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SCALING INVARIANCE OF THE MOTION OF HELICAL CURVES AND SOLITON EQUATIONS*

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Abstract

The scaling properties of the equations describing the motion of helical curves determine the scaling of the associated nonlinear evolution equations. Only a limited number of scaling-invariant evolution equations can be found.

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Hasimoto[1] has found an interesting connection between the nonlinear Schroedinger equation and the motion of helical curves. A generalization to the sine-Gordon and modified Korteweg-de Vries equations, and the relation to the associated inverse scattering formalism, was pointed out by Lamb.[2] He conjectures that other equations exhibiting soliton behaviour may also be related to the motion of helical space curves.

It is shown here that invariance under a scaling transformation[3] of the Serret-Frenet equations, which describe the spatial variations of a twisted curve, determines the scaling of the dependent variable in the scaling-invariant nonlinear evolution equation. Scaling of the assumed time dependence of the helix yields the scaling of time in the evolution equation. This equation is then completely determined by its scaling invariance properties. It is now possible to check whether the connection between helical curves and nonlinear evolution equations extends to other soliton equations.

The association of nonlinear equations with helical motion proceeds as follows.[1,2]

The Serret-Frenet equations are

$$\hat{t}_s = \kappa\hat{n}, \tag{1a}$$

$$\hat{b}_s = -\tau\hat{n}, \tag{1b}$$

$$\hat{n}_s = \tau\hat{b} - \kappa\hat{t}, \tag{1c}$$

where the subscript denotes partial differentiation with respect to the arc length s, and the functions $\kappa(s,t)$ and $\tau(s,t)$ are curvature and torsion respectively, which also depend on the time t. The tangent vector $\hat{t}$ is defined by the derivative of the position vector $\vec{X}(s,t)$,

$$\hat{t} \equiv \vec{X}_s(s,t), \tag{1d}$$

while $\hat{n}$ and $\hat{b}$ are the normal and binormal to the curve. With introduction of the complex vector $\vec{N}(s,t)$,

$$\vec{N} \equiv (\hat{n} + i\hat{b})\exp[i\int_{-\infty}^{s} ds'\,(\tau - \tau_0)] \,, \tag{2a}$$

and the complex scalar

$$\psi \equiv \kappa \exp\,[i\int_{-\infty}^{s} ds'\,(\tau - \tau_0)] \,, \tag{2b}$$

where τ_0 is the asymptotic value of the torsion, as $|s| \to \infty$, combination of eqs. (1a) - (1c) yields

$$\vec{N}_s + i\tau_0\vec{N} = -\psi\hat{t}, \tag{3a}$$

$$\hat{t}_s = \tfrac{1}{2}(\psi^*\vec{N} + \psi\vec{N}^*) \,. \tag{3b}$$

We will assume that $\psi \to 0$ as $|s| \to \infty$. The function ψ will be the dependent variable in the nonlinear evolution equation.

The norm-preserving variation of $\vec{N}$ and $\hat{t}$ in time, on the other hand, can be written as[1]

$$\vec{N}_t = iR\vec{N} + \gamma\hat{t} \,, \tag{4a}$$

$$\hat{t}_t = -\tfrac{1}{2}(\gamma^*\vec{N} + \gamma\vec{N}^*) \,, \tag{4b}$$

where $R(s,t)$ is real, and $\gamma(s,t)$ complex, while the time dependence of the position vector $\vec{X}$ can be expressed as

$$\vec{X}_t \equiv C^*\psi^*\vec{N} + C\psi\vec{N}^* + \theta\hat{t} \,, \quad C \equiv \tfrac{1}{2}(\zeta + i\eta) \,, \tag{4c}$$

where ζ, η, and θ are real functions of s and t yet to be determined.

The evolution equation for ψ is obtained by equating mixed second derivatives of $\vec{N}$ from (3) and (4),

$$\psi_t + \gamma_s + i(\tau_0\gamma - R\psi) = 0 , \tag{5a}$$

$$R_s = \tfrac{1}{2}i(\gamma\psi^* - \gamma^*\psi) . \tag{5b}$$

Furthermore, use of $\vec{X}_{st} = \vec{X}_{ts}$ gives[2]

$$-\tfrac{1}{2}\gamma = (C\psi)_s + i\tau_0 C\psi + \tfrac{1}{2}\theta\psi , \tag{6a}$$

$$\theta_s = \zeta|\psi|^2 , \tag{6b}$$

and using eq. (2b),

$$R_s = (\eta|\psi|^2)_s - \tfrac{1}{2}\eta|\psi|_s^2 - \theta_s\tau . \tag{6c}$$

The linear inverse scattering equations, which allow us to solve eq. (5a) for ψ analytically, follow from (3) and (6) as shown by Lamb.

Scaling invariance of an equation or function is defined[3] as invariance under the transformation from unprimed to primed variables:

$$s' = \alpha s , \tag{7a}$$

$$t' = \alpha^h t, \tag{7b}$$

and a corresponding transformation of the dependent variable. The scaling variable α and the exponent h are real. It is clear from eq. (1) that κ and τ scale as s^{-1}, and consequently the dependent variable ψ scales as

$$\psi(s,t) = \alpha\psi'(s',t') , \tag{7c}$$

while $\hat{n}$, $\hat{b}$, and thus $\vec{N}$, are invariant. This scaling of ψ prevents the Korteweg-de Vries equation $\psi_t + \psi\psi_x + \psi_{xxx} = 0$, for example, from being connected to helical curve motion. This equation is scaling-invariant when $\psi = \alpha^m\psi'$, with $m = 2$.

It is reasonable to require further that the defining relations (4) are also invariant under scaling. Then the other functions scale as

$$R(s,t) = \alpha^h R'(s',t'),\ \gamma(s,t) = \alpha^h \gamma'(s',t'), \tag{8a}$$

$$\eta(s,t) = \alpha^{h-2}\eta'(s',t'),\ \zeta(s,t) = \alpha^{h-2}\zeta'(s',t'), \tag{8b}$$

$$\theta(s,t) = \alpha^{h-1}\theta'(s',t') . \tag{8c}$$

The desired evolution equation (5a) must also be scaling invariant, since it follows from scaling-invariant equations (1) and (4). When we specify R, γ, η, ζ and θ as functionals of ψ, then eq. (5a) becomes a closed nonlinear equation of evolution for ψ. A further restriction, suggested by eq. (6), is to choose those functionals to be polynomials of ψ and its s-derivatives, with τ_0 appearing as a parameter, and the coefficients independent of s or t. Consequently, the evolution equation is also a polynomial. Each term of these polynomials must have the same scaling factor as that polynomial.

Indeed, the choice $h = 2$ and $h = 3$ respectively yield the nonlinear Schroedinger equation $i\psi_t + 2\psi_{ss} + |\psi|^2\psi = 0$, and the modified Korteweg-de Vries equation $\psi_t + \frac{3}{2}\psi^2\psi + \psi_{sss} = 0$. It is clear by inspection that the scaling (7) leaves these equations invariant. Furthermore, the functions R and γ are invariant: for example, for $h = 2$ we have[2] $R = |\psi|^2 - 2\tau_0^2$, and $\gamma = 2i\psi_s - 2\tau_0\psi$.

We now attempt to find the evolution equation with $h = 4$. We start with ζ with scaling power $h-2 = 2$. The most general form is a sum of all possible real terms, each with scaling factor α^2:

$$\zeta = c_1|\psi|^2 + c_2\tau_0^2 + c_3\tau_0(\psi + \psi^*) + ic_4\tau_0(\psi - \psi^*) + c_5(\psi_s + \psi_s^*)$$

$$+ ic_6(\psi_s - \psi_s^*) , \tag{9}$$

where the coefficients c_1 to c_6 are arbitrary real numbers, to be determined by eqs. (6b), (6a), and (5a). (A term such as $\tau_0^{-1}\psi_{ss}$ is excluded because τ_0 must be allowed to take any real value, including zero.) Eq. (6b) implies

$$\int_{-\infty}^{\infty} \zeta|\psi|^2 \alpha s = 0 , \tag{10}$$

for arbitrary ψ. Thus the coefficients c_1 to c_6 all vanish, as none of the terms cancel, or can be integrated to zero. Consequently, $\zeta = 0 = \theta_s$, but as the s-dependence in θ is only through $\psi(s,t)$, θ is independent of ψ, and can depend only on the constant parameter τ_0. Because θ scales with exponent $h-1 = 3$ [eq. (8c)], the most general form of θ is $\theta = a\tau_0^3$, where a is an arbitrary real constant. The functions η and R, with scaling exponents $h-2 = 2$ and $h = 4$ respectively, follow in a similar way from eq. (6c) as

$$\eta = b\,|\psi|^2 + c\tau_0^2 , \tag{11a}$$

$$R = \frac{3}{4}b\,|\psi|^4 + \frac{c}{2}\tau_0^2\,|\psi|^2 . \tag{11b}$$

Eq. (6a) now yields for γ:

$$\gamma = -ib(|\psi|^2\psi)_s - ic\,\tau_0^2\,\psi_s$$

$$+ \tau_0 b\,|\psi|^2\psi + (c - a)\tau_0^3\psi . \tag{11c}$$

Substituting R and γ in eq. (5a) yields

$$\psi_t - ib(|\psi|^2\psi)_{ss} - \frac{3}{4}b|\psi|^4\psi + 2b\,\tau_0(|\psi|^2\psi)_s + \tau_0^2[-ic\,\psi_{ss} + (ib - \frac{c}{2})|\psi|^2\psi]$$

$$+ (2c - a)\tau_0^3\psi_s + i(c - a)\tau_0^4\psi = 0 . \qquad (12)$$

The parameter τ_0 is the eigenvalue of the linear scattering equations,[2] and has to be determined by them. Consequently we must choose the constants a, b and c such that τ_0 disappears from eq. (12); hence the trivial result $a = b = c = 0$, $\psi_t = 0$.

Proceeding in a similar way the case $h = 5$ again yields the trivial equation $\psi_t = 0$. Going beyond $h = 5$ is increasingly tedious, as the number of terms in equations like eq. (9) increases rapidly, but our calculations suggest that no equations soluble by inverse scattering methods exist for $h \geq 4$.

In conclusion, it is suggested that only previously found[2] non-linear evolution equations for ψ with $h = 2$ or 3, can be connected to the motion of helical curves.

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