependent perturbations, we may choose the radiation gauge $\delta\phi \equiv 0$, whence $\delta\mathbf{E}(\mathbf{x},\omega) = i\omega\, \delta\mathbf{A}(\mathbf{x},\omega)$, and

$$\delta H(\mathbf{J},\boldsymbol{\theta};\omega) = i\,\omega^{-1} \int d^3x\; \mathbf{j}(\mathbf{x} \mid \mathbf{J},\boldsymbol{\theta}) \cdot \delta\mathbf{E}(\mathbf{x},\omega). \qquad (24b)$$

[For quasistatic perturbations, both terms of (24a) should be used.]

Now we can find

$$\delta\dot{\mathbf{J}} = -\,\partial\delta H/\partial\boldsymbol{\theta} = -\sum_{\boldsymbol{\ell}} i\,\boldsymbol{\ell}\; \delta H_{\boldsymbol{\ell}}(\mathbf{J};\omega) \exp(i\,\boldsymbol{\ell}\cdot\boldsymbol{\theta}), \qquad (25)$$

with

$$\delta H_{\boldsymbol{\ell}}(\mathbf{J};\omega) = i\,\omega^{-1} \int d^3x\; \mathbf{j}_{\boldsymbol{\ell}}(\mathbf{x} \mid \mathbf{J}) \cdot \delta\mathbf{E}(\mathbf{x},\omega). \qquad (24c)$$

In terms of Fourier components, the Vlasov equation (22) is

$$\delta f_{\boldsymbol{\ell}}(\mathbf{J};\omega) = [\boldsymbol{\ell}\cdot\boldsymbol{\omega}(\mathbf{J}) - \omega]^{-1}\, \delta H_{\boldsymbol{\ell}}(\mathbf{J};\omega)\boldsymbol{\ell} \cdot \partial f_0/\partial\mathbf{J} . \qquad (26)$$

Substituting this and (24c) into (21), we obtain the conductivity relation

$$\delta\langle \mathbf{j} \rangle(\mathbf{x},\omega) = \int d^3x'\, \boldsymbol{\sigma}(\mathbf{x},\mathbf{x}';\omega) \cdot \delta\mathbf{E}(\mathbf{x}',\omega), \qquad (27)$$

with

$$\underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}';\,\omega) = (2\pi)^3\, i\omega^{-1} \int d^3J \sum_{\underset{\sim}{\ell}} \underset{\sim}{j}^*_{\ell}(\underset{\sim}{x} \mid \underset{\sim}{J})\underset{\sim}{j}_{\ell}(\underset{\sim}{x}' \mid \underset{\sim}{J})$$

$$\times [\underset{\sim}{\ell} \cdot \underset{\sim}{\omega}(\underset{\sim}{J}) - \omega]^{-1}\, \underset{\sim}{\ell}\cdot\partial f_0/\partial \underset{\sim}{J} \,. \tag{28}$$

[To reduce this to the conductivity of a uniform field-free plasma, for purposes of comparison, let $\underset{\sim}{\ell} \to \underset{\sim}{k}$, $\underset{\sim}{J} \to \underset{\sim}{p}$, $\underset{\sim}{\omega} \to \underset{\sim}{v}$, and $\underset{\sim}{j}_{\ell}(\underset{\sim}{x} \mid \underset{\sim}{J}) \to e\, \underset{\sim}{v}\, \exp(-i\, \underset{\sim}{k}\cdot\underset{\sim}{x})$. Then $\underset{\approx}{\sigma}$ depends on $\underset{\sim}{x} - \underset{\sim}{x}'$, and (27) is a convolution integral.]

Formula (28) is meant to apply only for $\mathrm{Im}\,\omega > 0$, of course. In the limit $\mathrm{Im}\,\omega \to 0+$, the hermitian part of $\underset{\approx}{\sigma}$, which represents field dissipation, is

$$\underset{\approx}{\sigma}'(\underset{\sim}{x},\underset{\sim}{x}';\,\omega) = (2\pi)^3\, \omega^{-1} \int d^3J \sum_{\underset{\sim}{\ell}} \underset{\sim}{j}^*_{\ell}(\underset{\sim}{x} \mid \underset{\sim}{J})\underset{\sim}{j}_{\ell}(\underset{\sim}{x}' \mid \underset{\sim}{J})$$

$$\times (-\underset{\sim}{\ell} \cdot \partial f_0/\partial \underset{\sim}{J})\, \pi\, \delta(\omega - \underset{\sim}{\ell}\cdot\underset{\sim}{\omega}). \tag{29}$$

Dissipation means energy transfer from fields to particles; from (29), we see that this requires the resonance condition

$$\omega = \underset{\sim}{\ell} \cdot \underset{\sim}{\omega}(\underset{\sim}{J}) \equiv \ell_g\, \omega_g(\underset{\sim}{J}) + \ell_\varphi\, \omega_\varphi(\underset{\sim}{J}) + \ell_P\, \omega_P(\underset{\sim}{J}), \tag{30}$$

where $(\ell_g, \ell_\varphi, \ell_P)$ are three integers. (This is the generalization of $\omega = \underset{\sim}{k} \cdot \underset{\sim}{v}$ in the uniform plasma.) Note that this condition for energy transfer involves the mean frequencies $(\dot{\bar{\theta}}_g, \dot{\bar{\phi}})$, not the local ones $(\dot{\theta}_g, \dot{\varphi})$. That this is appropriate can be seen by considering the total change in, say, the gyrophase θ_g over several poloidal periods.

IV. NORMAL MODES

The linearized Maxwell equations are

$$\nabla \times \delta\underset{\sim}{E}(\underset{\sim}{x},\omega) + i\omega\, \delta\underset{\sim}{B}(\underset{\sim}{x},\omega) = 0,$$

$$\nabla \times \delta\underset{\sim}{B}(\underset{\sim}{x},\omega) - i\omega\, \delta\underset{\sim}{E}(\underset{\sim}{x},\omega) = 4\pi \int d^3x'\, \underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}';\,\omega) \cdot \delta\underset{\sim}{E}(\underset{\sim}{x},\omega).$$

They may be written concisely as

$$\underset{\approx}{K}(\omega) \cdot \delta\underset{\sim}{E}(\underset{\sim}{x},\omega) = -\,\delta\underset{\sim}{E}(\underset{\sim}{x},\omega)\,, \tag{31}$$

where the integro-differential operator $\underset{\approx}{K}(\omega)$ is defined by

$$\underset{\approx}{K}(\omega) \cdot \underset{\sim}{F}(\underset{\sim}{x}) \equiv -\,\omega^{-2}\, \nabla \times (\nabla \times \underset{\sim}{F}) + 4\pi i\omega^{-1} \int d^3x'\, \underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}';\,\omega)\cdot\underset{\sim}{F}(\underset{\sim}{x}'). \tag{32}$$

We assume that $\underset{\approx}{K}(\omega)$ is nearly a hermitian operator, so that the eigenfrequencies of (31) are nearly real. In Sec. III of Ref. 12, we have shown: (1) that the real parts ω_a of the eigenfrequencies, and the zero-order eigenfunctions $\underset{\sim}{E}^a(\underset{\sim}{x})$, are the solutions of

$$\underset{\approx}{K}'(\omega_a) \cdot \underset{\sim}{E}^a(\underset{\sim}{x}) = -\,\underset{\sim}{E}^a(\underset{\sim}{x}), \tag{33}$$

where $\underset{\approx}{K}'(\omega)$ is the hermitian part of $\underset{\approx}{K}(\omega)$, for ω real; (2) that the growth rates γ_a of the normal modes are given by

$$\gamma_a = -(4\pi)^{-1}\, \sigma_a\, \omega_a \int d^3x\, \underset{\sim}{E}^{a*}(\underset{\sim}{x}) \cdot \underset{\approx}{K}''(\omega_a) \cdot \underset{\sim}{E}^a(\underset{\sim}{x}), \tag{34}$$

where $\underset{\approx}{K}''(\omega)$ is the antihermitian part of $\underset{\approx}{K}(\omega)$, and the eigenfunctions are normalized to unit magnitude of wave energy:

$$(4\pi)^{-1}\,\omega_a \int d^3x\, \underset{\sim}{E}^{a*}(\underset{\sim}{x}) \cdot \partial \underset{\approx}{K}'(\omega)/\partial\omega\Big|_{\omega_a} \cdot \underset{\sim}{E}^{a}(\underset{\sim}{x}) \;=\; \sigma_a \;\equiv\; \pm 1\,. \tag{35}$$

The method of Landau and Lifshitz[13] may be used to show that for a normal mode $\underset{\sim}{E}(\underset{\sim}{x},t) = \mathcal{A}_a(t)\underset{\sim}{E}^a(\underset{\sim}{x})\exp(-i\omega_a t) + \text{c.c.}$, with the normalization (35), the wave energy is $W_a(t) = \sigma_a|\mathcal{A}_a(t)|^2$, and evolves at the rate

$$\frac{dW_a}{dt} = 2\,\gamma_a W_a\,. \tag{36}$$

To evaluate γ_a by (34), we express $\underset{\approx}{K}''$ in terms of $\underset{\approx}{\sigma}'$ by (32):

$$\gamma_a = -\sigma_a \iint d^3x\, d^3x'\, \underset{\sim}{E}^{a*}(\underset{\sim}{x}) \cdot \underset{\approx}{\sigma}'(\underset{\sim}{x},\underset{\sim}{x}';\omega_a) \cdot \underset{\sim}{E}^{a}(\underset{\sim}{x}),$$

and use (29) for $\underset{\approx}{\sigma}'$. We then find

$$\gamma_a = (2\pi)^3\,\sigma_a\,\omega_a^{-1}\int d^3J \sum_{\underset{\sim}{\ell}} \pi\,\delta[\omega_a - \underset{\sim}{\ell}\cdot\underset{\sim}{\omega}(\underset{\sim}{J})]\;\underset{\sim}{\ell}\cdot\frac{\partial f_0}{\partial \underset{\sim}{J}}\,\alpha_{\underset{\sim}{\ell}}{}^a(\underset{\sim}{J}), \tag{37}$$

where

$$\alpha_{\underset{\sim}{\ell}}{}^a(\underset{\sim}{J}) \equiv \left|\int d^3x\, \underset{\sim}{E}^a(\underset{\sim}{x}) \cdot \underset{\sim}{j}_{\underset{\sim}{\ell}}(\underset{\sim}{x}\,|\,\underset{\sim}{J})\right|^2 \tag{38}$$

represents the overlap between the eigenfunction of mode a and the ℓth Fourier amplitude of the particle current density for action $\underset{\sim}{J}$. For each positive frequency ω_a, there is a negative one $-\omega_a$ with the same growth rate [for $\omega_a \to -\omega_a$, let $\underset{\sim}{\ell} \to -\underset{\sim}{\ell}$ in (37)]; these are physically the same mode, so we may restrict our attention to positive ω_a. We note, but do not indicate explicitly, that $\alpha_{\underset{\sim}{\ell}}{}^a(\underset{\sim}{J})$ is nonvanishing only for those $\underset{\sim}{\ell}$ having ℓ_φ equal to the azimuthal mode number of $\underset{\sim}{E}^a(\underset{\sim}{x})$.

V. QUASILINEAR DIFFUSION

The continuity equation for $f(\underset{\sim}{J},\underset{\sim}{\theta};t)$ is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \underset{\sim}{\theta}}\cdot(\dot{\underset{\sim}{\theta}}\,f) + \frac{\partial}{\partial \underset{\sim}{J}}\cdot(\dot{\underset{\sim}{J}}\,f) = 0\,.$$

Upon averaging over $\underset{\sim}{\theta}$ (denoted here by $\langle\;\rangle$), this yields

$$\frac{\partial\langle f\rangle}{\partial t}(J;t) = -\frac{\partial}{\partial \underset{\sim}{J}}\cdot\langle\delta\dot{\underset{\sim}{J}}\,\delta f\rangle\,, \tag{39}$$

where δf is, to lowest order, the linear solution of Sec. III. It is here more convenient to write the solution of (22) as

$$\delta f(\underset{\sim}{J},\underset{\sim}{\theta};t) = -\int_0^t d\tau\;\delta\dot{\underset{\sim}{J}}(\underset{\sim}{J},\underset{\sim}{\theta} - \underset{\sim}{\omega}\tau;\,t-\tau)\cdot\partial f_0/\partial\underset{\sim}{J}\,, \tag{40}$$

with the usual dropping of the initial value term. Substituting this into (39), we obtain the diffusion-like equation:

$$\frac{\partial\langle f\rangle(\underset{\sim}{J};t)}{\partial t} = \frac{\partial}{\partial\underset{\sim}{J}}\cdot\left[\underset{\approx}{D}'(\underset{\sim}{J};t)\cdot\frac{\partial f_0}{\partial\underset{\sim}{J}}\right], \tag{41}$$

where

$$\underset{\approx}{D}'(\underset{\sim}{J};t) \equiv \int_0^\infty d\tau\langle\delta\dot{\underset{\sim}{J}}(\underset{\sim}{J},\underset{\sim}{\theta};t)\;\delta\dot{\underset{\sim}{J}}(\underset{\sim}{J},\underset{\sim}{\theta}-\underset{\sim}{\omega}\tau;\,t-\tau)\rangle, \tag{42}$$

and the usual Markov assumption has been made to justify the evaluation of f_0 at t and the extension of the upper limit to ∞.

To evaluate (42), we use (25) and (24a) for each factor $\delta\dot{\underset{\sim}{J}}$:

$$\underset{\approx}{D}'(\underset{\sim}{J};t) = \sum_{\underset{\sim}{\ell}} \underset{\sim}{\ell}\,\underset{\sim}{\ell} \int d^3x \int d^3x'\; \underset{\sim}{j}_{\underset{\sim}{\ell}}{}^{*}(\underset{\sim}{x}|\underset{\sim}{J})\; \underset{\sim}{j}_{\underset{\sim}{\ell}}(\underset{\sim}{x}'|\underset{\sim}{J}) :$$

$$\int_0^\infty d\tau\; \delta\underset{\sim}{A}(\underset{\sim}{x},t)\; \delta\underset{\sim}{A}(\underset{\sim}{x}',\, t-\tau)\exp(-i\,\underset{\sim}{\ell}\cdot\underset{\sim}{\omega}\tau). \tag{43}$$

We assume the perturbation to be a superposition of normal modes:

$$\delta\underset{\sim}{A}(\underset{\sim}{x},t) = \sum_a (i\omega_a)^{-1}\, \mathcal{a}_a(t)\, \underset{\sim}{E}^a(\underset{\sim}{x})\, \exp(-i\omega_a t) + \text{c.c.} \tag{44}$$

In evaluating (44) at $(t-\tau)$, we take the slowly varying amplitude at t, and call its contribution to (43) $\underset{\approx}{D}(\underset{\sim}{J};t)$ (without the prime). The correction can be shown[14] to contribute a reversible change to $\langle f \rangle$, of second order in the amplitudes; it may be interpreted as the particle contribution to the normal modes and is already included in the wave energy through (35). The zero-order part of $\langle f \rangle$ is f_0, which thus evolves as

$$\frac{\partial f_0(\underset{\sim}{J};\, t)}{\partial t} = \frac{\partial}{\partial \underset{\sim}{J}} \cdot \left[\underset{\approx}{D}(\underset{\sim}{J};\, t)\cdot \frac{\partial f_0}{\partial \underset{\sim}{J}}\right], \tag{45}$$

with $\underset{\approx}{D}$ evaluated from (43), (44), and (38) (and using $\underset{\sim}{j}_{-\underset{\sim}{\ell}} = \underset{\sim}{j}_{\underset{\sim}{\ell}}{}^{*}$):

$$\underset{\approx}{D}(\underset{\sim}{J};\, t) = \sum_a |W_a(t)|\, \omega_a^{-2} \sum_{\underset{\sim}{\ell}} \underset{\sim}{\ell}\,\underset{\sim}{\ell}\; 2\pi\, \delta[\omega_a - \underset{\sim}{\ell}\cdot\underset{\sim}{\omega}(\underset{\sim}{J})]\alpha_{\underset{\sim}{\ell}}{}^{a}(\underset{\sim}{J}). \tag{46}$$

The diffusion tensor (46) is a singular tensor field in the three-dimensional $\underset{\sim}{J}$-space, being nonzero only on the set of resonant surfaces $\{\underset{\sim}{\ell}\cdot\underset{\sim}{\omega}(\underset{\sim}{J}) = \omega_a\}$. Suppose that the number of these surfaces is large, so that the cells in action-space enclosed by them are small. Then small nonlinear effects[5] serve to spread out the delta-functions in $\underset{\approx}{D}$, so as to make $\underset{\approx}{D}$ a continuous tensor field.

A detailed study of the spreading and overlap of resonances has been made by Rosenbluth, Sagdeev, Taylor, and Zaslavski[6] for the problem of field line diffusion due to static magnetic perturbations. The entirely analogous problem of g.c. diffusion due to static electric or magnetic perturbations may be treated by using (24a) for δH, with no time-dependence, in evaluating (42), and setting $\ell_g = 0$ (so that magnetic moment is conserved). The resonance condition (1) then reduces to

$$\ell_\varphi\, \omega_\varphi(p_\varphi,\, J_P) + \ell_P\, \omega_P(p_\varphi,\, J_P) = 0\,,$$

determining a set of curves in the two-dimensional $p_\varphi - J_P$ space. As particle energy is conserved by a static perturbation, the g.c. random walk in this space must remain in the neighborhood of the curve $H_0(p_\varphi,\, J_P) =$ const. Only at the points of intersection of the resonance curves with the energy curve can diffusion occur. Hence a necessary condition for net diffusion is that the resonance widths overlap. Incidentally, we note that the case $\omega_a = 0$, $\ell_g \neq 0$ represents Arnol'd diffusion.[15]

Returning to the three-dimensional diffusion with time-dependent perturbation, we derive the entropy theorem. Defining

$$S(t) \equiv -\int d^3J \quad f_0(\underset{\sim}{J};\, t)\, \ln f_0 ,$$

we find

$$\dot{S}(t) = \int d^3J\ f_0^{-1}(\partial f_0/\partial \underset{\sim}{J})(\partial f_0/\partial \underset{\sim}{J}) : \underset{\approx}{D}(\underset{\sim}{J};t)$$

$$= \int d^3J\ f_0^{-1} \sum_a |W_a|\omega_a^{-2} \sum_{\underset{\sim}{\ell}} (\underset{\sim}{\ell}\cdot\partial f_0/\partial \underset{\sim}{J})^2 2\pi\delta(\omega_a - \underset{\sim}{\ell}\cdot\underset{\sim}{\omega})\alpha_{\underset{\sim}{\ell}}^{\ a}(\underset{\sim}{J}).$$

Since $\dot{S}$ is non-negative, the plasma continues to diffuse indefinitely, so long as the resonant surfaces in the action-space are populated. In the unlikely event that only the nonresonant cells bounded by the resonant surfaces are occupied, one can appeal to a number of effects (finite resonance widths, nonresonant instabilities, collisions) to repopulate the surfaces. Of course, if the plasma is stable, so that the wave energies vanish, there would be no quasilinear diffusion.

VI. ENERGY CONSERVATION

The diffusion equation (45), with the diffusion tensor (46), implies a conservation law which will be derived in this section. Let us consider the quantity

$$\mathcal{H}(t) \equiv \int d^3\theta \int d^3J \quad H_0(\underset{\sim}{J};\, t)\ f_0(\underset{\sim}{J};\, t), \tag{47}$$

which represents the sum of the unperturbed particle kinetic energies, twice the Coulomb interaction energy, and the Coulomb energy of interaction with the external Coulomb potential (if any). (As before, species summation is implicit.) The quasistatic

Hamiltonian H_0 changes adiabatically, as f_0 diffuses due to particle-wave interaction.

The rate of change of $\mathcal{H}$ due directly to the diffusion of f_0 is

$$(\dot{\mathcal{H}})_f = (2\pi)^3 \int d^3J \quad H_0 \frac{\partial}{\partial \underset{\sim}{J}} \cdot \left(\underset{\approx}{D} \cdot \frac{\partial f_0}{\partial \underset{\sim}{J}}\right)$$

$$= -(2\pi)^3 \int d^3J \quad \underset{\sim}{\omega} \cdot \underset{\approx}{D} \cdot \partial f_0/\partial \underset{\sim}{J} .$$

We substitute (46) for $\underset{\approx}{D}$, and find

$$(\dot{\mathcal{H}})_f = -\sum_a 2\,\gamma_a W_a ,$$

where formula (37) has been used for γ_a. Thus, by (36),

$$(\dot{\mathcal{H}})_f = -\frac{d}{dt}\sum_a W_a . \tag{48}$$

To evaluate the change of $\mathcal{H}$ due to the time-varying quasistatic Hamiltonian, it is more convenient to express (47) temporarily as

$$\mathcal{H}(t) = \int d^3r \int d^3p\ H_0(\underset{\sim}{r},\underset{\sim}{p};\, t)\ f_0(\underset{\sim}{r},\underset{\sim}{p};\, t),$$

where H_0 is given by (23), with $\underset{\sim}{A}_0$, ϕ_0. Then

$$(\dot{\mathcal{H}})_H = \int d^3r \int d^3p \ \frac{\partial H_0}{\partial t}\ f_0$$

$$= \int d^3\theta \int d^3J\ f_0(\underset{\sim}{J};\, t)\left\{-\int d^3x\ \underset{\sim}{j}(\underset{\sim}{x}|\underset{\sim}{J},\underset{\sim}{\theta}) \cdot \frac{\partial \underset{\sim}{A}_0}{\partial t}(\underset{\sim}{x},t) + \int d^3x\ \rho(\underset{\sim}{x}|\underset{\sim}{J},\underset{\sim}{\theta})\frac{\partial \phi_0}{\partial t}(\underset{\sim}{x},t)\right\},$$

by the analog of (24a). Thus, by (18),

$$(\dot{\mathcal{H}})_H = -\int d^3x \, \langle \mathbf{j} \rangle_0 (\mathbf{x},t) \cdot \frac{\partial \mathbf{A}_0}{\partial t} (\mathbf{x},t)$$

$$+ \int d^3x \, \langle \rho \rangle_0 (\mathbf{x},t) \frac{\partial \phi_0}{\partial t} (\mathbf{x},t) \, . \qquad (49)$$

Assuming that the plasma is quasi-neutral, the second term of (49) may be neglected. (If not, the succeeding formulas are easily modified.) The first term then yields

$$(\dot{\mathcal{H}})_H = \int d^3x \, \langle \mathbf{j} \rangle_0 (\mathbf{x},t) \cdot \mathbf{E}_0 (\mathbf{x},t)$$

$$= \frac{1}{4\pi} \int d^3x \, \mathbf{E}_0 \cdot \nabla \times \mathbf{B}_0 - \int d^3x \, \mathbf{j}^{ex} \cdot \mathbf{E}_0 \, , \qquad (50)$$

noting that the field $\mathbf{B}_0$ is produced by external currents $\mathbf{j}^{ex}$ and plasma currents $\langle \mathbf{j} \rangle_0$. Since radiation by quasistatic fields is negligible, the first term of (50) can be written as

$$\frac{1}{4\pi} \int d^3x \, \mathbf{B}_0 \cdot \nabla \times \mathbf{E}_0 = -\frac{d}{dt} \int d^3x \, [\mathbf{B}_0(\mathbf{x},t)]^2 / 8\pi \, . \qquad (51)$$

Combining (48), (50), and (51), we have

$$\frac{d}{dt} \left[\mathcal{H}(t) + \sum_a W_a + \int d^3x \, \frac{|\mathbf{B}_0|^2}{8\pi} \right] = -\int d^3x \, \mathbf{j}^{ex} \cdot \mathbf{E}_0 \, . \qquad (52)$$

This equation expresses the rate at which the "system energy" changes due to interaction with quasistatic external currents. The energy consists of three terms: (1) the unperturbed particle kinetic energies $\mathcal{H}$ (note that the Coulomb energies effectively cancel out by quasi-neutrality); (2) the wave

energies W_a, which include the perturbed field energies and the perturbed particle kinetic energies; (3) the unperturbed quasi-static magnetic field energy (the electric field energy being negligible).

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