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Damped Double Solitons in the Nonlinear Schrödinger Equation

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ABSTRACT

The nonlinear Schrödinger equation modified by a damping term is numerically investigated for initial conditions other than single solitons. With damping, colliding solitons still pass through each other, but the breather can change qualitatively into two continuously interacting but separated solitons. These results are consistent with a slow change in the inverse scattering eigenvalues due to the damping.

I. Introduction

The nonlinear Schrödinger equation, Eq. (1) below, arises as the envelope equation of a dispersive wave system which is almost monochromatic and weakly nonlinear.[1] For example, two plasma heating problems of current interest are approximated by this equation, in their nonlinear stage, viz. i) Langmuir turbulence when the background plasma is assumed in equilibrium with the ponderomotive pressure from the high-frequency fields,[2,3] and ii) a nonlinear stage of the mode-converted wave in Lower Hybrid heating of large tokamaks.[4]

When such a wave heats (transfers energy to) the particles of the plasma, a dissipation term appears in the nonlinear Schrödinger equation. Since the heating is slow the dissipation term is small and can be considered as a perturbation that, hopefully, leaves some qualitative properties of the solution unchanged. In Langmuir turbulence, for instance, the dissipation is wavenumber-dependent Landau damping,[2,3] while for the lower hybrid wave the damping is more difficult to obtain (see Ref. 5).

The nonlinear Schrödinger equation is one of a class of exactly solvable evolution equations. These equations have various properties in common, notably stable nonlinear wave solutions called solitons, and an infinite set of conservation laws.[6-8] It is well known[8] that a large enough initial condition in such an equation typically evolves into solitons. Thus it is necessary to study the effect of damping on single solitons, but this is not sufficient: for a more complete understanding one must find out how more general initial conditions behave[9] under damping.

In a previous paper[10] we treated single solitons with damping as perturbation, and established that single solitons damp in substantial agreement with a simple treatment based on their invariant shape and the first conservation law.[11-13] We discussed, for example, the influence on the damping rate of the exponent b in the damping law $\gamma_k = |k|^b$, and showed that the damping rate is a constant only for b = 0 and b = 2. Such a comparison between numerical computations and analytical considerations provides one example of construction and verification of possible soliton perturbation theories; after all, damping is just one particular perturbation.

A complete perturbation theory for soliton equations should not only predict the evolution of single solitons, but should ideally be able to treat arbitrary initial conditions. In an unperturbed soliton equation every initial condition develops into a background (radiation), which is supposed to disperse away and become unimportant over time, and into solitons, which stay around permanently (but even this unperturbed solution can usually not be calculated analytically).

The final solitons may have unequal velocities, in which case they exhibit pairwise collisions, or they may have equal velocities (in some equations such as the nonlinear Schrödinger equation), in which case they form a nonlinear superposition called breather.

In the last few years various soliton perturbation theories[14-16] have been developed. These theories assume that single solitons keep their shape, but adibatically change their parameters (amplitude and velocity). For more than one soliton they yield nonlinear relations between all parameters of the constitutent solitons, including the

intersoliton distance. These relations contain coefficients, spatial integrations over the soliton shape multiplied by the perturbation, that are almost intractable for other than single solitons. Therefore, we have not been able to extend our detailed analytical checkup on perturbed single solitons in Ref. 10 to double solitons. Instead, we attempt to numerically confirm the validity of one particular soliton perturbation theory based on the conservation laws. This approach is especially convenient for damping, and gave good results with relatively little effort for single solitons.

For the breather, a superposition of two solitons, we need two parameters: hence, besides the first we must use the third conservation law, in which the soliton parameters enter nonlinearly. For our purpose - the numerical verification of the two-time scale assumption which forms the basis of all soliton purturbation theories - this nonlinearity and the analytically prohibitive space integrations over soliton shape and perturbation present no special difficulty.

This paper, then, extends our previous work[10] on single solitons to the simplest two-soliton cases, namely to collisions of two equal solitons with opposite velocities, and to the simplest breather. The perturbation is again a simple damping of each Fourier mode with its own damping rate $\gamma_k = \varepsilon|k|^b$. We concentrate on the two simplest dampings, namely collisional damping, $b = 0$, and the damping $b = 2$, which introduces a small imaginary part in the coefficient of the dispersive term. But in contrast with the comparison with an analytical prediction we here compare to another numerical computation that uses the two-timescale assumption and the conservation laws.

In Section II we briefly discuss the inverse scattering transform and its eigenvalues, and give the relevant data on damping of single solitons. In Section III we treat colliding solitons. In Section IV we numerically study the damped breather in some detail, and show that its evolution is consistent with a two-timescale assumption. Section V we present our conclusions, including the generalization of these results to soliton perturbation theories.

II. Basics

Our nonlinear Schrödinger equation has the form

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{1}$$

Here $q(x,t)$ is a complex function of the real variables t (time) and x (space). A single soliton has the form

$$q_s(x,t) = 2\eta \ \operatorname{sech}[2\eta(x-4\,\xi t)] \exp i\,\theta, \tag{2}$$

with phase $\theta = 2\,\xi\,x - 4(\xi^2 - \eta^2)t.$

The parameter η determines the amplitude and inverse width of the soliton, and $4\,\xi$ is the velocity. This notation is not the simplest and deviates from previous use, but is appropriate for the inverse scattering transform whose notation we will employ.

Equation (1) has an infinite set of conservation laws. The first few are reminiscent of a particle mass in quantum mechanics,

$$I_1 = \int |q|^2\, dx, \tag{3}$$

the momentum, $I_2 = \int (q^* q_x - q q^*)\, dx$, and the energy,

$$I_3 = \int |q_x|^2 - |q|^4\, dx. \tag{4}$$

(all integrations are over the whole real axis $-\infty < x < \infty$). The higher conservation laws have no direct physical meaning, and are more complicated.

The inverse scattering transform shows that the complete nonlinear evolution of arbitrary initial conditions can be understood in terms of solitons, and a non-soliton part called radiation. The radiation part

is complicated, and we choose not to treat it here.[16]

The solitons each correspond to two parameters, the real part ξ and the imaginary part η, of the eigenvalue (ζ) in some linear scattering problem. In general it is difficult to find the eigenvalue for a given initial condition, but one can write down explicitly a full solution that corresponds to given eigenvalues, usually a complicated combination of exponentials which depend on the eigenvalues and on additional parameters that correspond to intial intersoliton distances and phases. The solutions are plotted in Figs. 1 and 2 for the two special cases we consider. Figure 1 shows $|q|^2$ for colliding solitons with initial condition

$$q(x,t = 0) = q_s(x - x_0,\ t = 0) + q_s(x + x_0,\ t = 0)\ \exp i\phi \qquad (5)$$

and q_s from Eq. (2). The parameters are $x_0 = 3$, or an intersoliton distance 6, amplitude $2\eta = 1.5$, and velocity $4\xi = \pm 0.75$. The initial phase difference ϕ is not visible in Fig. 1a for the initial condition, but the collision stage is very different for the two cases: for $\phi = 0$, in Fig. 1b, there is a large peak due to soliton overlap while for $\phi = \pi/2$, in Fig. 1c, the solitons are bouncing off of each other. The final state is similar to Fig. 1a. The full solution for the breather is

$$q(x,t) = 4i\ |\eta_1^2 - \eta_2^2|$$

$$\times \frac{\eta_1 \cosh 2\eta_2 x \exp i\Omega_1 t + \eta_2 \cosh 2\eta_1 x \exp i\Omega_2 t}{2(\eta_1 - \eta_2)^2 \cosh 2\eta_2 x \cosh 2\eta_1 x + 4\eta_1\eta_2\{\cosh[2x(\eta_1 - \eta_2)] + \cos \Omega t\}} \qquad (6)$$

where $\Omega_1 = 4\eta_1^2$, $\Omega_2 = 4\eta_2^2$ and $\Omega = \Omega_2 - \Omega_1$. Notice that the breather

amplitude $|q|^2$ is purely periodic, with the only time dependence entering through two occurrences of $\cos(\Omega t)$. Various stages for the breather evolution are given in Fig. 2, for the particular choices of eigenvalues $\eta_2 = 3/2$ and $\eta_1 = 1/2$. At $t = 0$ in Fig. 2a, $q(x,t = 0) = 2\ \mathrm{sech}(x)$, a single soliton whose amplitude is multiplied by two. This narrows slowly to the form plotted in Fig. 2b at $\Omega t = \pi/2$. Then the narrowing accelerates to the contracted breather stage given in Fig. 2c at $\Omega t = \pi$. Notice the amplitude of the peak, and the large values of the derivatives q_x. Breathers with eigenvalues other than 3/2 and 1/2 are qualitatively similar at $\Omega t = \pi$, but at $t = 0$ can show a double-humped shape exemplified by the dotted lines in Fig. 7b and c.

For the two cases that we consider with only two eigenvalues, the values of the conserved quantities are directly related to the eigenvalues:

$$I_1 = 4\ (\eta_1 + \eta_2), \tag{7a}$$

$$I_2 = 16\ (\eta_1\xi_1 + \eta_2\xi_2), \tag{7b}$$

$$I_3 = 16\ (\eta_1\xi_1^2 - \frac{1}{3}\eta_1^3 + \eta_1\xi_2^2 - \frac{1}{3}\eta_2^2). \tag{7c}$$

For colliding solitons $\eta_1 = \eta_2 = \eta$ and $\xi_1 = -\ \xi_2 = \xi$: hence $I_2 = 0$, and $I_3 = 32(\eta\xi^2 - \eta^3/3)$. The breather has eigenvalues with real parts equal to zero, but unequal imaginary parts: again, $I_2 = 0$ and $I_3 = -\ \frac{16}{3}\ (\eta_1^3 + \eta_2^3)$. Thus the eigenvalues can be found directly from the values of the conserved quantities, by a simultaneous solution of a first order and third order polynomial equation.

Now we introduce damping by adding to Eq. (1) an extra term $i\ FT^{-1}\ (\gamma_k\ q_k)$, where FT^{-1} denotes the inverse Fourier transform and k is the wavenumber. As discussed in Ref. 10, in the absence of

nonlinearity this term would damp each Fourier mode,

$q_k = \frac{1}{2\pi} \int q(x) \exp(-ikx)\, dx$, with its own damping decrement γ_k.

We consider the two simplest cases of the model damping $\gamma_k = \varepsilon |k|^b$, namely b = 0, (collisional damping) and b = 2, as a rough but convenient approximation to Landau damping. For b = 0, Eq. (1) acquires an extra term and becomes

$$iq_t + i\varepsilon q + q_{xx} + 2\,|q|^2\, q = 0, \tag{8}$$

while the case b = 2 just changes the coefficient of the dispersive term to a complex number,

$$iq_t + (1 - i\varepsilon)\, q_{xx} + 2|q|^2\, q = 0. \tag{9}$$

These seemingly innocuous changes in the equation have various and sometimes dramatic effects. Firstly, with the extra terms the inverse scattering transform does not apply, which is why we must use perturbation theory. Secondly, the quantities that are conserved under Eq. (1) are no longer conserved. The equations for these changes[10] become for the case b = 0:

$$dI_1/dt = -\,2\,\varepsilon\; I_1, \tag{10a}$$

and

$$dI_3 = -\,2\,\varepsilon \int |q_x|^2 - 2|q|^4\, dx, \tag{10b}$$

$$= -\,2\,[3I_3 - \int 2|q_x|^2 - |q|^4\, dx]. \tag{10c}$$

For b = 2 we obtain

$$dI_1/dt = -\,2\,\varepsilon \int |q_x|^2\, dx, \tag{11a}$$

and

$$dI_3/dt = -2\varepsilon\int |q_{xx}|^2 - 2[|q|_x^2]^2 + 4|q|^2|q_x|^2 dx, \tag{11b}$$

$$= -2\varepsilon\int |q_{xx}|^2 + |q|^2 (q^* q_{xx} + q q^*_{xx})\, dx. \tag{11c}$$

Even though I_1 and I_3 change in time proportional to ε we will still refer to them as conserved quantities. The invariant I_2 always remains zero for symmetric initial conditions.

We notice that Eq. (10a) describes an exponential decay for I_1 irrespective of the solution $q(x,t)$, but that the others do depend on the solution in some complicated way: The right hand sides of Eqs. (10b) - (11b) do not reduce to combinations of conserved quantities. Compare, for example, Eq. (10b) or (11a) with Eq. (3b), or Eq. (11b) with the next-higher conservation law[6]

$$I_5 = \int dx \left\{ |q_{xx}|^2 - [|q|_x^2]^2 - 6|q|^2 |q_x|^2 + 2|q|^6 \right\}.$$

However, the right hand sides are constant when the solution keeps a constant shape, that is, for a single soliton. For double solitons there is generally a strong dependence of the solution on time, and hence the expressions in Eqs. (10) and (11) are also time dependent. In this connection we recall from Ref. 10 that the decrease of I_1 and I_3 is consistent with the assumption of a stationary sech-shaped soliton with decreasing amplitude parameter. For $b = 0$ this is easy to see in Eq. (10c): The integral is zero, and I_3 is proportional to the third power of I_1. For double solitons this conclusion is no longer true, because its proof hinges on the explicit functional form, sech(x), of the single soliton.

Although it may not be apparent from Eqs. (11) we believe that the case b = 2 is particularly simple, partly on the basis of Eq. (9) but mostly because in this case an exact stationary soliton shape can be found.[10] For other values of the damping exponent b, however, including b = 0, there are shape changes of the soliton to second order in ε: these have a time-dependent effect on the damping rates that is readily noticeable.

For our purpose it is convenient to construct a simple perturbation theory on the basis of the conservation laws. We adopt a conventional two-timescale assumption: the constant parameters of the unperturbed problem, the inverse scattering eigenvalues in this case, change slowly in time when the perturbation, i.c. damping, is introduced. This assumption is very successful for single solitons, or single eigenvalues, as shown in Ref. 10. With damping single solitons approximately keep their shape, which reflects the continuing balance between nonlinearity and dispersion, but the solitons adapt their amplitude and width to agree with the change in the first conserved quantity.

Now we generalize to more eigenvalues, in which case the solutions—the breathers—are not stationary. But there is still a balance between nonlinearity, dispersion, and now also the time derivative (the time derivative does enter the single soliton balance but in a trivial way). This balance is only slightly affected by the damping which, however, causes the conserved quantities to change according to the exact equations (10) and (11). In some complicated nonlinear way the conserved quantities then determine the eigenvalues through Eq. (7), at least when the number of variables equals the number of equations. In principle, there is an infinite number of equations such as Eq. (7), and we are faced with an

overdetermined system. For a single soliton, we know that the change of a single variable is consistent with at least two of these equations:[10] In view of the exact solution for damping exponent $b = 2$, the damping single soliton is even consistent with an infinity of equations (7).

Generalizing to double solitons we determine the eigenvalues by the minimum number of conservation laws, and ignore the higher ones.

The two time scales in our approach are then: i) the slow timescale of order ε due to the damping, and ii) the natural timescale determined by the nonlinear Schrödinger equation (1) and the initial condition. For breathers, this timescale is $\pi/[4(\eta_2^2 - \eta_1^2)]$, the period Ω^{-1} in Eq. (6). For stationary single solitons the timescale degenerates to infinity.

Just as other soliton perturbation theories the present approach is analytically prohibitively complicated for breathers, especially for damping exponent $b = 2$ when Eq. (11) applies: since the right hand sides are not reducible to conserved quantities they must be evaluated with the explicit functional form Eq. (6). Numerically, however, the integrations are straightforward.

Our numerical procedure is then as follows. We numerically compute the integrals in Eqs. (10) or (11) at a particular time for given eigenvalues η_1 and η_2 using Eq. (6). Then we change the conserved quantities over a small timestep Δt according to Eqs. (10) and (11), and recompute the eigenvalues at the next time $t + \Delta t$ from Eq. (7). Note that in Eq. (6) we should replace the time dependence Ωt by $\int^t \Omega(t')dt'$ in the spirit of the two-timescale assumption.

velocity has decreased, they are still overlapping at the end of the run, as seen in Fig. 3c for $\phi = 0$.

The case of collisional damping, $b = 0$, is unexciting because now I_1 does not depend on soliton shape and just shows exponential decay, while only I_3 changes slightly in time similar to I_1 for the case $b = 2$. The solution in x-space is a widening and diminishing version of the undamped case, with no change in the velocity.

These computations suggest that it is reasonable to make a two-time-scale expansion of soliton dynamics. Unfortunately, the intersoliton distance, which codetermines the soliton shape, does not appear in the conservation laws, but must be approximated by a temporal integration of the velocity. We avoid this complication in our study of the breather, where this intersoliton distance appears to remain zero.

IV. The Damped Breather

What is the influence of damping on the characteristics of the breather? We recall that the breather in the absence of damping, shown in Fig. 2, is a purely periodic solution with period $T = 2\pi/(4\eta_2^2 - 4\eta_1^2)$: with the imaginary parts of the inverse scattering eigenvalues $\eta_2 = 3/2$ and $\eta_1 = 1/2$, $T = \pi/4$. At time $t = 0$ (modulo T) the breather reduces to a soliton multiplied by two, $q(x,t = 0) = 2\ \mathrm{sech}(x)$. At half-periods, $t = T/2$, modulo T, the breather contracts to a very high and consequently narrow state with large derivatives.

When collisional damping is introduced and Eq. (8) applies the invariant I_1 is exponential in time, in agreement with Eq. (10a). Figure 5 shows this evolution, and the behaviour of the eigenvalues η_1 and η_2 as computed from the conservation laws. The eigenvalues are symmetric around $I_1 = 4\ (\eta_1 + \eta_2)$, and have an overall exponential decay with wiggles superimposed. The wiggles are due to the enhanced decrease of I_3 when the breather is in the contracted state. The derivatives are then large (see Fig. 2c), and therefore the right hand side of Eq. (10b) is large.

In x-space the breather approximately returns to its original shape, just like an undamped breather, but with decreased amplitude. This explains the increase in the separation between the wiggles of Fig. 5.

The numerical results are rather more complicated for the case b = 2, which for a single soliton was the simplest. In Fig. 6(a) we plot the invariant I_1 (solid line) and the eigenvalues (dashed lines) as functions of time. Again we notice the approximately periodic decrease of I_1, but now there is a strong step-like dependence of I_1 on time. The relatively

slow decrease of I_1 between the "steps" corresponds to full periods, when the undamped breather would be sech-shaped, while the "steps" themselves come from the contracted breather state. However, when we measure the elapsed time between two successive steps of I_1 and correct for the increase in timescale due to the decreased amplitude we find no exact periodicity. A qualitative reason for this apparent lack of periodicity is evident in the space plots of $|q(x)|^2$ shown in Fig. 7b and c. With damping the breather does not return to its original shape, but instead it seems to split in two soliton-like shapes that overlap only moderately. Thus the breather period increases because the mutual attraction between the two constituent solitons in the breather diminishes as their overlap decreases.

In Ref. 10 we used the time derivative of I_1^{-2} as a measure of soliton shape. This quantity is plotted in Fig. 6b. The various humps, which correspond to the steep decline in I_1 but are normalized with I_1^3, have nearly equal maxima and shapes. This seems to indicate that at least the contracted breather is approximately scaling invariant. With increasing time there is, however, a definite increase of amplitude and a widening of these now slightly asymmetric peaks.

The increase in period and the shape changes of the breather can be understood from the eigenvalues η_1 and η_2, given in Fig. 6(a) by the dashed lines. The smaller eigenvalue hovers around the initial value 1/2, but the larger eigenvalue, initially 3/2, decreases with similar but larger steps than those of I_1 (I_3, not shown, has an even stronger time dependence). Thus, the difference between the eigenvalues decreases, and hence the period increases, since $T \propto (\eta_2^2 - \eta_1^2)^{-1}$ for an undamped breather. The double-

humped shape of the damped breather is less easily understood, because the analytical formula is complicated, but it can easily be shown numerically that such shapes indeed originate from two eigenvalues that are close together. The dashed lines in Fig. 7b and c give a plot of Eq. (8) with approximately the eigenvalues at that particular time. These dashed curves are further discussed later on.

At later times than shown here, or for larger dampings, the two imaginary parts η_1 and η_2 of the eigenvalues ζ coalesce. At this point the eigenvalues acquire real parts ξ, which means that the constituent solitons have obtained a velocity and asymptotically separate.

This is demonstrated by Fig. 8a for larger damping strength $\varepsilon = 0.1$. Initially the eigenvalues and I_1 behave qualitatively as in Fig. 4 for $\varepsilon = 0.05$, but at $t = 3.1$ they coalesce and develop real parts ξ. The magnitude of ξ, proportional to the velocity, is indicated by the difference between the broken lines, the sum of the eigenvalues, and the solid line. The solitons at $t = 5$ are given in Fig. 8b: They are clearly well separated, and could very well separate completely for larger times. Whether they actually separate is of little practical importance, because the soliton amplitude decreases rapidly for this damping strength $\varepsilon = 0.1$.

As long as the η's differ the eigenvalue ζ can not develop a real part for the following reason: suppose that with $\eta_1 \neq \eta_2$ there would be a real part ξ to ζ_1 and ζ_2 at some particular time t'. The ζ's must be of opposite sign, on account of $I_2 = 16\ (\eta_1\xi_1 + \eta_2\xi_2) = 0$. Now remove the damping for times greater than t', so that Eq. (1) is again satisfied and the given initial condition evolves in such a way that the eigenvalues remain constant. Because of their opposite velocities

the solitons that constitute the breather must separate to eventually form two disjoint solitons that are unequal, since the η's differ. Thus we would have an asymmetric final solution. This can not happen because the initial condition $q(x,t = 0) = 2\ \mathrm{sech}(x)$, and both equations (1) and (9) are invariant under reflection $x \to -x$.

The solid line in Fig. 9 shows the eigenvalues plotted against each other to further clarify the breather's shape changes. The important parameter here is the ratio of eigenvalues η_2/η_1. The eigenvalue η_1 for b = 2 in Fig. 8b oscillates in a narrow band around 1/2 while η_2 decreases until both eigenvalues ζ obtain real parts. For b = 0, Fig. 8a, the two eigenvalues oscillate around the straight line $\eta_2 = 3\ \eta_1$. Thus in this case the ratio of eigenvalues η_2/η_1 remains at its initial value 3, and corroborates the recurrence of the initially sech-shaped breather. Note that the oscillation amplitude of the wiggles remains approximately constant in time for b = 2, Fig. 9b, but increases for b = 0, Fig. 7a. As explained in Ref. 10, this increase is due to the growth of the damping term relative to the other terms for decreasing soliton amplitude. In contrast, when Eq. (9) applies, the case b = 2, the damping term is always ε times the dispersion, and the oscillation amplitude remains constant.

All these results are obtained from a numerical solution of Eqs. (8) and (9), and the eigenvalues are computed from values of the conservation laws which, as we observe, change in time with large steps. Now we must establish the validity of the two-timescale assumption that forms the basis of the available perturbation theories, including our own in Section II.

Ideally, we should do this by comparison of our results with analytical formula of the kind written down formally in Refs. 14-16, or in Section II. The analytical evaluation of such expressions for the breather is, however, prohibitively complicated and unrevealing. Therefore we compare instead with an additional numerical computation which assumes that at time t there exists a breather solution of the form given in Eq. (6), with eigenvalues $\eta_1(t)$ and $\eta_2(t)$ slowly changing functions of time as discussed in Section II.

Results from this computation are shown in Figs. (6), (7) and (9) by the dashed lines. They are in good agreement with those from a computation of the equations (8) or (9), given in the solid lines.

The breather shapes in Fig. 7(c) at t = 3 agree much better than those in Fig. 7b at t = 1. This is due to a small shift in the times between the two computations evident in Fig. 6b. At the stage t = 1 of breather evolution this shift produces a visible effect on $|q(x)|^2$, but at t = 3 where the eigenvalues are more equal and hence the period is larger the difference between the $|q(x)|^2$ is minimal.

The eigenvalue η_1 from the full computation is consistently larger than η_1 computed through the conservation laws, for equal η_2. We attribute this difference to second order shape changes of the breather in the full equation. These will tend to diminish the change in time of especially I_3 which, in turn, is mostly reflected in a smaller change of η_2. Therefore, η_2 in the full computation lags behind the corresponding value from the conservation laws, in which second order shape changes are excluded.

Our arguments here are patterned after those for single solitons in Section IV of Ref. 10, but for obvious reasons we do not attempt any quantitative analysis.

V. Conclusion

The good agreement between the two sets of computations demonstrates the correctness of our soliton perturbation theory, at least for the breather with two eigenvalues and with damping as perturbation. We recall that our approach is slightly restricted by the exclusion of radiation, and by the lack of intersoliton distance in the conservation laws. Our approach does have the essential feature of all soliton perturbation theories, namely the two timescale assumption. However, there is no particular reason besides numerical convenience and our familiarity with conservation laws to prefer their use over other approaches, nor is there anything special about breathers (again, except for convenience as noted earlier). In contrast to our method, in existing perturbation theories[14-16] damping is not singled out as a particularly suitable perturbation. Thus it seems that the two-timescale assumption that we have verified for damping will be valid for more general perturbations; such perturbations, then, would not destroy the existence or even change the value of the eigenvalues, but they may affect the soliton shape. An example is an extra term[9] $|q|^4 q$ in Eq. (1).

Hence we conclude that soliton perturbation theories, although justified, do not seem practical at present for anything but single solitons. Even our implementations of the two numerical methods compared in this paper used comparable amounts of computer time (5-10 seconds on a CDC7600, for the same time step $\Delta t = 0.005$ and number of grid points 128). Much additional work will be needed to develop additional approximations that increase the usefulness of soliton perturbation theories for multisolitons.

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FIGURE CAPTIONS

Fig. 1. Undamped soliton collision. Parameters are $2\eta = 1.5$ and $V = \pm 0.75$.

a) Initial condition,

b) collision stage at $t = 2.5$ with no initial phase difference $\phi = 0$,

c) collision stage at $t = 2.5$ with initial phase difference $\phi = \pi/2$.

Fig. 2. The undamped breather at three stages of its periodic development

a) $t = 0$,

b) $\Omega t = \pi/2$,

c) $\Omega t = \pi$.

Fig. 3. Damped soliton collision with damping parameters $b = 2$ and $\varepsilon = 0.1$.

a) Initial condition with eigenvalues $2\eta = 1.5$ and $2\xi = \pm 0.75$,

b) solution at $t = 2.5$ for $\phi = 0$,

c) solution at $t = 2.5$ for $\phi = \pi/2$.

Fig. 4. $I_1 = \int |q|^2 \, dx$ versus time for the cases of Fig. 1.

Fig. 5. $I_1 = \int |q|^2 \, dx$ and the eigenvalues η_1 and η_2 versus time for a damped breather, with damping parameters $b = 0$ and $\varepsilon = 0.1$.

Fig. 6. a) $I_1 = \int |q|^2 \, dx$ and the eigenvalues η_1 and η_2 for a damped breather, with damping parameters $b = 2$ and $\varepsilon = 0.05$.

b) The soliton shape measure dI_1^{-2}/dt versus time from Eq. (9) (solid line) and from the two-timescale assumption (broken line).

Fig. 7. Shapes of a damped breather, with damping parameters $b = 2$ and $\varepsilon = 0.05$.

a) Initial condition $q(x) = 2 \operatorname{sech}(x)$,

b) at $t = 1$, from Eq. (9) (solid line) and from the two-timescale assumption (dashed line),

c) at $t = 3$.

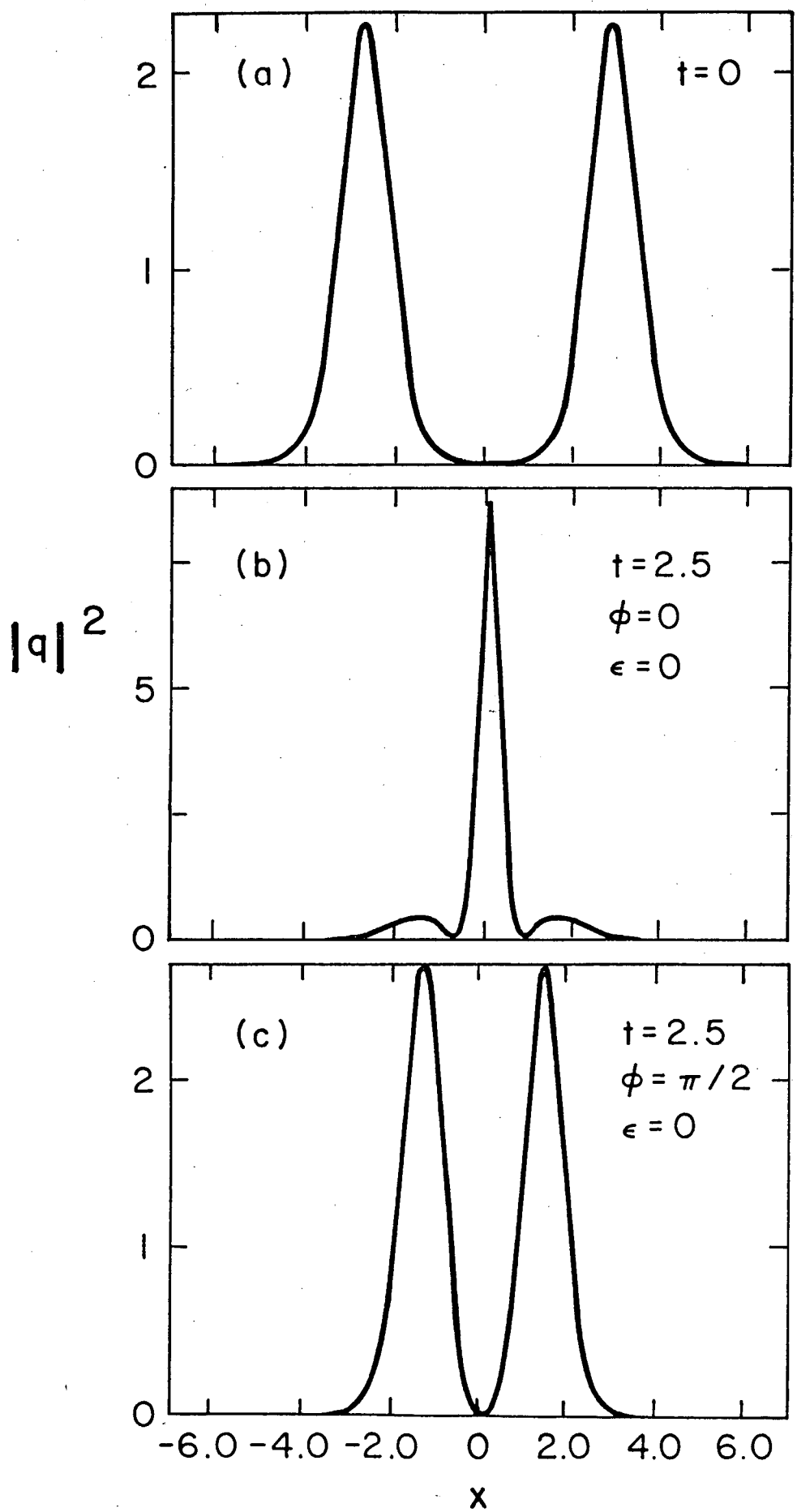
Fig. 8. a) $I_1 = \int |q|^2 \, dx$ and the imaginary part of the eigenvalues, η, and the real parts of the eigenvalues, ξ, for a damped breather with damping parameters $b = 2$ and $\pi = 0.1$,

b) the breather shape at $t = 5$.

Fig. 9. The breather eigenvalues η_1 and η_2 plotted against each other, from Eq. (9) (solid line) and the two-timescale assumption (broken line).

a) From Fig. 3,

b) from Fig. 4.



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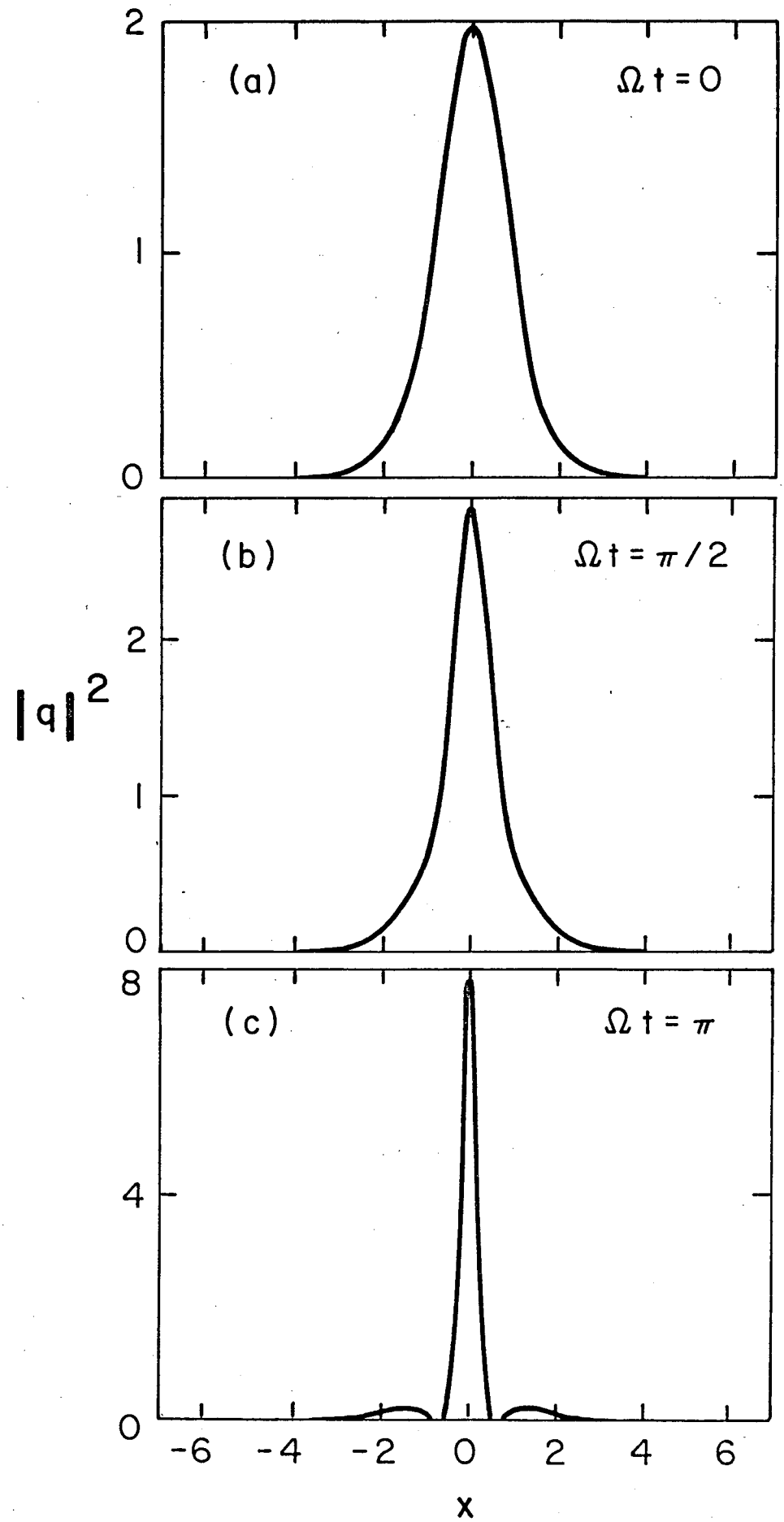

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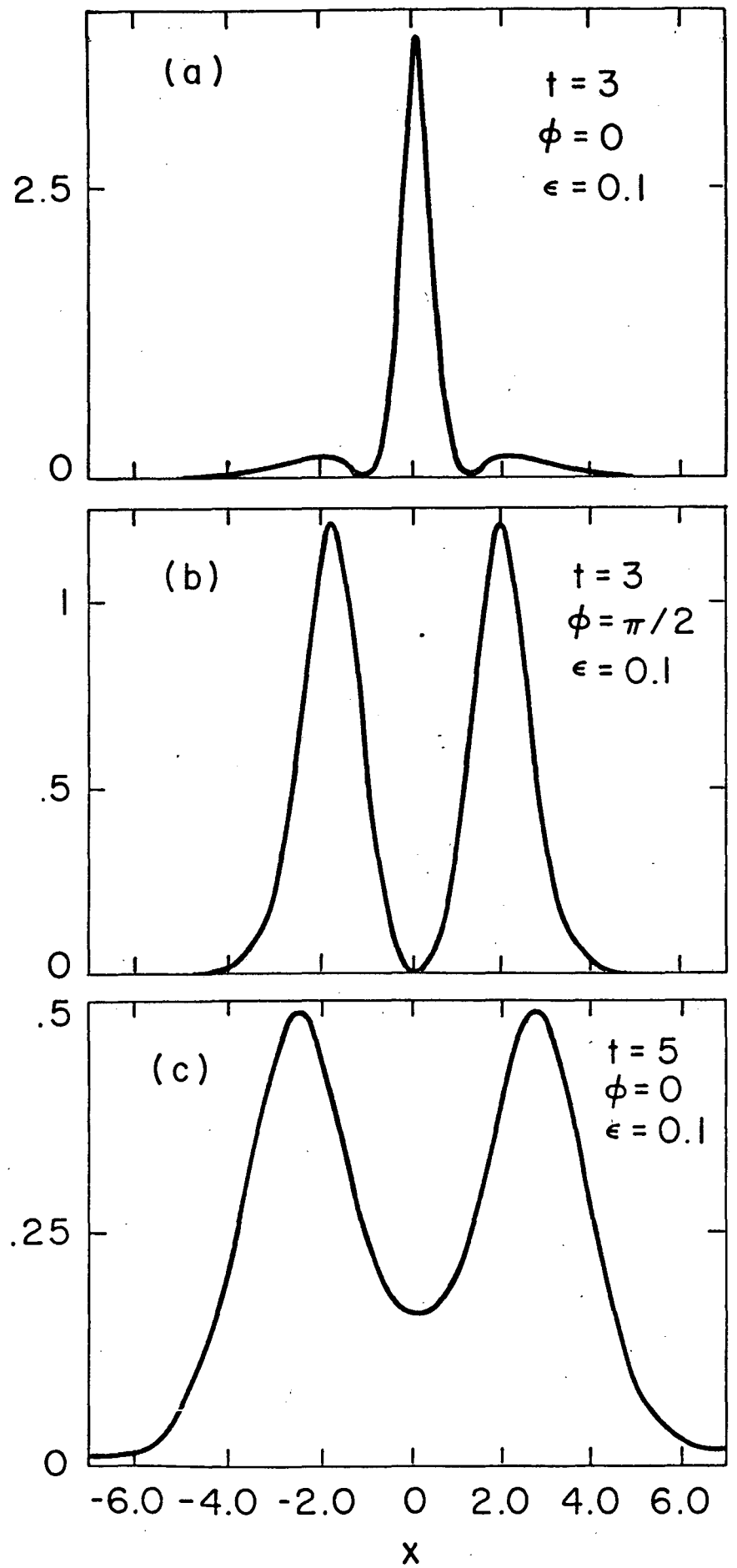

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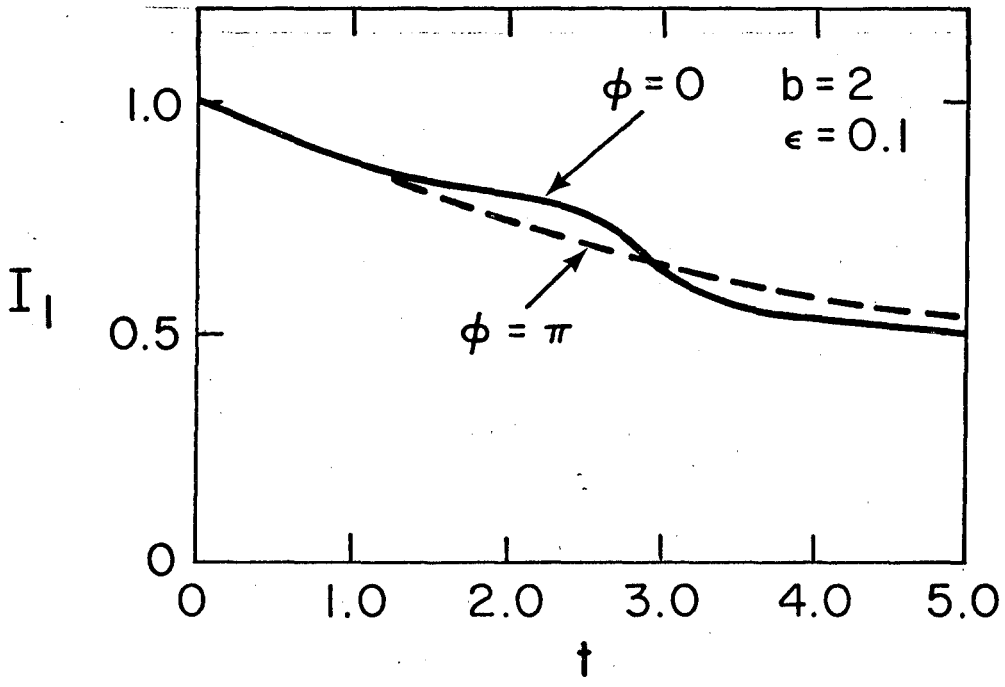

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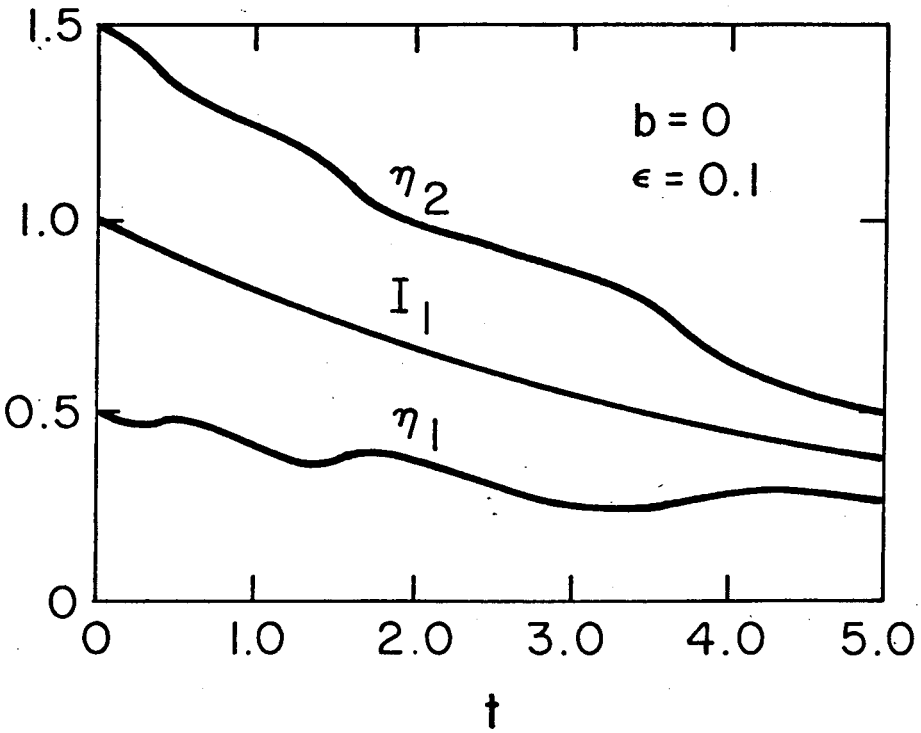

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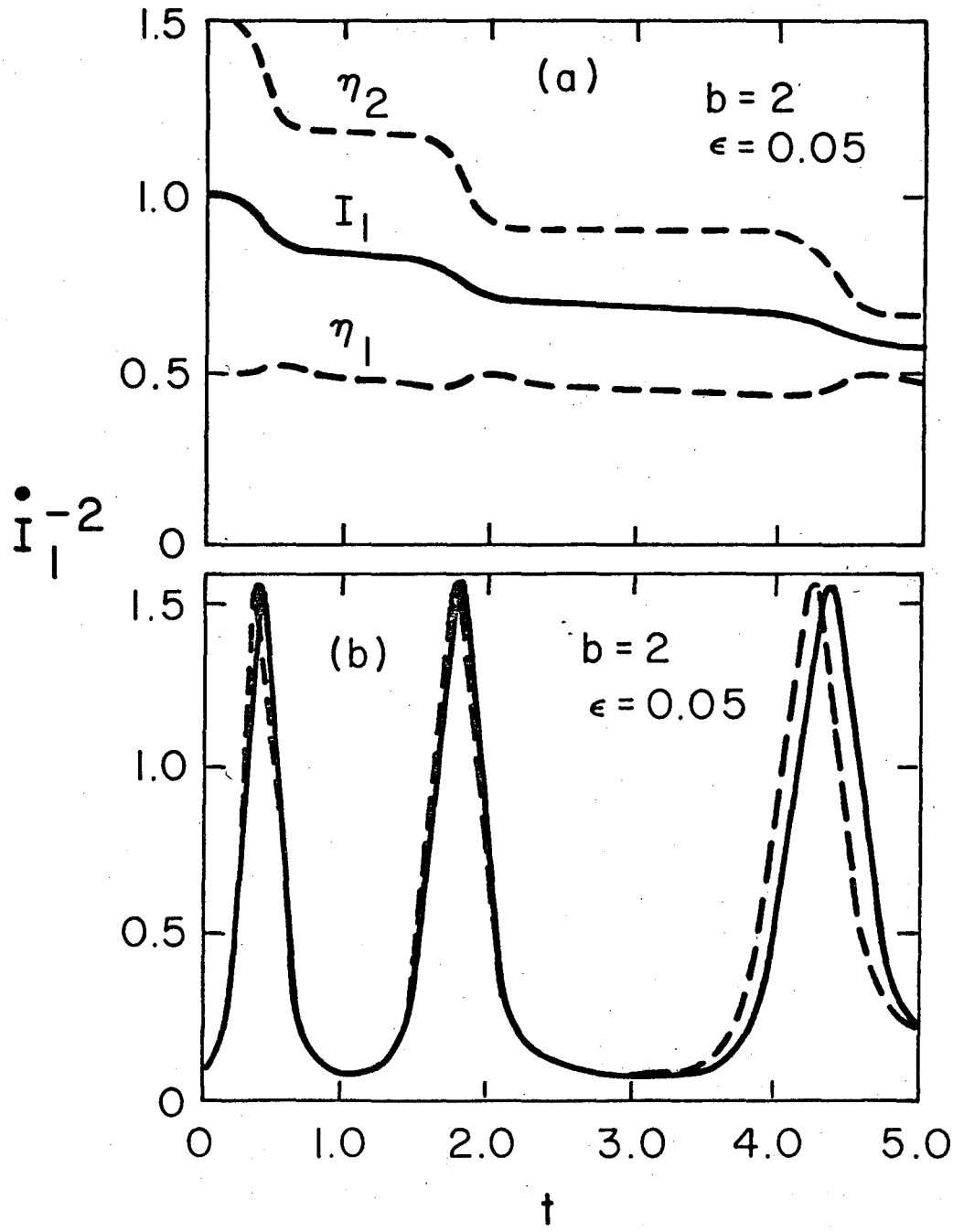

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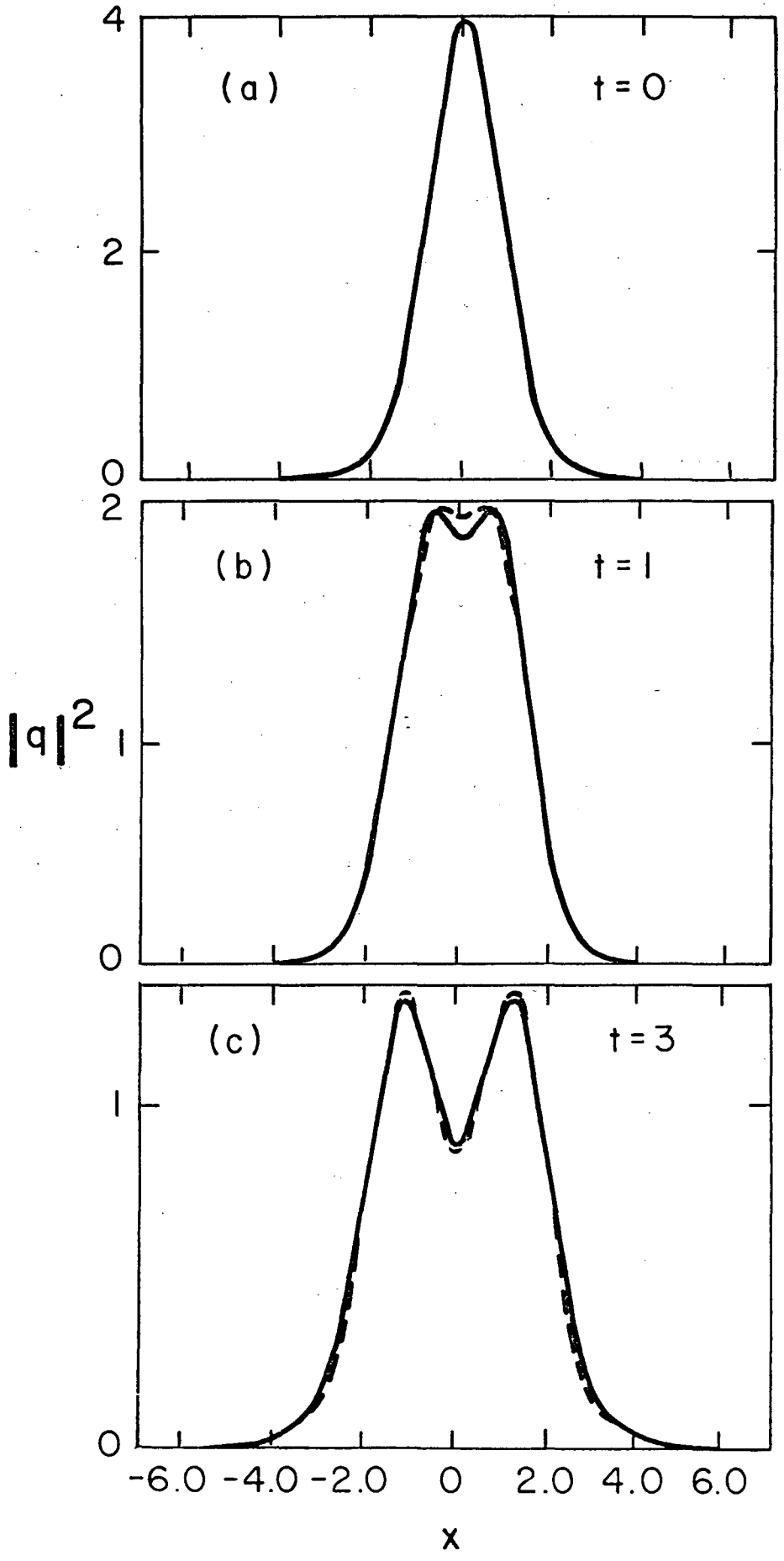

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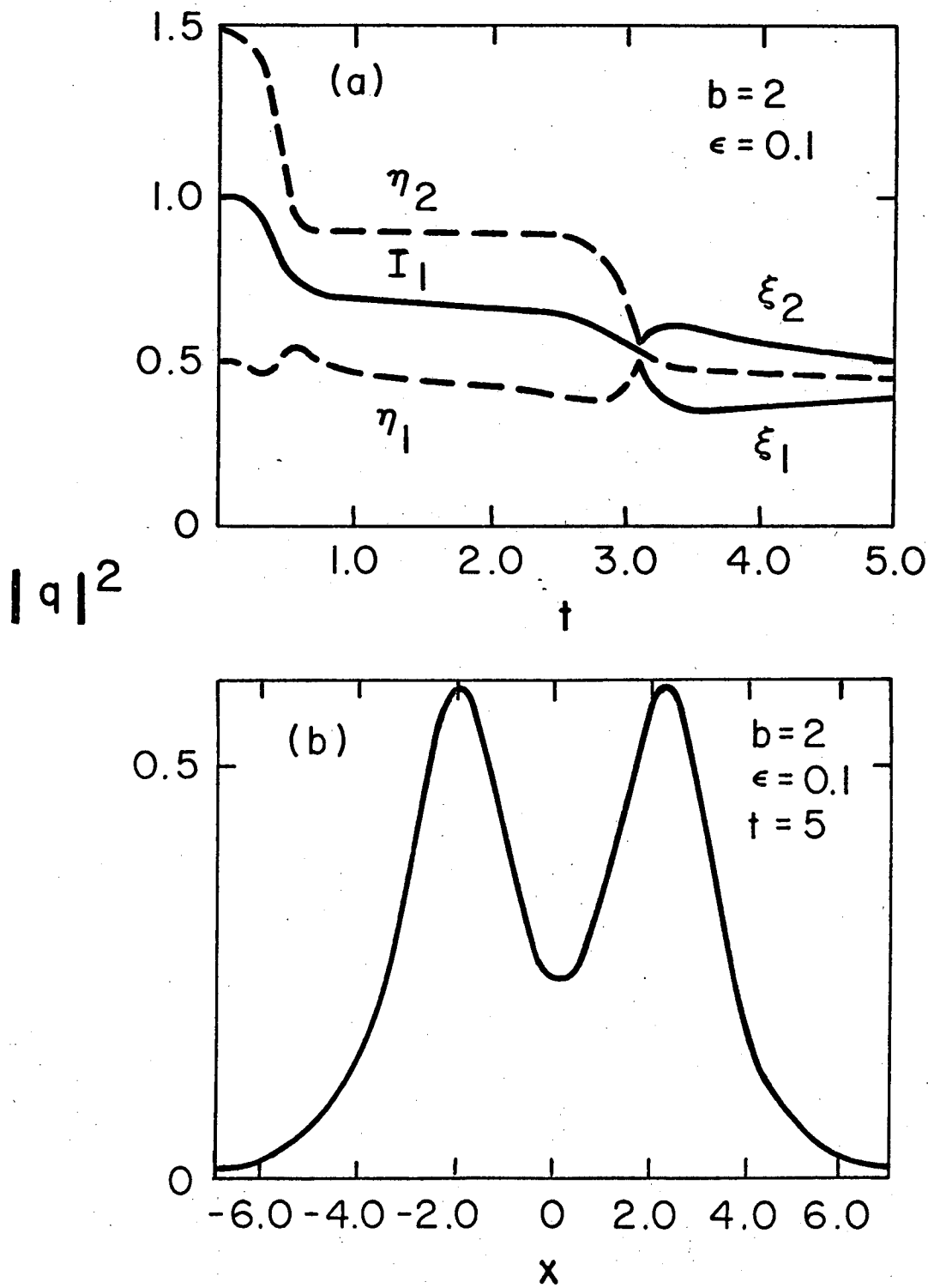

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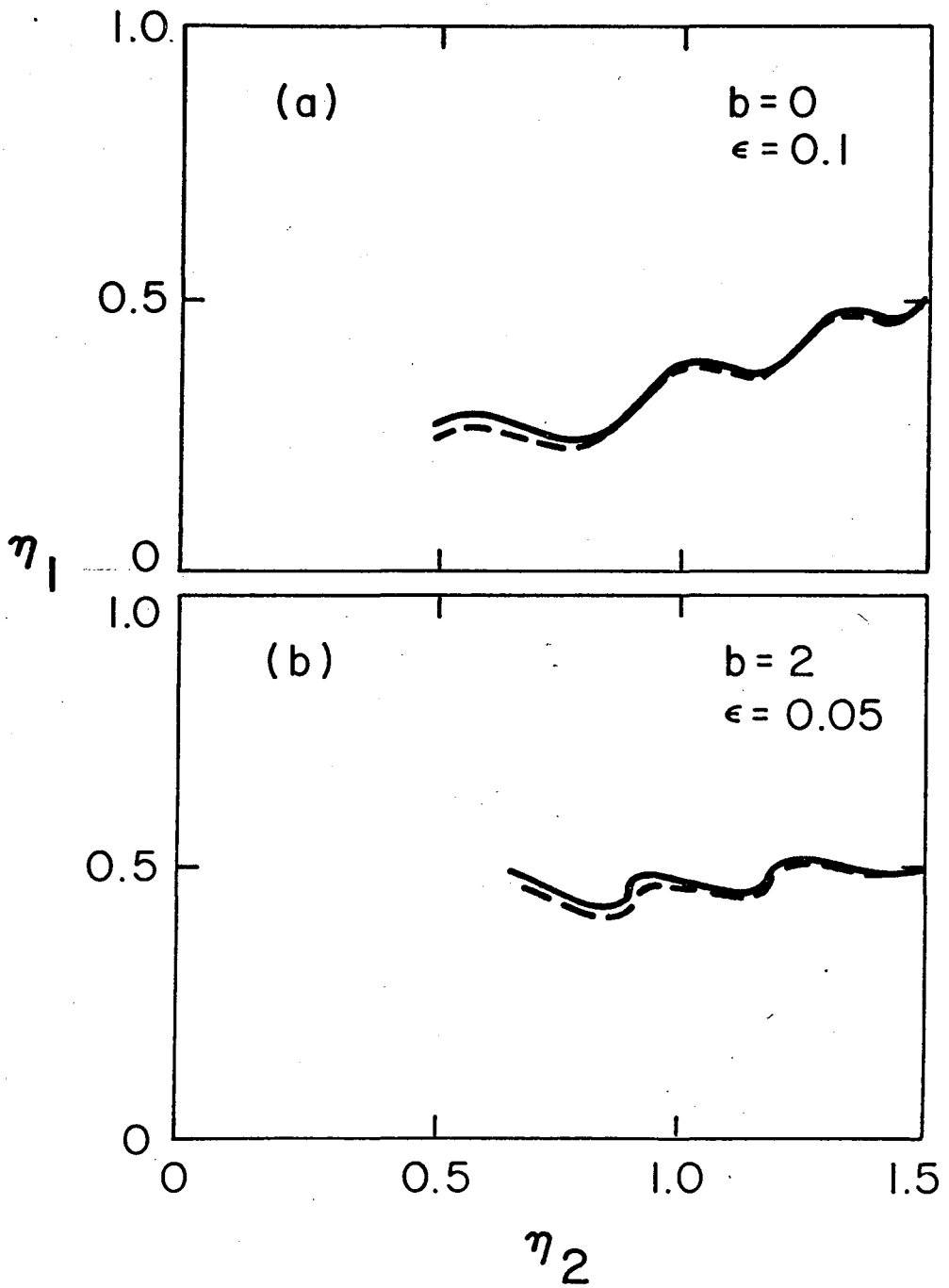

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