Submitted to Physics Letters B

4C-32 and 11C-34d

LBL-6476
Preprint c.1

RECEIVED
LAWRENCE
BERKELEY LABORATORY

AUG 1 1 1977

LIBRARY AND
DOCUMENTS SECTION

QUANTIZATION OF GAUGE THEORIES

Yoav Eylon

June 20, 1977

For Reference

Not to be taken from this room

Prepared for the U. S. Energy Research and
Development Administration under Contract W-7405-ENG-48

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

QUANTIZATION OF GAUGE THEORIES*

Yoav Eylon

Lawrence Berkeley Laboratory
University of California, Berkeley, California 94720

June 20, 1977

ABSTRACT

A path-integral procedure for quantizing gauge theories is proposed (on a heuristic level). The Hilbert space of physical states is constructed. Each physical state is represented by an infinite set of gauge equivalent configurations. All physical transition amplitudes are defined. In this approach, the "natural" value of the parameter θ is zero.

Consider a gauge theory with an arbitrary Lie group. Our purpose is to propose a quantization procedure by defining the physical Hilbert space and all transition amplitudes. In a non-gauge field theory, at each point $\vec{x}$ in 3-space, there is one degree of freedom, $\phi(\vec{x})$. Each field configuration $\phi(\vec{x})$, represents a state in the Hilbert space, which is the analogue of the state

* This work was supported by the U. S. Energy Research and Development Administration, under the auspices of the Division of Physical Research.

$|q_1,\ldots,q_n\rangle$ in quantum mechanics with n degrees of freedom. These states form a basis to the Hilbert space (in the Schrödinger representation), they are eigenstates of the $\hat{\phi}(\vec{x})$ operators and therefore of the potential energy $V\left[\phi(\vec{x})\right]$. ($V$ is the term in the Lagrangian which does not contain any time-derivatives). They are not eigenstates of the Hamiltonian. The transition amplitude from a state $\phi_o(\vec{x})$ at time t_o to $\phi_1(\vec{x})$ at t_1 is[1]:

$$\langle\phi_1(\vec{x})|\, e^{-iH(t_1-t_o)}|\phi_o(\vec{x})\rangle = \int D\left[\phi(\vec{x},t)\right] e^{iS\left[\phi(\vec{x},t)\right]} \qquad (1)$$

where $\phi(\vec{x},t_o) = \phi_o(\vec{x})$, $\phi(\vec{x},t_1) = \phi_1(\vec{x})$ and $S = \int_{t_o}^{t_1} L\, dt$.

In gauge theories we face the problem that different field configurations may be related by a gauge transformation. Our task is to overcome this difficulty in the spirit of the Faddeev-Popov ansatz.[2]

We want to construct our Hilbert space at any given time t. Since we freeze the time, the gauge transformations are given by a mapping $g(\vec{x})$ from 3-space to the group. The role of the gauge field A_μ is to compensate the derivatives of g in the x_μ direction. In the absence of the time direction, it is natural to choose the gauge $A_o = 0$ [3,4]. (In any case, A_o is not a canonical degree of freedom, since L does not contain its time derivative). Consider a given configuration $A_i(\vec{x})$. (We use $A_i(\vec{x}) = \sum_a A_i^a(\vec{x})\tau^a$, and from now on omit the space index i and color traces.) For any (time-independent) gauge transformation $g(\vec{x})$, we get a new configuration, which we denote by gA. (Note that at this point we cannot

decompose the mappings $g(\vec{x})$ into homotopy classes, since they can be continuously deformed into each other.)

Assumption 1: The set of all configurations gA (we fix A and run over all possible $g(\vec{x})$) corresponds to a single physical state $|\alpha\rangle$ in the Hilbert space.

Due to the group properties of the gauge transformations, the definition of the set is independent of the choice of the original A, as long as it belongs to the set. We represent the state $|\alpha\rangle$ by a dot (Fig. 1). The points u and v represent two configurations

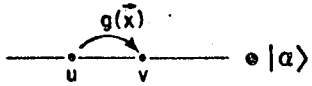

Figure 1

A^u and A^v in the $|\alpha\rangle$ set. They are related by some gauge transformation $g(\vec{x})$. The set of all configurations gA^u is schematically represented by the line to the left of the $|\alpha\rangle$ dot. A configuration A' which does not lie in the $|\alpha\rangle$ set, generates a new line gA', which represents the physical state $|\alpha'\rangle$. We put the $|\alpha'\rangle$ dot on the state axis, which comes out from the page through the $|\alpha\rangle$ dot. The line gA' lies to the left of the $|\alpha'\rangle$ dot. The scalar product in the Hilbert space is defined as $\langle\alpha'|\alpha\rangle = \int D[g(\vec{x})]\,\delta[gA' - A]$. This is the generalization of the $\delta[\phi'(\vec{x}) - \phi(\vec{x})]$ normalization in non-gauge theories.

Assumption 2: The transition amplitude from a physical state α_o at t_o to α_1 at t_1 has the form

$$\langle\alpha_1|e^{-iH(t_1-t_o)}|\alpha_o\rangle = \int D[\alpha(t)]e^{iS[\alpha(t)]} \qquad (2)$$

where the r.h.s. is still to be defined. $\alpha(t)$ is a physical trajectory: for any time t we choose a physical state (in contrast to configuration) $\alpha(t)$, with $\alpha(t_o) = \alpha_o$ and $\alpha(t_1) = \alpha_1$. In Fig. 2

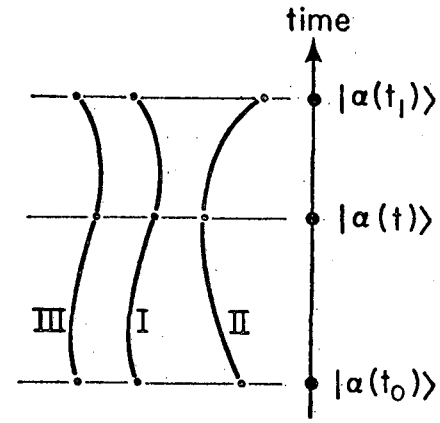

Figure 2

XBL 776-1186

we let time flow upwards. At each t we represent the physical states by dots on the state axis (which is perpendicular to the page). $\int D[\alpha(t)]$ is equivalent to summing over all curves (which go from α_o at t_o to α_1 at t_1) in the {state axis- time axis} plane. (This plane is perpendicular to the page.)

Let us consider a given physical trajectory $\alpha(t)$. (We draw it for simplicity on the plane of the page. This does not mean that α is independent of t). We now have to define the action $S[\alpha(t)]$. We may try to choose at each t ($t_o \le t \le t_1$) an arbitrary configuration $A(\vec{x},t)$ from the set corresponding to $\alpha(t)$ (trajectory I in Fig. 2), and set $S[\alpha(t)] = S[A(\vec{x},t)] \equiv \int_{t_o}^{t_1} dt \int d^3\vec{x}\,\mathcal{L}(\vec{x},t)$ where $\mathcal{L}$ ($\mathcal{L} = F_{\mu\nu}F^{\mu\nu}$) contains time derivatives. This definition is unacceptable since it depends on the choice of $A(\vec{x},t)$. If at each t we choose a different configuration, $g(\vec{x};t)A$, from the same set, we get a new trajectory (II in Fig. 2) which generally has a different action. (The reason is that the time t is just a parameter in $g(\vec{x};t)$ and the gauge transformations do not contain the $g^{-1}\partial_o g$ term which is necessary in order to make $\mathcal{L}$ gauge invariant.)

However, if $g(\vec{x};t)$ is independent of t, the new trajectory in configuration space will have the same action. We represent such new trajectory (III) by a trajectory which is parallel to I. To define a gauge independent action, we integrate over all trajectories in configuration space which correspond to the same physical trajectory $\alpha(t)$, and get $e^{iS[\alpha(t)]} = \int D[A] e^{iS[A]}$, where $A(\vec{x},t)$ is in the $\alpha(t)$ set. The only infinity we get due to gauge invariance is because of the equal action of parallel trajectories. To eliminate this infinity we fix $A(\vec{x},t_o)$ at some arbitrary configuration $A^o(\vec{x})$ which belongs to the α_o set. The resulting $S[\alpha(t)]$ is independent of the choice of $A^o(\vec{x})$. (Note that taking $A_o = 0$ and then choosing an arbitrary (time-independent) gauge transformation corresponds to a complete gauge fixing.)

Integrating over all possible physical trajectories $\alpha(t)$ we finally get

<u>Assumption 3</u>:

$$\langle \alpha_1 | e^{-iH(t_1-t_o)} | \alpha_o \rangle = \int D[A(\vec{x},t)] e^{iS[A(\vec{x},t)]} \qquad (3)$$

where $A(\vec{x},t_o) = A^o(\bar{x})$ and $A(\vec{x},t_1)$ is in the α_1 set.

Equation 3 is the main result (or assumption). We would like to get some more information by computing it for imaginary $(t_1 - t_o)$. In that case any configuration trajectory with infinite kinetic energy (or potential energy) will not contribute. The potential energy is $V(\alpha) = V[A(\vec{x})] = \int d^3\vec{x}\, F_{ij}F^{ij}$, and it is independent of the choice of $A(\vec{x})$. We can eliminate from the Hilbert space all states α with $V(\alpha) = \infty$ (but this is not essential for our arguments). The time-integrated kinetic energy T is

$\int_{t_o}^{t_1} dt \int d^3\vec{x}\, F_{oi}F^{oi}$ ($F_{oi} = \partial_o A_i$ in the $A_o = 0$ gauge). The minimal value of T (under the constraint $A(\vec{x},t_i) = A^i(\bar{x})$, $i = 0,1$) is[5] $T_{min} = \frac{1}{t_1 - t_o} \int d^3\bar{x}\, (\Delta A)^2$, where $\Delta A = A^1(\vec{x}) - A^o(\vec{x})$. We shall use the term "a good function" for a function $A(\vec{x})$ with $\int d^3\vec{x}\, A^2 < \infty$ (namely, $A \sim 1/r^{3/2+\varepsilon}$). We cannot interpolate two functions with a finite T if their difference is "bad". A state $|\alpha\rangle$ is good if its set contains at least one good A. The state with $V(\alpha) = 0$ (the classical vacuum) is good since its corresponding set contains the configuration $A(\vec{x}) = 0$ (It also contains bad configurations.). A good state cannot communicate with a bad state (namely, $T = \infty$ for any interpolating trajectory). From now on we ignore the bad states.

Let us choose at t_o a good $A^o(\vec{x})$, and let $A^1(\vec{x})$ be good. In Eq. 3 we have to integrate over all $g(\vec{x})A^1$. $g(\vec{x})$ is called good, if $g^{-1}\partial_i g$ is good. If $g(\vec{x})$ is bad, $g(\vec{x})A^1$ $(= g^{-1}A^1_i g + g^{-1}\partial_i g)$ is also bad since the first term cannot cancel the second. Therefore, we do not have to integrate over bad g's. Let us consider for a moment only those $g(\vec{x})$ which are smooth at infinity (namely, the limit $\lim_{r\to\infty} g(\Omega,r) = g(\Omega)$ exists). For $g(\Omega)$ which is not Ω independent, $g^{-1}\partial_i g \sim \frac{1}{r}$, and g is bad. The only good smooth g's are those with $g(\Omega) = \text{const}$. At that point we choose our group to be SU(2). All smooth mappings $g(\vec{x})$ with $g(\Omega) = \text{const}$. fall into homotopy classes[3] labelled by the integers. (If a non-smooth $g(\vec{x})$ is good it belongs to one of these classes). If $g^{(n)}$ belongs to the n class $(n \neq 0)$ we can find a finite-T trajectory from A^o to $g^{(n)}A^1$. However, this trajectory cannot be continuously deformed into the original one (from A^o to A^1)

without crossing an infinite-T barrier (this would define a continuous deformation of $g^{(n)}$ to $g(\vec{x}) = I$, which is impossible). The conclusion is that for the SU(2) group, the functional integral of Eq. 3 breaks into infinite number of components , labelled by the integers. Notice that the (classical) vacuum state did not play any special role. Consider now a transition from the vacuum to itself, and take $A^o = 0$. The $n = 0$ component contains the trajectory $A(\vec{x},t) = 0$. The $n = 1$ component includes the instanton solution[6] (in the $A_o = 0$ gauge). This proves that more than one component can be non-zero.

According to Eq. 3 we have to sum over all the components with an equal weight. Mathematically, this is equivalent to $\theta = 0$.[3,4]

One can modify the present approach in several ways in order to get a theory with $\theta \neq 0$:

a. Modify assumption 1, and break each set $g(\vec{x})A$ into many subsets (according to the behavior of $g(\vec{x})$ at infinity) and associate each subset with a different physical state. This leads to the Bloch-waves picture.[3,4]

b. Add the $\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$ term[4] to the Lagrangian. Since $F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu s^\mu$ with $s^\mu \propto T_r \epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2} A_\alpha \partial_\beta A_\gamma + \frac{1}{3} A_\alpha A_\beta A_\gamma\right]$, S_θ is given by the integral of s^μ over the surface of $\{t_o \leq t \leq t_1 \; ; \; |\vec{x}| < R\}$ (with $R \to \infty$). For a good trajectory $A(\vec{x},t)$, $S_\theta =$ $= \theta(W[A^1] - W[A^o])$ where[7] $W[A(\vec{x})] = \frac{1}{8\pi^2}\int d^3\vec{x}\; s^o(\vec{x})$. Since $W[g^{(n)}(\vec{x})\, A(\vec{x})] = W[A(\vec{x})] + n$, we get a relative phase of $e^{-i\theta n}$ between the different components of any transition amplitude.

Whether these modifications are "natural" or not is a matter of taste.

After completing the first draft, I became aware of a paper by Z. F. Ezawa[8] and rediscovered a paper by K. Cahill[9]. They both conclude that $\theta = 0$, using different (but related) approaches.

I would like to thank Eliezer Rabinovici for many helpful discussions. I thank Prof. S. Coleman for indicating a mistake in the first draft. Useful discussions with C. Callan, M. Chanowitz, M. Halpern, S. Mandelstam, C. Rebbi and P. Senjanovic are acknowledged.

REFERENCES

1. R. P. Feynman and A. R. Hibbs, "Quantum Mechanics and Path Integrals" (McGraw-Hill, 1965).
2. L. D. Faddeev and V. N. Popov, Phys. Lett. 25B (1967) 29.
3. R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172.
4. C. Callan, R. Dashen and D. Gross, Phys. Lett. 63B (1976) 334.
5. Y. Eylon and E. Rabinovici.
6. A Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. 59B (1975) 85.
7. R. Jackiw, "Quantum Meaning of Classical Field Theory" Review of Modern Physics, to be published.
8. Z. F. Ezawa, "Do pseudoparticles imply physically inequivalent multiple vacua in gauge theories" NBI-HE-77-4 (1977).
9. K. Cahill, "The subsidiary condition in Q. E. D. HUTP-77/B001 (1977).

FIGURE CAPTIONS

Fig. 1: A physical state and its corresponding set of configurations.

Fig. 2: Different trajectories in configuration space which correspond to one physical trajectory.

0 0 0 0 4 8 0 2 6 8 8

This report was done with support from the United States Energy Research and Development Administration. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the United States Energy Research and Development Administration.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720