SOME PROPERTIES OF CAPILLARY SURFACES ON ELLIPTICAL DOMAINS

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ABSTRACT

In this report we show there will exist a critical contact angle for an elliptical cross section if the ratio b/a of the minor and major semiaxes is less than 0.6116. (There is no solution of the capillary surface Eqs. (1) and (2) for contact angles less than the critical angle.) We also calculate a lower bound on γ_{crit} for various values of b/a. These bounds appear to be close to the values of γ_{crit} found by numerical solution of Eqs. (1) and (2).

1. DEFINITION OF THE PROBLEM

We consider the equilibrium free surface of a liquid partly filling a vertical cylinder for the case in which the surface height $u(x,y)$ is a single-valued smooth function of x and y, and in which there is sufficient liquid to cover the base of the cylinder entirely. The gravitational field is taken to be positive when directed vertically downward. The height then satisfies the equation

$$\nabla \cdot \left(\frac{1}{W} \nabla u\right) = \kappa u + 2H \tag{1}$$

$$W = (1 + |\nabla u|^2)^{1/2},$$

where ∇ is the two-dimensional operator $(\partial/\partial x, \partial/\partial y)$, $\kappa = \rho g/\sigma$ is the capillary constant, ρ is the difference in densities between the liquid and gas phases, g is the acceleration due to gravity, and σ is the gas-liquid surface tension. The constant $2H$ is determined by the cross-sectional shape of the cylinder, the volume of liquid, and the boundary condition satisfied by the free surface of the liquid at the cylinder wall.

The boundary condition for a free surface that makes a contact angle γ with the cylinder wall is

$$\frac{1}{W}\frac{\partial u}{\partial n} = \cos\gamma \quad \text{at the wall,} \tag{2}$$

where $\partial u/\partial n$ denotes the derivative of u with respect to the outward directed normal at the wall. Only the case of wetting liquids $0^\circ \leqslant \gamma < 90^\circ$, will be considered here. (The nonwetting case can be obtained from it directly by means of a simple transformation.)

2. CRITERION FOR THE EXISTENCE OF A CRITICAL CONTACT ANGLE FOR ELLIPTICAL CROSS SECTIONS

Let A denote the area and L the perimeter of the cross section of a cylinder. The following theorem holds [1,2].

<u>Theorem</u>. Suppose there is a point p on the boundary at which the curvature is greater than L/A; then there exists a critical contact angle such that there is no solution of Eqs. (1) and (2) in a neighborhood of p for $0 \leq \gamma < \gamma_{crit}$.

We shall apply this theorem to an elliptical cross section. For this application we must calculate the area and perimeter of an ellipse and the curvature at any point on it. The area of an ellipse is

$$A = \pi ab \tag{3}$$

where a and b are the semimajor and semiminor axes.

The ellipse is described by the equation

$$y(x) = b\,(1 - x^2/a^2)^{1/2} \quad . \tag{4}$$

The perimeter of the ellipse is

$$L = 4 \int_0^a [1 + (y')^2]^{1/2}\, dx$$

$$= 4 \int_0^a \frac{[1 - (1 - b^2/a^2)\, x^2/a^2]^{1/2}}{(1 - x^2/a^2)^{1/2}}\, dx \quad .$$

Let $x/a = \sin\theta$, and

$$m = 1 - b^2/a^2 \quad . \tag{5}$$

Then

$$L = 4aE(m) \quad , \tag{6}$$

where

$$E(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{\frac{1}{2}} \, d\theta \tag{7}$$

is the complete elliptic integral of the second kind. Values of L/A for $a = 1$ and various values of b are given in Table I to the indicated number of decimal places.

Table I. Values of L/A for $a = 1$ and various values of b.

b	m	L/A
0.2	0.96	6.688
0.25	0.9375	5.461
0.33	0.8911	4.290
0.4	0.84	3.663
0.5	0.75	3.084
0.6	0.64	2.708

These will be used in a later section of this report.

Next we shall calculate the curvature. The differential of arc length is

$$ds = [1 + (y')^2]^{1/2} \, dx \quad .$$

The unit vector $\hat{t}$ tangent to the curve $y(x)$ in two dimensions is

$$\begin{aligned} \hat{t} &= (dx/ds, \, dy/ds) \\ &= [1 + (y')^2]^{-1/2} \quad (1, y') \quad . \end{aligned} \tag{8}$$

The curvature C and the unit vector $\hat{n}$ normal to the curve $y(x)$ are defined by

$$C\hat{n} = d\hat{t}/ds$$

$$d\hat{t}/ds = [1 + (y')^2]^{-1/2} \quad d\hat{t}/dx \tag{9}$$

$$= y'' \; [1 + (y')^2]^{-2} \quad (-y',1) \; .$$

Therefore

$$\hat{n} = \mathrm{sgn}\,(y'') \; [1 + (y')^2]^{-1/2} \quad (-y',1) \tag{10}$$

and

$$C = |y'' \; [1 + (y')^2]^{-3/2}| \; . \tag{11}$$

Note that C also can be written

$$C = \left| \frac{d}{dx} \left\{ y' \; [1 + (y')^2]^{1/2} \right\} \right| \; . \tag{12}$$

Let $y(x)$ be an ellipse. Then

$$y' = -b^2x/a^2y$$

$$y'' = -b^4/a^2y^3$$

Thus

$$C = a^4b^4/(a^4y^2 + b^4x^2)^{3/2}$$

The curvature is maximum at the point $x = a$, $y = 0$ where the major axis intersects the ellipse. Thus

$$C_{max} = a/b^2 \; . \tag{13}$$

According to the theorem, there will exist a critical contact angle for ellipses with C_{max} greater than L/A, that is, for

$$b/a < \pi/4E(m) \; . \tag{14}$$

This inequality can be solved numerically. The result is that a critical angle will exist for ellipses with $b/a < 0.6116$.

3. A LOWER BOUND ON γ_{crit}

Consider the cross section of the cylinder shown in Fig. 1. Let A be the area of the entire cross section and L be its perimeter. The cross section is cut by a curve Γ, which intersects the perimeter at points p_1 and p_2, and whose length we also denote by Γ. Let A^* denote the part of the domain cut off by Γ and denote also the area of this part. Let L^* denote the part of the perimeter cut off by Γ and denote also the length of this part.

If a solution to Eqs. (1) and (2) exists for $\kappa = 0$ and contact angle γ, then integration of Eq. (1) over A^* gives

$$2HA^* = \int_{A^*} \nabla \cdot \left(\frac{1}{W} \nabla u\right) dA = \int_{\Gamma+L^*} \left(\frac{1}{W} \nabla u\right) \cdot \hat{n} \; dL$$

$$= L^* \cos \gamma + \int_{\Gamma} \left(\frac{1}{W} \nabla u\right) \cdot \hat{n} \; dL \; .$$

Now $|\hat{n} \cdot \nabla u| \leqslant |\nabla u| \leqslant W$; therefore

$$-\Gamma \leqslant \int_{\Gamma} \left(\frac{1}{W} \nabla u\right) \cdot \hat{n} \; dL \leqslant \Gamma \; .$$

Thus if a solution exists for contact angle γ, then for any curve Γ

$$-\Gamma \leqslant 2HA^* - L^* \cos \gamma \leqslant \Gamma \; . \tag{15}$$

This can be written

$$\frac{2HA^* - \Gamma}{L^*} \leqslant \cos \gamma \leqslant \frac{2HA^* + \Gamma}{L^*} \; . \tag{16}$$

Integration of Eq. (1) over the entire cross section gives

$$2HA = L \cos \gamma \; .$$

Eliminating H from Eq. (15) gives

$$-\Gamma \leq (L^* - A^*L/A) \cos\gamma \leq \Gamma .$$

For $0 \leq \gamma \leq 90°$ and $L^* - A^*L/A$ nonzero, this can be written

$$\cos\gamma \leq V , \tag{17}$$

where

$$V = \frac{\Gamma}{|L^* - A^*L/A|} . \tag{18}$$

Equations (16) and (17) are Lemma 6 and Corollary 3.2 of Ref. 2 for the special case of a two-dimensional domain.

Equation (17) holds for each contact angle for which a solution to Eqs. (1) and (2) exists. In particular it holds for γ_{crit}. Let V_o be the minimum of V with respect to all possible curves Γ. V_o is an upper bound for $\cos\gamma_{crit}$.

$$\cos\gamma_{crit} \leq V_o . \tag{19}$$

This gives a lower bound on γ_{crit}.

4. MINIMIZATION OF V FOR FIXED p_1 AND p_2

The minimization of V can be done in two steps. First, find Γ that minimizes V for a fixed pair of intersection points p_1 and p_2; then vary p_1 and p_2. It can be shown that the minimizing Γ is the arc of a circle [2].

Let R be the radius of the circular arc Γ, and let O be the center of the circle as shown in Fig. 2. Let 2y be the length of the chord from p_1 to p_2. Let A be the area of the region bounded by this chord and by Γ. Let 2θ be the angle $p_1\,O\,p_2$. Then the following relations hold.

$$\theta = \arcsin (y/R) \tag{20}$$

$$\Gamma = 2R\theta \tag{21}$$

$$A_1 = R^2\theta - yR \cos \theta \tag{22}$$

Let us hold p_1 and p_2 fixed and vary R. Then y is fixed, but θ, Γ, and A_1 vary. Also $\delta A^* = - \delta A_1$ because $A^* + A_1$ is fixed. Varying R in Eqs. (20)-(22) and eliminating $\delta\theta$ gives

$$\delta\Gamma = 2(\theta - \tan \theta)\ \delta R$$

$$\delta A^* = -2R(\theta - \tan \theta)\ \delta R$$

thus

$$\delta A^* = -R\ \delta\Gamma\ . \tag{23}$$

Let $(L^* - A^*L/A)$ be positive; then

$$(L^* - A^*L/A)\ \delta V = \delta\Gamma + (VL/A)\ \delta A^*\ . \tag{24}$$

Let V_p be the minimum value of V for fixed points p_1 and p_2, and let R_p be the minimizing radius. Setting $\delta V = 0$ in Eq. (24) and using Eq. (23) gives

$$R_p = A/(LV_p)\ . \tag{25}$$

5. MINIMIZATION OF V WITH p_1 AND p_2

Next, let us hold R fixed and vary p_1 and p_2. Let ϕ_1 denote the acute angle of intersection of Γ with the cross section at p_1, and let ϕ_2 denote the corresponding angle at p_2. Let δL^*_1 be the variation in L^* at p_1, and δL_2^* be the variation at p_2. Then

$$\delta\Gamma = \cos \phi_1\ \delta L_1^* + \cos \phi_2\ \delta L_2^* \tag{26}$$

$$\delta A^* = (\Gamma/2) \sin \phi_1\ \delta L_1^* + (\Gamma/2) \sin \phi_2\ \delta L_2^*\ . \tag{27}$$

Let $(L^* - A^*L/A)$ be positive; then

$$(L^* - A^*L/A)\ \delta V = \delta\Gamma + (VL/A)\ \delta A^* - V\ \delta L^* \quad . \tag{28}$$

Setting $\delta V = 0$ in Eq. (28) and using Eqs. (26) and (27) gives two equations for the coefficients of the independent variations $\delta L_1{}^*$ and $\delta L_2{}^*$.

$$\cos\phi_1 + U \sin\phi_1 = V \tag{29}$$

$$\cos\phi_2 + U \sin\phi_2 = V \quad , \tag{30}$$

where

$$U = VL\Gamma/2A \quad .$$

For a given value of V, L, Γ, and A, Eq. (29) has the solutions

$$\sin\phi_1 = \left\{U \pm [U^2 + (1 + U^2)(1 - V^2)]^{1/2}\right\}/(1 + U^2) \quad .$$

For $V < 1$, Eq. (29) has one positive solution for $\sin\phi_1$. For $V > 1$, it has two positive solutions; however, $V > 1$ gives no useful bound on $\cos\gamma_{crit}$, so we shall ignore this case. Thus if V has a relative minimum that is less than one, then ϕ_1 will equal ϕ_2 for the minimizing Γ.

Consider a cross section that is symmetric about some straight line, as is the ellipse of Fig. 3 about the x axis. Let the curve Γ have intersection points p_1 and p_2 on opposite sides of that line. If V has a relative minimum that is less than one, then the minimizing p_1 and p_2 will be symmetric about that line.

6. LOWER BOUND ON γ_{crit} FOR ELLIPTICAL CROSS SECTIONS

We apply the preceding results to the elliptical cross section shown in Fig. 3. The ellipse is

$$y(x) = b(1 - x^2/a^2)^{1/2} \quad . \tag{31}$$

Let p_1 be the point $[x_1, y(x_1)]$; it is sufficient to take p_2 to be $[x_1, - y(x_1)]$. Let A_2 be the area $A_1 + A^*$; then

$$A_2 = 2\int_{x_1}^{a} y(x)\ dx \tag{32}$$

$$A^* = A_2 - A_1 \tag{33}$$

$$L^* = 2\int_{x_1}^{a} [1 + (y')^2]^{1/2}\ dx \quad . \tag{34}$$

Equations (18), (20)-(22), (25), and (31)-(34) can be solved for V_p for each value of x_1. By calculating V_p for a sequence of values of x_1, we find the minimum value V_o.

The circular arc Γ must lie inside the ellipse. This is assured if R is greater than R_1. R_1 is the length of the line $O_1 p_1$ that is perpendicular to the ellipse at p_1 as shown in Fig. 3.

$$R_1^2 = [y(x_1)]^2 + b^4 x_1^2 / a^4 \tag{35}$$

$$R = \max\ (R_1, R_p) \quad . \tag{36}$$

The preceding equations give the following results for the lower bound on γ_{crit} for $a = 1$ and various values of b.

b	lower bound
0.20	35.12°
0.25	26.95°
0.33	16.88°
0.40	10.32°
0.50	3.71°
0.60	0.12°

These bounds appear to be close to the values of γ_{crit} found by numerical solution of Eqs. (1) and (2) [3]. The lower bound as a function of b is shown in Fig. 4.

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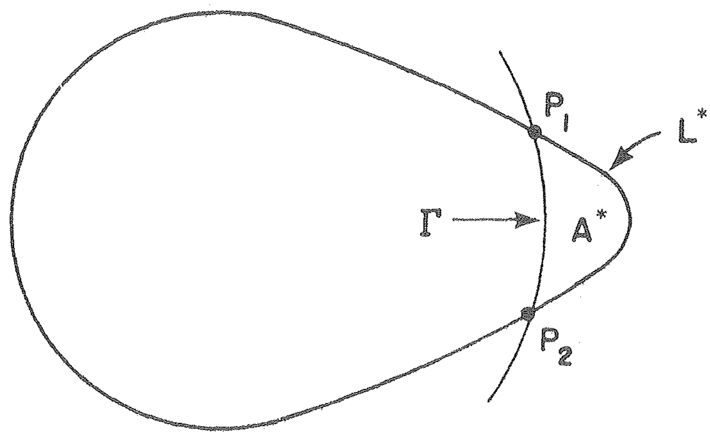

Fig. 1. Cross section of a cylinder. (XBL 776-949)

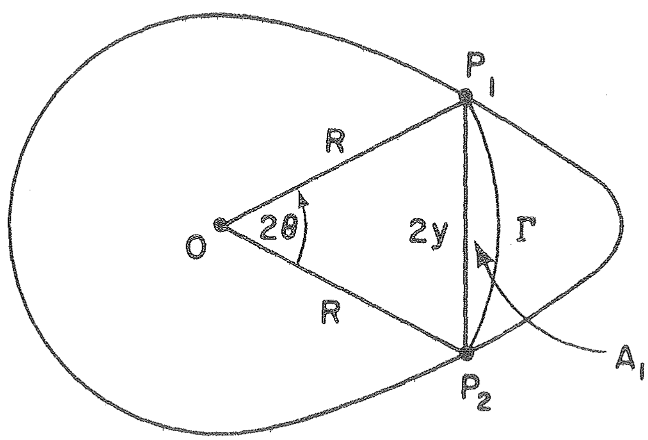

Fig. 2. Cross section of a cylinder with variables defined. (XBL 776-949)

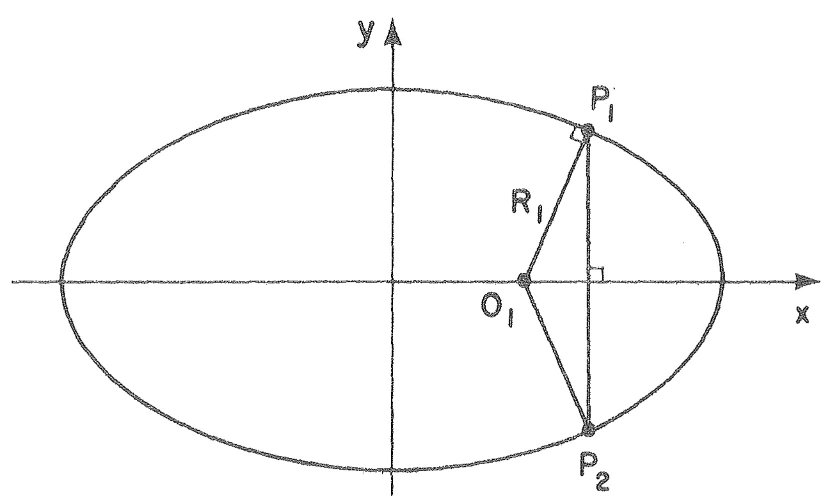

Fig. 3. Cross section of an ellipse. (XBL 776-949)

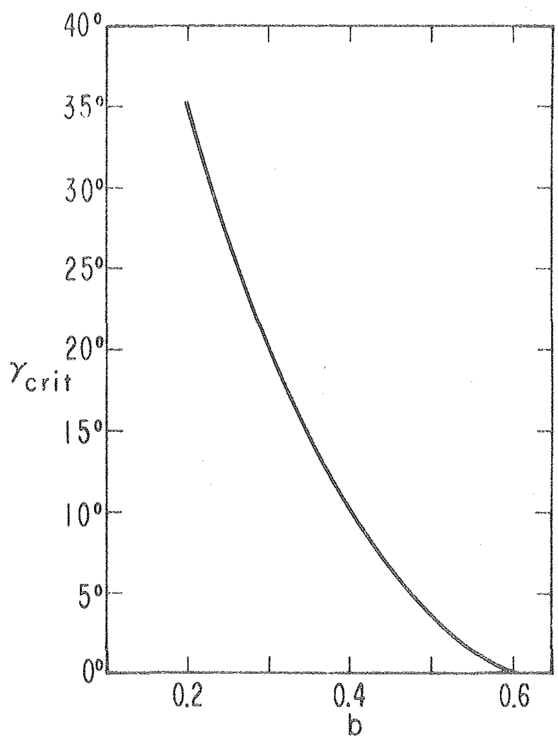

Fig. 4. Lower bound on γ_{crit} as a function of the ratio b of the semiminor and semimajor axes. (XBL 776-1117)

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