EVOLUTION OF THE AMPLITUDE DISTRIBUTION FUNCTION

FOR A BEAM SUBJECTED TO STOCHASTIC COOLING

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I. INTRODUCTION. The suggestion of S. van der Meer(1) for stochastic cooling or feedback damping of a circulating charged particle beam offers promise of increasing the luminosity of a storage ring and may be a particularly attractive technique if antiprotons are to be employed as one of the beams in such a device. Encouraging initial tests of such a system have been reported from CERN by P. Bramham et al.,(2) and further tests are in progress in that Laboratory.(3)

The original report of van der Meer(1) considered the repeated use of a kicker to suppress the transverse phase-space displacement of the centroid of a group of particles detected at a pick-up station situated up-stream (e.g., by $S\lambda_\beta/4$),(1) and the report estimated the expected rate of damping of the mean-square oscillation amplitude. In the present report we extend this analysis so as to provide information on the manner in which the character of the amplitude distribution function may be affected by the damping procedure mentioned above. It is believed that information concerning the evolution of the form of the distribution function may be of particular interest in cases in which a "halo" is imposed on the distribution by injection of a group of particles to supplement those in a beam that has already been subjected to appreciable feedback damping. Results of the analytic work will be illustrated, and compared with the results of simulation computations.

For consistency with the approach of van der Meer, we continue to assume that the kicker truly results in a zero transverse phase-space displacement for the centroid of the group of particles to which it is applied--although with a single pick-up device, capable of detecting spatial displacements only, the time scale of the damping process in fact may be doubled. We further ignore such potentially significant complications as imperfect amplifier performance, extraneous noise, or loss of particles to the chamber walls, and we restrict the analysis to the case in which complete "mixing" (or phase decoherence) is assumed to occur between successive applications of the correction procedure.

II. ANALYSIS. A single application of the full van der Meer correction leads to new particle amplitudes A_i' given, for N particles, by

$$A_i'^2 = A_i^2 - (2/N)A_i\Sigma_j A_j \cos(\phi_i-\phi_j) + (1/N^2)\Sigma_j\Sigma_k A_j A_k \cos(\phi_j-\phi_k). \qquad (1)$$

Thus, for random relative phases and $N >> 1$, the average change of the A_i^2 is expected to be

$$<\Delta(A^2)> = -(2/N) <A^2> + (1/N) <A^2> = -(1/N) <A^2>, \qquad (2)$$

as given by van der Meer.(1) Accordingly, with $u = A^2$, $\tau = t/N$, and time (t) measured in units of the time between successive corrections,

$$d<u>/d\tau = -<u> \qquad (3)$$

with the solution $<u> = C \exp(-\tau)$ [where $C = <u>|_{t=0}$] (4)

--regardless of the form of the initial distribution, provided only that complete phase mixing occurs between successive corrections. A similar analysis [Appendix A] can be performed for a beam considered to be composed of (say) two groups for which the evolution of their individual mean square amplitudes is of interest.

A binominal development of Eq.(1) to obtain $\Delta(u_i P)$ suggests the relations

REA

$$d\langle u^p\rangle/d\tau = -2p\langle u^p\rangle + p^2\langle u\rangle\langle u^{p-1}\rangle, \tag{5}$$

at least for integer $p \geqslant 0$, thus providing a soluble sequence of ordinary differential equations for the (even) amplitude moments [with $\langle u\rangle$, corresponding to $p=1$, given by Eq.(4)]. A distribution function $f(u;\tau)$, of squared amplitude that satisfies the partial differential equation

$$\partial f/\partial\tau = 2\,\partial(uf)/\partial u + C\exp(-\tau)\,\partial(u\,\partial f/\partial u)/\partial u \tag{6}$$

will be found (by integration over the distribution and the assumption of reasonable characteristics for f and for $\partial f/\partial u$ at the limits) to be consistent with the moment equation (5).[4] Numerical or analytic solution of Eq.(6) thus may provide a useful means for predicting the evolution of the form of a prescribed initial distribution and indeed (Sect.III) has been found in test examples to provide results consistent with simulation computations.

A formal analytic solution to Eq.(6) can be written in terms of Laguerre polynominals in the form[6,7]

$$f(u;\tau) = \langle u\rangle^{-1}\exp(-v)\sum_{m=0}\alpha_m\exp(-m\tau)\,L_m(v) \tag{7a}$$

(as can be readily confirmed, term-by-term, by reference only to the Laguerre differential equation), where we have written

$$v = u/\langle u\rangle \text{ and } \langle u\rangle \text{ is as given by Eq.(4).} \tag{7b}$$

With the adoption of this solution, the coefficients α_m are to be evaluated in terms of the initial distribution function (making use of the weighted orthonormality of the Laguerre polynominals) as[8]

$$\alpha_m = \int_0^\infty f(u;0)\,L_m(u/C)\,du. \tag{7c}$$

The formal solution, Eq.(7a), is attractive, and informative, in that it immediately suggests that as time increases (and the higher order factors $\exp(-m\tau)$ become increasingly small), the form of the distribution $f(u;\tau)$ will approach a pure exponential function, of width characterized by $\langle A^2\rangle = \langle u\rangle = C\exp(-\tau)$ -- as was found in initial simulation computations. We note, however, the alternative closed form solution given in[7].

III. EXAMPLE. As an example we consider the evolution of the two-group distribution function

$$f(u;0) = n_1\exp(-u/C_1) + n_2\exp(-u/C_2), \tag{8a}$$

with $n_1 + n_2 = 1$ and the initial mean square amplitude then given by

$$C = \langle u\rangle|_{\tau=0} = n_1C_1 + n_2C_2. \tag{8b}$$

Such an initial distribution may typify a beam composed of a core and a halo component, of which mention has been made in the Introduction. Simulation computations performed with the initial distribution specified by Eq.(8a) indicate the expected melding of the groups to form ultimately a composite group of simple exponential form whose mean square amplitude continues to damp in the expected manner ($\langle u\rangle = C\exp(-\tau)$). Figures 1a-d illustrate this behavior, with results for the individual groups indicated by dashed lines and results for the total distribution shown by a solid line. [Note that, because of the shrinkage of amplitude as the damping progresses, we have plotted $\langle u\rangle f(u;\tau)$ vs. $u/\langle u\rangle$.]

Results in agreement with those depicted on Figs.1a-d are obtained through use of the formula given in[7] for $f(u;\tau)$. With the initial distribution considered here, this formula gives[9] (with $\langle u\rangle = C\exp(-\tau)$)

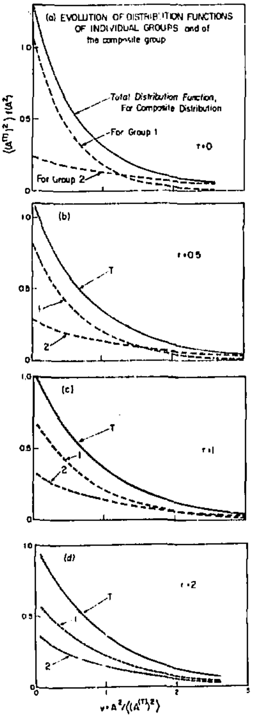

Fig.1. $n_1 = 0.6$, $n_2 = 0.4$

$C_1 = 5.0$, $C_2 = 15.0$ $(C=9.0)$.

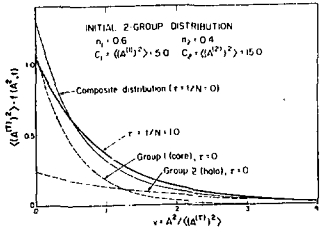

Fig.2. Composite A^2 distribution for $\tau = 0$ and $\tau = 1$.

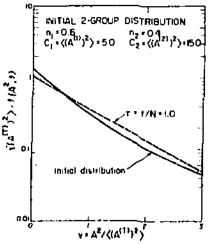

Fig.3. Approach of A^2 distribution to an exponential form.

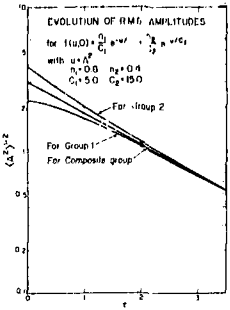

Fig.4. Convergence of the root-mean-square amplitudes to a common value.

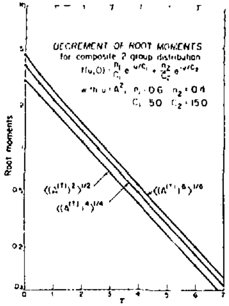

Fig.5. Evolution of root moments of composite distribution, to approach constant ratios.

$$f(u;\tau) = \frac{e^{-u/\langle u\rangle}}{\langle u\rangle}\left[n_1 \frac{C}{(1-e^{-\tau})C + e^{-\tau}C_1} \exp\left(-\frac{C-C_1}{C}\frac{u}{(1-e^{-\tau})C + e^{-\tau}C_1}\right) + n_2 \frac{C}{(1-e^{-\tau})C + e^{-\tau}C_2} \exp\left(-\frac{C-C_2}{C}\frac{u}{(1-e^{-\tau})C + e^{-\tau}C_2}\right)\right] \tag{9}$$

The distribution $f(u;\tau)$ can also be computed, with identical results, from Eqs.(7) in cases for which the convergence of Eq.(7a) permits numerical evaluation.[10] The change of form of the distribution function for the composite beam is directly shown, by a comparison of results for $\tau = 0$ and for $\tau = 1.0$, on Fig.2. The approach of this distribution function to an exponential form is most clearly apparent from the semi-logarithmic plot of Fig.3.

The behavior of the mean square amplitudes of the individual groups is most readily computed from the results presented in Appendix A. The convergence of the associated root-mean-square amplitudes, for the individual groups and for the composite group, to a common value is illustrated graphically in Fig.4. Similarly, $\langle (A^{(\ell)})^4 \rangle^{1/4}$ and higher root-moments approach constant ratios, characteristic of an exponential distribution function $f(u;t)$ as illustrated in Fig.5.

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APPENDIX A

Evolution of the Mean Square Amplitudes of the Individual Groups of a Two-Group Distribution

For a distribution regarded as comprised of two groups,

$$\Delta[(A_{i_1}^{(1)})^2] = -(2/N)A_{i_1}^{(1)}\left[\Sigma_{j_1} A_{j_1}^{(1)} \cos(\phi_{i_1}^{(1)}-\phi_{j_1}^{(1)}) + \Sigma_{j_2} A_{j_2}^{(2)} \cos(\phi_{i_1}^{(1)}-\phi_{j_2}^{(2)})\right] + (1/N^2)\Sigma_j\Sigma_k A_j A_k \cos(\phi_j-\phi_k)$$

for the i_1^{th} particle of Group 1. The random phase assumption then leads to

$$\frac{d}{dt}\langle (A^{(1)})^2\rangle = -(2/N)\langle (A^{(1)})^2\rangle + (1/N^2)[N^{(1)}\langle (A^{(1)})^2\rangle + N^{(2)}\langle (A^{(2)})^2\rangle]$$

or

$$\frac{d}{d\tau}\langle (A^{(1)})^2\rangle = -2\langle (A^{(1)})^2\rangle + [n_1\langle (A^{(1)})^2\rangle + n_2\langle (A^{(2)})^2\rangle]$$

(where, as in the text, $N^{(1)} = n_1 N$ and $N^{(2)} = n_2 N$), and similarly for $d\langle (A^{(2)})^2\rangle/d\tau$. Accordingly, with C_1, C_2 denoting the initial respective mean square amplitudes of groups 1 and 2, we may write the solution of these equations as

$$\langle (A^{(1)})^2\rangle = [C_1 + n_2(C_2-C_1)(1-e^{-\tau})]e^{-\tau}, \quad \langle (A^{(2)})^2\rangle = [C_2 + n_1(C_1-C_2)(1-e^{-\tau})]e^{-\tau}.$$

APPENDIX B

Equation (6) as a Fokker-Planck Equation

With $f(u;\tau)$ denoting the distribution function for $u = A^2$ and $\psi(u,\delta u)$ denoting the probability of an increment δu to the quantity u in a time interval δt,

$$f(u;t + \delta t) = \int_{-u}^{\infty} f(u-\delta u;t)\ \psi(u-\delta u,\delta u)\ d(\delta u),$$

as is characteristic of a Markoff process. A Taylor development of this relation then leads to

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial u}[f \cdot <\delta u>] + \frac{1}{2}\frac{\partial^2}{\partial u^2}[f \cdot <(\delta u)^2>],$$

where the quantities $<\delta u>$ and $<(\delta u)^2>$ are functions of u that represent averages (over the permissible range of δu) of changes or squared changes of u expected per unit interval δt.

In the present application, with δu for an $i^{\underline{th}}$ particle given by

$$\delta u = \delta(A^2) = -(2/N)A_i\Sigma_j A_j \cos(\phi_i-\phi_j) + (1/N^2)\Sigma_m\Sigma_n A_m A_n \cos(\phi_m-\phi_n),$$

the presumption of random phase leads to

$$<\delta u> = -2A^2/N + <A^2>/N = -(2/N)u + <u>/N.$$

Similarly,

$$<\delta(u^2)> = -(4/N)u^2 + (4/N)u <u>.$$

Accordingly,

$$<(\delta u)^2> \equiv <\delta(u^2)> -2u <\delta u> = (2/N)u <u>.$$

[It may be worth noting that we have found $<(\delta u)^p>$ to be zero through order 1/N for all integer $p > 2$.] The partial differential equation then becomes

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial u}\left[-2\frac{uf}{N} + \frac{<u>f}{N}\right] + \frac{1}{2}\frac{\partial^2}{\partial u^2}\left(2\frac{u<u>f}{N}\right)$$

or

$$\frac{\partial f}{\partial \tau} = 2\frac{\partial}{\partial u}(uf) + <u>\frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial u}\right),$$

wherein (as given originally by van der Meer[1]) $<u>$ may be taken [consistently with Eq.(6)] to be given by Eq.(4) of the text.

REFERENCES AND NOTES

1. S. van der Meer, CERN Internal Report CERN/ISR-PO/72-31 (CERN, Geneva, Switzerland; 1972)--cited in Ref.2.

2. P. Bramham, G. Carron, H. G. Hereward, K. Hübner, W. Schnell, and L. Thorndahl Nuclear Instr.and Meth., 125, 201-202 (1975).

3. G. Carron, L. Faltin, W. Schnell, and L. Thorndahl, Bull. Am. Phys. Soc. 22 (No.2), 148 (Feb. 1977)--Abstract G6, 1977 Particle Acc. Conf.

4. An alternative derivation of the partial differential equation for $f(u;\tau)$ [Eq.(6)] can be obtained through use of the Fokker-Planck equation[5]--see Appendix B.

5. S. Chandrasekhar, Rev. Mod. Phys, 15, 1-89 (1943); reprint in N. Wax (Ed.), Selected Papers on Noise and Stochastic Processes (Dover Publications, Inc., New York; 1954), pp. 3-91.

6. In finding the solution shown by Eq.(7a) we originally commenced with the moment equations (5) and introduced an amplitude distribution function $g(A;\tau)$ $[=2A\, f\,(A^2;\tau)]$ that should satisfy the partial differential equation

$$\partial g/\partial\tau = \partial(Ag)/\partial A + (C/4)\exp(-\tau)\,\partial[A\partial(g/A)/\partial A]/\partial A.$$

We then wrote $z = (A^2/C)\exp(\tau)$ and regarded $g(A;\tau)$ as a function $G(z;\tau)$ to obtain a partial differential equation in which none of the coefficients was explicitly τ-dependent. We next replaced G by the dependent variable $S = [(1/\sqrt{z})\exp(z-\tau/2)]G$ to obtain a partial differential equation that, by separation of variables, led to a solution in terms of Laguerre polynominals. Transcription of this solution into the original variables led to a result equivalent to Eq.(7a). For numerical solution of the partial differential equation, it may be convenient to introduce the independent variable $w = A\exp(\tau/2)$ and to employ as the dependent variable a function $H(w;\tau) = [\exp(-\tau/2)]g$. The partial differential equation for H is

$$\partial H/\partial\tau = (1/2)\,\partial(wH)/\partial w + (C/4)\,\partial[w\partial(H/w)/\partial w]/\partial w$$

--again an equation in which none of the coefficients is τ-dependent--and it is expedient to seek solutions that have the formal character of being odd with respect to w.

7. An alternative, closed-form solution may be written

$$f(u;\tau) = \langle u\rangle^{-1}\exp(-u/\langle u\rangle)\,\frac{\exp[-(1-e^{-\tau})^{-1}u/C]}{1-e^{-\tau}}\times$$
$$\int_0^\infty \exp[-(e^{\tau}-1)^{-1}x/C]\; I_0\,(2(1-e^{-\tau})^{-1}\sqrt{ux}/C)\,f(x;0)\,dx,$$

where I_0 is the zero-order modified Bessel function of the first kind--see I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals. . .(Academic Press, New York; 1965), Sec. 8.976 (1), p.1038 (with $\alpha = 0$) to relate this solution to that proposed by Eq.(7) in the text.

8. Since $f(u)$ is normalized to unity and $L_0(u/C) = 1$, $\alpha_0 = 1$. Also, since the initial value of $\langle u\rangle$ is C and $L_0(u/C)-L_1(u/C) = u/C$, we find $\alpha_0-\alpha_1 = 1$ and, hence, $\alpha_1 = 0$.

9. Note that $\int_0^\infty e^{-\beta x}\, I_0\,(\gamma\sqrt{x})\,dx = (1/\beta)\exp(\tfrac{1}{4}\gamma^2/\beta)$.

10. For the example of Sec. III, evaluation of Eq.(7c) leads to $\alpha_m = n_1(1-C_1/C)^m + n_2(1-C_2/C)^m$--see Gradshteyn and Ryzhik (cited in [7]), Sec. 7.414 (6), p. 844. The resultant Eq.(7a) may not have suitable convergence characteristics for small τ under certain circumstances however--thus consider, for example, an initial distribution (8a) with $n_1 = 0.75$, $n_2 = 0.25$, $C_1 = 4.0$, and $C_2 = 16.0$ ($C = 7.0$), for which the factor $1-C_2/C = -9/7$ and the coefficients α_m ultimately increase essentially in geometrical progression (with alternating sign).

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