NONLINEAR WAVE EVOLUTION IN VLASOV PLASMA:

A LIE-TRANSFORM ANALYSIS

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ABSTRACT

Nonlinear wave evolution in Vlasov plasma is analyzed using the Lie transform, a powerful mathematical tool which is applicable to Hamiltonian systems. The first part of this thesis is an exposition of the Lie transform. Dewar's general Lie transform theory is explained and is used to construct Deprit's Lie transform perturbation technique. The basic theory is illustrated by simple examples.

Two general and remarkable results are obtained by applying Lie transform theory to the Vlasov equation. The first is a useful relation between the ponderomotive Hamiltonian (the kinetic generalization of the ponderomotive potential) and the linear susceptibility of Vlasov plasma. This general relation then yields a second result, a general formula for the second-order self-consistent quasistatic density perturbation produced by a wave packet. For a plasma with one species of single charged ions this formula is $\delta n(x) = -(4\pi)^{-1}[|\underset{\sim}{\mathcal{E}}(\underset{\sim}{x})|^2 - |\underset{\sim}{\mathcal{B}}(\underset{\sim}{x})|^2]/(T_e+T_i)$. This formula, which includes kinetic effects, applies to both magnetized and unmagnetized plasma.

Lie transform theory is then applied to two problems, the first of which is to find the nonlinear wave equation that governs waves in unmagnetized plasma. We derive the formula of Mora and Pellat for the wave-generated quasistatic magnetic field, and via systematic perturbation theory we find the equation which gives the nonlinear evolution of the wave amplitude. This equation unifies previous work: it includes the ponderomotive effects of Kaw, Schmidt, and Wilcox, the self-precession effect of Arons and Max, and the (previously neglected) effect of self Faraday rotation by the quasistatic magnetic field. The

calculation shows that the self Faraday rotation effect is comparable to the self precession effect. This fact implies that a measurement of the polarization rotation acquired by a nonlinear electromagnetic wave in passing through a plasma does not determine the self-generated quasistatic magnetic field.

The second problem considered is the propagation of a test wave in the system consisting of a plasma and a large amplitude electromagnetic wave. We show that the change in the polarization of the test wave, as it propagates through the laser-irradiated plasma, consists of several effects. In addition to Faraday rotation by the laser-generated quasistatic magnetic field, there are other effects which are not simply a rotation of the test-wave polarization. For example, the test wave polarization may change from plane to elliptical. These results indicate that one may obtain unreliable results from the technique of measuring quasistatic magnetic fields through the Faraday rotation of test waves.

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1. Introduction

This thesis is an application of the Lie transform to the analysis of nonlinear wave evolution in Vlasov plasma. The Lie transform is a new mathematical tool which simplifies the analysis of a Hamiltonian system, such as the Vlasov equation. Here the Lie transform is used to discuss nonlinear wave evolution, by which we mean the nonlinear development of the amplitude of a single wave or wave packet. We do not consider other nonlinear processes, such as three wave coupling or induced scattering.

1.1 Motivation and Background

There are many practical reasons for studying the nonlinear evolution of waves in plasma. The laser-fusion schemes require a large-amplitude, and therefore nonlinear, wave to interact with plasma. Filamentation, a possible outcome of nonlinear wave evolution, may play an important role in the laser-fusion scheme.[1] Radiofrequency heating[2,3] of magnetically confined plasma to fusion temperature also requires the presence of a large amplitude wave in plasma. Morales and Lee[4] have suggested that the lower-hybrid heating scheme may be limited by filamentation, which leads to intense localized electric fields. They note,[5] "The interaction of plasma particles with these fields may alter the transport of energy to the interior of the plasma, . . ." Of course, if energy transport is altered for the worse, the lower-hybrid heating scheme will have to be abandoned. Another reason for studying nonlinearity is to understand the evolution of a linearly unstable system. For example, Aamodt, Lee, Liu, and Rosenbluth[6] have shown that the nonlinear gyrofrequency shift causes the ion-cyclotron-loss-cone mode to

saturate.

In addition one might study nonlinear wave evolution in order to understand exactly solvable nonlinear wave equations, such as the Korteweg-deVries equation,[7,8] the nonlinear Schrödinger equation,[9] and the modified Korteweg-de Vries equation.[10,11] All of these equations are derived by invoking tain approximations. If we relax these approximations do we destroy exact solvability?

The ponderomotive potential concept,[12] which arises in nonlinear wave evolution, deserves special study. The ponderomotive potential concept provides an elegant way of describing the average motion of a particle in a high-frequency wave. There have been several proposals for using the ponderomotive potential to confine plasma[13] either entirely or partially.[13-15] Here the idea is not that nonlinearity may cause instability (such as filamentation) or saturate a linear instability, but that waves might be driven to large amplitudes so that nonlinearity may be exploited to advantage. It is natural to try to determine when the ponderomotive potential concept breaks down, and what concept replaces it; e.g., must kinetic effects be described by a diffusion operator?[16] Or is there a more elegant and convenient formalism?

Early work on nonlinear wave evolution concentrated on plane waves, i.e., no amplitude modulation. Montgomery and Tidman[17] applied systematic perturbation theory to the cold-plasma equations. They computed the nonlinear conductivity of the plasma and the nonlinear frequency shift of transverse waves. Later, Sluijter and Montgomery[18] showed that relativistic effects are important. A recent example of nonlinear plane-wave calculation is the work of Arons and Max,[19] who discussed

electromagnetic wave propagation in a relativistic Vlasov plasma.

The necessity of allowing for amplitude modulation was demonstrated by Kaw, Schmidt, and Wilcox.[1] When the wave amplitude has a gradient, no matter how small, there is a quasistatic density response $\delta n \approx (8\pi)^{-1} (e^2/m\omega^2) (E^2)/(T_e+T_i)$. The change in the plasma conductivity due to this density response is much larger than the nonlinear conductivity found by earlier workers.[17,18] However, the analysis of Kaw, Schmidt, and Wilcox is not systematic. They simply modify the linear susceptibility by replacing the unperturbed density n_o by the nonlinearly corrected density $n_o+\delta n$. One cannot obtain the previously known nonlinear conductivity from the formalism of Kaw, Schmidt, and Wilcox.

This heuristic method of Kaw, Schmidt, and Wilcox for including nonlinearity has since been used by others. Morales and Lee[4] modified the equation which describes the linear propagation of lower-hybrid waves by changing the unperturbed density to the nonlinearly corrected density. Aamodt, Lee, Liu, and Rosenbluth[16] used this method to include the nonlinear gyrofrequency shift in the evolution of ion cyclotron waves. This heuristic method has great appeal because it has a simple interpretation and it is easy to use. However, the heuristic method is not systematic, and so it may leave out important effects.

A systematic perturbation method for including nonlinearity has appeared in the literature (see e.g. ref. 20). This method assumes a perturbation which is a sum of two waves with nearly equal wave-vectors and frequencies, e.g., $\phi = a \exp(i\underset{\sim}{k}_1\cdot\underset{\sim}{x}-i\omega_1 t) + b \exp(i\underset{\sim}{k}_2\cdot\underset{\sim}{x}-i\omega_2 t) + c.c.$ Such a perturbation can be rewritten in the form of a sinusoidal modulation of a single wave: $\phi = [a \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}-i\Omega t) + b \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{x}+i\Omega t)] \exp$

$(i\underset{\sim}{k}_0\cdot\underset{\sim}{x}-i\omega_0 t)$ + c.c, where $\underset{\sim}{k}_0 = (\underset{\sim}{k}_1+\underset{\sim}{k}_2)/2$, $\omega_0 = (\omega_1+\omega_2)/2$, $\underset{\sim}{k} = \underset{\sim}{k}_1-\underset{\sim}{k}_0$, and $\Omega = \omega_1-\omega_0$. By systematic perturbation theory one can derive the nonlinear dispersion relation $\Omega(\underset{\sim}{k})$. Unfortunately this method cannot describe non-sinusoidal modulations, such as solitons. (Since the equations are nonlinear, superposition does not apply.)

All of these analyses are characterized by the neglect of collisions. *This neglect can usually be justified a posteriori by showing* that the mean free path is long compared to a filamentation length or by showing that the collision frequency is small compared to the nonlinear frequency shift. Such justification is rarely provided in the literature, but it can usually be produced with a modest effort.

We summarize the present understanding of nonlinear wave evolution as follows. The nonlinear evolution of plane waves and of sinusoidally modulated plane waves has been analyzed by systematic perturbation theory, but one must be able to treat non-sinusoidal modulations in order to include solitons and filamentation. To date the analyses of non-sinusoidal modulations have been heuristic. Thus we are motivated to analyze nonlinear wave evolution by some new procedure, which is systematic and which can include arbitrary amplitude modulations. Furthermore, since collisions are not important, while kinetic effects are, our procedure must involve the analysis of the Vlasov-Maxwell equations.

Since we must analyze the Vlasov equation, which is governed by a Hamiltonian, it is advantageous to use Hamiltonian (or canonical) perturbation theory. Recently, Hori[21] introduced Lie-transform perturbation theory, a great improvement over the older Poincaré-Von Zeipel method.[22] Deprit[23] further improved Hori's formulation so that it could

be conveniently applied to time-dependent systems. Deprit also derived recursion relations between the terms in the Lie series. Then Dewar[24] improved Deprit's work by eliminating the need for power series expansions.

Dewar[25] applied canonical perturbation theory to the Vlasov equation. At that time he was not aware of Lie transforms, and so he used the older Poincaré-Von Zeipel technique. In that work Dewar reformulated quasilinear theory. Since that time, Johnston[26] used Dewar's technique to discuss induced scattering, and Johnston, Kaufman, and Johnston[27] used Dewar's technique to derive three-wave coupling coefficients for Vlasov plasma. These papers proved the worth of canonical perturbation theory, even in the Poincaré-Von Zeipel form. It follows that Lie transform perturbation theory should further simplify the analysis of the Vlasov equation. (We note that Kawakami[28] applied Hori's Lie-transform theory to the Vlasov equation, and Dewar[24] applied Deprit's Lie transform theory to the Vlasov equation.)

Thus we have a problem and a new method for analyzing this problem. The problem is to present a systematic discussion of nonlinear wave evolution, including ponderomotive effects and kinetic effects. The new method is the Lie transform.

1.2 Synopsis

Lie transform theory is used throughout this thesis with the exception of Chapter 6. Accordingly, we begin with an exposition of Lie transform theory in Chapters 2-5. Chapter 2 is a discussion of preliminary material. Here we discuss the basic ideas of Hamiltonian theory. Chapter 3 contains a review of Dewar's Lie transform theory. We

introduce the Lie transform, we show how it is generated by a phase function $w(\underset{\sim}{q},\underset{\sim}{p})$, and we show how to find the new Hamiltonian from the Lie transform. Finally, we show how to construct Deprit's Hamiltonian perturbation theory from Dewar's Lie transform theory. (Historically the development was in the opposite order.)

In sections 2.3, 3.4, and 4.4 and Chapter 5 we discuss peripheral issues, which are not important to the later development, and which may be skipped in a first reading of this thesis. Section 2.3 is a discussion of the simplest (but secular) perturbation theory, which is equivalent to the usual time dependent perturbation theory of quantum mechanics.[30] In section 3.4 we show how to prove Noether's theorem by using Lie transforms. In section 4.4 we show that Kolmogorov's (superconvergent) technique[31] does not lead to a simpler perturbation theory in practice, as has been suggested by Howland.[32] Finally, in Chapter 5 we give a short discussion of the various conventions used in Lie transform theory.

In Chapter 6 we begin the discussion of nonlinear wave evolution. The goal is to go as far as possible without specifying the medium, except to say that it has a linear response tensor $\underset{\approx}{D}(\underset{\sim}{k},\omega) \equiv 1 + \underset{\approx}{\chi}(\underset{\sim}{k},\omega) - k^2c^2/\omega^2 + \underset{\sim}{k}\underset{\sim}{k}c^2/\omega^2$. We first consider wave energy and wave momentum. We prove that these concepts apply not only to normal modes ($\det \underset{\approx}{D} = 0$) but also to any nearly monochromatic perturbation. The techniques used in this proof are then applied to collisionless plasma to show that wave energy and wave momentum can be divided into contributions from the electromagnetic field and from the separate species. Next we consider the nonlinear evolution of wave packets. We show how to systematically

derive nonlinear wave equations in terms of the nonlinear current. (The nonlinear current itself is computed in later chapters to complete the derivation of the nonlinear wave equation.) We also outline the heuristic method of references 1, 4, and 6 for including nonlinearity and we show how it is related to the systematic method.

In Chapter 7 we discuss general aspects of nonlinear wave evolution in Vlasov plasma. We first write the Vlasov equation in Hamiltonian form. Then we use Lie transforms to eliminate the rapid oscillations from the problem. As a result we obtain a closed set of equations for the evolution of the wave packet amplitude, the nonlinear fields, and the oscillation-center distribution, i.e., the transformed Vlasov function which has only slow variations.

A quantity of great importance to this analysis is the ponderomotive Hamiltonian, which governs the evolution of the oscillation center distribution. The ponderomotive Hamiltonian is the kinetic generalization of the ponderomotive potential, since it contains the momentum dependence of the ponderomotive force. In section 7.4 we prove a very general theorem, which relates the ponderomotive Hamiltonian to the linear susceptibility. A special case of this theorem is the expression $\psi_s = -(4\pi m_s)^{-1}\underset{\sim}{\mathcal{E}}_1^* \cdot \underset{\sim}{\chi}_s(\omega)\cdot\underset{\sim}{\mathcal{E}}_1$ which relates the ponderomotive potential ψ_s of species s to the cold plasma susceptibility of species s, where $\underset{\sim}{\mathcal{E}}_1$ is the amplitude of the oscillating field, $\underset{\sim}{E}_1(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)\exp(-i\omega t) + c.c$ (For unmagnetized plasma, simply use $\chi_s = 4\pi n_s e^2/(m_s\omega^2)$ to obtain the standard result[12] $\psi_s = e^2|\underset{\sim}{\mathcal{E}}_1|^2/(m_s\omega^2)$.) The general theorem is applicable to any Vlasov system (hot, magnetized, inhomogeneous, time-dependent) upon excluding resonant particles.

In Chapter 8 we use these methods to analyze nonlinear wave evolution in unmagnetized uniform Vlasov plasma. Our systematic analysis unifies previous work in that it contains the self-generated magnetic field of Mora and Pellat,[33] the ponderomotive effects found by Kaw, Schmidt, and Wilcox,[1] and the self-precession effect of Arons and Max.[19] In addition it contains the (previously neglected) self Faraday rotation which is caused by the self-generated magnetic field. Of course, one could have inclnded self Faraday rotation by using the heuristic method, i.e., one simply modifies the linear susceptibility to include the self-generated magnetic field. However, this method cannot include the self-precession effect. Since the self-precession effect and the self Faraday rotation effect are comparable, we conclude (in Chapter 8) that the heuristic method for including nonlinearity is wrong in that it leaves out effects which are comparable to those it keeps.

The fact that self precession and self Faraday rotation are comparable led to the study of Chapter 9. Here we determine how a test wave propagates through a laser irradiated plasma. We expect the laser-generated magnetic field to cause Faraday rotation of the test wave, but in analogy with Chapter 8 we also expect to find additional terms which cause polarization precession. In fact, we find many other effects which can completely dominate Faraday rotation. These other, more complicated, effects do not consist of just polarization precession; they can also cause the wave polarization to change, for example, from plane to elliptical.

In Chapter 10 we discuss some aspects of nonlinear wave propagation in magnetized plasma. We derive the ponderomotive Hamiltonian

for a magnetized particle. Here the elegance of the Hamiltonian methods becomes apparent, since the ponderomotive Hamiltonian contains not only the ponderomotive force,[12] and the ponderomotive drifts (with kinetic corrections), but also the nonlinear gyrofrequency shift, which plays an important role in the nonlinear saturation of the ion-cyclotron-loss-cone mode.[6] Next we find a general formula for the quasistatic response to the ponderomotive Hamiltonian. For a single ion species (with $e_i = -e_e$) plasma, this formula is $\delta n = (4\pi)^{-1}(|\underset{\sim}{\mathcal{E}}_1|^2 - |\underset{\sim}{\mathcal{B}}_1|^2)/(T_e+T_i)$. This formula applies to magnetized and unmagnetized, Vlasov plasma if (1) $|\nabla \ln|\mathcal{E}_1||<<\underset{\sim}{k}$ and (2) resonant particle effects are unimportant. Finally, we use the heuristic method (recognizing its limitations) to include this density perturbation in the propagation of wave packets.

2. Notation and Background

In this chapter we lay the foundation for later work. We briefly discuss the elements of Hamiltonian mechanics: Poisson brackets and canonical transformations. This discussion is not intended as a substitute for those found in the textbooks on this subject [1-3, 13,14]. It is mostly a repetition of standard material, but rewritten using operator notation, which proves very convenient for discussing Lie transforms.

Using the operator notation we discuss the time development transformation generated by a Hamiltonian. We show how it can be used to solve Liouville's equation. Finally we discuss how it can be found using secular perturbation theory, in analogy with quantum mechanics.

2.1 Canonical Transformations

We work in a 2N dimensional space. The variables are collectively denoted by the vector $\underset{\sim}{z}$. The components of $\underset{\sim}{z}$ are $z_1, \ldots, z_N = q_1, \ldots, q_N$ and $z_{N+1}, \ldots, z_{2N} = p_1, \ldots, p_N$. The Poisson bracket relations are contained in the matrix γ defined by

$$\gamma_{ij} \equiv \{z_i, z_j\} = \begin{cases} 1 \text{ for } j = i+N \\ -1 \text{ for } i = j+N \\ 0 \text{ otherwise} \end{cases} \tag{2.1.1}$$

By inspection we see that the matrix γ is antisymmetric and invertible:

$$\gamma_{ij} = -\gamma_{ji} \tag{2.1.2}$$

$$-\sum_j \gamma_{ij}\, \gamma_{jk} = \delta_{ik} \tag{2.1.3}$$

We now consider a time dependent mapping of the 2N variables by the vector function $\underset{\sim}{Z}(\underset{\sim}{z},t)$. We call this mapping canonical if the new variables have the same Poisson bracket relations as the old. The Poisson bracket of two functions f and g is generally given by

$$\{f, g\} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} = \sum_{\substack{i=1\\ j=1}}^{2N} \frac{\partial f}{\partial z_i}\, \gamma_{ij}\, \frac{\partial g}{\partial z_j} . \tag{2.1.4}$$

Thus, the mapping $\underset{\sim}{Z}(\underset{\sim}{z},t)$ is canonical if

$$\{Z_i, Z_j\} = \{z_i, z_j\} \tag{2.1.5}$$

or

$$\sum_{k,\ell} \frac{\partial Z_i}{\partial z_k}\, \gamma_{k\ell}\, \frac{\partial Z_j}{\partial z_\ell} = \gamma_{ij}$$

We take it to be well known that a canonical mapping preserves the Hamiltonian equations of motion.

Given any canonical mapping $\underset{\sim}{Z}(\underset{\sim}{z},t)$ of the phase space variables, we define a corresponding canonical transformation operator T(t) which operates on functions of phase space. The action of this operator is that it evaluates the function at the mapped point. If f = Tg, then

$$f(\underset{\sim}{z},t) \equiv g(\underset{\sim}{Z}(\underset{\sim}{z},t),\ t). \tag{2.1.6}$$

A particular example of this is when $g(\underset{\sim}{z},t) = z_i$, one of the canonical variables. (Strictly speaking we mean the function z_i, not the variable; if there were only one variable z, this would be the identity function, g(z)= z.) By our rule for Tg, we have

$$T\underset{\sim}{z} = \underset{\sim}{Z}(\underset{\sim}{z},t) \quad . \tag{2.1.7}$$

We see that the mapping is obtained by applying the transformation to the variables (but thought of as functions).

We also introduce an operator to denote the Poisson bracket. To every phase space function $f(\underset{\sim}{z},t)$ we assign an operator L_f whose action on a function g is given by

$$L_f\, g = \{f,g\} \quad . \tag{2.1.8}$$

Some authors have defined L_f with a minus sign relative to this choice. We have made this choice so that Jacobi's identity in operator form is easy to remember:

$$[L_f,\ L_g] = L_{\{f,g\}} \ . \tag{2.1.9}$$

In the above equation the square brackets denote the commutator of the two operators.

We note that the mapping $f \to L_f$ is linear and unique, but it is not invertible. If we are given two operators $L_f = L_g$, we cannot conclude that f=g. Since in this case $L_f z_j = L_g z_i$, we have

$$\frac{\partial f}{\partial z_j} = \frac{\partial g}{\partial z_j} \quad \text{for all } j. \tag{2.1.10}$$

Thus we conclude that the difference (f-g) has no functional dependence on the canonical variables, but it may depend on time or other parameters.

To illustrate how the use of the operator notation simplifies equations, we consider the statement that Poisson bracket relations are preserved by canonical transformations. In the non operator notation the equation is written

$$\left\{f[\underset{\sim}{Z}(\underset{\sim}{z},t),t],g[\underset{\sim}{Z}(\underset{\sim}{z},t),t]\right\} = \{f,g\}\Big|_{\underset{\sim}{Z}(\underset{\sim}{z},t),t} . \tag{2.1.11}$$

In words, transforming the functions and then forming the Poisson bracket gives the same result as first forming the Poisson bracket and then transforming. In operator notation this statement is written

$$L_{Tf}\, Tg = TL_f g . \tag{2.1.12}$$

Since this is true for any g, we have the operator relation,

$$TL_f = L_{Tf}T \quad \text{or} \quad TL_f T^{-1} = L_{Tf} , \tag{2.1.13}$$

for any canonical transformation T, and any phase space function f.

When evaluating a Poisson bracket, such as the one on the left of (2.1.11), we will need to use the chain rule to calculate derivatives. To avoid confusion, we introduce a very explicit notation for derivatives. By the symbol

$$\frac{\partial g}{\partial z_m}\Big|_{\underset{\sim}{Z}(\underset{\sim}{z},t),t,}$$

we mean: take the derivative of the function $g(\underset{\sim}{z},t)$ with respect to the variable z_m, then substitute $\underset{\sim}{Z}(\underset{\sim}{z},t)$ and t into the argument slots of this new function. We use this cumbersome notation since it is unambiguous as explained in Appendix A. To illustrate the use of this notation we apply the chain rule to Eq. (2.1.6):

$$\frac{\partial f}{\partial t} = \sum_m \frac{\partial g}{\partial z_m}\bigg|_{\underline{Z}(\underline{z},t),t} \cdot \frac{\partial Z_m}{\partial t} + \frac{\partial g}{\partial t}\bigg|_{\underline{Z}(\underline{z},t),\ t} \quad . \tag{2.1.14}$$

Whenever we do not explicitly show the arguments of a function, they are assumed to be $(\underline{z},t)$.

Finally, we want to introduce the concept of an infinitesimal canonical transformation. Such a transformation is of the form

$$T = I + \varepsilon L_g \tag{2.1.15}$$

where g is any phase function, and ε is near zero. This transformation is called an infinitesimal canonical transformation of order ε, since it is canonical to order ε^2. To see this, consider the Poisson bracket of $\{Z_i(\underline{z},t),\ Z_j(\underline{z},t)\}$. It is given by

$$\begin{aligned}\{Z_i(\underline{z},t),\ Z_j(\underline{z},t)\} \quad &= \quad \{z_i,\ z_j\} + \varepsilon\left\{z_i,\{g,\ z_j\}\right\} \\ &\quad + \varepsilon\left\{\{g,\ z_i\},\ z_j\right\} + \varepsilon(\varepsilon^2) \quad .\end{aligned} \tag{2.1.16}$$

Using (2.1.1), and Jacobi's identity, this can be put in the form

$$\{Z_i,\ Z_j\} = \gamma_{ij} + \mathcal{O}(\varepsilon^2) \ , \tag{2.1.17}$$

and is therefore canonical to order ε^2.

2.2 The Time Development Operator

In Hamiltonian mechanics we solve for the trajectories of a given Hamiltonian. A trajectory is found by starting at some position $\underset{\sim}{z}_0$, and integrating forward in time using Hamilton's equations, $\dot{\underset{\sim}{z}} = \{\underset{\sim}{z},h\}$. An alternative point of view is that Hamilton's equations specify a mapping $\underset{\sim}{\mathcal{Z}}(\underset{\sim}{z},t)$, which gives the position of a particle at time t for a particle which was at $\underset{\sim}{z}$ at t=0. The equations which define the mapping are

$$\frac{\partial \mathcal{Z}_i}{\partial t} = \sum_j \gamma_{ij} \frac{\partial h}{\partial z_j}\bigg|_{\underset{\sim}{\mathcal{Z}}(\underset{\sim}{z},t),t} \tag{2.2.1}$$

and

$$\mathcal{Z}_i(\underset{\sim}{z},0) = z_i \quad . \tag{2.2.2}$$

If one knows the mapping, one knows all of the trajectories. Hamiltonian mechanics therefore consists of finding the mappings associated with given Hamiltonians.

In this section we develop this point of view. We start with this mapping and construct the corresponding transformation. We show how the time derivative of the transformation is related to the Hamiltonian. We show how this transformation can be used to solve the inhomogeneous Liouville equation (2.2.12). Finally we give an example of a time development transformation.

The importance of this section is that it formalizes the method of solving the inhomogeneous Liouville equation by integrating along particle orbits. We give the explicit solution in terms of the time development operator. Of course, our expression is useful only when the time development operator is explicitly known, i.e. for solvable Hamiltonians. However, this is a situation which occurs repeatedly in perturbation theory. In perturbation

theory we have a solvable Hamiltonian h_o. The effects of the perturbation Δh are found by integrations along the particle orbits of h_o.

One can show that the mapping $\mathcal{Z}(\underset{\sim}{z},t)$ is canonical by showing that the time derivative of the Poisson bracket $\{\mathcal{Z}_i, \mathcal{Z}_j\}$ vanishes. We do not show this here since it is a standard topic in most texts (e.g. ref. [1] ch. 7). We call this mapping the time development mapping. The corresponding transformation M(t) is the time development transformation. M(t) is defined as follows. The relation f = Mg means

$$f(\underset{\sim}{z}, t) = g(\mathcal{Z}(\underset{\sim}{z}, t), t). \tag{2.2.3}$$

The effect of the mapping M is to develop the observables in time. As an example, suppose we are given the potential energy $V(\underset{\sim}{z},t)$ at time t as a function of the particle's position at time t. By applying M, we get a new function defined by

$$\mathcal{V}(\underset{\sim}{z},t) \equiv (MV)(\underset{\sim}{z},t) = V\left(\mathcal{Z}(\underset{\sim}{z}, t), t\right). \tag{2.2.4}$$

The new function $\mathcal{V}$ is the potential energy at time t of a particle which was at phase point $\underset{\sim}{z}$ initially (i.e. at t=0). Another example is where $g = z_i$, one of the variables, but thought of as a function, as in Eq. (2.1.7). In this case $Mz_i = \mathcal{Z}_i(\underset{\sim}{z}, t)$.

Since the mapping $\mathcal{Z}(\underset{\sim}{z}, t)$ is related to h by (2.2.1), we expect the time derivative of M to be related to h. We define the time derivative of the operator M by the usual procedure:

$$\frac{\partial M}{\partial t} g \equiv \lim_{\tau \to 0} \frac{M(t+\tau)g - M(t)g}{\tau}. \tag{2.2.5}$$

This definition insures that the product rule for differentiation will hold, $\frac{\partial}{\partial t}(Mg) = \frac{\partial M}{\partial t} g + M \frac{\partial g}{\partial t}$. In our case, where the action of M is given by (2.2.4), we can use the chain rule to show that the time derivative of M is given by

$$\left(\frac{\partial M}{\partial t} g\right)(\underset{\sim}{z},t) = \sum_i \frac{\partial g}{\partial z_i}\bigg|_{\underset{\sim}{\mathcal{Z}}(\underset{\sim}{z},t),\ t} \frac{\partial \mathcal{Z}_i}{\partial t} \quad . \tag{2.2.6}$$

Inserting (2.2.1) into this expression, we get

$$\left(\frac{\partial M}{\partial t} g\right)(\underset{\sim}{z},t) = \sum_{ij} \frac{\partial g}{\partial z_i}\bigg|_{\underset{\sim}{\mathcal{Z}}(\underset{\sim}{z},t),\ t} \gamma_{ij} \frac{\partial h}{\partial z_j}\bigg|_{\underset{\sim}{\mathcal{Z}}(\underset{\sim}{z},t),\ t} \quad . \tag{2.2.7}$$

Since g is arbitrary, (2.2.7) is the same as the operator equality

$$\frac{\partial M}{\partial t} = - ML_h \quad . \tag{2.2.8}$$

Whenever we are given a relation such as (2.2.8), we say that the transformation M is generated by the function h. To see why, we use the expression $\big(M(t+\tau) - M(t)\big)/\tau + \mathcal{O}(\tau)$ for the time derivative of M in equation (2.2.9). The result,

$$M(t + \tau) = M(t)\left(I - \tau L_h + \mathcal{O}(\tau^2)\right), \tag{2.2.9}$$

shows that we generate the operator $M(t + \tau)$ from the operator $M(t)$ by multiplication of $M(t)$ by the infinitesimal canonical transformation $I - \tau L_h$

The inverse operator $M^{-1}(t)$ has properties similar to those of M. We define M^{-1} by the same route. We first consider the inverse mapping $\underset{\sim}{\mathcal{Z}}^{-1}(\underset{\sim}{z},t)$, which gives the initial position of a particle which is at $\underset{\sim}{z}$ at time t. Then we define M^{-1} by

$$(M^{-1}g)(\underset{\sim}{z},\ t) \equiv g(\underset{\sim}{\mathcal{Z}}^{-1}(\underset{\sim}{z},\ t),\ t). \tag{2.2.10}$$

By definition we have $M^{-1}M = MM^{-1} = I$. By differentiating this expression we get

$$\frac{\partial M^{-1}}{\partial t} = L_h M^{-1} \quad . \tag{2.2.11}$$

In our discussion, the time t=0 has been a special time. However, we could have instead used time t_o as the initial time. The time development transformation from t_o to t is given by $M(t)M^{-1}(t_o)$, i.e. we transform back

from t_o and then forward to t.

We can now show how to solve the inhomogeneous Liouville equation;

$$\frac{\partial f}{\partial t} + \{f, h\} = g, \tag{2.2.12}$$

with the operator M. The equation occurs over and over in perturbation theory. There the function h is the unperturbed Hamiltonian, the function g is the perturbation, and f, which we are solving for, defines the canonical transformation used in perturbation theory. To solve (2.2.12) we introduce the function $f' = Mf$. Using (2.2.8) we see that f' satisfies

$$\frac{\partial f'}{\partial t} = Mg \ . \tag{2.2.13}$$

This equation is now trivially integrated giving

$$f'(t) = \int_{t_o}^{t} d\tau \quad M(\tau)g(\tau) + f'(t_o) \ . \tag{2.2.14}$$

Inserting $f' = Mf$ into (2.2.14) gives

$$f(t) = M^{-1}(t) \int_{t_o}^{t} d\tau M(\tau)g(\tau) + M^{-1}(t)M(t_o)f(t_o) \tag{2.2.15}$$

as the general solution of (2.2.12). This method of solution is commonly known as finding f by integrating g along the particle orbits of h.

A special case of particular importance is when g=0. In this case (2.2.12) reduces to the Liouville equation, and its solution (from 2.2.15) is (for t_o=0):

$$f(t) = M^{-1}(t)f(0)$$

or

$$f(\underline{z}, t) = f(\underline{z}^{-1}(\underline{z},t), 0). \tag{2.2.16}$$

We see in (2.2.16) that to find a distribution at a later time, we apply M^{-1}. That is, M(t) is the time development operator for observables, but $M^{-1}(t)$ is the time development operator for distributions.

When the Hamiltonian h is time independent, many of the results simplify. In particular, L_h and M commute in this case.

$$L_h M(t) = M(t) L_h \tag{2.2.17}$$

To see this, we use the fact that the time development transformation preserves the Hamiltonian, $M(t)h = h$, if h is time independent. From this we have

$$L_{M(t)h} = L_h \ , \tag{2.2.18}$$

which yields (2.2.17) upon insertion of (2.1.13).

Using (2.2.17) we can prove the relation

$$M(t - t') = M(t)M^{-1}(t'). \tag{2.2.19}$$

On physical grounds, this looks true. If h is time independent, the transformation which develops observables from time 0 to time $t-t'$, on the left side of Eq. (2.2.19), must be the same as the transformation which develops observables in time t' to $t' + (t-t')$, the right side of Eq. (2.2.19). If h is time independent, M can depend only on the time difference. Using (2.2.19), we can simplify (2.2.15). Inserting gives

$$\begin{aligned} f(t) &= \int_{t_o}^{t} d\tau M(\tau - t) g(\tau) + M^{-1}(t - t_o) f(t_o) \\ &= \int_{t_o - t}^{o} d\tau M(\tau) g(t + \tau) + M^{-1}(t - t_o) f(t_o). \end{aligned} \tag{2.2.20}$$

For an illustration of these ideas, we consider the free particle Hamiltonian $h = \frac{1}{2} p^2$ for one degree of freedom. Using (2.2.1), we see that the time development mapping is

$$Q(q,p,t) = q + pt \tag{2.2.21a}$$

$$P(q,p,t) = p \quad . \tag{2.2.21b}$$

Now we consider the density function in x-space given a particle's phase space coordinate (q,p): $\rho(x,t;\ q,p) = \delta(x - q)$. By applying the time development operator to ρ, we get a new function $\rho' = M\rho$,

$$\rho'(x, t; q,p) = \delta(x - q - pt). \tag{2.2.22}$$

Whereas ρ gives the density at time t as a function of the coordinate at time t, ρ' gives the density at time t as a function of the initial (t=0) phase space coordinates. One must always keep in mind that $\underset{\sim}{z}$ or (q, p) are dummy variables.

Now suppose we have a particle distribution which is gaussian in q and p at t=0

$$f(q, p, 0) = e^{-\frac{1}{2}p^2 - \frac{1}{2}q^2} \tag{2.2.23}$$

To find the distribution at time t, we develop f in time using M^{-1} as in (2.2.16). The inverse transformation is trivially solved for, $\zeta^{-1} = q - pt$ and $\xi^{-1} = p$, so we get

$$\begin{aligned} f(q,p,t) &= M^{-1}(t)\ f(q,p,0) \\ &= e^{-\frac{1}{2}p^2 - \frac{1}{2}(q - pt)^2} \end{aligned} \tag{2.2.24}$$

2.3 Short Time Solution for the Time Development Operator; Secular Perturbation Theory.

The operator representation M of the time development transformation of the previous section looks very similar to the time development transformation U in quantum mechanics [12]. In quantum mechanics the operator U is unitary and its time derivative is found by multiplication by the antihermitian operator -iH,

$$\frac{\partial U}{\partial t} = -iHU \tag{2.3.1}$$

In Hamiltonian mechanics the time derivative of the operator M^{-1} is found by multiplication by the antihermitian operator L_h as in (2.2.11). This analogy leads us to construct expressions for the time development operator M in terms of time ordered products as in quantum mechanics.

We employ the method of successive approximations to find M. We assume M is of the form

$$M(t) = I + M_1(t) + M_2(t) + \ldots, \tag{2.3.2}$$

where the terms are ordered in powers of the Hamiltonian. Inserting this into (2.2.8) gives

$$\frac{\partial M_n}{\partial t} = -M_{n-1}\, L_h, \tag{2.3.3}$$

which is easily integrated, resulting in

$$M_n(t) = -\int_0^t d\tau M_{n-1}(\tau) L_{h(\tau)} \quad . \tag{2.3.4}$$

By repeating the process in (2.3.4) we arrive at the general formula for the nth term:

$$M_n = (-1)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n L_{h(t_n)} \cdots L_{h(t_1)} \; . \tag{2.3.5}$$

This expression is exactly analogous to the quantum mechanical expression. As in quantum mechanics, we can introduce the time ordering operator $\mathcal{T}$ which rearranges operators putting the one at earliest time first. Then the expression for M_n becomes

$$M_n = \frac{1}{n!} \mathcal{T}\left(-\int_0^t d\tau L_{h(\tau)}\right)^n , \tag{2.3.6}$$

allowing M to be written

$$M = \mathcal{T} \exp\left(-\int_0^t d\tau L_{h(\tau)}\right) . \tag{2.3.7}$$

This form for M is of course purely formal. In writing (2.3.7) we mean that one sums up the separate terms M_n found by (2.3.5). Only in very special cases can this sum be done in closed form. An example is the time independent Hamiltonian $H = \frac{1}{2}p^2$ for which we have

$$M = \exp\left(tp \frac{\partial}{\partial q}\right). \tag{2.3.8}$$

This is just the Taylor operator for a shift in q by the amount tp.

$$Mf = g \quad \Rightarrow g(q,p,t) = f(q + tp, p, t) \tag{2.3.9}$$

In complicated cases, the sum is not known. Practicality requires that we evaluate only the first few terms of M, and hence we get a solution which is valid only for short times.

This same technique can be used to construct a perturbation theory. We consider the Hamiltonian $h = h_o + h_1$, where h_o is the solved unperturbed Hamiltonian, and h_1 is a perturbation. If the effects of h_1 are small, then we expect the total time development transformation M to differ little from the unperturbed time development transformation $M_o(t)$. Thus we write

$$M(t) = m(t)M_o(t), \tag{2.3.10}$$

and we expect m(t) to be close to the identity.

Differentiating M, and using the known time derivative of M, we find

$$\frac{\partial M}{\partial t} = -ML_h = \frac{\partial m}{\partial t} M_o + m \frac{\partial M_o}{\partial t} \quad . \tag{2.3.11}$$

Now we use $L_h = L_{h_o} + L_{h_1}$ and the known time derivative of M_o in (2.3.11). The result is

$$\begin{aligned} \frac{\partial m}{\partial t} &= -m\, M_o L_{h_1} M_o^{-1} \\ &= -m\, L_{M_o h_1} \quad . \end{aligned} \tag{2.3.12}$$

The last step follows from (2.1.12).

Noting the analogy between (2.3.12) and (2.2.8), we find m just as we found M. The result is

$$m = \mathcal{T}_c \exp\left(-\int_0^t d\tau\, L_{M_o(\tau)h_1(\tau)}\right) \quad . \tag{2.3.13}$$

This form of perturbation theory is analogous to the Dyson Wick Feynman formalism in quantum mechanics. In quantum mechanics, the theory ultimately breaks down due to divergences. In classical mechanics the theory breaks down because of secular terms. In quantum mechanics, the divergences are regularized using a renormalization technique. In classical mecbanics we treat the secular terms by isolating them. We use a transformation theory to get rid of rapidly oscillating terms, and we study the resonant terms individually. The development of a convenient transformation theory is tbe subject of the remainder of this paper.

3. Continuous Families of Canonical Transformations

We now consider canonical mappings which depend on an additional parameter θ. That is, the mapping $\underset{\sim}{Z}(\underset{\sim}{z}, t, \theta)$ is canonical for all values of t and for θ in some domain. Further, we require that $\underset{\sim}{Z}(\underset{\sim}{z}, t, \theta)$ be continuous, and twice differentiable in all variables simultaneously. We call this set of mappings (all θ values) a continuous family of canonical transformations.

In this chapter we will examine the consequences of the parametric dependence of $\underset{\sim}{Z}(\underset{\sim}{z}, t, \theta)$. We first establish the fact that the mapping can be specified by a single function $w(\underset{\sim}{z}, t, \theta)$, in place of the 2N functions $\underset{\sim}{Z}(\underset{\sim}{z}, t, \theta)$. This fact is proven elsewhere [1, 16], but we include it here for completeness. In the second section, we derive the expression for the transformed system. For the case where w is a power series in θ, this was first done by Deprit [8]. Later, Dewar [10] found the general expression for the new Hamiltonian when w is not a power series in θ.

The last two sections are illustrations of the theory. Section 3 consists of a simple example where all of the operators can be found explicitly. In section 4 we use the theory to prove the symmetry-equals-invariant theorem, known as Noether's theorem in Lagrangian mechanics.

3.1 The Generating Function

An important property of canonical transformations is that they can be specified by a single function. This property is helpful since it means that one can work with this single function rather than the 2N functions $\underset{\sim}{Z}(\underset{\sim}{z},t)$. In Hamilton-Jacobi theory this function is $F(\underset{\sim}{q},\underset{\sim}{P},t)$ [1,3], the generating function of mixed variables.

Given a family of canonical mappings $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$, we will show that there exists a function $w(\underset{\sim}{z},t,\theta)$ satisfying

$$\frac{\partial Z_i}{\partial\theta} = \left\{Z_i,\ w(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),\ t,\ \theta)\right\}$$

$$= \sum_j \gamma_{ij} \frac{\partial w}{\partial z_j}\Bigg|_{\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),\ t,\ \theta} \quad . \tag{3.1.1}$$

This property is shown in ref. [1] sec. VII-2, but we include a proof for completeness. If w has appropriate properties (say $w \in C_\infty$), we can specify $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ as the unique solution of (3.1.1) for some given boundary conditions. In our work we will always require that the mapping reduce to the identity when $\theta=0$ giving the boundary condition

$$\underset{\sim}{Z}(\underset{\sim}{z},t,0) = \underset{\sim}{z} \quad . \tag{3.1.2}$$

We note that equation (3.1.1) is analogous to (2.2.1). Just as the Hamiltonian h generates the time development mapping $\mathcal{Z}$ in the variable t, the function w generates the mapping $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ in the variable θ. Following this analogy, we introduce the canonical transformation T obtained from the mapping $\underset{\sim}{Z}$, just as M came from $\mathcal{Z}$. From (2.2.8) we see that T must satisfy

$$\frac{\partial T}{\partial\theta} = - TL_w \quad . \tag{3.1.3}$$

We also note that T^{-1} must satisfy

$$\frac{\partial T^{-1}}{\partial \theta} = L_w T^{-1} \quad , \tag{3.1.4}$$

which is analogous to (2.2.11). Finally, any function f satisfying

$$\frac{\partial f}{\partial \theta} + \{f,w\} = g \tag{3.1.5}$$

can be solved using

$$f(\theta) = T^{-1}(\theta)\int_{\theta_o}^{\theta} d\theta' \; T(\theta')g(\theta') + T^{-1}(\theta)T(\theta_o)f(\theta_o) \tag{3.1.6}$$

which is analogous to (2.2.15).

These equations are meaningful only if we can prove the existence of w. To do this we consider the vector $\underset{\sim}{v}$ given by

$$v_j = -\sum_i \gamma_{jk} \frac{\partial Z_k}{\partial \theta}\bigg|_{\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),\; t,\; \theta} \quad . \tag{3.1.7}$$

Suppose we can show that v_j is the gradient of a function w,

$$v_j = \frac{\partial w}{\partial z_j} = -\sum_i \gamma_{jk} \frac{\partial Z_k}{\partial \theta}\bigg|_{\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t,\theta} \; . \tag{3.1.8}$$

Then, by multiplying (3.1.8) by γ_{ij}, summing over j, and using (2.1.3) we get

$$\frac{\partial Z_i}{\partial \theta}\bigg|_{\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t,\theta} = \sum_j \gamma_{ij} \frac{\partial w}{\partial z_j} \quad . \tag{3.1.9}$$

Transforming the variable in (3.1.9) gives (3.1.1). We see that we must prove that $\underset{\sim}{v}$ is a gradient of a function w. Of course, knowing that $\underset{\sim}{v}$ is a gradient, we can find w very simply by integration:

$$w = \int^{\underset{\sim}{z}} \sum_m v_m \, dz_m \quad . \tag{3.1.10}$$

To prove that $\underset{\sim}{v}$ is a gradient, we will show that the integrability condition holds. This amounts to showing that the quantity

$$\frac{\partial v_i}{\partial z_j} = - \sum_{km} \gamma_{ik} \frac{\partial^2 Z_k}{\partial z_m \partial \theta} \Bigg|_{\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t,\theta} \cdot \frac{\partial Z^{-1}{}_m}{\partial z_j} \tag{3.1.11}$$

is symmetric in i and j. This proof follows from some special properties of the derivative matrix of a canonical transformation. This proof is algebraically tedious, and one need not follow it to understand the rest of this paper, so the reader may choose to skip the rest of this section.

One property which we wish to show is that the derivative matrix of the inverse mapping $(\partial Z^{-1}{}_i/\partial z_j)$ is simply related to the derivative matrix of the forward mapping $(\partial Z_i/\partial z_j)$. By differentiating the following identity,

$$z_i = Z_i\left(\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t,\theta\right) , \tag{3.1.12}$$

it follows that the two matrices are inverses:

$$\delta_{ik} = \sum_j \frac{\partial Z_i}{\partial z_j} \Bigg|_{\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t,\theta} \cdot \frac{\partial Z^{-1}{}_j}{\partial z_k} . \tag{3.1.13}$$

However, we also note that by multiplying (2.1.6) by γ_{jk} and summing over j we get

$$\delta_{ik} = \sum_j \frac{\partial Z_i}{\partial z_j} \left(- \sum_{\ell m} \gamma_{j\ell} \frac{\partial Z_m}{\partial z_\ell} \gamma_{mk} \right) . \tag{3.1.14}$$

Comparing (3.1.13) to (3.1.14), we see (since $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ is invertible) that

$$\frac{\partial Z^{-1}{}_j}{\partial z_k} = - \sum_{\ell m} \gamma_{j\ell} \frac{\partial Z_m}{\partial z_\ell} \Bigg|_{\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t,\theta} \cdot \gamma_{mk} , \tag{3.1.15}$$

relating the inverse mapping derivative matrix to the forward mapping derivative matrix.

From the Poisson bracket relations (2.1.6) we can prove the other identity needed to show the symmetry of (3.1.11). By differentiating (2.1.6) with respect to θ we get

$$\sum \frac{\partial^2 z_k}{\partial\theta\partial z_m} \gamma_{m\ell} \frac{\partial z_n}{\partial z_\ell} = -\sum_{m\ell} \frac{\partial z_k}{\partial z_m} \gamma_{m\ell} \frac{\partial^2 z_n}{\partial\theta\partial z_\ell} . \tag{3.1.16}$$

Now we use (3.1.16) and (3.1.15) to prove the symmetry of (3.1.11). We first insert (3.1.15) into (3.1.11) obtaining

$$\frac{\partial v_i}{\partial z_j} = \sum_{k\ell mn} -\left(\gamma_{ik} \frac{\partial^2 z}{\partial\theta\partial z_\ell} \cdot \gamma_{\ell m} \frac{\partial z_n}{\partial z_m} \gamma_{nj}\right)\Bigg|_{\underline{z}^{-1}(\underline{z},t,\theta),t,\theta} . \tag{3.1.17}$$

Then we use (3.1.16) to put this in the form

$$\frac{\partial v_i}{\partial z_j} = \sum_{k\ell mn} \left(\gamma_{ik} \frac{\partial z_k}{\partial z_\ell} \gamma_{\ell m} \frac{\partial^2 z_n}{\partial z_m \partial\theta} \gamma_{nj}\right)\Bigg|_{\underline{z}^{-1}(\underline{z},t,\theta),t,\theta} . \tag{3.1.18}$$

By rearranging the dummy indices and using the antisymmetry property of γ_{ij}, we get

$$\frac{\partial v_i}{\partial z_j} = \sum_{k\ell mn} -\left(\gamma_{jk} \frac{\partial^2 z_n}{\partial z_\ell \partial\theta} \gamma_{\ell m} \frac{\partial z_n}{\partial z_m} \gamma_{ni}\right)\Bigg|_{\underline{z}^{-1}(\underline{z},t,\theta),t,\theta} . \tag{3.1.19}$$

Comparing this to (3.1.17) we see that the integrability condition holds, and so w does exist.

3.2. The New Hamiltonian

The reason we introduce transformation theory is that we hope to be able to transform to a new system where the Hamiltonian has a simpler form. Upon solving the equations in the simpler system, we can transform back to obtain the solutions in the original system.

The new Hamiltonian K must have the property of giving the equations of motion for the transformed variables $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$:

$$\dot{Z}_i = \left\{Z_i,\ K\left(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t,\theta\right)\right\} \quad . \tag{3.2.1}$$

In this equation, overdot refers to the time derivative along a trajectory. Using the old Hamiltonian, we can also find Z_i:

$$\dot{Z}_i = \frac{\partial Z_i}{\partial t} + \left\{Z_i,\ h(\underset{\sim}{z},t)\right\} \quad . \tag{3.2.2}$$

Introducing the function $H(\underset{\sim}{z},t,\theta) = h(\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t)$, we can write (3.2.2) in the form

$$\dot{Z}_i = \left\{Z_i,\ H\left(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t,\theta\right)\right\} + \frac{\partial Z_i}{\partial t} \quad . \tag{3.2.3}$$

Hence we need only find a function R such that

$$\frac{\partial Z_i}{\partial t} = \left\{Z_i,\ R\left(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t,\theta\right)\right\} , \tag{3.2.4}$$

than the new Hamiltonian K is given by $K = H + R$.

That such an R exists is evident from the following considerations. $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ is a canonical transformation for all t and θ. Since it is canonical for all θ, we were able to conclude that w exists satisfying (3.1.1). Likewise, since $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ is canonical for all t, we conclude that an R exists satisfying (3.2.4). Knowing that R exists, we can be more secure about finding it.

To find R, we consider equations (3.1.1) and (3.2.4) written in operator form.

$$\frac{\partial T}{\partial \theta} = - TL_w \tag{3.2.5}$$

$$\frac{\partial T}{\partial t} = - TL_R \tag{3.2.6}$$

Now we equate the second partials of T calculated from the two above equations.

$$\frac{\partial T}{\partial t} L_w + T L_{\frac{\partial w}{\partial t}} = \frac{\partial T}{\partial \theta} L_R + T L_{\frac{\partial R}{\partial \theta}} \tag{3.2.7}$$

By using (3.2.5) and (3.2.6) again we get

$$L_{\frac{\partial R}{\partial \theta}} + L_{\{R,w\}} = L_{\frac{\partial w}{\partial t}} \quad . \tag{3.2.8}$$

From equation (3.2.8) we would like to deduce the relationship between R and w. As discussed in section 2.1, we can only deduce

$$\frac{\partial R}{\partial \theta} + \{R,w\} = \frac{\partial w}{\partial t} + g \quad , \tag{3.2.9}$$

where g is any function independent of $\underset{\sim}{z}$. However, we can set g equal to zero here, since R has relevance only through its Poisson bracket relations as in (3.2.4). The addition of a function independent of $\underset{\sim}{z}$ to R does not affect any Poisson bracket relations. Setting $g = 0$, we get

$$\frac{\partial R}{\partial \theta} + \{R,w\} = \frac{\partial w}{\partial t} \quad . \tag{3.2.10}$$

Following the discussion in the last section, this equation can be integrated using (3.1.6). We use $\theta_o = 0$ since then T reduces to the identity implying that $R(\theta_o = 0)$ vanishes. This gives the result for R:

$$R(\theta) = T^{-1}(\theta) \int_o^\theta d\theta' \; T(\theta') \frac{\partial w}{\partial t} (\theta') \; . \tag{3.2.11}$$

If we now use $K = H + R$, we get the new Hamiltonian.

$$K(\theta) = T^{-1}(\theta)h + T^{-1}(\theta)\int_o^\theta d\theta' T(\theta') \frac{\partial w}{\partial t} (\theta'). \tag{3.2.12}$$

This result was first obtained by Dewar [10], but with different conventions.

To compare (3.2.12) with more familiar equations, we consider the case where the transformation is time independent. In this case $\frac{\partial w}{\partial t} = 0$, so

$$K(\theta) = T^{-1}(\theta)h \quad . \tag{3.2.13}$$

Using the definition of T, this can be written

$$K\left(\underline{Z}(\underline{z},t,\theta),t,\theta\right) = h(\underline{z},t,\theta), \tag{3.2.14}$$

which simply says that the new Hamiltonian evaluated at the mapped positions equals the old Hamiltonian evaluated at the old positions.

In the time dependent case, the remainder function R must equal $\partial F/\partial t$, the time derivative of the generating function of mixed variables, which appears in Hamilton-Jacobi theory. Using this fact, one can derive the relation between w and F. The relation between w and F was first found by Dewar by a different method [11].

A final point which we wish to discuss is the transformation of the Liouville equation. If f is the old density and F (not the generating function) is the new density, they must be related by

$$F = T^{-1}f, \tag{3.2.15}$$

since the new density at the new point equals the old density at the old point. Since f satisfies

$$\frac{\partial f}{\partial t} + \{f,h\} = 0 \ , \tag{3.2.16}$$

we can show that F satisfies

$$\frac{\partial F}{\partial t} + \{F,K\} = 0, \tag{3.2.17}$$

as it must if F is the density in the transformed system. One shows this by combining (3.2.6) and (3.2.12) to obtain

$$\frac{\partial T}{\partial t} = -TL_K + L_h T \ , \tag{3.2.18}$$

and using this expression in differentiating f = TF with respect to time.

3.3 An Illustration

To illustrate this theory, we consider a specific transformation. We consider the transformation given by the generating function

$$w = -\theta t p^2 \ . \tag{3.3.1}$$

We must first find the variable mapping $Q(q, p, t, \theta)$, $P(q, p, t, \theta)$. This mapping is found by solving (3.1.1), which becomes in this case

$$\frac{\partial Q}{\partial \theta} = -2\theta t \ P \tag{3.3.2a}$$

and

$$\frac{\partial P}{\partial \theta} = 0 \tag{3.3.2b}$$

These equations are trivially solved giving

$$Q = q - \theta^2 t p \tag{3.3.3a}$$

$$P = p \tag{3.3.3b}$$

We note that the boundary conditions (3.1.2) have been satisfied.

The operator T corresponding to (3.3.3) acts on a function f as follows. If $Tf = g$, then

$$g(q, p, t, \theta) = f(q - \theta^2 t p, p, t, \theta). \tag{3.3.4}$$

We could also use the operator notation to deduce (3.3.4). The Lie operator in this case is given by

$$L_w = + 2\theta t p \frac{\partial}{\partial q} \ . \tag{3.3.5}$$

The equation for T is

$$\frac{\partial T}{\partial \theta} = -T \ 2\theta t p \frac{\partial}{\partial q} \ . \tag{3.3.6}$$

This equation is trivially solved giving

$$\begin{aligned} T &= \exp\left(-\theta^2 t p \frac{\partial}{\partial q}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (-\theta^2 t p)^n \left(\frac{\partial}{\partial q}\right)^n \ , \end{aligned} \tag{3.3.7}$$

and this is just the Taylor series expression for the shift operator in (3.3.4)

We now calculate the new Hamiltonian using (3.2.12) in the case where the original system is a free particle system, $h = p^2/2m$. In this case we see that $T^{-1}h = h$ and $T\frac{\partial w}{\partial t} = \frac{\partial w}{\partial t}$, so that K is easily calculated giving

$$K = \frac{1}{2m}p^2 - \frac{1}{2}\theta^2 p^2 \tag{3.3.8}$$

We see that the motion in the new system is free particle motion with the new mass given by

$$M(\theta) = \frac{m}{1 - \theta^2 m} \quad . \tag{3.3.9}$$

We see that we can transform away the Hamiltonian entirely in this simple case, by choosing $\theta^2 = \frac{1}{m}$. In this new system q and p are constant. To find the motion in the original system, we first apply the inverse transformation

$$Q^{-1} = q + pt/m \tag{3.3.10a}$$

$$P^{-1} = p$$

Then we substitute the trajectories in the transformed system $q_K(t) = q_o$ and $p_K(t) = p_o$ to get the trajectories in the original system

$$q_h(t) = q_K(t) + \frac{1}{m}t\,p_K(t) = q_o + \frac{1}{m}tp_o \tag{3.3.11a}$$

$$p_h(t) = p_K(t) = p_o \quad . \tag{3.3.11b}$$

3.4 The Symmetry-Equals-Invariant Theorem (Noether's theorem)

Noether's theorem (ref. [1] p. 83) in Lagrangian dynamics states that if one is given a family of transformations which leaves the form of the Lagrangian unchanged except for the addition of a total time derivative, then one can construct an invariant of the motion. Conversely, given an invariant of the motion, one can construct a family of transformations which leave the Lagrangian unchanged except for the addition of a total time derivative.

Since there is a direct correlation between Lagrangian dynamics and Hamiltonian dynamics, one expects a similar theorem to hold in Hamiltonian mechanics. This theorem has in fact been discussed by Whittaker [2] and Anderson [15]. The advantage of the Hamiltonian formulation is that the relevant transformations are the time dependent canonical transformations, the natural transformations which occur in Hamiltonian mechanics. In contrast, the Lagrangian theorem requires the use of path transformations (see ref. [1] p. 83) not just time dependent coordinate transformations.

However, one drawback of the theorems presented in refs. [2] and [15] is that they consider only infinitesimal transformations. With the finite transformation theory presented earlier, this can be remedied, as was done in ref. [16]. Here we present this theorem using the operator notation of this paper.

Given a system which evolves according to a Hamiltonian h, we say that h has a symmetry if there exists a transformation family $T(\theta)$ which leaves the Hamiltonian unchanged up to the addition of a phase space independent function:

$$K = h + f \tag{3.4.1}$$

where

$$\frac{\partial f}{\partial z_i} = 0 \quad \text{for all } i. \tag{3.4.2}$$

The reason we allow the addition of f is that it will not affect the motion in the new system, since its Poisson bracket with any function vanishes.

We define an invariant to be any function g whose total time derivative along a trajectory vanishes:

$$\frac{\partial g}{\partial t} + \{g,h\} = 0. \tag{3.4.3}$$

We will prove that given a symmetry, one can construct an invariant, and given an invariant, one can construct a symmetry.

First we suppose that we are given a symmetry. Using (3.4.1) in (3.2.12), we have

$$h + f = T^{-1}(\theta)h + T^{-1}(\theta)\int_0^\theta d\theta' T(\theta') \frac{\partial w}{\partial t}(\theta') \quad , \tag{3.4.4}$$

where w is constructed from T as discussed in 3.1. We premultiply this equation by T, and note that Tf = f, since f does not depend on the phase space variables.

$$T(\theta)h + f = h + \int_0^\theta d\theta' T(\theta') \frac{\partial w}{\partial t}(\theta') \tag{3.4.5}$$

Differentiating (3.4.5) with respect to θ, and premultiplying by T^{-1} gives

$$\frac{\partial w}{\partial t} - \frac{\partial f}{\partial \theta} + \{w,h\} = 0. \tag{3.4.6}$$

Thus the function $g(z,t,\theta) = w - \int^t d\tau\, \partial f/\partial\theta$ is an invariant for all θ.

To prove the other half of the theorem, we assume we have an invariant g. From this invariant we construct a transformation T(θ) using w = g. The new Hamiltonian K is then given by

$$K(\theta) = T^{-1}(\theta)h + T^{-1}(\theta)\int_0^\theta d\theta' T(\theta') \frac{\partial g}{\partial t} \quad . \tag{3.4.7}$$

Premultiplying T and differentiating with respect to θ gives

$$-T\, L_g K + T \frac{\partial K}{\partial \theta} = T \frac{\partial g}{\partial t} \quad , \tag{3.4.7}$$

which implies that

$$\frac{\partial K}{\partial \theta} + \{K, g\} = \frac{\partial g}{\partial t} \quad . \tag{3.4.8}$$

We now use the fact that g is an invariant. This gives

$$\frac{\partial (K-h)}{\partial \theta} + \{K-h, g\} = 0, \tag{3.4.9}$$

since h is independent of θ. We define b by $b = T\ (K-h)$. In terms of b, (3.4.9) reads

$$\frac{\partial b}{\partial \theta} = 0 \quad . \tag{3.4.10}$$

Since $b = 0$ at $\theta = 0$, b vanishes for all θ, from which we deduce $K = h$, proving the theorem.

As stated here, the theorem is more powerful than it is in its usual form. Here we find that the generating function g is an invariant for all values of θ, whereas usually only g at $\theta = 0$ is shown to be an invariant.

We are used to thinking of the explicitly time independent invariants such as momentum and angular momentum. This theorem shows that symmetries are associated even with time dependent invariants such as the function

$$g = p \cos\omega_0 t + q \sin\omega_0 t \tag{3.4.11}$$

for the harmonic oscillator $h = \frac{1}{2}\omega_0(q^2 + p^2)$. In general a Hamiltonian in 2N dimensional phase space has 2N functionally independent invariants, and hence its symmetry Lie group is 2N dimensional.

4. Deprit Perturbation Theory

In this section we show how to construct a perturbation theory from the general transformation theory of the previous chapter. Historically the development was in the opposite direction. Deprit constructed power series representations of the transformation, and later Dewar gave the general representation applicable even when the transformation is not analytic.

The basic idea of the perturbation theory is that the Hamiltonian which we wish to solve consists of a solvable term plus unsolvable terms which are ordered in a small parameter. We equate this small parameter to the parameter θ of a Lie transform which is to be determined. We then expand the Lie transform in powers of the parameter θ, and use the expansions in equation (3.2.12). We end up with an equation for each order relating the new Hamiltonian to the transformation and to the old Hamiltonian.

The next stage in the transformation theory is to pick the transformation, thereby choosing the form of the new Hamiltonian. Ideally we would like to pick the transformation so that K vanishes in all the higher orders. However, this choice is not always best since it may lead to secular terms or small denominators in the transformation, making it useless for a discussion of long time effects. The scheme we adopt is then to transform away as many terms as possible. The slowly varying terms, which give rise to secularities and small denominators, must be kept in the new Hamiltonian. Th's scheme usually simplifies the analysis to some degree since at least the rapidly varying terms are transformed away. Sometimes this scheme in effect solves the problem when the slowly varying terms depend only on the momenta.

The structure of this chapter is as follows. In the first section we derive the power series expressions for the Lie transform theory. Then we illustrate the method of choosing the transformations outlined in the previous

paragraph by doing two examples. However, it must be noted that the described method of choosing is not the only one. A nice feature of the Deprit perturbation theory is that one can use any method to select the transformation. The important thing is to choose the transformation so that the new Hamiltonian is as easy to analyze as possible. Finally we include a section on superconvergence, showing how it occurs in Lie transform theory. In this section we also discuss the practical usage of the superconvergence technique.

4.1 Deprit Perturbation Series Relations

This perturbation theory relies on the power series expansions of the Lie transforms. In this section we assume that the Hamiltonian h is ordered in a parameter which we equate to the Lie transform parameter. We then insert the power series expansions into (3.2.12) to obtain the expression for the nth order term in the new Hamiltonian.

The starting point is the assumption that the objects h,w,K, and T can all be expanded in power series. For h,K, and T we have

$$h(\underset{\sim}{z},t,\theta) = \sum_{n=0}^{\infty} \theta^n h_n(\underset{\sim}{z},t), \tag{4.1.1}$$

$$K(\underset{\sim}{z},t,\theta) = \sum_{n=0}^{\infty} \theta^n K_n(\underset{\sim}{z},t), \tag{4.1.2}$$

and

$$T(t,\theta) = \sum_{n=0}^{\infty} \theta^n T_n(t) \ . \tag{4.1.3}$$

However, for w we assume a slightly different form

$$w(\underset{\sim}{z},t,\theta) = \sum_{n=0}^{\infty} \theta^n w_{n+1}(\underset{\sim}{z},t) \ . \tag{4.1.4}$$

The reason for this choice is that w occurs in (3.2.12) along with an integral over $d\theta$ which effectively boosts the order of w by 1 in all the equations.

We also assume that the transformation is close to the identity. This is necessary for perturbation theory to work at all. This implies that $T_o = I$, putting $\theta = 0$. It also implies $K_o = h_o$ since in (3.2.12) the integral is at least of order θ.

We first find the form for the operator T to all orders. We start with the relation

$$\frac{\partial T}{\partial \theta} = -T\, L_w . \tag{4.1.5}$$

To use this relation we will need to know the series for L_w. Since the map $g \to L_g$ is linear, we have

$$L_w = \sum_{n=0}^{\infty} \theta^n L_{w_{n+1}} \tag{4.1.6}$$

We abbreviate the operator L_{w_n} by L_n. Now we simply take the series representations for L_w and T and insert them into (4.1.5). As a result we calculate the following recursion relation:

$$T_n = \frac{-1}{n} \sum_{m=0}^{n-1} T_m\, L_{n-m} . \tag{4.1.7}$$

By iterating this relation we arrive at

$$T_n = \sum_{\substack{(m_1,\ldots,m_r) \\ n>m_1>m_2>\ldots>m_r>0}} \left(\frac{-1}{n}\right)\left(\frac{-1}{m_1}\right) \cdots \left(\frac{-1}{m_r}\right) L_{m_r} \cdots L_{m_2-m_3} L_{m_1-m_2} L_{n-m_1} . \tag{4.1.8}$$

The sum is over all sets of integers $m_1,\ldots,m_r$ satisfying $n>m_1>m_2>\ldots>m_r>0$. This expression was first derived by Littlejohn [17].

From the expression $\frac{\partial T^{-1}}{\partial \theta} = L_w T^{-1}$, we can derive similar relations for T^{-1} The results are

$$T^{-1}_n = \frac{1}{n} \sum_{m=0}^{n-1} L_{n-m}\, T^{-1}_m \tag{4.1.9}$$

which gives

$$T_n^{-1} = \sum_{\substack{(m_1,\ldots,m_r) \\ n>m_1>m_2>\ldots> m_r>0}} \left(\frac{1}{n}\right)\left(\frac{1}{m_1}\right) \cdots \left(\frac{1}{m_r}\right) L_{n-m_1} L_{m_1-m_2} \cdots L_{m_r} . \tag{4.1.10}$$

To find the expression for K, we could insert the series for T, w, and h into (3.2.12). If this were done, it would be better to choose the conventions of ref. [16]. However, by manipulating (3.2.12), one can derive the power series

for K from the resulting equation [17]. If one finds K by this method it is better to use the conventions given here, as discussed in Ch. 5.

Premultiplying (3.2.12) by T and differentiation with respect to θ gives

$$\frac{\partial T}{\partial \theta} K + T \frac{\partial K}{\partial \theta} = \frac{\partial h}{\partial \theta} + T \frac{\partial w}{\partial t} \quad . \tag{4.1.11}$$

Using (3.2.5), and premultiplying by T^{-1} then gives

$$\frac{\partial w}{\partial t} = \frac{\partial K}{\partial \theta} - L_w K - T^{-1} \frac{\partial h}{\partial \theta} \quad . \tag{4.1.12}$$

Inserting the series expressions for K, T, and h gives (in n^{th} order)

$$\frac{\partial w_n}{\partial t} = n\, K_n - \sum_{m=0}^{n-1} L_{n-m} K_m - \sum_{m=1}^{n} T^{-1}_{n-m}\, m\, h_m \quad . \tag{4.1.13}$$

By writing out the first term of the first sum, and the last term of the sum we get the final answer (note $K_o = h_o$):

$$\frac{\partial w_n}{\partial t} + \left\{ w_n, h_o \right\} = n\, K_n - n\, h_n - \sum_{m=1}^{n-1} \left(L_{n-m} K_m + m\, T^{-1}_{n-m}\, h_m \right) \quad . \tag{4.1.14}$$

This is the form with which we work when doing n^{th} order perturbation theory. After calculating through order (n-1), we know all of the quantities in the sum in the above equation. We also know h_n. We then pick K_n in a manner described in the next section. Finally we find w_n by integrating the right side of (4.1.14) along a trajectory. At this point all of the quantities are known through order n, hence we know the motion through n^{th} order, and we can begin the order (n+1) calculation.

For future reference, we give the formulas for the various quantities through fourth order.

$$T_o = I \tag{4.1.15a}$$

$$T_1 = -L_1 \tag{4.1.15b}$$

$$T_2 = -\frac{1}{2} L_2 + \frac{1}{2} L_1^2 \tag{4.1.15c}$$

$$T_3 = -\frac{1}{3} L_3 + \frac{1}{6} L_2 L_1 + \frac{1}{3} L_1 L_2 - \frac{1}{6} L_1^3 \tag{4.1.15d}$$

$$T_4 = -\frac{1}{4} L_4 + \frac{1}{12} L_3 L_1 + \frac{1}{8} L_2^2 + \frac{1}{4} L_1 L_3$$

$$- \frac{1}{24} L_2 L_1^2 - \frac{1}{12} L_1 L_2 L_1 - \frac{1}{8} L_1^2 L_2 + \frac{1}{24} L_1^4 \tag{4.1.15e}$$

$$T_o^{-1} = I \tag{4.1.16a}$$

$$T_1^{-1} = L_1 \tag{4.1.16b}$$

$$T_2^{-1} = \frac{1}{2} L_2 + \frac{1}{2} L_1^2 \tag{4.1.16c}$$

$$T_3^{-1} = \frac{1}{3} L_3 + \frac{1}{6} L_1 L_2 + \frac{1}{3} L_2 L_1 + \frac{1}{6} L_1^3 \tag{4.1.16d}$$

$$T_4^{-1} = \frac{1}{4} L_4 + \frac{1}{12} L_1 L_3 + \frac{1}{8} L_2^2 + \frac{1}{4} L_3 L_1$$

$$+ \frac{1}{24} L_1^2 L_2 + \frac{1}{12} L_1 L_2 L_1 + \frac{1}{8} L_2 L_1^2 + \frac{1}{24} L_1^4 \tag{4.1.16e}$$

We note that $T_n^{-1} = T_n^+$, where the dagger indicates hermitian conjugate. To find the hermitian conjugate of a product of operators, we reverse the order of the product, and then take the hermitian conjugate of each of the multipliers:

$$(AB)^+ = B^+ A^+ . \tag{4.1.17}$$

Since $L_n^+ = -L_n$ (L_n is antihermitian), we see that $T_n^{-1} = T_n^+$ is true in (4.1.16). This fact is proven in general by noting that the differential equations for T and T^{-1} are hermitian conjugates.

The equations for K_n to fourth order are:

$$K_o = h_o \tag{4.1.17a}$$

$$\frac{\partial w_1}{\partial t} + \{w_1, h_o\} = K_1 - h_1 \tag{4.1.17b}$$

$$\frac{\partial w_2}{\partial t} + \{w_2, h_o\} = 2(K_2 - h_2) - L_1(K_1 + h_1) \tag{4.1.17c}$$

$$\frac{\partial w_3}{\partial t} + \{w_3, h_o\} = 3(K_3 - h_3) - L_1(K_2 + 2h_2) - L_2(K_1 + \tfrac{1}{2}h_1) - \tfrac{1}{2}L_1^2\, h_1 \tag{4.1.17d}$$

$$\frac{\partial w_4}{\partial t} + \{w_4, h_o\} = 4(K_4 - h_4) - L_1(K_3 + 3h_3) - L_2(K_2 + h_2) - L_1^2 h_2 - L_3(K_1 + \tfrac{1}{3}h_1) - \tfrac{1}{6}(L_1L_2 + 2L_2L_1 + L_1^3)h_1 \,. \tag{4.1.17e}$$

In passing, we note that Littlejohn [17] has written a computer program to find these equations to arbitrary order.

4.2 The Anharmonic Oscillator

In this section we apply the Deprit perturbation technique to a sample problem as an illustration. The problem we consider is that of a particle moving in a potential with cubic and fifth power force nonlinearities. The Hamiltonian in this case is given by

$$h = \frac{1}{2}\ p^2 + \frac{1}{2}\omega_o^2 q^2 + \frac{\varepsilon}{4}\omega_o^2 q^4 + \frac{\varepsilon^2}{8} a\omega_o^3 q^6 \tag{4.2.1}$$

We solve for the motion near the bottom of the well where the nonlinearity is small.

The unperturbed Hamiltonian is a simple harmonic oscillator. For convenience we use the (unperturbed) action angle variables defined by

$$p = -\sqrt{2\omega_o j}\ \sin\phi \tag{4.2.2a}$$

$$q = \sqrt{2j/\omega_o}\ \cos\phi \tag{4.2.2b}$$

In terms of these variables the Hamiltonian takes the form

$$h = \omega_o j + \varepsilon j^2 \cos^4\phi + \varepsilon^2 a j^3 \cos^6\phi \quad . \tag{4.2.3}$$

The ordering scheme for this Hamiltonian h is of course

$$\begin{aligned} h_o &= \omega_o j \\ h_1 &= j^2 \cos^4\phi \\ h_2 &= a j^3 \cos^6\phi \\ h_n &= 0 \quad \text{for } n > 2 \end{aligned} \tag{4.2.4}$$

We equate the Lie transform parameter θ to the small parameter ε.

The first step in perturbation theory is to solve the unperturbed problem; i.e. we must find the time development operator. In this case the time development mapping is

$$\mathcal{S}(\phi, j, t) = \phi + \omega_o t \tag{4.2.5a}$$

$$\mathcal{J}(\phi, j, t) = j \quad . \tag{4.2.5b}$$

The time development mapping M_o is therefore given by

$$f = M_o g \Rightarrow f(\phi,j,t) = g(\phi + \omega_o t, j, t). \tag{4.2.6}$$

To do first order perturbation theory we consider the first order Deprit equation (4.1.17b). In this case we have

$$\frac{\partial w_1}{\partial t} + \{w_1, h_o\} = K_1(\phi,j,t) - j^2\cos^4\phi,$$

$$= K_1 - j^2\left(\frac{3}{8} + \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi\right). \tag{4.2.7}$$

We can solve for w_1 using (2.2.20). Since we are looking for any solution to (4.2.7), we use the indefinite integral:

$$w_1 = -\int^t d\tau\left[j^2\left(\frac{3}{8} + \frac{1}{2}\cos\left[2\phi + \omega_o(\tau - t)\right] + \frac{1}{8}\cos\left[4\phi + 4\omega_o(\tau - t)\right]\right) - K_1\left(j, \phi + \omega_o(\tau - t), \tau\right)\right] \tag{4.2.8}$$

We would like to pick $K_1 = 0$, but since h_1 has a nonzero time average this would cause w_1 to be secular (unbounded in time). Therefore we pick K_1 to be the time average of h_1 along a particle orbit:

$$K_1(j) = \frac{\omega_o}{2\pi}\int_0^{2\pi/\omega_o} d\tau h_1(j, \phi + \omega_o\tau)$$

$$= \frac{3}{8} j^2 \quad . \tag{4.2.9}$$

Inserting this result into (4.2.8) gives

$$w_1 = -j^2\left(\frac{1}{32}\sin 4\phi + \frac{1}{4}\sin 2\phi\right)/\omega_o \quad . \tag{4.2.10}$$

We have completed our calculation to first order since we have w_1 and K_1. Introducing the transformed variables Φ and J we know that they evolve in time according to

$$J = J_o \tag{4.2.11}$$

$$\Phi = \Phi_o + (\omega_o + \frac{3}{4}\varepsilon J_o)t, \tag{4.2.12}$$

since to first order the transformed Hamiltonian is

$$K(J) = \omega_o J + \frac{3}{8}\varepsilon J^2 \ . \tag{4.2.13}$$

Also, by applying the inverse transform (4.1.16) we know that the old and new variables are related by $z_1 = Z_1 + \{w_1, Z_1\}$ giving

$$\phi = \Phi + \varepsilon J\left(\frac{1}{16}\sin 4\Phi + \frac{1}{2}\sin 2\Phi\right)/\omega_o \tag{4.2.14}$$

$$j = J - \varepsilon J^2\left(\frac{1}{8}\cos 4\Phi + \frac{1}{2}\cos 2\Phi\right)/\omega_o . \tag{4.2.15}$$

Inserting the time development of J and Φ (4.2.11) and (4.2.12) into (4.2.14) and (4.2.15) gives the time development of j and ϕ.

Going to second order, we insert our results for K_1 and w_1 into (4.1.17c). The result is

$$\frac{\partial w_2}{\partial t} + \{w_2, h_o\} = 2\left[K_2 - \frac{aj^3}{32}\left(10 + 15\cos 2\phi + 6\cos 4\phi + \cos 6\phi\right) + \frac{j^3}{64\omega_o}\left(17 + 33\cos 2\phi + 6\cos 4\phi - 2\cos 6\phi\right)\right] . \tag{4.2.16}$$

Since the average along a particle orbit is essentially a phase average, we can pick off the terms in K_2 by inspection.

$$K_2(j) = \left(\frac{5a}{16} - \frac{17}{64\omega_o}\right)j^3 \tag{4.2.17}$$

We could now find w_2 from (4.2.16) and proceed to third order.

To understand the physical meaning of this transformation, we note that these transformations do not change the topology of the variables. Φ and ϕ are both defined modulo 2π. Hence we can calculate the action I of the oscillator

$$I = \frac{1}{2\pi}\oint p(q)dq = \frac{1}{2\pi}\int_0^{2\pi} J(\Phi)d\Phi \quad . \tag{4.2.18}$$

Since J is constant along an orbit, it follows that the action is J. The transformation T takes the action angle variables of the unperturbed oscillator to the action angle variables of the perturbed system.

The physical relationship which one usually desires for an oscillator is the frequency versus energy function. The numerical value of the Hamiltonian is unchanged in a time independent transformation, so we can get this relationship by eliminating J from the expressions for the new Hamiltonian and the frequency:

$$K = \omega_o J + \frac{3}{8}\varepsilon J^2 + \varepsilon^2\left(\frac{5a}{16} - \frac{17}{64\omega_o}\right)J^3 \tag{4.2.19}$$

$$\omega = \omega_o + \frac{3}{4}\varepsilon J + \varepsilon^2\left(\frac{15a}{16} - \frac{51}{64\omega_o}\right)J^2 . \tag{4.2.20}$$

To second order in ε this result is

$$\omega = \omega_o + \frac{3}{4}\varepsilon\left(\frac{E}{\omega_o}\right) + \varepsilon^2\left(\frac{E}{\omega_o}\right)^2\left(\frac{15a}{16} - \frac{69}{64\omega_o}\right) \tag{4.2.21}$$

We have set K = E.

We now compare our results to a known solvable system, a pendulum. In this case the Hamiltonian is

$$h = \frac{1}{2}p^2 + \omega_o^2(1 - \cos q) \tag{4.2.22}$$

$$= \frac{1}{2}p^2 + \frac{1}{2}\omega_o^2 q^2 - \frac{1}{4!}\omega_o^2 q^4 + \frac{1}{6!}\omega_o^2 q^6 . \tag{4.2.23}$$

Comparing with (4.2.1), we see that $\varepsilon = -\frac{1}{6}$ and $a = \frac{2}{5\omega_o}$. Perturbation theory thus gives

$$\omega = \omega_o\left(1 - \frac{1}{8}\frac{E}{\omega_o^2} - \frac{5}{256}\left(\frac{E}{\omega_o^2}\right)^2\right) \tag{4.2.24}$$

The exact result (ref. [18], p. 112) is

$$\omega = \omega_o\left[2K\left(\sqrt{\frac{E}{2\omega_o^2}}\right)\right]^{-1} , \tag{4.2.25}$$

which yields (4.2.24) upon expansion (K is the complete elliptic integral).

A point to note in this example is that we generate the exact solution as a power series in ε. It could happen that there are singularities a $\varepsilon=0$, e.g. $\omega(\varepsilon)$ in (4.2.20) could contain a term like e^{-1/ε^2}. Perturbation theory

cannot pick these terms up since analyticity in ε is assumed from the outset. But, if the exact solution is a power series, we should be able to obtain it term by term using perturbation theory.

4.3 The Parametrically Driven Anharmonic Oscillator

We now show what happens when we add a time dependent perturbation to the problem of the previous section. The particular perturbation we choose is a parametric time dependence, so that h has the form

$$h = \frac{1}{2} p^2 + \frac{1}{2} \omega_o^2 (1 + \alpha \sin\Omega t) q^2 + \frac{\varepsilon}{4} \omega_o^2 q^4 \tag{4.3.1}$$

Before embarking on this problem, we discuss the ordering scheme. If we wish to be careful, in any calculation we must assign an order to each of the various terms in a Hamiltonian. Then we can approach the question of errors. However, in most work in physics only very rough error estimates are given. Secondly, a natural small parameter may not appear obvious. For example, in the problem of the last section we could have set $\varepsilon=1$ and followed the same perturbation calculation. The Deprit equations would keep track of the orders, and we would have to keep in mind that our solution is good only when the term we treat as a perturbation is small. In the previous calculation this would mean small energy. Hence in doing a calculation it is not necessary that h_2 have an ε^2 in front of it, instead we know that upon doing a calculation to first order, errors of the size of h_2 remain.

In the above Hamiltonian we have two small parameters, α and ε. We want to solve it in the case where the time dependent perturbation is smaller than the nonlinearity, so we choose

$$\begin{aligned} h_o &= \frac{1}{2} p^2 + \frac{1}{2} \omega_o^2 q^2 \\ h_1 &= \frac{\varepsilon}{4} \omega_o^2 q^4 \\ h_2 &= \frac{1}{2} \alpha \omega_o^2 \sin\Omega t \; q^2 \\ h_n &= 0 \quad \text{for } n>0. \end{aligned} \tag{4.3.2}$$

In doing this calculation we set $\theta=1$, letting the Deprit perturbation equations keep track of the ordering.

We again switch to the more convenient variables. We have done the first order analysis in the previous section, and so we know h_1 and K_1. Here we start with h_2 (and the more convenient variables).

$$h_2 = \frac{\alpha\omega_o j}{2}\left[\sin\Omega t + \frac{1}{2}\sin(\Omega t + 2\phi) + \frac{1}{2}\sin(\Omega t - 2\phi)\right] \tag{4.3.3}$$

This term modifies (4.2.16) to be

$$\frac{\partial w_2}{\partial t} + \{w_2, h_o\} = 2K_2 - \alpha\omega_o j\left[\sin\Omega t + \frac{1}{2}\sin(\Omega t + 2\phi) + \frac{1}{2}\sin(\Omega t - 2\phi)\right] + \frac{j^3}{\omega_o}\left(\frac{17}{32} + \frac{51}{16}\cos 2\phi + \frac{3}{4}\cos 4\phi - \frac{1}{8}\cos 6\phi\right). \tag{4.3.4}$$

The analysis of the last four terms was done in the previous section. Since (4.3.4) is linear in w_2 and K_2, we can find w_2 and K_2 by analyzing the equation

$$\frac{\partial w_2}{\partial t} + \{w_2, h_o\} = 2K_2 - \alpha\omega_o j\left[\sin\Omega t + \frac{1}{2}\sin(\Omega t + 2\phi) + \frac{1}{2}\sin(\Omega t - 2\phi)\right]. \tag{4.3.5}$$

Adding the results of this analysis to the w_2 and K_2 of the previous section will give the total w_2 and K_2.

In analyzing (4.3.5) we first set $K_2 = 0$ and integrate to find w_2. This gives

$$w_2 = +\alpha\omega_o j\left(\frac{\cos\Omega t}{\Omega} + \frac{1}{2}\frac{\cos(\Omega + 2\phi)}{\Omega + 2\omega_o} + \frac{1}{2}\frac{\cos(\Omega t - 2\phi)}{\Omega - 2\omega_o}\right). \tag{4.3.6}$$

We see that w_2 is very large if either Ω, $\Omega + 2\omega_o$, or $\Omega - 2\omega_o$ is small. This is the familiar problem of small denominators. For $\Omega \gg \omega_o$ we pick w_2 as in (4.3.6), but we must treat the other cases individually.

Here we treat the case where $\Omega \sim 2\omega_o$. Then we choose K_2 to cancel the only term which we can not transform away. Thus we have

$$w_2 = \alpha\omega_o j \left(\frac{\cos\Omega t}{\Omega} + \frac{1}{2}\frac{\cos(\Omega t + 2\phi)}{\Omega + 2\omega_o}\right) \tag{4.3.7}$$

$$K_2 = \frac{\alpha\omega_o j}{4} \sin(\Omega t - 2\phi) \quad . \tag{4.3.8}$$

Collecting all the terms in K including those parts of K_2 calculated in the previous section we have

$$K = \omega_o j + \frac{3}{2}\varepsilon j^2 - \frac{17}{64}\varepsilon^2 \frac{j^3}{\omega_o} + \frac{\alpha\omega_o j}{4}\sin(\Omega t - 2\phi). \tag{4.3.9}$$

In this case we have not been able to transform to a Hamiltonian independent of the angle , but we have simplified the analysis by eliminating two terms. In fact, this Hamiltonian can now be solved by transforming to the new variables

$$I = j \tag{4.3.10}$$

$$\theta = \frac{\Omega}{2} t + \phi \quad . \tag{4.3.11}$$

Using this transformation K becomes a time independent one degree of freedom Hamiltonian,

$$K = \omega_o I + \frac{3}{2}\varepsilon I^2 - \frac{17}{64}\varepsilon^2 \frac{I^3}{\omega_o} + \frac{\alpha\omega_o I}{4}\sin 2\theta, \tag{4.3.12}$$

which can now be solved by quadrature.

4.4 Superconvergence; Kolmogorov's Technique

In this section we discuss the phenomenon of superconvergence. This phenomenon is known to occur when one is doing Hamiltonian perturbation theory using the generating functions of mixed variables $F(\underset{\sim}{q},\underset{\sim}{P},t)$. Here we show how the same phenomenon occurs when doing Hamiltonian perturbation theory with Lie transforms.

In the Poincare-Von Zeipel technique [4,5] a mixed variable generating function $F(\underset{\sim}{q},\underset{\sim}{P},t)$ is introduced and chosen, order by order, to cancel the phase dependent part (rapidly varying part) of the Hamiltonian. Kolmogorov [6] improved this scheme by introducing the method of successive transformations, here after known as Kolmogorov's technique. In this scheme one transfoms to successive systems. After each transformation the Hamiltonian becomes solvable to higher order. The improvement arising in this technique is that after doing the nth transformation one has solved the problem to order $\varepsilon^{(2n)}$. Another way of putting it is as follows. Let δh_n be the rapidly varying part of the Hamiltonian after the nth transformation. Then the terms of the sequence δh_1, δh_2, $\delta h_3 \ldots$ are of orders ε^2, ε^4, $\varepsilon^8, \ldots$. This phenomenon is termed superconvergence. It does not by itself imply convergence.

In this paper we show how superconvergence arises in Lie transform theory. We show explicitly that the successive errors go like ε^2, ε^4, $\varepsilon^8, \ldots$ when combining Lie transform theory with Kolmogorov's technique. This has been shown previously in the case of time independent transformations [9]. Here we extend these results to the case of time dependent transformations. We also draw new conclusions concerning the use of Kolmogorov's technique in practical calculations.

We begin by assuming we have an ordered Hamiltonian which is solvable in zeroth order.

$$h(\underset{\sim}{z},t) = \sum \varepsilon^n h_n(\underset{\sim}{z},t). \tag{4.4.1}$$

Transforming to appropriate coordinates, the zeroth order Hamiltonian is written entirely in terms of the momenta.

$$h_o(\underset{\sim}{z},t) = h_o(\underset{\sim}{p}). \tag{4.4.2}$$

As usual, the vector $\underset{\sim}{z}$ represents all of the phase space variables.

We will introduce successive Lie transforms to successive new Hamiltonians. We distinguish the various transforms by a left subscript. In particular, we introduce the transform ${}_1T$. According to the usual method, the variable mapping is given by

$$\begin{aligned} {}_1\underset{\sim}{Z}(\underset{\sim}{z},t) &= \underset{\sim}{z} - \varepsilon\left\{{}_1w_1, \underset{\sim}{z}\right\} + \frac{1}{2}\varepsilon\left\{{}_1w_1, \{{}_1w_1, \underset{\sim}{z}\}\right\} \\ &\quad - \frac{1}{2}\varepsilon\{{}_1w_2, \underset{\sim}{z}\} + \cdots \\ &= {}_1T\,\underset{\sim}{z}\,, \end{aligned} \tag{4.4.3}$$

where the ${}_1w_i$'s are yet unchosen .

Following the general Lie transform scheme, we choose the generating function ${}_1w_i$ by examining the equations for the transformed Hamiltonian ${}_1K = \sum_n \varepsilon^n {}_1K_n(\underset{\sim}{z},t)$. To second order these equations are

$${}_1K_o = h_o \tag{4.4.4}$$

$$\frac{\partial\, {}_1w_1}{\partial t} + \{{}_1w_1, h_o\} = {}_1K_1 - h_1 \tag{4.4.5}$$

$$\frac{\partial\, {}_1w_2}{\partial t} + \{{}_1w_2, h_o\} = 2({}_1K_2 - h_2) - \{{}_1w_1, h_1 + {}_1K_1\}\ . \tag{4.4.6}$$

In the ordinary Lie transform method, we choose the new Hamiltonian ${}_1K$ order by order to make the right hand side of these equations have zero average along a particle orbit. Then the ${}_1w_i$'s, which are found by integration of the right hand side along an orbit of h_o, will be nonsecular.

In the successive transformation scheme, we choose the new Hamiltonian differently. The general rule is that we calculate the generating function only in those orders which can be done nonrecursively. In particular, in first order we can pick, ${}_1K_1 = \overline{h}_1$ (overbar denotes orbit average) and find ${}_1w_1$ to calculate the Poisson bracket on the right hand side of (4.4.6). Hence we choose ${}_1w_r = 0$ for $n > 1$, and we use ${}_1w_1$ and the transformation equations, e,g. (4.4.6), to find ${}_1K_n$ for $n > 1$.

The result is that we now have the hamiltonian ${}_1K$ which is solvable to first order. ${}_1K_1 = {}_1K_1(p)$ since ${}_1K_1$ was chosen to be the average of h_1. We want to apply a second transformation, but before doing so, we rearrange the terms in our Hamiltonian. We pick our "new-old" Hamiltonian ${}_1h = \sum \varepsilon^n {}_1h_n$ to be given by

$$ {}_1h_o = {}_1K_o + \varepsilon_1K_1 \tag{4.4.7} $$

$$ {}_1h_1 = 0 \tag{4.4.8} $$

$$ {}_1h_2 = {}_1K_2 \tag{4.4.9} $$

$$ {}_1h_3 = {}_1K_3 \tag{4.4.10} $$

$$ \vdots $$

That is, we absorb all of the solvable parts in the unperturbed Hamiltonian.

Now we apply a second transformation ${}_2T$ to the above Hamiltonian. Since there is no first order perturbation to transform away, ${}_2T$ vanishes in first order. Using this fact, we get the following equations relating the new Hamiltonian to the generating function of ${}_2T$.

$$ \frac{\partial {}_2w_2}{\partial t} + \{{}_2w_2, {}_1h_o\} = 2({}_2K_2 - {}_1h_2) \tag{4.4.11} $$

$$ \frac{\partial {}_2w_3}{\partial t} + \{{}_2w_3, {}_1h_o\} = 3({}_2K_3 - {}_1h_3) \tag{4.4.12} $$

$$\frac{\partial\, {}_2w_4}{\partial t} + \{{}_2w_4, {}_1h_o\} = 4({}_2K_4 - {}_1h_4) - \{{}_2w_2, {}_1h_2 + {}_2K_2\} \tag{4.4.13}$$

Now we pick ${}_2K$ so that the right hand sides have zero orbit average along ${}_1h_o$. We note that Eqs. (4.4.11) and (4.4.12) can be solved nonrecursively, but to solve (4.4.13), we must first know the results from (4.4.11). Hence we select ${}_2w_2$ and ${}_2w_3$ by the usual method, we set ${}_2w_n = 0$ for $n > 3$, and we use the higher order transform equations to find ${}_2K_n$ for $n > 3$.

We remark on the ordering system here. By regrouping terms in (4.4.7), we have messed up the ordering scheme. We have added an order ε^1 term to an order ε^o term and called the result order ε^o. This does not invalidate our results. When we calculate ${}_2w_2$ in (11) it is good to second order, but since ${}_1h_o$ has a first order part, ${}_2w_2$ has higher than ε^2 effects included in it.

Let's review what has been accomplished. First we introduced an order ε transformation ${}_1T$. Then we regrouped terms in the Hamiltonian and transformed away order ε^2 and order ε^3 oscillations using ${}_2T$. If we regroup terms again, we will have ${}_2h_n = 0$ for $n = 1,2,3$. It follows that the perturbations in ${}_2h$ are of order ε^4.

To construct an induction proof we use the equation (4.1.14). We assume the unsolvable terms in the Hamiltonian ${}_{n-1}h$ are of orders higher than or equal to $\varepsilon^{(2^n)}$. By regrouping terms we get ${}_{n-1}h_m = 0$ for $1 \leq m < 2^n$. We apply the transformation ${}_nT$. We must have ${}_nw_1, \ldots, {}_nw_{(2^n-1)} = 0$; ${}_nT_1, \ldots, {}_nT_{(2^n-1)} = 0$; and ${}_nK_1, \ldots, {}_nK_{(2^n-1)} = 0$. Simply put, if there is no perturbation to $(2^n-1)^{th}$ order, then there is no new transformation nor new Hamiltonain to $(2^n-1)^{th}$ order. From (4.1.14), this implies

$$\frac{\partial\, {}_nw_m}{\partial t} + \{{}_nw_m, h_o\} = m({}_nK_m - {}_{n-1}h_m) \tag{4.4.14}$$

for $m < 2^{n+1} - 1$, meaning that we can choose our transformation nonrecursively to order $\varepsilon^{(2^{n+1})}$ or higher. This completes the induction proof.

Kolmogorov's technique is very appealing since the errors in the perturbation theory are of ordera ε, ε^2, ε^4, ε^8,... after each step rather than the usual ε, ε^2, ε^3, ε^4,... . One should be closer to the right answer after n steps of Kolmogorov's technique than after n steps of the usual method. To theoretical physicists Kolmogorov's technique is appealing for another reason. In calculating the orbit integrals Kolmogorov's technique uses an improved unperturbed Hamiltonian. The inclusion of higher order terms in the unperturbed Hamiltonian is called renomalization, and it is believed to be important in a number of fields such as quantum field theory, statistical mechanics, and plasma physics.

A point to he noted about Kolmogorov's technique is that terms of order ε^3, ε^5, ε^6,... do not vanish. Instead they are being absorbed in lower order terms. For example, the term of order three is included in the generating function ${}_2w = {}_2w_2 + \varepsilon_2 w_3$. Although Kolmogorov's technique has fewer terms than the standard method, each term is much more complicated.

There are reasons for believing that Kolmogorov's technique is not the most practical method of attacking problems. To see one reason, compare Kolmogorov's technique to the standard Deprit method in fourth order theory. Kolmogorov's technique requires knowing ${}_2h$ (${}_2h_o$ plus perturbations) at this point, and ${}_2h$ must be found by multiplying three infinite series: ${}_2h = {}_2T^{-1} \cdot {}_1T^{-1} \cdot h$. In the standard Deprit method, only two infinite series are ever multiplied, c.f. (4.1.14).

Another criticism of Kolmogorov's technique is that it can introduce spurious small denominators. An example of this is the one degree of freedom Hamiltonian $h = \frac{1}{2} p^2 + \frac{1}{2} \omega_o^2 q^2 + \Delta U(\varepsilon, q)$ for which the transformed Hamiltonian

is exponential in the true action J:

$$K(J) = \omega_o(1-e^{-\varepsilon J})/\varepsilon = \omega_o J - \frac{\varepsilon}{2}\omega_o J^2 + \ldots \quad . \tag{4.4.15}$$

If we use perturbation theory to find K from h (as in section 4.2), we would have K to first order in ε upon doing the first order transformation. Using the standard method we have a fixed ordering, and so we just generate the power series in (4.4.15) term by term. In Kolmogorov's technique we change the ordering by absorbing the solvable terms in the unperturbed Hamiltonian. After the first transformation we would get

$$_1h_o(j) = \omega_o j - \frac{\varepsilon}{2}\omega_o j^2 \quad . \tag{4.4.16}$$

Recalling section 4.2, we note that each time we integrated along a particle orbit we had a $1/\omega_o$ in our expressions for the w_n's. If we instead use the unperturbed Hamiltonain of (4.4.16), we end up with a factor of $1/_1\omega_o = 1/_1\omega_o = 1/(\omega_o - \varepsilon\omega_o j)$. We see that Kolmogorov's technique fails for $j = 1/\varepsilon$ and perhaps beyond, because it gives a small denominator.

The reason Komogorov's technique is worse than the standard method in this case is not hard to see. The improved frequency $_1\omega_o$ is closer than ω_o to the final result ω in the range $0 < J \leq 2/\varepsilon$, but the improved frequency is much farther than ω_o from the final result for $2/\varepsilon \leq J < \infty$. Furthermore the improved frequency is qualitatively different: it has a root for finite J. Neither ω_o nor ω are ever zero in the range $0 \leq J < \infty$.

We conclude that Kolmogorov's technique is not necessarily the most practical way of obtaining answers. This does not contradict the fact that Kolmogorov [6], Arnold [20], and Moser [21] all used this technique to show convergence of certain invariant tori. In their proof the concern was about getting very good expression in small regions of phase space. In the interval $0 \leq J \leq 1$, the improved Hamiltonian $_1h_o$ does give a much better value for the frequency than does the unperturbed Hamiltonian $h_o = \omega_o j$.

5. Discussion of the Choice of Conventions

The primary purpose of this paper has been to give an expository account of Hamiltonian perturbation theory using the Lie transformation theory of Dewar, and from it to derive Deprit's perturbation theory. Furthermore, we have illustrated the theories with examples. However, a secondary purpose of this paper has been to introduce conventions which are convenient in practical applications. In this section we outline how and why the conventions presented here differ from previous ones.

We first mention that there are two minus signs in the definition of the operator L_f. The more obvious one is that (2.1.8) differs from the corresponding definitions in for example references [8] and [10]. We chose the definition (2.1.8) to rid Jacobi's identity of a minus sign. In the case where L_f is defined by

$$L_f g = \{g,f\}, \tag{2.1.8'}$$

Jacobi's identity is

$$[L_f, L_g] = - L_{\{f,g\}} . \tag{2.1.9'}$$

A second minus sign occurs because of a difference in the meanings of coordinates and momenta in the different literatures. In the celestial mechanics literature [7-9] the unperturbed Hamiltonian depends only on the coordinates. Here and elsewhere [1, 16, 18, 19] the unperturbed Hamiltonian is chosen to depend only on the momenta. This is the natural choice when dealing with nearly free particle motion where momentum is conserved in the unperturbed system. Of course, when momenta and coordinates are interchanged, the sign of the Poisson bracket changes.

In choosing the transformation T in terms of L_w, there are a number of possibilities. Hori [7] defines T by simple exponentiation: $T \equiv \exp(L_w)$. Deprit's choice [8], that T satisfies a differential equation, proves more useful in two respects. One is able to derive recursion relations for the successive terms, and one is able to do time dependent transformations in a straightforward manner. With Hori's choice one does time dependent theory by first enlarging the phase space with time as a new coordinate. Dewar [10] improved Deprit's work by not requiring T to be a power series. Basically, we have followed Dewar's formalism, but we differ in minor ways on the form of the differential equation. Our choice is to make the evolution of T in θ by L_w exactly analogous to the evolution of the time development operator M in t by L_h.

Another convention we must choose is whether T should transform from old to new coordinaces as done here, or from new to old as in [10]. The choice made here simplifies the perturbation equations, in particular Eq. (4.1.14). In perturbation theory we average (4.1.14) along a particle orbit to find K_n. Since the K_n's are averages and the w_n's have no averages, this gives

$$K_n = \overline{h}_n + \sum_{m=1}^{n-1} \frac{m}{n} \overline{T^{-1}_{n-m} h_m} . \tag{5.1}$$

Had we made the opposite choice, we would have effectively interchanged h and K, and the result for K_n would be

$$K_n = \overline{h_n} - \frac{1}{n} \sum_{m=1}^{n-1} \left(\overline{L_{n-m} h_m} + m \overline{T^{-1}_{n-m} K_m} \right). \tag{5.1'}$$

By our choice, we avoid the extra set of terms in (5.1').

We also write our expansions differently. In previous works, the expansion (4.1.4) is of the form

$$w = \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n w_{n+1} \quad . \tag{4.1.4'}$$

That choice does not allow Eq. (4.1.14) to have such a simple appearance. The effect of that choice is to insert n!'s in all the equations. The result is that Deprit's equations (e.g. ref. [8] eq. (27)) have binomial coefficients whereas ours have simple fractions.

In summary, we have changed the conventions in minor ways. We have presented the conventions we have found to be most useful in practical calculations.

6. Wave Packets: The Meaning of Wave Energy and Momentum and the Effect of Nonlinear Currents on Wave-Packet Evolution

In this chapter we prepare the way for later work by discussing the aspects of wave packet evolution which are universal to all uniform dispersive media. The goal is to go as far as possible using only the concept of the linear response tensor $\underset{\approx}{D}(\underset{\sim}{k},\omega)$. Past work along these lines includes expressions for the wave energy[1] and wave momentum[2] of normal mode wave packets, and it includes derivations of the geometric optics of weakly nonuniform systems.[3,4]

We begin by discussing the technique of expanding in Fourier space. This technique is useful for analyzing wave packet perturbations, i.e. perturbations which have a narrow spectrum in Fourier $(\underset{\sim}{k},\omega)$ space. In section 6.2 we use this technique to generalize the concepts of wave energy and wave momentum. We derive expressions for the wave energy and wave momentum, which apply to perturbations that are not normal modes. Exploiting these expressions allows us to write the wave energy and wave momentum as sums of contributions from the electromagnetic field and the individual species of a Vlasov plasma.

Then we analyze wave packet evolution using the method of variation of parameters.[5] We used this method to systematically derive the nonlinear wave equation for the electric field amplitude, given the nonlinear currents. We also discuss a heuristic method for deriving nonlinear wave equations. This method, which may leave out important effects, has seen much use in the literature[6-9] because it is easy to apply. In this chapter we compare the general aspects of the heuristic

method and the systematic method. Later in the thesis we show in specific examples how the heuristic method leaves out important effects.

6.1 Expansion in Fourier Space: A Useful Technique for Analyzing Wave Packets

The problem we consider is how to find approximate local relations in real space for quantities which are related by simple multiplication in Fourier space [as, for example, the current and the electric field in a linear dielectric are related by the conductivity: $\underset{\sim}{j}(\underset{\sim}{k},\omega) = \underset{\approx}{\sigma}(k,\omega)\cdot\underset{\sim}{E}(\underset{\sim}{k},\omega)$]. We solve this problem for the case where the quantities (e.g. $\underset{\sim}{j}$ and $\underset{\sim}{E}$) have a narrow spectrum in Fourier space by using the technique of expansion in Fourier space. Here we describe this technique and apply it to find the local relations between $\underset{\sim}{j}$ and $\underset{\sim}{E}$, p and $\underset{\sim}{E}$, and $\underset{\sim}{B}$ and $\underset{\sim}{E}$ in a homogeneous dielectric.

Since $\underset{\sim}{x}$ and t are treated equivalently in this section and much of the next, we introduce the notation[10] of special relativity. The four-vector x has contravariant components $x^{0-3} = (t,\underset{\sim}{x})$ and covariant components $x_{0-3} = (t,-\underset{\sim}{x})$. Similarly the four-vector k has contravariant components $k^{0-3} = (\omega,\underset{\sim}{k})$ and covariant components $k_{0-3} = (\omega,-\underset{\sim}{k})$. Also, in this section and section 6.2 only, we use the Einstein summation convention for repeated indices.

Let us consider an electric field of wave packet form, i.e.

$$\underset{\sim}{E}(x) = \underset{\sim}{\mathcal{E}}(x)e^{-i\kappa_\mu x^\mu}+c.c. \quad . \tag{6.1.1}$$

The dominant k-vector for this electric field is $k^{0-3} \equiv (\Omega,\underset{\sim}{\kappa})$, and the

slowly varying amplitude is $\underset{\sim}{\mathcal{E}}(x)$. The Fourier transform of E is given by

$$\begin{aligned}\underset{\sim}{E}(k) &= \int d^4x\, e^{ik_\mu x^\mu}\, \underset{\sim}{E}(x) \\ &= \int d^4x \left[e^{i(k_\mu-\kappa_\mu)x^\mu}\, \underset{\sim}{\mathcal{E}}(x) + e^{i(k_\mu+\kappa_\mu)x^\mu}\, \underset{\sim}{\mathcal{E}}^*(x) \right] \\ &= \underset{\sim}{\mathcal{E}}(k-\kappa) + \underset{\sim}{\mathcal{E}}^*(k+\kappa). \end{aligned} \tag{6.1.2}$$

In our notation system the Fourier transform $\underset{\sim}{E}(k)$ is distinguished from $\underset{\sim}{E}(x)$ only by the argument. Furthermore, $\underset{\sim}{\mathcal{E}}^*(k)$ denotes the Fourier transform of $\underset{\sim}{\mathcal{E}}^*(x)$, not the complex conjugate of $\underset{\sim}{\mathcal{E}}(k)$. Since $\underset{\sim}{\mathcal{E}}(x)$ varies slowly in x, $\underset{\sim}{\mathcal{E}}(k)$ is highly peaked at k=0. Hence the first term in Eq. (6.1.2) is highly peaked at $k = \kappa$, and the second at $k = -\kappa$.

In Fourier space the current is related to the electric field by matrix multiplication:

$$\begin{aligned}\underset{\sim}{j}(k) &= \underset{\approx}{\sigma}(k)\cdot\underset{\sim}{E}(k) \\ &= \underset{\approx}{\sigma}(k)\cdot\underset{\sim}{\mathcal{E}}(k-\kappa) + \underset{\approx}{\sigma}(k)\cdot\underset{\sim}{\mathcal{E}}^*(k+\kappa). \end{aligned} \tag{6.1.3}$$

Inverse Fourier transforming this equation we find $j(\underset{\sim}{x})$.

$$\underset{\sim}{j}(x) = \int \frac{d^4k}{(2\pi)^4} \; e^{-ik_\mu x^\mu} \; \underset{\approx}{\sigma}(k)\cdot\underset{\sim}{\mathcal{E}}(k-\kappa)$$

$$+ \int \frac{d^4k}{(2\pi)^4} \; e^{+ik_\mu x^\mu} \; \underset{\approx}{\sigma}(k)\cdot\underset{\sim}{\mathcal{E}}^*(k+\kappa)$$

$$= \int \frac{d^4k}{(2\pi)^4} \; e^{-i(k_\mu+\kappa_\mu)x^\mu} \; \underset{\approx}{\sigma}(k+\kappa)\cdot\underset{\sim}{\mathcal{E}}(k)$$

$$+ \int \frac{d^4k}{(2\pi)^4} \; e^{-i(k_\mu-\kappa_\mu)x} \; \underset{\approx}{\sigma}(k-\kappa)\cdot\underset{\sim}{\mathcal{E}}^*(k) \qquad (6.1.4)$$

The last step follows from changing the variable of integration. Pulling the rapidly varying phase out of the integrals we find

$$\underset{\sim}{j}(x) = \underset{\sim}{\mathcal{J}}(x)\, e^{-i\kappa_\mu x^\mu} + \text{c.c.}, \qquad (6.1.5)$$

where

$$\underset{\sim}{\mathcal{J}}(x) = \int \frac{d^4k}{(2\pi)^4} \; e^{-ik_\mu x^\mu} \; \underset{\approx}{\sigma}(k+\kappa)\cdot\underset{\sim}{\mathcal{E}}(k). \qquad (6.1.6)$$

To evaluate this integral we invoke the highly peaked nature of $\underset{\sim}{\mathcal{E}}(k)$. This peaked nature allows us to appropriate $\underset{\approx}{\sigma}(k+\kappa)$ near k=0 by the first few terms in the Taylor series

$$\underset{\approx}{\sigma}(k+\kappa) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(k_\mu \frac{\partial}{\partial \kappa_\mu} \right)^n \underset{\approx}{\sigma}(\kappa). \qquad (6.1.7)$$

Inserting this expression into Eq. (6.1.6) we obtain

$$\underset{\sim}{\mathcal{J}}(x) = \int \frac{d^4k}{(2\pi)^4} \; e^{-ik_\mu x^\mu} \sum_{n=0}^{\infty} \frac{1}{n!} \left(k_\mu \frac{\partial}{\partial \kappa_\mu} \right)^n \underset{\approx}{\sigma}(\kappa) \cdot \underset{\sim}{\mathcal{E}}(k)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial \kappa_\mu} \right)^n \underset{\approx}{\sigma}(\kappa) \cdot \underset{\sim}{\mathcal{E}}(k)$$

$$= \exp\left[i \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial \kappa_\mu} \right] \underset{\approx}{\sigma}(\kappa) \cdot \underset{\sim}{\mathcal{E}}(x) \quad . \tag{6.1.8}$$

Approximating this expression by the first two terms we obtain

$$\underset{\sim}{j}(x) = e^{-ik_\alpha x^\alpha} \left(1 + i \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial \kappa_\mu} \right) \underset{\approx}{\sigma}(\kappa) \cdot \underset{\sim}{\mathcal{E}}(x) + \text{c.c.}, \tag{6.1.9}$$

a local relation between $\underset{\sim}{j}(x)$ and $\underset{\sim}{\mathcal{E}}(x)$.

The critical step in obtaining this local relation is the approximation of the sum (6.1.7) by its first two terms. The validity of this approximation depends on how much $\underset{\approx}{\sigma}(k+\kappa)$ varies over the narrow region in k space where $\underset{\sim}{\mathcal{E}}(k)$ is significant.

The relations between ρ and $\underset{\sim}{E}$, and $\underset{\sim}{B}$ and $\underset{\sim}{E}$ are found in the same way. From $\rho(k) = \underset{\sim}{k} \cdot \underset{\sim}{j}(k)/\omega = \underset{\sim}{k} \cdot \underset{\sim}{\sigma}(k) \cdot \underset{\sim}{E}(k)/\omega$ we obtain

$$\rho(x) = e^{-i\kappa_\mu x^\mu} \left(1 + i \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial \kappa_\mu} \right) \left[\underset{\sim}{\kappa} \cdot \underset{\approx}{\sigma}(\kappa) \cdot \underset{\sim}{\mathcal{E}}(x)/\Omega \right] + \text{c.c.}, \tag{6.1.10}$$

and from Faraday's law $\underset{\sim}{B}(k) = c\underset{\sim}{k} \times \underset{\sim}{E}(k)/\omega$ we find

$$\underset{\sim}{B}(x) = e^{-i\kappa_\mu x^\mu}\left(1+i\,\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \kappa_\mu}\right)\left[c\underset{\sim}{\kappa}\times\underset{\sim}{\mathcal{E}}(x)/\Omega\right]+c.c. \qquad (6.1.11)$$

In summary, we have found local expressions between $\underset{\sim}{j}$ and $\underset{\sim}{E}$, ρ and $\underset{\sim}{E}$, and $\underset{\sim}{B}$ and $\underset{\sim}{E}$ by the method of expansions in Fourier space.

6.2 Wave Energy and Momentum

We consider a homogeneous time independent dielectric (such as a magnetized plasma or a polarized crystal) which is excited by an external current $\underset{\sim}{j}_e(x)$. By computing the average rate at which the external current source loses energy and momentum, we derive expressions for the energy and momentum of the dielectric. Our expressions are generalizations of the well known wave energy and momentum formulas,[1,2] in that our expressions apply as well to perturbations which are not normal modes. In addition our generalization technique allows us to uniquely divide the wave momentum and energy into a contribution from the electromagnetic field and contributions from each species in a multispecies collisionless plasma.

In general a dielectric responds both linearly and nonlinearly to an external current $\underset{\sim}{j}_e(x)$. The linear (in $\underset{\sim}{j}_e$) response includes the electric field $\underset{\sim}{E}_1(x)$ and the associated dielectric charges and currents. In Fourier space the linear response satisfies

$$\frac{i\omega}{4\pi}\ \underset{\approx}{D}(k)\cdot\underset{\sim}{E}_1(k) = \underset{\sim}{j}_e(k) \tag{6.2.1}$$

where $\underset{\approx}{D} \equiv \underset{\approx}{I}\left(1-|\underset{\sim}{k}|^2c^2/\omega^2\right) + \underset{\sim}{k}\,\underset{\sim}{k}\,c^2/\omega^2 + 4\pi i\underset{\approx}{\sigma}(k)/\omega$ is known here as the linear response tensor. Assuming $\underset{\sim}{j}_e(k)$ is highly peaked near $k=\kappa$, the response $\underset{\sim}{E}_1(k)$ is also, and we can use the method of expansion in Fourier space. We find [essentially Eq. (6.1.9) with $\underset{\approx}{\sigma} \to i\omega\underset{\approx}{D}\ 4\pi$]:

$$\underset{\sim}{j}_e(x) = e^{-i\kappa_\mu x^\mu}\left(1+i\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \kappa_\mu}\right)\left[\frac{i\Omega}{4\pi}\underset{\approx}{D}(\kappa)\cdot\underset{\sim}{\mathcal{E}}_1(x)\right]+\text{c.c.}, \qquad (6.2.2)$$

where $\underset{\sim}{\mathcal{E}}_1$ is the slowly varying amplitude of $\underset{\sim}{E}_1(x)$.

To find the energy in the dielectric we consider the negative of the average power density $\overline{-\underset{\sim}{E}(x)\cdot\underset{\sim}{j}_e(x)}$ transferred to the external current $\underset{\sim}{j}_e$ by the electric field $\underset{\sim}{E}$. Expanding $\underset{\sim}{E}$ in orders of $\underset{\sim}{j}_e$ we have

$$\overline{-\underset{\sim}{E}\cdot\underset{\sim}{j}_e} = -\overline{\underset{\sim}{E}_o\cdot\underset{\sim}{j}_e} - \overline{\underset{\sim}{E}_1\cdot\underset{\sim}{j}_e} - \overline{\underset{\sim}{E}_2\cdot\underset{\sim}{j}_e} \qquad (6.2.3)$$

through second order. The first term in the above expression vanishes since $\underset{\sim}{E}_o$ is stationary and $\underset{\sim}{j}_e$ is oscillating. The second-order electric field $\underset{\sim}{E}_2$, which appears in the last term, is a sum of terms which are either nearly stationary or oscillating at the second harmonic ($k\sim 2\kappa$), as we shall see in the next section. Hence $\underset{\sim}{E}_2\cdot\underset{\sim}{j}_e$ has only oscillating terms at the the first and third harmonics and it vanishes upon averaging. Therefore Eq. (6.2.3) reduces to

$$-\overline{\underset{\sim}{E}\cdot\underset{\sim}{j}_e} = -\overline{\underset{\sim}{E}_1\cdot\underset{\sim}{j}_e} \quad . \qquad (6.2.4)$$

We now insert $\underset{\sim}{j}_e$ from Eq. (6.2.2) and $\underset{\sim}{E}_1$ into Eq. (6.2.4) obtaining

$$\begin{aligned}
-\overline{\underset{\sim}{E}\cdot\underset{\sim}{j}_e} = &-\frac{i\Omega}{4\pi}\,\underset{\sim}{\mathcal{E}}^*(x)\cdot\left[\underset{\approx}{D}(\kappa) - \underset{\approx}{D}^+(\kappa)\right]\cdot\underset{\sim}{\mathcal{E}}_1(x) \\
&+\underset{\sim}{\mathcal{E}}_1^*(x)\cdot\left[\frac{\partial}{\partial\kappa_\mu}\frac{\Omega}{4\pi}\underset{\approx}{D}(\kappa)\right]\cdot\frac{\partial}{\partial x^\mu}\underset{\sim}{\mathcal{E}}_1(x) \\
&+\left[\frac{\partial}{\partial x^\mu}\underset{\sim}{\mathcal{E}}_1^*(x)\right]\cdot\left[\frac{\partial}{\partial\kappa_\mu}\frac{\Omega}{4\pi}\underset{\approx}{D}^+(\kappa)\right]\cdot\underset{\sim}{\mathcal{E}}_1(x), \qquad (6.2.5)
\end{aligned}$$

where $\underset{\approx}{D}^{+}(\kappa)$ is the adjoint of the matrix $\underset{\approx}{D}(\kappa)$. Introducing the hermitian and antihermitian parts of $\underset{\approx}{D} \equiv \underset{\approx}{D}_h + i\underset{\approx}{D}_a$, Eq. (6.2.5) can be written in the form

$$-\overline{\underset{\sim}{E}\cdot\underset{\sim}{j}_e} = \frac{\Omega}{2\pi}\,\underset{\sim}{\mathcal{E}}_1^*(x)\cdot\underset{\approx}{D}_a(\kappa)\cdot\underset{\sim}{\mathcal{E}}_1(x) + \frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial \kappa_\mu}\left[\frac{\Omega}{4\pi}\,\underset{\sim}{\mathcal{E}}_1^*(x)\cdot\underset{\approx}{D}_h(\kappa)\cdot\underset{\sim}{\mathcal{E}}(x)\right]$$
$$+\left\{ i\underset{\sim}{\mathcal{E}}_1^*(x)\left[\frac{\partial}{\partial \kappa_\mu}\,\frac{\Omega}{4\pi}\,\underset{\approx}{D}_a(\kappa)\right]\cdot\frac{\partial}{\partial x^\mu}\,\underset{\sim}{\mathcal{E}}_1(x) + \text{c.c.}\right\}. \qquad (6.2.6)$$

The antihermitian part of $\underset{\approx}{D}$ represents dissipation, which we assume small [e.g. $O(\varepsilon)$]. We have also assumed wave-packet perturbations, so that the operator $(\partial/\partial\kappa_\mu)(\partial/\partial x^\mu)$ is small [e.g. $O(\delta)$]. Thus the last term in Eq. (6.2.6) is much smaller than the first two [e.g. $O(\varepsilon\delta)$] and may be neglected, and reducing Eq. (6.2.6) to

$$-\overline{\underset{\sim}{E}\cdot\underset{\sim}{j}_e} = \frac{\Omega}{2\pi}\,\underset{\sim}{\mathcal{E}}_1^*(x)\cdot\underset{\approx}{D}_a(\kappa)\cdot\underset{\sim}{\mathcal{E}}_1(x) + \frac{\partial}{\partial t}\,\frac{\partial}{\partial \Omega}\left[\frac{\Omega}{4\pi}\,\underset{\sim}{\mathcal{E}}_1^*(x)\cdot\underset{\approx}{D}_h(\kappa)\cdot\underset{\sim}{\mathcal{E}}_1(x)\right]$$
$$-\nabla\cdot\frac{\partial}{\partial \underset{\sim}{\kappa}}\left[\frac{\Omega}{4\pi}\,\underset{\sim}{\mathcal{E}}_1^*(x)\cdot\underset{\approx}{D}_h(\kappa)\cdot\underset{\sim}{\mathcal{E}}_1(x)\right]. \qquad (6.2.7)$$

This equation has a simple interpretation. The quantity $-\overline{\underset{\sim}{E}\cdot\underset{\sim}{j}_e}$ is power density supplied to the dielectric by the external current. This quantity equals the sum of three terms. The first is the rate at which the dielectric dissipates energy. The second is the time derivative of the reversibly stored wave energy density. The last term is the divergence of the energy flux density. We denote the wave energy density by W:

$$W(x) \equiv \frac{1}{4\pi}\, \underset{\sim}{\mathcal{E}}_1^*(x)\cdot\left[\frac{\partial}{\partial\Omega}\, \Omega \underset{\approx}{D}_h(\kappa)\right]\cdot \underset{\sim}{\mathcal{E}}_1(x). \tag{6.2.8}$$

Integrating Eq. (6.2.7) over all space and neglecting dissipation ($\underset{\approx}{D}_a = 0$) we find

$$-\int d^3\underset{\sim}{x}\; \overline{\underset{\sim}{E}(x)\cdot \underset{\sim}{j}_e(x)} = \frac{d}{dt}\int d^3\underset{\sim}{x}\, W(x). \tag{6.2.9}$$

That is, in the absence of dissipation, the time derivative of the spatial integral of W equals the total power transferred to the plasma.

We stress that nowhere in this derivation have we used $\underset{\approx}{D}(\kappa)\cdot\underset{\sim}{\mathcal{E}}_1(x) = 0$, i.e. that we are driving a normal mode of oscillation. Of course, in that case, Eq. (6.2.9) gives

$$W(x) = \frac{\Omega}{4\pi}\, \underset{\sim}{\mathcal{E}}_1^*(x)\cdot\frac{\partial}{\partial\Omega}\, \underset{\approx}{D}_h(\kappa)\cdot\underset{\sim}{\mathcal{E}}_1(x). \tag{6.2.10}$$

By an identical analysis we derive the equation

$$\begin{aligned}
\overline{-(\rho_e \underset{\sim}{E} + \underset{\sim}{j}_e \times \underset{\sim}{B}/c)} = {} & \frac{\underset{\sim}{\kappa}}{2\pi}\, \underset{\sim}{\mathcal{E}}_1^*\cdot\underset{\approx}{D}_a\cdot\underset{\sim}{\mathcal{E}}_1 . \\
& + \frac{1}{4\pi}\frac{\partial}{\partial t}\left[\frac{\underset{\sim}{\kappa}}{\Omega^2}\, \frac{\partial}{\partial\Omega}\, \underset{\sim}{\mathcal{E}}_1^*\cdot\Omega^2\underset{\approx}{D}_h\cdot\underset{\sim}{\mathcal{E}}_1 - \frac{1}{\Omega}\, \underset{\sim}{\mathcal{E}}_1\underset{\sim}{\mathcal{E}}_1^*\cdot\underset{\approx}{D}_h\cdot\underset{\sim}{\kappa} - \frac{1}{\Omega}\, \underset{\sim}{\mathcal{E}}_1^*\underset{\sim}{\kappa}\cdot\underset{\approx}{D}_h\cdot\underset{\sim}{\mathcal{E}}_1\right] \\
& - \frac{1}{4\pi}\, \nabla\cdot\left[\frac{\partial}{\partial\underset{\sim}{\kappa}}\left(\underset{\sim}{\kappa}\, \underset{\sim}{\mathcal{E}}_1^*\cdot\underset{\approx}{D}_h\cdot\underset{\sim}{\mathcal{E}}_1\right) - 2I\underset{\approx}{\mathcal{E}}_1\underset{\approx}{D}_h\cdot\underset{\sim}{\mathcal{E}}_1^* + \underset{\sim}{\mathcal{E}}_1^*\cdot\underset{\approx}{D}_h\underset{\sim}{\mathcal{E}}_1 \right. \\
& \qquad\qquad \left. + \underset{\sim}{\mathcal{E}}_1\cdot\underset{\approx}{D}_h\underset{\sim}{\mathcal{E}}_1^*\right]
\end{aligned} \tag{6.2.11}$$

Here the arguments of the functions are the same as in Eq. (6.2.7),

i.e. $\underset{\approx}{D}(\kappa)$, $\underset{\sim}{\xi}_1(x)$. The interpretation of this equation parallels the preceding interpretation of Eq. (6.2.7). We define the wave momentum density G by

$$\underset{\sim}{G}(x) \equiv \frac{1}{4\pi}\left[\frac{\underset{\sim}{\kappa}}{\Omega^2}\ \frac{\partial}{\partial\Omega}\ \underset{\sim}{\xi}_1^*\cdot\Omega^2\underset{\approx}{D}_h\cdot\underset{\sim}{\xi}_1 - \frac{1}{\Omega}\ \underset{\sim}{\xi}_1\underset{\sim}{\xi}_1^*\cdot\underset{\approx}{D}_h\cdot\underset{\sim}{\kappa} - \frac{1}{\Omega}\ \underset{\sim}{\xi}_1^*\ \underset{\sim}{\kappa}\cdot\underset{\approx}{D}_h\cdot\underset{\sim}{\xi}_1\right] \quad (6.2.12)$$

From (6.2.11) we can prove that when dissipation is absent, the time derivative of the spatial integral of $\underset{\sim}{G}$ is the force exerted on the plasma by the external current density $\underset{\sim}{j}_e$ and charge density ρ_e. Computing the derivative in Eq. (6.2.12) and requiring the perturbation to be a normal mode, i.e. $\underset{\approx}{D}_h\cdot\underset{\sim}{\xi}_1=0$, we obtain the standard expression,[2]

$$\underset{\sim}{G}(x) = \frac{\underset{\sim}{\kappa}}{4\pi}\ \underset{\sim}{\xi}_1^*(x)\cdot\frac{\partial}{\partial\Omega}\underset{\approx}{D}_h(\kappa)\cdot\underset{\sim}{\xi}_1(x). \quad (6.2.13)$$

Now we specialize to the case of a collisionless multispecies plasma. The average power density put into species s is $P^s = \overline{\underset{\sim}{E}\cdot\underset{\sim}{j}^s}$. To second order we have

$$P^s = \overline{\underset{\sim}{E}_0\cdot\underset{\sim}{j}_2^s} + \overline{\underset{\sim}{E}_1\cdot\underset{\sim}{j}_1^s} + \overline{\underset{\sim}{E}_2\cdot\underset{\sim}{j}_0^s} \quad (6.2.14)$$

In general species s may have nonzero $\underset{\sim}{j}_0^s$ and $\underset{\sim}{j}_2^s$, and all three terms in the above expression contribute to the power density. However, we may define the "wave energy density" W^s of species s to be that

energy density which comes from the term $\overline{\underset{\sim}{E}_1 \cdot \underset{\sim}{j}_1^s}$, which is bilinear in the wave amplitude $[\underset{\sim}{j}_1^s(k) = \underset{\approx}{\sigma}^s(k) \cdot \underset{\sim}{E}_1(k)]$. Using the same method of analysis as before (except for $i\omega D/4\pi \to \underset{\approx}{\sigma}^s = -i\omega \underset{\approx}{\chi}^s/4\pi$) we find

$$\overline{\underset{\sim}{E}_1 \cdot \underset{\sim}{j}_1^s} = \frac{\Omega}{2\pi} \underset{\sim}{\xi}_1^*(x) \cdot \underset{\approx}{\chi}_a^s(\kappa) \cdot \underset{\sim}{\xi}_1(x) + \frac{i\Omega}{4\pi} \underset{\sim}{\xi}_1^*(x) \cdot \underset{\approx}{\chi}_h^s(k) \cdot \underset{\sim}{\xi}_1(x) \Big]$$

$$- \nabla \cdot \frac{\partial}{\partial \underset{\sim}{\kappa}} \left[\frac{\Omega}{4\pi} \underset{\sim}{\xi}_1^*(x) \cdot \underset{\approx}{\chi}_h^s(\kappa) \cdot \underset{\sim}{\xi}_1(x) \right]. \tag{6.2.15}$$

Thus the wave energy of species s is defined to be

$$W^s(k) = \frac{1}{4\pi} \frac{\partial}{\partial \Omega} \underset{\sim}{\xi}_1^*(x) \cdot \Omega \underset{\approx}{\chi}_h^s(\kappa) \cdot \underset{\sim}{\xi}_1(x)$$

$$= - \frac{\partial}{\partial \Omega} \underset{\sim}{\xi}_1^*(x) \cdot \underset{\approx}{\sigma}_a^s(\kappa) \cdot \underset{\sim}{\xi}_1(x). \tag{6.2.16}$$

Similarly the wave momentum density of species s is defined by

$$\underset{\sim}{\mathcal{G}}^s(x) = \frac{1}{4\pi} \left[\frac{\underset{\sim}{\kappa}}{\Omega^2} \frac{\partial}{\partial \Omega} \underset{\sim}{\xi}_1^* \cdot \Omega^2 \underset{\approx}{\chi}_h^s \cdot \underset{\sim}{\xi}_1 - \frac{1}{\Omega} \underset{\sim}{\xi}_1^* \cdot \underset{\approx}{\chi}_h^s \cdot \underset{\sim}{\kappa} - \frac{1}{\Omega} \underset{\sim}{\xi}_1^* \cdot \underset{\approx}{\chi}_h^s \cdot \underset{\sim}{\kappa} \right] \tag{6.2.17}$$

It is easy to verify that summing the wave energy (momentum) density of all the species as given by Eq. (6.2.16) [Eq.(6.2.17)] and the electromagnetic energy (momentum) density $\overline{\underset{\sim}{E}_1 \cdot \underset{\sim}{E}_1 + \underset{\sim}{B}_1 \cdot \underset{\sim}{B}_1}/8\pi$ $(\underset{\sim}{E}_1 \times \underset{\sim}{B}_1/4\pi c)$ gives the total wave energy (momentum) of the system as given by Eq. (6.2.8) [Eq. (6.2.12)]. Thus we have achieved a natural division of the total wave energy density and wave momentum density into contributions from the electromagnetic field and the separate species of a collisionless plasma.

In conclusion we stress the limitations of our results. We have shown that the spatial integral of W is the total energy supplied to the dielectric by the external current, but we have not shown that W gives the true local energy density. This must be decided by a nonlinear calculation. We shall see in chapter 8 that in general W is not the true local energy density. Similar statements hold for $\underset{\sim}{G}$.

6.3 Derivation of Nonlinear Wave Equations Using the Method of Harmonic Balance

The study of waves in a dielectric divides naturally into two parts: (1) computing the charge density response $\delta\rho(\underset{\sim}{x},t)$ and the current density response $\delta\underset{\sim}{j}(\underset{\sim}{x},t)$ to an electromagnetic field perturbation $\delta\underset{\sim}{E}(\underset{\sim}{x},t)$ and $\delta\underset{\sim}{B}(\underset{\sim}{x},t)$, and (2) solving for perturbations $\delta\underset{\sim}{E}$ and $\delta\underset{\sim}{B}$ which, together with the responses $\delta\rho$ and $\delta\underset{\sim}{j}$, satisfy Maxwell's equations,

$$\nabla\cdot\delta\underset{\sim}{E} = 4\pi\delta\rho \tag{6.3.1}$$

$$\nabla\cdot\delta\underset{\sim}{B} = 0 \tag{6.3.2}$$

$$\nabla\times\delta\underset{\sim}{E} = -c^{-1}\partial(\delta\underset{\sim}{B})/\partial t \tag{6.3.3}$$

$$\nabla\times\delta\underset{\sim}{B} = c^{-1}[4\pi\delta\underset{\sim}{j} + \partial(\delta\underset{\sim}{E})/\partial t] \ . \tag{6.3.4}$$

(Of course, the unperturbed system must also satisfy Maxwell's equations.) In this section we discuss the second part of the problem. We assume the response to be known, and we show how to find wave-packet perturbations that are consistent with Maxwell's equations.

To begin, we discuss the concept of linear response. In general, the fully nonlinear current response of a spatially homogeneous time independent medium to the perturbing field can be written in the form

$$\delta \underset{\sim}{j}(\underset{\sim}{x},t) = \int d^3x' \int_{-\infty}^{t} dt' \; \underset{\approx}{\sigma}(\underset{\sim}{x}-\underset{\sim}{x}',t-t')\cdot\delta\underset{\sim}{E}(\underset{\sim}{x}',t')$$

$$+ \int d^3x' \int_{-\infty}^{t} dt' \int d^3x'' \int_{-\infty}^{t} dt'' \; \underset{\approx}{\sigma}(\underset{\sim}{x}-\underset{\sim}{x}',t-t',\underset{\sim}{x}-\underset{\sim}{x}'',t-t''):\delta\underset{\sim}{E}(\underset{\sim}{x}',t')\delta\underset{\sim}{E}(\underset{\sim}{x}'',t'')$$

$$+ \int d^3x' \int_{-\infty}^{t} dt' \int d^3x'' \int_{-\infty}^{t} dt'' \int d^3x''' \int_{-\infty}^{t} dt''' \; \underset{\approx}{\sigma}(\underset{\sim}{x}-\underset{\sim}{x}',t-t',\underset{\sim}{x}-\underset{\sim}{x}'',t-t'',$$

$$\underset{\sim}{x}-\underset{\sim}{x}''',t-t''')\vdots\; \delta\underset{\sim}{E}(\underset{\sim}{x}',t')\delta\underset{\sim}{E}(\underset{\sim}{x}'',t'')\delta\underset{\sim}{E}(\underset{\sim}{x}''',t''') + \ldots \; . \tag{6.3.5}$$

We see that the current can be separated into linear (λ) current,

$$\delta \underset{\sim}{j}_{\lambda}(\underset{\sim}{x},t) \equiv \int d^3x' \int_{\infty}^{t} dt' \; \underset{\approx}{\sigma}(\underset{\sim}{x}-\underset{\sim}{x}',t-t')\cdot\delta\underset{\sim}{E}(\underset{\sim}{x}',t') \tag{6.3.6}$$

and nonlinear (ν) current,

$$\delta \underset{\sim}{j}_{\nu}(\underset{\sim}{x},t) \equiv \delta \underset{\sim}{j}(\underset{\sim}{x},t) - \delta \underset{\sim}{j}_{\lambda}(\underset{\sim}{x},t) \; . \tag{6.3.7}$$

Since Eq. (6.3.6) has the convolution form, we may Fourier transform to obtain

$$\delta \underset{\sim}{j}_{\lambda}(\underset{\sim}{k},\omega) = \underset{\approx}{\sigma}(\underset{\sim}{k},\omega)\cdot\delta\underset{\sim}{E}(\underset{\sim}{k},\omega) \quad , \tag{6.3.8}$$

where

$$\underset{\approx}{\sigma}(\underset{\sim}{k},\omega) \equiv \int d^3s \int_0^{\infty} d\tau \; e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{s}-\omega t)} \; \underset{\approx}{\sigma}(\underset{\sim}{s},\tau) \quad . \tag{6.3.9}$$

This concept of linear versus nonlinear response is not to be confused with ordering. By ordering we mean that the perturbations can be written as series,

$$\delta \underset{\sim}{E}(\underset{\sim}{x},t) = \underset{\sim}{E}_1(\underset{\sim}{x},t) + \underset{\sim}{E}_2(\underset{\sim}{x},t) + \underset{\sim}{E}_3(\underset{\sim}{x},t) + \ldots \tag{6.3.10}$$

$$\delta \underset{\sim}{B}(\underset{\sim}{x},t) = \underset{\sim}{B}_1(\underset{\sim}{x},t) + \underset{\sim}{B}_2(\underset{\sim}{x},t) + \ldots \tag{6.3.11}$$

$$\delta \underset{\sim}{j}(\underset{\sim}{x},t) = \underset{\sim}{j}_1(\underset{\sim}{x},t) + \underset{\sim}{j}_2(\underset{\sim}{x},t) + \ldots \tag{6.3.12}$$

$$\delta \rho(\underset{\sim}{x},t) = \rho_1(\underset{\sim}{x},t) + \rho_2(\underset{\sim}{x},t) + \ldots , \tag{6.3.13}$$

in some small parameter (to be determined in each specific system). If we insert Eqs. (6.3.10) and (6.3.12) into Eq. (6.3.5), we note, for example, that we have a <u>linear</u> second-order current,

$$\underset{\sim}{j}_{2\lambda}(\underset{\sim}{x},t) = \int d^3x' \int_{-\infty}^{t} dt' \; \underset{\approx}{\sigma}(\underset{\sim}{x}-\underset{\sim}{x}',\; t-t') \cdot \underset{\sim}{E}_2(\underset{\sim}{x}',t') \quad , \tag{6.3.14}$$

and a <u>nonlinear</u> second-order current,

$$\underset{\sim}{j}_{2\nu}(\underset{\sim}{x},t) = \int d^3x' \int_{-\infty}^{t} dt' \int d^3x'' \int_{-\infty}^{t} d^3t'' \; \underset{\approx}{\sigma}(\underset{\sim}{x}-\underset{\sim}{x}',t-t',\underset{\sim}{x}-\underset{\sim}{x}'',t-t'') :$$

$$\underset{\sim}{E}_1(\underset{\sim}{x}',t') \; \underset{\sim}{E}_1(\underset{\sim}{x}'',t'') \quad . \tag{6.3.15}$$

The concept of linear current can be used to rewrite Maxwell's equations. Using Eq. (6.3.10) we write Eq. (6.3.4) in the form,

$$\nabla \times \delta \underset{\sim}{B}(\underset{\sim}{x},t) - \frac{1}{c}[4\pi\delta \underset{\sim}{j}_\lambda(\underset{\sim}{x},t) + \frac{\partial}{\partial t}\, \delta \underset{\sim}{E}(\underset{\sim}{x},t)] = \frac{4\pi}{c}\, \delta \underset{\sim}{j}_\nu(\underset{\sim}{x},t) \quad , \tag{6.3.16}$$

which we Fourier transform to obtain

$$i\underset{\sim}{k} \times \delta \underset{\sim}{B}(\underset{\sim}{k},\omega) = [4\pi\delta \underset{\sim}{j}_\lambda(\underset{\sim}{k},\omega) - i\omega\delta \underset{\sim}{E}(\underset{\sim}{k},\omega)]/c = 4\pi\delta \underset{\sim}{j}_\nu(\underset{\sim}{k},\omega)/c \quad .$$

The left-hand side of this equation can be written in terms of $\delta \underset{\sim}{E}$ by using (6.3.8) and Faraday's law (6.3.3) in Fourier space: $\delta \underset{\sim}{B}(\underset{\sim}{k},\omega) = ck \times \delta \underset{\sim}{E}(\underset{\sim}{k},\omega)/\omega$. We find

$$\omega \underset{\approx}{D}(\underset{\sim}{k},\omega) \cdot \delta \underset{\sim}{E}(\underset{\sim}{k},\omega) = -4\pi i \delta \underset{\sim}{j}_\nu(\underset{\sim}{k},\omega) \quad , \tag{6.3.17}$$

where

$$\underset{\approx}{D}(\underset{\sim}{k},\omega) \equiv \underset{\approx}{I}(1-k^2c^2/\omega^2) + \underset{\sim}{k}\underset{\sim}{k}c^2/\omega^2 + 4\pi i\underset{\approx}{\sigma}(\underset{\sim}{k},\omega)/\omega \quad . \tag{6.3.18}$$

We first analyze Eq. (6.3.17) for <u>linear</u> wave-packet perturbations. We take $\underset{\sim}{E}_1$ to be of wave-packet form,

$$\underset{\sim}{E}_1(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)e^{i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \text{c.c.} \quad , \tag{6.3.19}$$

where $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)$ is slowly varying, we neglect the nonlinear current in Eq. (6.3.17), and so we obtain

$$\omega\underset{\approx}{D}(\underset{\sim}{k},\omega)\cdot\left[\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) + \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{k}+\underset{\sim}{k}_0,\omega+\omega_0)\right] = 0 \quad . \tag{6.3.20}$$

(We now use $\underset{\sim}{k}_0,\omega_0$ to denote the central wave vector and frequency of the perturbation, and we no longer use relativistic notation.) Since $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)$ is slowly varying, $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k},\omega)$ is highly peaked at the origin of $(\underset{\sim}{k},\omega)$ space. If we consider the region of Fourier space where $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)$ is significant, i.e. $\underset{\sim}{k},\omega \approx \underset{\sim}{k}_0,\omega_0$, we find that $\underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{k}+\underset{\sim}{k}_0,\omega+\omega_0) \to \underset{\sim}{\mathcal{E}}_1^*(2\underset{\sim}{k}_0,2\omega_0)$ is insignificant there, and Eq. (6.3.20) reduces to

$$\omega\underset{\approx}{D}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) = 0 \quad . \tag{6.3.21}$$

Dividing the tensor $\underset{\approx}{D}$ into hermitian and antihermitian parts $\underset{\approx}{D} = \underset{\approx}{D}_h + i\underset{\approx}{D}_a$, we rewrite the above equation:

$$\omega\underset{\approx}{D}_h(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) = 4\pi i\underset{\approx}{\sigma}_h(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) \quad . \tag{6.3.22}$$

In deriving this equation we used $\omega\underset{\approx}{D}_a = 4\pi\underset{\approx}{\sigma}_h$. The right side of this equation represents the linear dissipation of the wave. Here we assume dissipation to be small so that it can be treated in the usual way (see ref. 6 for example). Since we also assume small nonlinearity, the two

effects can be combined additively. However, here we neglect dissipation entirely, since it has been thoroughly discussed in the literature. Thus we now consider

$$\omega \underset{\approx}{D}_h(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) = 0 \quad . \tag{6.3.23}$$

To analyze this last equation, we introduce the diagonal representation of the hermitian matrix $\underset{\approx}{D}_h(\underset{\sim}{k},\omega)$:

$$\underset{\approx}{D}_h(\underset{\sim}{k},\omega) = \sum_{\alpha=1}^{3} D_\alpha(\underset{\sim}{k},\omega)\ \hat{u}_\alpha(\underset{\sim}{k},\omega)\ \hat{u}^*_\alpha(\underset{\sim}{k},\omega) \quad . \tag{6.3.24}$$

The eigenvalues D_α in this equation are real. The complex eigenvectors $\hat{u}_\alpha$ form an orthonormal basis:

$$\hat{u}^*_\alpha(\underset{\sim}{k},\omega)\cdot\hat{u}_\beta(\underset{\sim}{k},\omega) = \delta_{\alpha\beta} \quad .$$

If we expand $\underset{\sim}{\mathcal{E}}_1$ is this orthonormal basis,

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) \equiv \sum_{\alpha=1}^{3} \mathcal{E}_1^\alpha(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)\ \hat{u}_\alpha(\underset{\sim}{k},\omega) \ , \tag{6.3.25}$$

then Eq. (6.3.23) reduces to

$$\omega D_\alpha(\underset{\sim}{k},\omega)\ \mathcal{E}_1^\alpha(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) = 0 \qquad \text{for } \alpha = 1,2,3 \quad . \tag{6.3.26}$$

We are looking for wave packet solutions of Eq. (6.3.26). Mathematically this means that at least one of the D_α's has a root $\omega_\alpha(\underset{\sim}{k})$, which we take to be real, in the neighborhood of $\underset{\sim}{k}_0,\omega_0$. (In a degenerate case, such as transverse waves in unmagnetized plasma, two or more D_α's have roots $\omega_\alpha(\underset{\sim}{k})$ in the neighborhood of $\underset{\sim}{k}_0,\omega_0$.) We use S to denote the set of indices α for which $D_\alpha(\underset{\sim}{k},\omega)$ has a root near $\underset{\sim}{k}_0,\omega_0$, and we use $\bar{S}$ to denote the complement of S. For α an element of $\bar{S}(\alpha\in\bar{S})$, the only way to satisfy Eq. (6.3.25) is by requiring

$$\mathcal{E}_1^{\alpha}(\underset{\sim}{k}-\underset{\sim}{k}_0 \quad \omega_0) = 0 \qquad \text{for } \alpha\in\bar{S} \quad . \tag{6.3.27}$$

That is, that part of the linear electric field with polarization $\hat{u}_\alpha$ must vanish if $\alpha\in\bar{S}$. On the other hand, let us examine the case $\alpha\in S$. Since now D_α has a root $\omega_\alpha(\underset{\sim}{k})$,

$$D_\alpha[\underset{\sim}{k},\omega_\alpha(\underset{\sim}{k})] = 0 \qquad \text{for } \alpha\in S \quad , \tag{6.3.28}$$

in the neighborhood of $\underset{\sim}{k}_0,\omega_0$, we expand ωD_α in $[\omega-\omega_\alpha(\underset{\sim}{k})]$:

$$\omega D_\alpha(\underset{\sim}{k},\omega) \approx [\omega-\omega_\alpha(\underset{\sim}{k})]\,\frac{\partial}{\partial\omega}\,[\omega D_\alpha(\underset{\sim}{k},\omega)]\Big|_{\omega=\omega_\alpha(\underset{\sim}{k})} \qquad \text{for } \alpha\in S \tag{6.3.29}$$

in Eq. (6.3.26), and we obtain

$$(\omega-\omega_\alpha)\overline{\omega D_\alpha}(k)\,\mathcal{E}_1^{\alpha}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)=0 \qquad \text{for } \alpha\in S \tag{6.3.30}$$

where

$$\overline{\omega D_\alpha}(\underset{\sim}{k}) \equiv \frac{\partial}{\partial\omega}\,[\omega D_\alpha(\underset{\sim}{k},\omega)]\Big|_{\omega=\omega_\alpha(\underset{\sim}{k})} \qquad \text{for } \alpha\in S \quad . \tag{6.3.31}$$

As shorthand, we use β to denote indices in the set S ($\beta\in S$) and μ to denote indices in the set $\bar{S}$ ($\mu\in\bar{S}$).

We assume $\overline{\omega D_\beta}(\underset{\sim}{k})$ to be nonzero in the neighborhood of $\underset{\sim}{k}_0$ and divide Eq. (6.3.30) through by $\overline{\omega D_\beta}(\underset{\sim}{k})$ (if $\overline{\omega D_\beta}(\underset{\sim}{k})$ did vanish, we would go to the next derivative):

$$[\omega-\omega_\beta(\underset{\sim}{k})]\mathcal{E}_1^{\beta}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) = 0 \quad . \tag{6.3.32}$$

Then we Fourier transform over ω to obtain

$$\left[i\,\frac{\partial}{\partial t} + \omega_0 - \omega_\beta(\underset{\sim}{k})\right]\mathcal{E}_1^{\beta}(\underset{\sim}{k}-\underset{\sim}{k}_0,t) = 0 \quad . \tag{6.3.33}$$

To return to the variable $\underset{\sim}{x}$, we first expand $\omega_\beta(\underset{\sim}{k})$ about $\underset{\sim}{k}_0$:

$$\omega_\beta(\underset{\sim}{k}) = \omega_\beta(\underset{\sim}{k}_0) + (\underset{\sim}{k}-\underset{\sim}{k}_0)\cdot\underset{\sim}{v}_\beta + \tfrac{1}{2}(\underset{\sim}{k}-\underset{\sim}{k}_0)(\underset{\sim}{k}-\underset{\sim}{k}_0):\underset{\approx}{d}_\beta \quad , \tag{6.3.34}$$

where

$$\underset{\sim}{v}_\beta \equiv \left.\frac{\partial\omega_\beta}{\partial\underset{\sim}{k}}\right|_{\underset{\sim}{k}_0} \tag{6.3.35}$$

and

$$\underset{\approx}{d}_\beta \equiv \left.\frac{\partial^2\omega_\beta}{\partial\underset{\sim}{k}\partial\underset{\sim}{k}}\right|_{\underset{\sim}{k}_0} , \tag{6.3.36}$$

and then Fourier transform over $\underset{\sim}{k}$, thereby obtaining

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \underset{\approx}{d}_\beta:\nabla\nabla + \omega_0-\omega_\beta(\underset{\sim}{k}_0)\right]\mathcal{E}_1^\beta(\underset{\sim}{x},t) = 0 \quad . \tag{6.3.37}$$

Usually we choose $\omega_0 = \omega_\beta(\underset{\sim}{k}_0)$ in this equation, so that it reduces to

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \underset{\approx}{d}_\beta:\nabla\nabla\right]\mathcal{E}_1^\beta(\underset{\sim}{x},t) = 0 \quad . \tag{6.3.38}$$

We have derived equations for the evolution of the set $\{\mathcal{E}_1^\alpha(\underset{\sim}{x},t)$. If $D^\alpha(\underset{\sim}{k},\omega)$ has no root near $(\underset{\sim}{k}_0,\omega_0)$, then $\mathcal{E}_1^\alpha = 0$, but if $D^\alpha(\underset{\sim}{k},\omega)$ has a root near $(\underset{\sim}{k}_0,\omega_0)$, then $\mathcal{E}_1^\alpha$ satisfies (6.3.38). The next step is to determine how $\{\mathcal{E}_1^\alpha(\underset{\sim}{x},t)\}$ is related to the electric field amplitude $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)$.

To find this relation we inverse Fourier transform,

$$\underset{\sim}{E}_1(\underset{\sim}{x},t) = \int\frac{d^3kd}{(2\pi)^4}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}\, \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) + \text{c.c.}$$

$$= \int\frac{d^3kd\omega}{(2\pi)^4}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)} \sum_{\alpha=1}^{3} \hat{u}_\alpha(\underset{\sim}{k},\omega)\mathcal{E}_1^\alpha(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) + \text{c.c.} \tag{6.3.39}$$

using Eq. (6.3.25). Pulling out the rapidly varying phase,

$$\underset{\sim}{E}_1(\underset{\sim}{x},t) = e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega_0 t)} \int \frac{d^3k d\omega}{(2\pi)^4} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)} \sum_{\alpha=1}^{3} \hat{u}_\alpha(\underset{\sim}{k}+\underset{\sim}{k}_0,\omega+\omega_0)\mathcal{E}_1^\alpha(\underset{\sim}{k},\omega)$$

$$+ \text{c.c.} \quad , \tag{6.3.40}$$

by making the change $\underset{\sim}{k} \to \underset{\sim}{k} + \underset{\sim}{k}_0$ in the integration variable, we find the first-order amplitude to be given by

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) = \int \frac{d^3k d\omega}{(2\pi)^4} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)} \sum_{\alpha=1}^{3} \hat{u}_\alpha(\underset{\sim}{k}+\underset{\sim}{k}_0,\omega+\omega_0)\mathcal{E}_1^\alpha(\underset{\sim}{k},\omega) \ . \tag{6.3.41}$$

Since $\mathcal{E}_1^\alpha$ is highly peaked near the origin of Fourier space, we can calculate this integral by expansion in Fourier space, c.f. Sec. 6.1:

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \frac{\partial}{\partial\omega_0}\frac{\partial}{\partial t} - i\nabla\cdot\frac{\partial}{\partial \underset{\sim}{k}_0} \right)^n \sum_{\alpha=1}^{3} \hat{u}_\alpha(\underset{\sim}{k}_0,\omega_0)\mathcal{E}_1^\alpha(\underset{\sim}{x},t) \ . \tag{6.3.42}$$

Through first order in $(\partial/\partial t,\nabla)$, this equation gives

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) = \sum_{\alpha=1}^{3} \left[\mathcal{E}_1^\alpha(\underset{\sim}{x},t)\hat{u}_\alpha(\underset{\sim}{k}_0,\omega_0) + i \frac{\partial \mathcal{E}_1^\alpha}{\partial t}(\underset{\sim}{x},t) \frac{\partial}{\partial\omega_0} \hat{u}_\alpha(\underset{\sim}{k}_0,\omega_0) \right.$$

$$\left. - i\nabla\mathcal{E}_1^\alpha(\underset{\sim}{x},t)\cdot \frac{\partial}{\partial \underset{\sim}{k}} \hat{u}_\alpha(\underset{\sim}{k}_0,\omega_0) \right] \ . \tag{6.3.43}$$

Thus we see that there are two steps in the linear analysis. The first is finding the evolution of $\mathcal{E}_1^\alpha$ as given by Eqs. (6.3.27) and (6.3.88). The second step is using Eq. (6.3.42) to determine $\underset{\sim}{\mathcal{E}}_1$.

This analysis requires two assumptions. The first is that $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)$ is highly peaked in Fourier space in the neighborhood of $\underset{\sim}{k}_0,\omega_0$. This assumption is needed to deduce Eq. (6.3.4) from Eq. (6.3.3), to use the expansion (6.3.18) for $\omega_\beta(\underset{\sim}{k})$, and to find the amplitude

$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)$ from $\mathcal{E}_1^{\alpha}(\underset{\sim}{x},t)$ in Eqs. (6.3.25) and (6.3.26). The meaning of this assumption is that the amplitude $\underset{\sim}{\mathcal{E}}(\underset{\sim}{x},t)$ must vary slowly in space and time (how slowly depends on the wave in question). The second assumption is that $\underset{\sim}{k}_0,\omega_0$ is near a root of ωD_β, so that we can make the approximation (6.3.29). This assumption means that we are considering normal mode oscillations of a plasma.

Let us consider the particular case of transverse waves in cold unmagnetized plasma, for which $\omega D_\beta = \omega - (k^2c^2 + \omega_p^2)/\omega$ and $\omega_\beta(\underset{\sim}{k}) = (\omega_p^2 + k^2c^2)^{\frac{1}{2}}$. It is easy to show that to use the expression (6.3.29), we must have $[\omega-\omega_\beta(\underset{\sim}{k})] \ll \omega_\beta(\underset{\sim}{k})$, i.e. $\partial\underset{\sim}{\mathcal{E}}/\partial t \ll \omega_0\underset{\sim}{\mathcal{E}}$. One can also show that to neglect the third derivative in Eq. (6.3.34) and to keep the second derivative, we must have $c^2(\underset{\sim}{k}-\underset{\sim}{k}_0)\cdot\underset{\sim}{k}_0 \ll \omega_0^2$. This condition amounts to $|\nabla\underset{\sim}{\mathcal{E}}| \ll (\omega_0^2/k_0^2c^2)|\underset{\sim}{k}_0\underset{\sim}{\mathcal{E}}| = (1 + \omega_p^2/k_0^2c^2)|\underset{\sim}{k}_0\underset{\sim}{\mathcal{E}}|$.

Now we turn to second-order theory. Since the first order field has the wave packet form (6.3.19), we expect the second-order nonlinear current to have the form:

$$\underset{\sim}{j}_{2\nu}(\underset{\sim}{x},t) = \underset{\sim}{j}_{2\nu 0}(\underset{\sim}{x},t) + \left[\underset{\sim}{\mathcal{J}}_{2\nu 2}(\underset{\sim}{x},t)\, e^{2i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \text{c.c.}\right] . \qquad (6.3.44)$$

That is, the electric field in Eq. (6.3.15) beats to produce a second-order zeroth-harmonic nonlinear current $j_{2\nu 0}$, and a second-order second-harmonic nonlinear current,

$$\underset{\sim}{j}_{2\nu 2}(\underset{\sim}{x},t) \equiv \underset{\sim}{\mathcal{J}}_{2\nu 2}(\underset{\sim}{x},t)\, e^{2i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \text{c.c.} \qquad (6.3.45)$$

We expect the nonlinear current (6.3.44) to drive electric fields at the same harmonics:

$$\underset{\sim}{E}_2(\underset{\sim}{x},t) = \underset{\sim}{E}_{20}(\underset{\sim}{x},t) + \underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{x},t)\, e^{i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + c.c \quad . \tag{6.3.46}$$

We Fourier transform (6.3.44) and (6.3.46) and insert the results into the second-order part of Eq. (6.3.17) to find

$$\omega\underset{\approx}{D}(\underset{\sim}{k},\omega)\cdot\left[\underset{\sim}{E}_{20}(\underset{\sim}{k},\omega) + \underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{k}-2\underset{\sim}{k}_0,\omega-2\omega_0) + \underset{\sim}{\mathcal{E}}^*_{22}(\underset{\sim}{k}+2\underset{\sim}{k}_0,\omega+2\omega_0)\right]$$

$$= -4\pi i\left[\underset{\sim}{j}_{2\nu 0}(\underset{\sim}{k},\omega) + \underset{\sim}{\mathcal{J}}_{2\nu 2}(\underset{\sim}{k}-2\underset{\sim}{k}_0,\omega-2\omega_0) + \underset{\sim}{\mathcal{J}}^*_{2\nu 2}(\underset{\sim}{k}+2\underset{\sim}{k}_0,\omega+2\omega_0)\right] . \tag{6.3.47}$$

We split this equation into three equations by applying the same reasoning that was used in going from Eq. (6.3.20) to Eq. (6.3.21). For example, for $\underset{\sim}{k},\omega$ near the origin, only the first term on each side of Eq. (6.3.47) is significant. The other terms vanish, since they are highly peaked near either $(2\underset{\sim}{k}_0,2\omega_0)$ or $(-2\underset{\sim}{k}_0,-2\omega_0)$. Thus we find

$$\omega\underset{\approx}{D}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{E}_{20}(\underset{\sim}{k},\omega) = -4\pi i\, \underset{\sim}{j}_{2\nu 0}(\underset{\sim}{k},\omega) \quad . \tag{6.3.48}$$

On the other hand, for $\underset{\sim}{k},\omega$ near $2\underset{\sim}{k}_0,2\omega_0$, only the middle term on each side is important, and so we find

$$\omega\underset{\approx}{D}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{k}-2\underset{\sim}{k}_0,\omega-2\omega_0) = -4\pi i\, \underset{\sim}{\mathcal{J}}_{2\nu 2}(\underset{\sim}{k}-2\underset{\sim}{k}_0,\omega-2\omega_0) \quad . \tag{6.3.49}$$

This last expression may be solved for $\underset{\sim}{\mathcal{E}}_{22}$,

$$\underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{k}-2\underset{\sim}{k}_0,\omega-2\omega_0) = -\frac{4\pi i}{\omega}\,\underset{\approx}{D}^{-1}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{J}}_{2\nu 2}(\underset{\sim}{k}-2\underset{\sim}{k}_0,\omega-2\omega_0), \tag{6.3.50}$$

and then expanded in Fourier space to obtain a local expression for $\underset{\sim}{\mathcal{E}}_{22}$:

$$\underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{x},t) = -4\pi i\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\,\frac{\partial}{\partial t}\,\frac{\partial}{\partial 2\omega_0} - i\nabla\cdot\frac{\partial}{\partial 2\underset{\sim}{k}_0}\right)^n \underset{\approx}{D}^{-1}(2\underset{\sim}{k}_0,2\omega_0)\cdot\underset{\sim}{\mathcal{J}}_{2\nu 2}(\underset{\sim}{x},t)/(2\omega_0). \tag{6.3.51}$$

To lowest order this expression reduces to

$$\underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{x},t) = \frac{-4\pi i}{2\omega_0}\,\underset{\approx}{D}^{-1}(2\underset{\sim}{k}_0,2\omega_0)\cdot\underset{\sim}{\mathcal{J}}_{2\nu 2}(\underset{\sim}{x},t) \quad . \tag{6.3.52}$$

The other expression (6.3.37) is not so easily converted to $\underset{\sim}{x}$-t space by expanding in Fourier space, since $\underset{\approx}{D}\,(\underset{\sim}{k},\omega)$ may have singularities for $\underset{\sim}{k},\omega \to 0$. For example, one term in $\underset{\approx}{D}(\underset{\sim}{k},\omega)$ is $(\underset{\approx}{I}-\hat{\underset{\sim}{k}}\hat{\underset{\sim}{k}})k^2c^2/\omega^2$. Instead it is simpler to return to Maxwell's equations,

$$\begin{aligned}
\nabla\cdot\underset{\sim}{E}_{20} &= 4\pi(\rho_{2\lambda 0} + \rho_{2\nu 0}) \\
\nabla\cdot\underset{\sim}{B}_{20} &= 0 \\
\nabla\times\underset{\sim}{E}_{20} &= -\frac{1}{c}\frac{\partial \underset{\sim}{B}_{20}}{\partial t} \\
\nabla\times\underset{\sim}{B}_{20} &= \frac{4\pi}{c}\,(\underset{\sim}{j}_{2\lambda 0} + \underset{\sim}{j}_{2\nu 0}) + \frac{1}{c}\frac{\partial \underset{\sim}{E}_{20}}{\partial t} \quad ,
\end{aligned} \tag{6.3.53}$$

in the problems considered in the later chapters of this thesis.

We proceed to third order. We use the harmonic structure of $\underset{\sim}{E}_1$ and $\underset{\sim}{E}_2$ to infer that $\underset{\sim}{j}_{3\nu}$, which is bilinear in $\underset{\sim}{E}_1$ and $\underset{\sim}{E}_2$ or trilinear in $\underset{\sim}{E}_1$, has the form :

$$\underset{\sim}{j}_{3\nu}(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)\, e^{i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \underset{\sim}{\mathcal{J}}_{3\nu 3}(\underset{\sim}{x},t)\, e^{3i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \text{c.c.}, \tag{6.3.54}$$

and similarly for $\underset{\sim}{E}_3$:

$$\underset{\sim}{E}_3(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_{31}(\underset{\sim}{x},t)\, e^{i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \underset{\sim}{\mathcal{E}}_{33}(\underset{\sim}{x},t)\, e^{3i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \text{c.c.}, \tag{6.3.55}$$

We insert $\underset{\sim}{E}$ up to third order and $\underset{\sim}{j}$ up to third order into Eq. (6.3.17), and we separate the first-harmonic terms to obtain

$$\omega \underset{\sim}{D}(\underset{\sim}{k},\omega)\cdot[\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) + \underset{\sim}{\mathcal{E}}_{31}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)]$$

$$= 4\pi i \underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0). \tag{6.3.56}$$

(The third-harmonic terms can of course by handled in the same fashion as were the second-order second-harmonic terms.) To handle the first harmonic terms we must be careful since (as has been assumed) one of the D_α's vanishes at or near $\underset{\sim}{k}_0,\omega_0$, and so $\underset{\sim}{D}(\underset{\sim}{k}_0,\omega_0)$ is not invertible.

We use the diagonalization (6.3.24) of $\underset{\sim}{D}$ to write (6.3.56) in the form

$$\omega D_\alpha(\underset{\sim}{k},\omega)\left[\mathcal{E}_1^\alpha(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) + \mathcal{E}_{31}^\alpha(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)\right]$$

$$= -4\pi i\ \hat{u}_\alpha^*(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) \quad . \tag{6.3.57}$$

We recall that $D_\alpha(\underset{\sim}{k},\omega)$ has a root near $\underset{\sim}{k}_0,\omega_0$ for $\alpha\in s$ (or $\alpha = \beta$), but $D_\alpha(\underset{\sim}{k},\omega)$ does not have a root near $\underset{\sim}{k}_0,\omega_0$ for $\alpha\in\bar{s}$ (or $\alpha = \mu$). In the latter case we can solve Eq. (6.3.57) by simple division:

$$\mathcal{E}_1^\mu(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) + \mathcal{E}_{31}^\mu(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) =$$

$$-\frac{4\pi i}{\omega D_\mu(\underset{\sim}{k},\omega)}\ \hat{u}_\mu^*(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) \quad . \tag{6.3.58}$$

We note now that we have an ambiguity. We have two variables $\mathcal{E}_1^\mu$ and $\mathcal{E}_{31}^\mu$ to describe the first-harmonic amplitude. In this case the right side of Eq. (6.3.58) is a third order quantity, so to preserve our ordering scheme we choose $\mathcal{E}_1^\mu = 0$ and

$$\mathcal{E}_{31}^\mu(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) = \frac{-4\pi i\ \hat{u}_\mu^*(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)}{\omega D_\mu(\underset{\sim}{k},\omega)} \quad . \tag{6.3.59}$$

Of course, we may expand in Fourier space and transform back to real

space to obtain

$$\mathcal{E}^{\mu}_{31}(\underset{\sim}{x},t) = 4\pi i\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\frac{\partial}{\partial t}\frac{\partial}{\partial\omega_0} - i\nabla\cdot\frac{\partial}{\partial \underset{\sim}{k}_0}\right)^n \frac{\hat{u}^*_{\mu}(\underset{\sim}{k}_0,\omega_0)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)}{\omega_0 D_{\mu}(\underset{\sim}{k}_0,\omega_0)} \quad . \tag{6.3.60}$$

which is

$$\mathcal{E}^{\mu}_{31}(\underset{\sim}{x},t) = -4\pi i\,\frac{\hat{u}^*_{\mu}(\underset{\sim}{k}_0,\omega_0)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)}{\omega_0 D_{\mu}(\underset{\sim}{k}_0,\omega_0)} \tag{6.3.61}$$

to lowest order.

We now consider the other case, $\alpha = \beta$, where ωD_{β} does have a root $\omega_{\beta}(\underset{\sim}{k})$ near $\omega_0, \underset{\sim}{k}_0$. Since ωD_{β} vanishes we cannot simply divide through by ωD_{β}. Instead, we use the expansion (6.3.29) in Eq. (6.3.56) to obtain

$$\left[\omega-\omega_{\beta}(\underset{\sim}{k})\right]\left[\mathcal{E}^{\beta}_{1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) + \mathcal{E}^{\beta}_{31}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)\right] =$$

$$- 4\pi i\,\frac{\hat{u}^*_{\beta}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)}{\overline{\omega D_{\beta}(\underset{\sim}{k})}} \tag{6.3.62}$$

Again the right side of this equation is a third-order quantity. However, on the left hand side we have the factor $\omega-\omega_{\beta}(\underset{\sim}{k})$ which represents the nonlinear frequency shift, a quantity which is of some order greater than zero in $\underset{\sim}{\mathcal{E}}_1$. This time we preserve our ordering scheme by choosing $\mathcal{E}^{\beta}_{31} = 0$ and

$$\left[\omega-\omega_{\beta}(\underset{\sim}{k})\right]\mathcal{E}^{\beta}_{1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0) = \frac{-4\pi i\;\hat{u}^*_{\beta}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{k}-\underset{\sim}{k}_0,\omega-\omega_0)}{\overline{\omega D_{\beta}(\underset{\sim}{k})}} \tag{6.3.63}$$

We use the expression (6.3.34), and we expand the above equation in Fourier space to obtain the real space equation:

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \tfrac{1}{2}\underset{\sim}{d} :\nabla\nabla\right]\mathcal{E}_1^\beta(\underset{\sim}{x},t) = -4\pi i\sum_{n=0}^{\infty}\frac{1}{n!}\, i\left(\frac{\partial}{\partial t}\frac{\partial}{\partial\omega_0} - i\nabla\cdot\frac{\partial}{\partial\underset{\sim}{k}_0}\right)\cdot$$

$$\hat{u}_\beta^*(\underset{\sim}{k}_0,\omega_0)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)/\,\overline{\omega D_\beta}(\underset{\sim}{k}_0)\ . \qquad (6.3.64)$$

The above equation is valid only with the right side calculated to lowest order on $(\nabla,\frac{\partial}{\partial t})$,

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \tfrac{1}{2}\underset{\sim}{d}_\beta :\nabla\nabla\right]\mathcal{E}_1^\beta(\underset{\sim}{x},t) = -4\pi i\ \hat{u}_\beta^*(\underset{\sim}{k}_0,\omega_0)\cdot\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)/\,\overline{\omega D_\beta}(\underset{\sim}{k}_0)\ , \qquad (6.3.65)$$

since we have computed the left side only to lowest order in $\omega-\omega_\beta(k)$.

Thus we have derived the nonlinear evolution of $\underset{\sim}{E}$ in terms of the nonlinear currents. We use Eqs. (6.3.65) and 6.3.61) to determine the first-harmonic quantities $\mathcal{E}_1^\alpha(\underset{\sim}{x},t)$ and $\mathcal{E}_{31}^\alpha(\underset{\sim}{x},t)$. Then we expand in Fourier space, as in Eq. (6.3.42), to determine the first-harmonic amplitude.

6.4 Derivation of Nonlinear Wave Evolution Using a Heuristic (But Sometimes Incomplete) Method

We are about to describe a method for obtaining nonlinear wave equations. This method has been used in one form or another to discuss the nonlinear evolution of transverse waves in unmagnetized[6,7] plasma, lower-hybrid waves in magnetized plasma,[8] and ion cyclotron waves in magnetized plasma.[9] This method has appeal because (1) it corresponds to a simple physical picture, and (2) it involves less work than would a systematic calculation. Unfortunately this method sometimes gives an incomplete answer; it may leave out effects of comparable size to the

effects it includes. This method seems to have no name in the literature, so we give it one: the "heuristic method for including nonlinearity," or in short: the "heuristic method."

The heuristic method begins with Eq. (6.3.37) which we rewrite here:

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \underset{\approx}{d}_\beta:\nabla\nabla\right]\underset{\sim}{\mathcal{E}}_1^\beta(\underset{\sim}{x},t) = [\omega_\beta(\underset{\sim}{k}_0)-\omega_0]\underset{\sim}{\mathcal{E}}_1^\beta(\underset{\sim}{x},t) \ . \tag{6.4.1}$$

This equation is strictly valid only for linear perturbations. However, we apply it to nonlinear perturbations in the following manner. In general, the dispersion relation $\omega_\beta(\underset{\sim}{k}_0)$ depends on a number of parameters $\underset{\sim}{\lambda}$. These parameters include the background density, magnetic field and so on. In the absence of the wave these background parameters have the value $\underset{\sim}{\lambda}_0$. However, to second order in the wave amplitude these quantities change by the amount $\underset{\sim}{\lambda}_2 \equiv \underset{\sim}{\lambda}-\underset{\sim}{\lambda}_0$.

To account for this nonlinear change we use $\omega_\beta(\underset{\sim}{k}_0,\underset{\sim}{\lambda} = \underset{\sim}{\lambda}_0+\underset{\sim}{\lambda}_2)$ on the right side of Eq. (6.4.1) rather than $\omega_\beta(\underset{\sim}{k}_0,\underset{\sim}{\lambda}_0)$. This gives

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \tfrac{1}{2}\underset{\approx}{d}_\beta:\nabla\nabla\right]\underset{\sim}{\mathcal{E}}_1^\beta(\underset{\sim}{x},t) = \left[\underset{\sim}{\lambda}_2\cdot\frac{\partial}{\partial\underset{\sim}{\lambda}_0}\,\omega_\beta(\underset{\sim}{k}_0,\underset{\sim}{\lambda}_0)\right]\underset{\sim}{\mathcal{E}}_1^\beta(\underset{\sim}{x},t) \tag{6.4.2}$$

upon expanding $\omega_\beta(\underset{\sim}{k}_0,\underset{\sim}{\lambda}_0+\underset{\sim}{\lambda}_2)$ and choosing $\omega_\beta(\underset{\sim}{k}_0,\underset{\sim}{\lambda}_0) = \omega_0$. This is <u>not</u> a derivation. In an ad hoc manner we have inserted a nonlinear effect, the frequency shift due to the change in the background state, into an equation which is only linearly valid.

As an aid to understanding the above equation, we present the following quasi-derivation. We again use the idea of the background state depending on a number of parameters $\underset{\sim}{\lambda}$, but now we apply this idea to the conductivity. That is, due to $\underset{\sim}{\lambda}_2$, the conductivity is modified to

$$\underset{\approx}{g}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0+\underset{\sim}{\lambda}_2) = \underset{\approx}{g}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0) + \underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0} \underset{\approx}{g}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0) \ . \tag{6.4.3}$$

By applying the second term in the conductivity to an electric field $\underset{\sim}{E}(\underset{\sim}{x},t) = \hat{u}_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\mathcal{E}_1^\beta(\underset{\sim}{x},t)\exp\left[i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)\right] + \text{c.c.}$, we find the nonlinear current

$$\underset{\sim}{j}_{3\nu 1} = \underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)\ e^{i(\underset{\sim}{k}_0\cdot\underset{\sim}{x}-\omega_0 t)} + \text{c.c.} \quad , \tag{6.4.4}$$

where

$$\underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t) = \left[\underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0} \underset{\approx}{g}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\right]\cdot\hat{u}_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\mathcal{E}_1^\beta(\underset{\sim}{x},t) \quad . \tag{6.4.5}$$

We insert this expression into Eq. (6.3.65) to obtain the nonlinear equation for $\mathcal{E}_1^\beta$:

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \tfrac{1}{2}\underset{\approx}{d}_\beta:\nabla\nabla\right]\mathcal{E}_1^\beta(\underset{\sim}{x},t) = -4\pi i\ \hat{u}_\beta^*(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\cdot$$

$$\left[\underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0} \underset{\approx}{g}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\right]\cdot\hat{u}_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\mathcal{E}_1^\beta(\underset{\sim}{x},t)/\ \overline{\omega D}_\beta(\underset{\sim}{k}_0,\lambda_0) \quad . \tag{6.4.6}$$

To put Eq. (6.4.6) in the form (6.4.2) we perform a series of manipulations. We first note that

$$\underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0} \underset{\approx}{g}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0) = \underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0} \frac{\omega_0}{4\pi i} \underset{\approx}{D}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0) \ , \tag{6.4.7}$$

since the electromagnetic part of $\underset{\approx}{D}$ does not depend on any changeable parameters. Secondly we note that

$$\hat{u}_\beta^*(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\cdot\left(\underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0}\right)\underset{\approx}{D}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\cdot\hat{u}_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)$$

$$= \left(\underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0}\right)\hat{u}_\beta^*(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\cdot\underset{\approx}{D}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\cdot\hat{u}_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)$$

$$= \underset{\sim}{\lambda}_2\cdot \frac{\partial}{\partial \underset{\sim}{\lambda}_0} D_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0) \quad , \tag{6.4.8}$$

since $\underset{\approx}{D}(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\cdot\hat{u}_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0) = 0$. By using these relations we reduce Eq. (6.4.6) to

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\beta\cdot\nabla\right) + \tfrac{1}{2}\underset{\approx}{\omega}_\beta : \nabla\nabla\right]\mathcal{E}_1^\beta(\underset{\sim}{x},t) = -\left[\underset{\sim}{\lambda}_2\cdot\frac{\partial}{\partial\underset{\sim}{\lambda}_0}\,\omega_0 D_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0)\right]$$

$$\mathcal{E}_1^\beta(\underset{\sim}{x},t)\,/\,\overline{\omega D_\beta}(\underset{\sim}{k}_0,\lambda_0). \tag{6.4.9}$$

Finally we expand the equation,

$$\omega_\beta(\underset{\sim}{k}_0,\lambda)D_\beta(\underset{\sim}{k}_0,\omega_\beta(\underset{\sim}{k}_0,\underset{\sim}{\lambda}),\underset{\sim}{\lambda}) = 0 \quad , \tag{6.4.10}$$

to prove

$$\overline{\omega D_\beta}(\underset{\sim}{k}_0,\underset{\sim}{\lambda}_0)\underset{\sim}{\lambda}_2\cdot\frac{\partial}{\partial\underset{\sim}{\lambda}_0}\omega_\beta(\underset{\sim}{k}_0,\underset{\sim}{\lambda}_0) = -\,\omega_0\underset{\sim}{\lambda}_2\cdot\frac{\partial}{\partial\underset{\sim}{\lambda}_0}D_\beta(\underset{\sim}{k}_0,\omega_0,\underset{\sim}{\lambda}_0) \quad . \tag{6.4.11}$$

From Eqs. (6.4.9) and (6.4.11) we deduce Eq. (6.4.2).

Thus we have a quasi-derivation of Eq. (6.4.2). This is <u>not</u> a valid derivation, since the initial hypothesis, that the nonlinear current $j_{3\nu 1}$ is given by Eqs. (6.4.4) and (6.4.5), can not be proven. In fact, as we show in section 8.4, the heuristic method leaves out terms which are comparable to the terms in includes. Nevertheless, Eq. (6.4.2) has great appeal because it has a simple interpretation and it is easy to use. The interpretation is that the important effect of nonlinearity is the change in the background state. Eq. (6.4.2) is easy to use, since one need only compute $\underset{\sim}{\lambda}_2$, a second-order quantity, rather than $j_{3\nu 1}$, the third-order quantity needed for the systematic formalism of the last section.

7. Application of Lie Transforms to Vlasov Systems: General Aspects

The development of the transformation theory of Chapters 2-5 relied on the Lie structure of the canonical transformation group. With minor modifications one ought to be able to develop an equivalent theory for other Hamiltonian systems such as classical Hamiltonian field theory, quantum mechanics, and quantum field theory. In this chapter we discuss the application of Lie transform theory to the Vlasov-Maxwell equations.

In the Vlasov-Maxwell system of equations the density $f_s(\underset{\sim}{x},\underset{\sim}{v},t)$ of particles of species s evolves nonrelativistically according to

$$\frac{\partial f_s}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f_s + \frac{e_s}{m_s}\left[\underset{\sim}{E}(\underset{\sim}{x},t)+(\underset{\sim}{v}/c)\times\underset{\sim}{B}(\underset{\sim}{x},t)\right]\cdot\frac{\partial f_s}{\partial v} = 0 \ , \tag{7.0.1}$$

where the charge and mass of these particles are e_s and m_s. From f_s we compute the charge and current density of species s via

$$\rho_s(\underset{\sim}{x},t) = e_s\int d^3v\ f_s(\underset{\sim}{x},\underset{\sim}{v},t) \tag{7.0.2}$$

and

$$\underset{\sim}{j}_s(\underset{\sim}{x},t) = e_s\int d^3v\ \underset{\sim}{v}\ f_s(\underset{\sim}{x},\underset{\sim}{v},t) \ . \tag{7.0.3}$$

The evolution of the self-consistent electric and magnetic fields $E(x,t)$ and $B(x,t)$ in (7.0.1) is given by Maxwell's equations:

$$\nabla\cdot\underset{\sim}{E} = 4\pi\sum_s \rho_s \tag{7.0.4}$$

$$\nabla\cdot\underset{\sim}{B} = 0 \tag{7.0.5}$$

$$\nabla\times\underset{\sim}{E} = -\frac{1}{c}\frac{\partial\underset{\sim}{B}}{\partial t} \tag{7.0.6}$$

$$\nabla \times B = \frac{4\pi}{c} \sum_s j_s + \frac{1}{c} \frac{\partial \underset{\sim}{E}}{\partial t} . \tag{7.0.7}$$

We apply Lie transforms to this set of equations following previous work using Hamiltonian perturbation theory[1-4] for discussing the Vlasov-Maxwell equations. We first write (7.0.1) in Hamiltonian form. We then use Lie transforms to analyze (7.0.1) for given electricmagnetic fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$. Finally, we require $\underset{\sim}{E}$ and $\underset{\sim}{B}$ to satisfy Maxwell's equations; this part of the calculation is done without Lie transforms.

Obviously, simplification would result if we could apply Lie transform techniques to the entire syatem (7.0.1-7). To do this we would need a Hamiltonian or, equivalently, a Lagrangian for the entire system. A Lagrangian description exists for the system (7.0.1-7) in Lagrangian variables,[5] but we have found this description cumbersome because of its use of Lagrangian coordinates. A Lagrangian description using Eulerian coordinates[6] exists, but we find this description cumbersome in its use of the auxiliary field $\underset{\sim}{\xi}$, the displacement vector. In addition, since this description involves an expansion in $\underset{\sim}{\xi}$, the position of a particle in the perturbed system relative to its position in the unperturbed system, this description would not apply to ponderomotive effects.[7] Ponderomotive effects can cause a particle to bounce off a wave packet. After bouncing, the time average of $\dot{\underset{\sim}{\xi}}$ is nonzero, and $\underset{\sim}{\xi}$ increases without bound, invalidating the expansion.

Previous applications of Hamiltonian perturbation theory to the Vlasov-Maxwell equations have concentrated on plane waves in uniform media. Dewar[1] derived the quasilinear evolution equations of a longitudinal unmagnetized plasma using the Poincaré-Von Zeipel perturbation

method[8] for Hamiltonian systems. Johnston[2] used the same method to derive the induced scattering coefficients of plane waves in magnetized and unmagnetized plasma, and Johnston, Kaufman, and Johnston used this method for discussing three wave coupling coefficients.[3] Lie transforms were first used in discussing the Vlasov-Maxwell equations by Dewar[9] in his formulation of turbulence theory.

This thesis is directed toward the description of the nonlinear evolution of a single normal mode of finite extent, in which case ponderomotive effects are important. In this chapter we discuss the general aspects of this problem. We begin in section 7.1 with the Hamiltonization of equation (7.0.1). In section 7.2 we outline the general scheme for calculating linear and nonlinear effects. In section 7.3 we present the general calculation of the linear susceptibility of Vlasov plasma, and in section 7.4 we prove a relation between the linear susceptibility of a Vlasov plasma and the ponderomotive Hamiltonian $K_{2\nu 0}$, a quantity of major importance in nonlinear wave evolution.

7.1 Hamiltonian Description of Vlasov Systems

In this section we rewrite the Vlasov equation in terms of canonical variables. The evolution is then given by a Hamiltonian and is therefore amenable to analysis by Lie transforms. We also describe the ordering of the various quantities, since the ordering of the Hamiltonian formulation differs from the usual case.

We first note that the relativistic Vlasov equation is equivalent to Liouville's equation for $f^s(\underset{\sim}{q},\underset{\sim}{p},t)$,

$$\frac{\partial f^s}{\partial t} + \{f^s, h^s\} = 0 \ , \tag{7.1.1}$$

with the Hamiltonian

$$h^s(\underset{\sim}{q},\underset{\sim}{p},t) = \left\{m_s^2c^4 + [c\underset{\sim}{p} - e_s\underset{\sim}{A}(\underset{\sim}{q},t)]^2\right\}^{\frac{1}{2}} + e_s\Phi(\underset{\sim}{q},t) \ , \tag{7.1.2}$$

where e_s and m_s are the charge and mass of species s. To derive Liouville's equation from the Vlasov equation one must use the expression

$$\begin{aligned}\underset{\sim}{v}^s(\underset{\sim}{q},\underset{\sim}{p},t) &= \dot{\underset{\sim}{q}} = \partial h^s/\partial \underset{\sim}{p} \\ &= [\underset{\sim}{p}-e_s\underset{\sim}{A}(\underset{\sim}{q},t)/c]/\left\{m_s^2 + [\underset{\sim}{p}-e_s\underset{\sim}{A}(\underset{\sim}{q},t)/c]^2/c^2\right\}^{\frac{1}{2}} \ ,\end{aligned} \tag{7.1.3}$$

which gives the velocity $\underset{\sim}{v}$ in terms of the canonical variables $\underset{\sim}{q},\underset{\sim}{p}$. The fact that the velocity is not an independent variable is important when one computes the current density,

$$\underset{\sim}{j}^s(\underset{\sim}{x},t) = \int d^6z \ \underset{\sim}{\eta}_s(\underset{\sim}{x}|\underset{\sim}{z},t) \ f^s(\underset{\sim}{z},t) \ , \tag{7.1.4}$$

where

$$\underset{\sim}{\eta}_s(\underset{\sim}{x}|\underset{\sim}{z},t) \equiv e_s\underset{\sim}{v}^s(\underset{\sim}{q},\underset{\sim}{p},t)\delta(\underset{\sim}{x}-\underset{\sim}{q}) . \tag{7.1.5}$$

The quantity $\underset{\sim}{\eta}_s(\underset{\sim}{x}|\underset{\sim}{z},t)$, the current density at $\underset{\sim}{x}$ due to a particle at $\underset{\sim}{z}$ at time t, depends on the vector potential in this canonical formalism. On the other hand, computing the charge density is simpler:

$$\rho^s(\underset{\sim}{x},t) = \int d^6z \ \mu_s(\underset{\sim}{x}|\underset{\sim}{z},t) \ f^s(\underset{\sim}{z},t) \tag{7.1.6}$$

where

$$\mu_s(\underset{\sim}{x}|\underset{\sim}{z},t) \equiv e_s\delta(\underset{\sim}{x}-\underset{\sim}{q}) \ . \tag{7.1.7}$$

Eqs. (7.1.1), (7.1.4), and (7.1.6) together with Maxwell's equations are the canonical form of the Vlasov-Maxwell system.

Much of this work is species independent, and so it is convenient

to drop the species label s. Furthermore, we choose units such that $m = c = 1$. Of course, we revert to ordinary units and restore species labels whenever confusion might arise. Thus we have

$$h(\underset{\sim}{q},\underset{\sim}{p},t) = \left|1 + [\underset{\sim}{p}-e\underset{\sim}{A}(\underset{\sim}{q},t)]^2\right|^{\frac{1}{2}} + e\Phi(\underset{\sim}{q},t), \tag{7.1.8}$$

$$\underset{\sim}{v}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{\underset{\sim}{p}-e\underset{\sim}{A}(\underset{\sim}{q},t)}{\left|1+[\underset{\sim}{p}-e\underset{\sim}{A}(\underset{\sim}{q},t)]^2\right|^{\frac{1}{2}}} . \tag{7.1.9}$$

and

$$\eta(\underset{\sim}{x}|\underset{\sim}{z},t) = e\ \underset{\sim}{v}(\underset{\sim}{q},\underset{\sim}{p},t)\delta(x-q) . \tag{7.1.10}$$

In nonlinear theory we expect the electromagnetic field, and hence the potentials $\underset{\sim}{A}$ and Φ, to have terms of all orders (in some small parameter):

$$\Phi(\underset{\sim}{x},t) = \sum_{n=0}^{\infty} \Phi_n(\underset{\sim}{x},t) \tag{7.1.11}$$

$$\underset{\sim}{A}(\underset{\sim}{x},t) = \sum_{n=0}^{\infty} \underset{\sim}{A}_n(\underset{\sim}{x},t) . \tag{7.1.12}$$

We want to determine how this ordering is reflected in the Hamiltonian formalism. Of course, to zeroth order we have

$$h_0(\underset{\sim}{q},\underset{\sim}{p},t) = \left|1 + [\underset{\sim}{p}-e\underset{\sim}{A}_0(\underset{\sim}{q},t)]^2\right|^{\frac{1}{2}} + e\Phi_0(\underset{\sim}{q},t) , \tag{7.1.13}$$

$$\underset{\sim}{v}_0(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{\underset{\sim}{p}-e\underset{\sim}{A}_0(\underset{\sim}{q},t)}{\left|1+[\underset{\sim}{p}-e\underset{\sim}{A}_0(\underset{\sim}{q},t)]^2\right|^{\frac{1}{2}}} \tag{7.1.14}$$

and

$$\eta_0(\underset{\sim}{q},\underset{\sim}{p},t) = e\ \underset{\sim}{v}_0(\underset{\sim}{q},\underset{\sim}{p},t)\delta(\underset{\sim}{x}-\underset{\sim}{q}) \ . \tag{7.1.15}$$

To find the higher order terms in the Hamiltonian we note that

$$h(\underset{\sim}{q},\underset{\sim}{p},t) = h_0(\underset{\sim}{q},\underset{\sim}{p}-e\delta\underset{\sim}{A}(\underset{\sim}{q},t),t) + e\delta\Phi(\underset{\sim}{q},t) \ , \tag{7.1.16}$$

where

$$\delta\underset{\sim}{A}(\underset{\sim}{q},t) \equiv \underset{\sim}{A}(\underset{\sim}{q},t) - \underset{\sim}{A}_0(\underset{\sim}{q},t) \tag{7.1.17}$$

and

$$\delta\Phi(\underset{\sim}{q},t) \equiv \Phi(\underset{\sim}{q},t) - \Phi_0(\underset{\sim}{q},t) \ . \tag{7.1.18}$$

We Taylor expand Eq. (7.1.16) to obtain

$$h(\underset{\sim}{q},\underset{\sim}{p},t) = \sum_{n=0}^{\infty} \frac{1}{n!} (-e\delta\underset{\sim}{A}\cdot\frac{\delta}{\delta p})^n\ h_0(\underset{\sim}{q},\underset{\sim}{p},t) + e\delta\Phi(q,t) \ . \tag{7.1.19}$$

By inserting the expansion (7.1 12) into Eq. (7.1.19) we obtain

$$h(\underset{\sim}{q},\underset{\sim}{p},t) = \sum_{n=0}^{\infty} h_n(\underset{\sim}{q},\underset{\sim}{p},t) \ , \tag{7.1.20}$$

where

$$h_1 = -e\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1 + e\Phi_1 \ , \tag{7.1.21}$$

$$h_2 = -e\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2 + e\Phi_2 + \frac{1}{2}e^2\gamma_0^{-1}[\underset{\sim}{A}_1\cdot\underset{\sim}{A}_1-(\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^2] \ , \tag{7.1.22}$$

$$h_3 = -e\underset{\sim}{v}_0\cdot\underset{\sim}{A}_3 + e\Phi_3 + e^2\gamma_0^{-1}[\underset{\sim}{A}_1\cdot\underset{\sim}{A}_2-\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2]$$
$$+ \frac{1}{2}e^3\gamma_0^{-2}[\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1|\underset{\sim}{A}_1|^2 - (\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^3] \ ,$$

$$\vdots$$

and $$\gamma_0(\underset{\sim}{q},\underset{\sim}{p},t) \equiv \left\{1 + [\underset{\sim}{p}-e\underset{\sim}{A}_0(\underset{\sim}{q},t)]^2\right\}^{\frac{1}{2}} \ . \tag{7.1.23}$$

The higher order terms in $\underset{\sim}{v}$ and $\underset{\sim}{\eta}$ are found by using Eqs. (7.1.3) and (7.1.10) order by order. The results are

$$\underset{\sim}{v}_1 = -e[\underset{\sim}{A}_1 - \underset{\sim}{v}_0 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1]\gamma_0^{-1} \ , \tag{7.1.24}$$

$$\underset{\sim}{v}_2 = -e\gamma_0^{-1}[\underset{\sim}{A}_2 - \underset{\sim}{v}_0 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2] - \frac{1}{2}e^2\gamma_0^{-2}[\underset{\sim}{v}_0 \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1$$

$$+ 2\underset{\sim}{A}_1 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 - 3\underset{\sim}{v}_0 (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2] \ , \tag{7.1.25}$$

$$\underset{\sim}{v}_3 = -e\gamma_0^{-1}[\underset{\sim}{A}_3 - \underset{\sim}{v}_0 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_3] \quad {}^2\gamma_0^{-2}[\underset{\sim}{v}_0 \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2 + \underset{\sim}{A}_1 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2 + \underset{\sim}{A}_2 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 - 3\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2]$$

$$- \frac{1}{2}e^3\gamma_0^{-3}[-\underset{\sim}{A}_1 \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 + 3\underset{\sim}{A}_1 (\underset{\sim}{v} \cdot \underset{\sim}{A}_1)^2 + 3\underset{\sim}{v}_0 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 - 5\underset{\sim}{v}_0 (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^3] \tag{7.1.26}$$

$$\vdots$$

and $$\underset{\sim}{\eta}_n(\underset{\sim}{q},\underset{\sim}{p},t) = e\ \underset{\sim}{v}_n(\underset{\sim}{q},\underset{\sim}{p},t)\delta(\underset{\sim}{x}-\underset{\sim}{q}) \ . \tag{7.1.27}$$

We see that the velocity $\underset{\sim}{v}$ and the current density $\underset{\sim}{\eta}$ have terms of all orders. Thus, if we have the solution of Liouville's equation order by order,

$$f(\underset{\sim}{q},\underset{\sim}{p},t) = \sum_{n=0}^{\infty} f_n(\underset{\sim}{q},\underset{\sim}{p},t) \ , \tag{7.1.28}$$

and we want to know the current, we must use

$$j(\underset{\sim}{x},t) = \int d^6z \sum_{n=0}^{\infty} \eta_n(\underset{\sim}{x}|\underset{\sim}{z},t) \sum_{m=0}^{\infty} f_m(\underset{\sim}{z},t) \ , \tag{7.1.29}$$

of which the nth order part is

$$j_n(\underset{\sim}{x},t) = \int d^6z \sum_{m=0}^{n} \underset{\sim}{\eta}_{n-m}(\underset{\sim}{x}|\underset{\sim}{z},t)\ f_m(\underset{\sim}{z},t) . \tag{7.1.30}$$

However, since $\Pi(\underset{\sim}{x}|\underset{\sim}{z},t)$ has no higher order terms, the higher order contributions to the charge density are given by

$$\rho_n(\underset{\sim}{x},t) = \int d^6z\ \Pi(\underset{\sim}{x}|\underset{\sim}{z},t)\ f_n(\underset{\sim}{z},t) \ . \tag{7.1.31}$$

In summary, we have written the Vlasov equation in terms of canonical variables, and we have described the ordering of the resulting Hamiltonian system.

7.2 Calculation of Linear and Nonlinear Effects

In this section we outline the general method of using Lie transforms to analyze nonlinear wave evolution. We start by noting general considerations which apply to any problem, but then, to discuss nonlinear effects, we specialize to the case of wave packet propagation in uniform plasma, and we assume that resonant particles are absent (or unimportant). Much of this discussion may seem vague to the reader who had not studied any concrete examples. Therefore the reader is urged to study Chapter 8 in parallel with this section. The specific calculations of Chapter 8 and the general statements of this section are more easily understood in combination.

To begin, we must have a thorough knowledge of the unperturbed system. In particular, we must know the unperturbed Hamiltonian (7.1.13), its time development operator $M_0(t)$, which satisfies

$$\frac{\partial M_0}{\partial t} = -M_0(t)\ L_{h_0(t)} \tag{7.2.1}$$

and the imperturbed Vlasov distribution $f_0(\underset{\sim}{z},t)$ which satisfies

$$\frac{\partial f_0}{\partial t} + \{f_0, h_0\} = 0\ . \tag{7.2.2}$$

Let us add perturbations $\delta\underset{\sim}{E}$ and $\delta\underset{\sim}{B}$ to this system. The problem is to calculate the change δf of the distribution, from which we can calculate the charge density perturbation $\delta\rho$ and the current density perturbation $\delta\underset{\sim}{j}$. Our method for finding δf is somewhat indirect. We

introduce a Lie transform T (and therefore a generating function w) and a function F such that

$$f = TF \ . \tag{7.2.3}$$

From section 3.2, Eqs. (3.2.12, 15-18), we know that if F satisfies

$$\frac{\partial F}{\partial t} + \{F,K\} = 0 \ , \tag{7.2.4}$$

where K is given by

$$K(\theta) = T^{-1}(\theta)h + T^{-1}(\theta)\int_0^\theta d\theta' T(\theta') \frac{\partial w}{\partial t}(\theta') \ , \tag{7.2.5}$$

then f satisfies Liouville's equation (7.1.1). Now, we choose T to make the new Hamiltonian K, which is given by (7.2.5), as simple as possible. Next, we solve for F in that system, i.e. Eq. (7.2.4). Finally, we transform back via Eq. (7.2.3) to obtain f, the desired solution. Actually, this method proceeds order by order, so we will need to use (7.1.15a-e) for T and (7.1.17a-e) in place of Eq. (7.2.5).

In zeroth order, the Lie transform reduces to the identity, i.e., Eqs. (7.2.3-5) become

$$f_0 = F_0 \tag{7.2.6}$$

$$\frac{\partial f_0}{\partial t} + \{F_0,K_0\} = 0 \tag{7.2.7}$$

$$K_0 = h_0 \ . \tag{7.2.8}$$

We proceed to first order. The first step is finding suitable vector and scalar potentials for the perturbation $\underset{\sim}{E}_1$ and $\underset{\sim}{B}_1$. Inserting these potentials into (7.1.21) we obtain h_1. The usual way of finding the response f_1 is to solve the linearized Vlasov equation,

$$\frac{\partial f_1}{\partial t} + \{f_1,h_0\} + \{f_0,h_1\} = 0 \ , \tag{7.2.9}$$

by integration along a trajectory. In contrast, the Lie transform method involves using Eqs. (7.2.3-5), which through first order are:

$$f_1 = [(1+T_1+\ldots)(F_0+F_1+\ldots)]_1 \tag{7.2.10}$$

$$= -\{w_1,f_0\} + F_1 ,$$

$$\frac{\partial F_1}{\partial t} = \{F_1,K_0\} + \{F_0,K_1\} = 0 , \tag{7.2.11}$$

and Eq. (7.1.17b). We choose w_1 to make K_1 in Eq. (7.1.17b) as simple as possible, we solve (7.2.11) for F_1, and then we use (7.2.10) to find f_1.

The Lie transform scheme is flexible in that it does not tell us how to choose K_1 or w_1. We choose one of these functions to our advantage, and then we solve Eq. (7.1.17b) for the other. For example, suppose we choose $w_1 = 0$. Then (4.1.17b) gives $K_1 = h_1$, and so F_1 satisfies (from Eq. (7.2.11)),

$$\frac{\partial F_1}{\partial t} + \{F_1,h_0\} + \{f_0,h_1\} = 0 . \tag{7.2.12}$$

Since $f_1 = F_1$, when $w_1 = 0$, we are left with the same equation (7.2.12) as we originally had (7.2.9). That is, setting $w_1 = 0$ means we have set the transformation equal to the identity (through first order).

Suppose instead we choose $K_1 = 0$. In this case Eq. (7.1.17b) gives

$$\frac{\partial w_1}{\partial t} + \{w_1,h_0\} = -h_1 , \tag{7.2.13}$$

which can be integrated along a trajectory to obtain w_1:

$$w_1(\underset{\sim}{z},t) = -\int^{0} d\tau\, M_0(\tau)h_1(t+\tau) . \tag{7.2.14}$$

(We use the indefinite integral since we can use any solution of Eq.

(7.2.13).) Since $K_1 = 0$, F_1 now satisfies

$$\frac{\partial F_1}{\partial t} + \{F_1, h_0\} = 0 \ . \tag{7.2.15}$$

To find the correct particular solution of Eq. (7.2.15), we note that w_1 vanishes when h_1 vanishes, and f_1 should vanish when h_1 vanishes. Hence we deduce $F_1 = 0$ from Eq. (7.2.10), so that

$$f_1 = -\{w_1, f_0\} \ . \tag{7.2.16}$$

Thus we have seen the two extremes, choosing $w_1 = 0$ and choosing $K_1 = 0$. More generally we may choose to have neither K_1 nor w_1 vanish.

So how do we choose K_1 (or w_1)? We benefit from choosing $K_1 = 0$, since then the linear problem is solved. In fact some of the nonlinear effects, such as the L_1^2 term in the transformation T, are known once we know w_1. However, we cannot choose $K_1 = 0$ if h_1 varies slowly along an orbit since then the integration (7.2.14) gives secular terms, and the result for w_1 would be valid only for short times. Hence the prescription we adopt is to choose T to transform away the oscillations and to keep the slowly varying terms in K. This is called the oscillation center prescription;[1] F is called the oscillation-center distribution, and K is called the oscillation-center Hamiltonian.

Of course, to say that the linear problem is solved we must be able to explicitly compute the integral (7.2.14) to find w_1. After all, we can always write the solution to Eq. (7.2.9) as an integral along a trajectory. For now we simply state that the integral (7.2.14) can be done explicitly if the phrase $(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)$ of the perturbation $\underset{\sim}{E}_1(\underset{\sim}{q},t) = \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{q},t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t)$ varies sufficiently rapidly along an orbit. Since this qualification depends on details of the unperturbed

motion, we leave further discussion for later chapters.

No matter how f_1 is found, it can be inserted into Eq. (7.1.29) to find the linear current,

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = \int d^6z\,[\eta_0(\underset{\sim}{x}|\underset{\sim}{z})f_1(\underset{\sim}{z},t) + \eta_1(\underset{\sim}{x}|\underset{\sim}{z})f_0(\underset{\sim}{z},t)] \quad . \tag{7.2.17}$$

As we show in the next section, we can extract, from the above equation, the conductivity $\underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',t,t')$, which gives $\underset{\sim}{j}_1(\underset{\sim}{x},t)$ from $\underset{\sim}{E}_1(\underset{\sim}{x},t)$ via

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = \int d^3x' \int^t dt'\; \underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',t,t')\cdot\underset{\sim}{E}_1(\underset{\sim}{x}',t') \quad . \tag{7.2.18}$$

For a spatially homogeneous and time-independent medium, for which $\underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',t,t') = \underset{\approx}{\sigma}(\underset{\sim}{x}-\underset{\sim}{x}',0,t-t',0)$, we can compute the Fourier-space conductivity,

$$\underset{\approx}{\sigma}(\underset{\sim}{k},\omega) \equiv \int d^3s \int_0^\infty d\tau\; \underset{\approx}{\sigma}(\underset{\sim}{s},0,\tau,0)e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{s}-\omega\tau)} \quad . \tag{7.2.19}$$

The conductivity $\underset{\approx}{\sigma}$ can be inserted in the expression for the linear response tensor $\underset{\approx}{D} = \underset{\approx}{I}\,(1-k^2c^2/\omega^2) + \underset{\sim}{k}\underset{\sim}{k}c^2/\omega^2 + 4\pi i\,\sigma(\underset{\sim}{k},\omega)/\omega$, and so we can use the theory of section 6.3 to describe linear wave packet propagation.

Now we turn to nonlinear theory. From this point on we restrict ourselves to a certain class of nonlinear problems. We consider wave packets (6.1.1) in a spatially-homogeneous and time-independent medium, and we require that linear resonant-particle damping is unimportant. (Linear and nonlinear resonant-particle damping of plane waves have been treated by Dewar[1] and Johnston.[2]) The absence of resonant particles allows us to choose $K_1 = 0$, that is, we require h_1 (i.e.,

$\exp[i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t])$ to be rapidly varying along a trajectory. Our goal is to achieve an understanding of ponderomotive effects: nonlinear effects which arise because the wave amplitude varies in space and time.

Since we have assumed the linear fields E_1 and B_1 to have a slowly varying amplitude and a rapidly varying phase $(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)$, all of the first-order quantities have this form. This obviously applies to the vector potential A_1, and the Hamiltonian h_1. Furthermore, it applies to w_1, as we show explicitly in the particular problems treated in Chapters 8-10 of this thesis. Therefore, Eq. (7.2.18) implies that f_1 also has a slowly varying amplitude and a rapidly varying phase $(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)$.

To second-order the Lie transform equations (7.2.3-5) are

$$f_2 = F_2 + \frac{1}{2}\{w_1,\{w_1,h_0\}\} - \frac{1}{2}\{w_2,f_0\} , \tag{7.2.20}$$

$$\frac{\partial F_2}{\partial t} + \{F_2,h_0\} + \{f_0,K_2\} = 0 , \tag{7.2.21}$$

and

$$K_2 - \frac{1}{2}\left(\frac{\partial w_2}{\partial t} + \{w_2,h_0\}\right) = h_2 + \frac{1}{2}\{w_1,h_1\} \tag{7.2.22}$$

for $K_1 = 0$. Eq. (7.1.29) gives the second-order current,

$$\underset{\sim}{j}_2 = \int d^6z\ (\underset{\sim}{\eta}_0 f_2 + \underset{\sim}{\eta}_1 f_1 + \underset{\sim}{\eta}_2 f_0) . \tag{7.2.23}$$

Now the question is how to choose w_2 to make K_2 as simple as possible.

Before answering this question, we discuss some of the concepts of the formalism of section 6.3. The first concept is linear response versus nonlinear response. The second-order Hamiltonian, which is given by Eq. (7.1.22), and which appears in Eq. (7.2.22), can be separated into a linear (λ) part and nonlinear (ν) part, $h_2 = h_{2\nu} + h_{2\lambda}$,

which are given by

$$h_{2\lambda} = - e\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2 + e\Phi_2 \tag{7.2.24}$$

and

$$h_{2\nu} = \frac{1}{2} e^2 \gamma_0^{-1} [\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 - (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2] \quad . \tag{7.2.25}$$

The same is true for the second-order current function $\underset{\sim}{\eta}_2 = \underset{\sim}{\eta}_{2\lambda} + \underset{\sim}{\eta}_{2\nu}$:

$$\eta_{2\lambda}(\underset{\sim}{x}|\underset{\sim}{z},t) = -e^2 \gamma_0^{-1} [\underset{\sim}{A}_2 - \underset{\sim}{v}_0 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2] \delta(\underset{\sim}{x} - \underset{\sim}{q}) \tag{7.2.26}$$

$$\eta_{2\nu}(\underset{\sim}{x}|\underset{\sim}{z},t) = - \frac{1}{2} e^3 \gamma_0^{-2} [\underset{\sim}{v}_0 \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 + 2\underset{\sim}{A}_1 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 - 3\underset{\sim}{v}_0 (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2] \tag{7.2.27}$$

Similarly we can separate all the second-order quantities into a linear part and a nonlinear part. The linear part Eqs. (7.2.20-23) is

$$f_{2\lambda} = F_{2\lambda} - \frac{1}{2} \{w_{2\lambda}, f_0\} \tag{7.2.28}$$

$$\frac{\partial F_{2\lambda}}{\partial t} + \{F_{2\lambda}, h_0\} + \{f_0, K_{2\lambda}\} = 0 \tag{7.2.29}$$

$$K_{2\lambda} - \frac{1}{2}\left(\frac{\partial w_{2\lambda}}{\partial t} + \{w_{2\lambda}, h_0\}\right) = h_{2\lambda} \tag{7.2.30}$$

$$\underset{\sim}{j}_{2\lambda} = \int d^6 z (\eta_0 f_{2\lambda} + \underset{\sim}{\eta}_{2\lambda} f_0) \ , \tag{7.2.31}$$

and the nonlinear part of Eqs. (7.2.20-23) is

$$f_{2\nu} = F_{2\nu} + \frac{1}{2} \{w_1, \{w_1, f_0\}\} - \frac{1}{2} \{w_{2\nu}, f_0\} \tag{7.2.32}$$

$$\frac{\partial F_2}{\partial t} + \{F_{2\nu}, h_0\} + \{f_0, K_{2\nu}\} = 0 \tag{7.2.33}$$

$$K_{2\nu} - \frac{1}{2}\left(\frac{\partial w_{2\nu}}{\partial t} + \{w_{2\nu}, h_0\}\right) = h_{2\nu} + \frac{1}{2} \{w_1, h_1\} \tag{7.2.34}$$

$$\underset{\sim}{j}_{2\nu} = \int d^6 z \ (\eta_0 f_{2\nu} + \underset{\sim}{\eta}_1 f_1 + \underset{\sim}{\eta}_{2\nu} f_0) \quad . \tag{7.2.35}$$

The second concept we introduce is harmonic number. Since each of the linear quantities has a rapidly varying phase, each of the second-order quantities has terms which oscillate at the zeroth harmonic (i.e., they vary slowly) and terms which oscillate at the second harmonic. Accordingly, we write

$$h_2 = h_{20} + h_{22} \tag{7.2.36}$$

$$h_{2\nu} = h_{2\nu 0} + h_{2\nu 2} \tag{7.2.37}$$

$$\{w_1,h_1\} = \{w_1,h_1\}_0 + \{w_1,h_1\}_2 \tag{7.2.38}$$

$$\vdots \quad ,$$

and we may further separate Eqs. (7.2.20-35) into zeroth-harmonic parts and second-harmonic parts.

Now we return to the question of solving these equations, i.e., how do we choose w_2? We begin by discussing the second-harmonic nonlinear terms. We recall the assumption that $\exp(i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t)$ varies rapidly along a trajectory, and hence h_1 can be transformed away. It follows that the second harmonics, which have the factor $\exp(2i\underset{\sim}{k}\cdot\underset{\sim}{q}-2i\omega t)$, vary even more rapidly, and hence they may be transformed away. Thus we choose

$$K_{2\nu 2} = 0 \ , \tag{7.2.39}$$

which implies $F_{2\nu 2} = 0$, and we solve for $w_{2\nu 2}$ by integrating (7.2.34) along a trajectory,

$$-\frac{1}{2} w_{2\nu 2} = \int^0 d\tau \ M_0(\tau)\,[h_{2\nu 2}(t+\tau) + \tfrac{1}{2}\{w_1(t+\tau),h_1(t+\tau)\}_2] \quad . \tag{7.2.40}$$

The remarks concerning the solution (7.2.14) for w_1 also apply here; to

have truly accomplished anything, we must be able to compute the integral (7.2.40) explicitly. Again, this is possible, as we show in later chapters, because of the rapidly varying phase. From $w_{2\nu 2}$ and Eq. (7.2.32) we compute $f_{2\nu 2}$.

$$f_{2\nu 2} = \frac{1}{2}\{w_1,\{w_1,f_0\}\}_2 - \frac{1}{2}\{w_{2\nu 2},f_0\} \tag{7.2.41}$$

Then we use $f_{2\nu 2}$ in (7.2.35) to find $\underset{\sim}{j}_{2\nu 2}$.

$$\underset{\sim}{j}_{2\nu 2} = \int d^6z\ [\underset{\sim}{\eta}_0 f_{2\nu 2} + (\underset{\sim}{\eta}_1 f_1)_2 + \underset{\sim}{\eta}_{2\nu 2} f_0] \tag{7.2.42}$$

At this point we use the formalism of section 6.3, specifically Eq. (6.3.52), to compute the second-order second-harmonic field from $\underset{\sim}{j}_{2\nu 2}$. Knowing the second-order fields, we are ready to compute the second-order second-harmonic linear quantities.

This computation proceeds in an analogous manner, but from Eq. (7.2.28). Since second-harmonic linear terms also have the factor $\exp(2i\underset{\sim}{k}\cdot\underset{\sim}{q}-2i\omega t)$, we may set $K_{2\lambda 2} = 0$ and, therefore, $F_{2\lambda 2} = 0$. We solve (7.2.30) by integration along a trajectory

$$-\frac{1}{2} w_{2\lambda 2} = \int^0 d\tau\ M_0(\tau)\ h_{2\lambda 2}(t+\tau)\ , \tag{7.2.43}$$

and we use this solution in Eq. (7.2.28) to find

$$f_{2\lambda 2} = -\frac{1}{2}\{w_{2\lambda 2},f_0\}\ . \tag{7.2.44}$$

This result can be inserted into (7.2.31) to find

$$\underset{\sim}{j}_{2\lambda 2} = \int d^6z\ (\underset{\sim}{\eta}_0 f_{2\lambda 2} + \underset{\sim}{\eta}_{2\lambda 2} f_0)\ . \tag{7.2.45}$$

The result of these manipulations is the following. If the condition, that the integrals (7.2.40) and (7.2.43) can be computed explicitly, is satisfied, we have solved the second-order second-harmonic problem. (As we see in later sections, this condition holds when the amplitude $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)$ varies slowly.) We then have explicit formulas (7.2.40-45) and (6.3.52) for the second-order second-harmonic quantities.

Now we turn to the second-order zeroth-harmonic quantities, which satisfy

$$f_{2\nu 0} = F_{2\nu 0} - \frac{1}{2}\{w_{2\lambda 0}, f_0\} \tag{7.2.46}$$

$$\frac{\partial F_{2\lambda 0}}{\partial t} + \{F_{2\lambda 0}, h_0\} + \{f_0, K_{2\lambda 0}\} = 0 \tag{7.2.47}$$

$$K_{2\lambda 0} - \frac{1}{2}\left(\frac{\partial w_{2\lambda 0}}{\partial t} + \{w_{2\lambda 0}, h_0\}\right) = h_{2\lambda 0} \ , \tag{7.2.48}$$

$$\underset{\sim}{j}_{2\lambda 0} = \int d^6z \ (\underset{\sim}{\eta}_0 f_{2\lambda 0} + \eta_{2\lambda 0} f_0) \ , \tag{7.2.49}$$

and

$$f_{2\nu 0} = F_{2\nu 0} + \frac{1}{2}\{w_1\{w_1, f_0\}\}_0 - \frac{1}{2}\{w_{2\nu 0}, f_0\} \tag{7.2.50}$$

$$\frac{\partial F_{2\nu 0}}{\partial t} + \{F_{2\nu 0}, h_0\} + \{f_0, K_{2\nu 0}\} = 0 \tag{7.2.51}$$

$$K_{2\nu 0} - \frac{1}{2}\left(\frac{\partial w_{2\nu 0}}{\partial t} + \{w_{2\nu 0}, h_0\}\right) = h_{2\nu 0} + \frac{1}{2}\{w_1, h_1\}_0 \tag{7.2.52}$$

$$\underset{\sim}{j}_{2\nu 0} = \int d^6z \ [\underset{\sim}{\eta}_0 f_{2\nu 0} + (\underset{\sim}{\eta}_1 f_1)_0 + \underset{\sim}{\eta}_{2\nu 0} f_0] \ . \tag{7.2.53}$$

In general we cannot transform away the second-order Hamiltonian, i.e., we cannot choose $K_{20} = 0$, since the terms in the above equations are slowly varying. As a result we have not completely determined the solution by using Lie transforms, but we have simplified the problem. The coupled equations (6.3.53) and (7.2.46-53), which describe the evolution of the zeroth-harmonic quantities, have only slowly varying terms; the rapid oscillations, the second-harmonics, have been transformed away. This is a simplification even if we must resort to numerical techniques. When equations have only slow time variations, they can be numerically integrated for a much longer time.

One term, $K_{2\nu 0}$, in these equations deserves special attention. $K_{2\nu 0}$ is called the ponderomotive Hamiltonian, since it depends on the linear fields and it governs the motion of the oscillation-centers (i.e. the evolution of F_{20}). $K_{2\nu 0}$ is the kinetic generalization of the ponderomotive potential ($e^2\mathcal{E}_1/m\omega^2$ in the unmagnetized case) since $K_{2\nu 0}$ contains the momentum dependence of the average nonlinear force. In the last section of this chapter we investigate the general aspects of $K_{2\nu 0}$ more thoroughly. In later chapters we investigate $K_{2\nu 0}$ in detail in the special cases considered.

We proceed to third order, for which the Lie transform equations (7.2.3-5) are

$$f_3 = T_3 f_0 + T_1 F_2 + F_3 \ , \tag{7.2.54}$$

$$\frac{\partial F_3}{\partial t} + \{F_3, h_0\} + \{F_0, K_3\} = 0 \ , \tag{7.2.55}$$

and

$$K_3 - \frac{1}{3}\left(\frac{\partial w_3}{\partial t} + \{w_3,h_0\}\right) = h_3 + \frac{1}{3}\{w_1,K_2+2h_2\}$$

$$+ \frac{1}{6}\{w_2,h_1\} + \frac{1}{6}\{w_1,\{w_1,h_1\}\}, \tag{7.2.56}$$

where T_3 is given by Eq. (4.1.15d). We have used $K_1 = F_1 = 0$ to obtain these equations. The third-order current is given by

$$\underset{\sim}{j}_3 = \int d^6z\ (\underset{\sim}{\eta}_0 f_3 + \underset{\sim}{\eta}_1 f_2 + \underset{\sim}{\eta}_2 f_1 + \underset{\sim}{\eta}_3 f_0) \quad . \tag{7.2.57}$$

The harmonic structure of the first-order and second-order quantities implies that the third order quantities oscillate at the first harmonic and at the third harmonic, and so we may transform away the third-order Hamiltonian, i.e. we choose $K_3 = 0$. This implies $F_3 = 0$. Thus, the third order equations reduce to

$$f_3 = T_3 f_0 + T_1 F_2 \ , \tag{7.2.58}$$

$$-\frac{1}{3} w_3 = \int_0^0 d\tau\ M_0(\tau) \left[h_3 + \frac{1}{3}\{w_1,K_2+2h_2\} + \frac{1}{6}\{w_2,h_1\}\right.$$

$$\left. + \frac{1}{6}\{w_1,\{w_1,h_1\}\}\right]\Bigg|_{t+\tau} \quad , \tag{7.2.59}$$

and Eq. (7.2.57).

Before analyzing these equations we note that they can be simplified. We denote the time derivative along a trajectory by the symbol d_0,

$$d_0 w_3 \equiv \frac{\partial w_3}{\partial t} + \{w_3,h_0\} \quad , \tag{7.2.60}$$

so that Eq. (7.2.56) can be written

$$-\frac{1}{3}\, d_0 w_3 = h_3 + \frac{1}{3}\{w_1, K_2 + 2h_2\} + \frac{1}{6}\{w_2, h_1\} + \frac{1}{6}\{w_1, \{w_1, h_1\}\} \quad . \tag{7.2.61}$$

The third term in this expression was be written

$$\begin{aligned} \frac{1}{6}\{w_2, h_1\} &= -\frac{1}{6}\{w_2, d_0 w_1\} \\ &= -\frac{1}{6}\, d_0\{w_2, w_1\} + \frac{1}{6}\{d_0 w_2, w_1\} \\ &= -\frac{1}{6}\, d_0\{w_2, w_1\} - \frac{1}{3}\{h_2 - K_2 + \frac{1}{2}\{w_1, h_1\}, w_1\} \end{aligned} \tag{7.2.62}$$

upon using Eqs. (7.1.17b,c) with $K_1 = 0$. Inserting this into Eq. (7.2.61) we find

$$-\frac{1}{3}\, d_0 w_3 = h_3 + \frac{1}{3}\{w_1, 3h_2 + \{w_1, h_1\}\} - \frac{1}{6}\, d_0\{w_2, w_1\} \quad . \tag{7.2.63}$$

Thus we can split $w_3 = w_3' + w_3''$ into two terms,

$$w_3'' = \frac{1}{2}\{w_2, w_1\} \tag{7.2.64}$$

and

$$-\frac{1}{3}\, d_0 w_3' = h_3 + \frac{1}{3}\{w_1, 3h_2 + \{w_1, h_1\}\} \quad , \tag{7.2.65}$$

one of which is already integrated. This also simplifies T_3. Using Eq. (4.1.15d) and the above results, we find

$$T_3 f_0 = -\frac{1}{3}\{w_3', f_0\} + \frac{1}{2}\{w_1, \{w_2, f_0\}\} - \frac{1}{6}\{w_1, \{w_1, \{w_1, f_0\}\}\} \quad . \tag{7.2.66}$$

Thus, to find f_3 we use integrate (7.2.65) along a trajectory to find w_3',

$$-\frac{1}{3}\, w_3' = \int^{0} d\tau\, M_0(\tau) \left[h_3 + \frac{1}{3}\{w_1, 3h_2 + \{w_1, h_1\}\}\right]\Bigg|_{t+\tau}$$

then we use w_3' in Eq. (7.2.66), and we insert that result into Eq. (7.2.58).

We now proceed with the analysis of the third-order equations. We separate the various quantities into linear and nonlinear parts. The linear equations are

$$h_{3\lambda} = -e\underset{\sim}{v}_0\cdot\underset{\sim}{A}_3 + e\Phi_3 \quad , \tag{7.2.68}$$

$$\underset{\sim}{\eta}_3 = -e^2\gamma_0^{-1}(\underset{\sim}{A}_3 - \underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{A}_3)\delta(\underset{\sim}{x}-\underset{\sim}{q}) \quad , \tag{7.2.69}$$

$$f_{3\lambda} = -\frac{1}{3}\{w'_{3\lambda}, f_0\} \quad , \tag{7.2.70}$$

$$-\frac{1}{3}w'_{3\lambda} = \int^0 d\tau\, M_0(\tau)h_{3\lambda}(t+\tau) \quad , \tag{7.2.71}$$

and

$$\underset{\sim}{j}_{3\lambda} = \int^0 d^6z(\underset{\sim}{\eta}_0 f_{3\lambda} + \underset{\sim}{\eta}_{3\lambda}f_0) \quad . \tag{7.2.72}$$

The nonlinear equations are

$$h_{3\nu} = e^2\gamma_0^{-1}\left[\underset{\sim}{A}_1\cdot\underset{\sim}{A}_2 - \underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2\right] + \frac{1}{2}e^3\gamma_0^{-2}\left[\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{A}_1\cdot\underset{\sim}{A}_1 - (\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^3\right] \quad , \tag{7.3.73}$$

$$\underset{\sim}{\eta}_{3\nu} = \left\{-e^3\gamma_0^{-2}\left[\underset{\sim}{v}_0\underset{\sim}{A}_1\cdot\underset{\sim}{A}_2 + \underset{\sim}{A}_1\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2 + \underset{\sim}{A}_2\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1 - 3\underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2\right] - \frac{1}{2}e^4\gamma_0^{-3}\left[-\underset{\sim}{A}_1\underset{\sim}{A}_1\cdot\underset{\sim}{A}_1 + 3\underset{\sim}{A}_1(\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^2 + 3\underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{A}_1\cdot\underset{\sim}{A}_1 - 5\underset{\sim}{v}_0(\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^3\right]\right\}\cdot \delta(\underset{\sim}{x}-\underset{\sim}{q}) \quad , \tag{7.2.74}$$

$$f_{3\nu} = -\frac{1}{3}\{w'_{3\nu}, f_0\} + \frac{1}{2}\{w_1,\{w_2,f_0\}\} - \frac{1}{6}\{w_1,\{w_1,\{w_1,f_0\}\}\} - \{w_1,F_2\} \quad , \tag{7.2.75}$$

$$-\frac{1}{3}w'_{3\nu} = \int^0 d\tau\, M_0(\tau)\left[h_{3\nu} + \frac{1}{3}\{w_1, 3h_2 + \{w_1,h_1\}\}\right]\Big|_{t+\tau} \quad , \tag{7.2.76}$$

and

$$\underset{\sim}{j}_{3\nu} = \int d^6z(\underset{\sim}{\eta}_0 f_{3\nu} + \underset{\sim}{\eta}_1 f_2 + \underset{\sim}{\eta}_2 f_1 + \underset{\sim}{\eta}_{3\nu} f_0)$$

$$= e\int d^3p(\underset{\sim}{v}_0 f_{3\nu} + \underset{\sim}{v}_1 f_2 + \underset{\sim}{v}_2 f_1 + \underset{\sim}{v}_{3\nu} f_0) \quad . \tag{7.2.77}$$

Now we discuss the use of these equations. Assuming that the integral (7.2.76) can be computed explicitly, Eqs. (7.2.73-77) give the nonlinear third-order current in terms of the lower order fields $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ and the oscillation-center distrubution F_2. The first-harmonic part of these currents can be inserted into Eqs. (6.3.65) and (6.3.59) to obtain the evolution equation for the first-harmonic amplitude and to obtain the third-order first-harmonic electric field. The third-harmonic nonlinear current can be used to obtain the third-harmonic third-order electric field. Finally we insert the third-order fields into Eq. (7.2.68), and we use Eqs. (7.2.69-72) to compute the third-order linear response.

In summary, we have used Lie transforms to remove the oscillations in the Vlasov-Maxwell equations to third order in the first-harmonic amplitude. As a result, we have a closed set of equations which determine the evolution of the first-harmonic amplitude, the oscillation-center distribution, and the second-order slowly varying fields. Equation (6.3.65) determines the evolution of the first-harmonic amplitude in terms of the third-order nonlinear current, which depends on the oscillation-center distrihution and the second-order slowly varying fields through Eqs. (7.2.73-77). The oscillation-center distribution evolves according to Eqs. (7.2.47) and (7.2.51), and the second-order slowly varying fields evolve according to Eq. (6.3.53).

We now proceed with the analysis of the third-order equations. We separate the various quantities into linear and nonlinear parts. The linear equat.) are

$$h_{3\lambda} = -e\underset{\sim}{v}_0\cdot\underset{\sim}{A}_3 + \Phi_3 \quad , \tag{7.2.68}$$

$$\underset{\sim}{\eta}_3 = -e^2\gamma_0^{-1}(\underset{\sim}{A}_3 - \underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{A}_3)\delta(\underset{\sim}{x}-\underset{\sim}{q}) \quad , \tag{7.2.69}$$

$$f_{3\lambda} = -\frac{1}{3}\{w'_{3\lambda},f_0\} \quad , \tag{7.2.70}$$

$$-\frac{1}{3}w'_{3\lambda} = \int^0 d\tau\, M_0(\tau)h_{3\lambda}(t+\tau) \quad , \tag{7.2.71}$$

and

$$\underset{\sim}{j}_{3\lambda} = \int^0 d^6z(\underset{\sim}{\eta}_0 f_{3\lambda} + \underset{\sim}{\eta}_{3\lambda}f_0) \quad . \tag{7.2.72}$$

The nonlinear equations are

$$h_{3\nu} = e^2\gamma_0^{-1}\left[\underset{\sim}{A}_1\cdot\underset{\sim}{A}_2 - \underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2\right] + \frac{1}{2}e^3\gamma_0^{-2}\left[\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{A}_1\cdot\underset{\sim}{A}_1 - (\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^3\right] \quad , \tag{7.3.73}$$

$$\underset{\sim}{\eta}_{3\nu} = \left\{-e^3\gamma_0^{-2}\left[\underset{\sim}{v}_0\underset{\sim}{A}_1\cdot\underset{\sim}{A}_2 + \underset{\sim}{A}_1\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2 + \underset{\sim}{A}_2\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1 - 3\underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{v}_0\cdot\underset{\sim}{A}_2\right] - \frac{1}{2}e^4\gamma_0^{-3}\left[-\underset{\sim}{A}_1\underset{\sim}{A}_1\cdot\underset{\sim}{A}_1 + 3\underset{\sim}{A}_1(\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^2 + 3\underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1\underset{\sim}{A}_1\cdot\underset{\sim}{A}_1 - 5\underset{\sim}{v}_0(\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)^3\right]\right\}\cdot \delta(\underset{\sim}{x}-\underset{\sim}{q}) \quad , \tag{7.2.74}$$

$$f_{3\nu} = -\frac{1}{3}\{w'_{3\nu},f_0\} + \frac{1}{2}\{w_1,\{w_2,f_0\}\} - \frac{1}{6}\{w_1,\{w_1,\{w_1,f_0\}\}\} - \{w_1,h_2\} \quad , \tag{7.2.75}$$

$$-\frac{1}{3}w'_{3\nu} = \int^0 d\tau\, M_0(\tau)\left[h_{3\nu} + \frac{1}{3}\{w_1,3h_2 + \{w_1,h_1\}\}\right]\Big|_{t+\tau} \quad , \tag{7.2.76}$$

and

$$\underset{\sim}{j}_{3\nu} = \int d^6z(\underset{\sim}{\eta}_0 f_{3\nu} + \underset{\sim}{\eta}_1 f_2 + \underset{\sim}{\eta}_2 f_1 + \underset{\sim}{\eta}_{3\nu} f_0)$$

$$= e\int d^3p(\underset{\sim}{v}_0 f_{3\nu} + \underset{\sim}{v}_1 f_2 + \underset{\sim}{v}_2 f_1 + \underset{\sim}{v}_{3\nu} f_0) \quad . \tag{7.2.77}$$

Now we discuss the use of these equations. Assuming that the integral (7.2.76) can be computed explicitly, Eqs. (7.2.73–77) give the nonlinear third-order current in terms of the lower order fields $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ and the oscillation-center distrubution F_2. The first-harmonic part of these currents can be inserted into Eqs. (6.3.65) and (6.3.59) to obtain the evolution equation for the first-harmonic amplitude and to obtain the third-order first-harmonic electric field. The third-harmonic nonlinear current can be used to obtain the third-harmonic third-order electric field. Finally we insert the third-order fields into Eq. (7.2.68), and we use Eqs. (7.2.69–72) to compute the third-order linear response.

In summary, we have used Lie transforms to remove the oscillations in the Vlasov-Maxwell equations to third order in the first-harmonic amplitude. As a result, we have a closed set of equations which determine the evolution of the first-harmonic amplitude, the oscillation-center distribution, and the second-order slowly varying fields. Equation (6.3.65) determines the evolution of the first-harmonic amplitude in terms of the third-order nonlinear current, which depends on the oscillation-center distribution and the second-order slowly varying fields through Eqs. (7.2.73–77). The oscillation-center distribution evolves according to Eqs. (7.2.47) and (7.2.51), and the second-order slowly varying fields evolve according to Eq. (6.3.53).

7.3 Kubo Relations for the Vlasov Conductivity

Kubo[10] gave a general prescription for finding linear susceptibilities of time-independent Hamiltonian systems. Furthermore, Kubo showed that the linear susceptibility of a classical Hamiltonian system in thermal equilibrium is generally related to the expectation value of the product of the corresponding observables. This is known as the fluctuation-dissipation theorem. However, Kubo's results do not apply to the Vlasov conductivity, since in this case the presence of a linear perturbation affects not only the evolution of the state f but also the form of the observable $\underset{\sim}{\eta}$. Hence we are motivated to generalize Kubo's results to Vlasov theory. We determine the conductivity of a <u>time-dependent</u> Vlasov system. Then we specialize to the case where the unperturbed distribution f_0 depends only on the Hamiltonian h_0 (this is more general than thermal equilibrium) in order to prove a generalized fluctuation-dissipation theorem.

From section 7.2 we know that the linear current response j_1 of a Vlasov plasma to a perturbation h_1 is determined by Eqs. (7.2.9) and (7.2.17). In the present analysis we assume that f_0 and h_0 represent a time-dependent system for which the time development operator $M_0(t)$ is known. Then we use Eq. (2.2.15) to solve Eq. (7.2.9):

$$f_1(t) = M_0^{-1}(t) \int_{-\infty}^{t} dt' \; M_0(t')\{h_1(t'), f_0(t')\} \quad . \tag{7.3.1}$$

In using Eq. (2.2.15) we have assumed that h_1 vanishes for $t \to -\infty$.

We use the radiation gauge ($\Phi_1 = 0$) for the perturbation. From Eqs. (7.1.21 & 15) we find

$$h_1(\underset{\sim}{z},t) = -\int d^3x'\ \eta_0(\underset{\sim}{x}'|\underset{\sim}{z},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x}',t) \quad . \tag{7.3.2}$$

We insert (7.3.2) into (7.3.1) and the result into (7.2.17) to obtain

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = -e^2\underset{\sim}{A}_1(\underset{\sim}{x},t)\cdot\int d^3p\ f_0(\underset{\sim}{x},\underset{\sim}{p},t)\gamma_0^{-1}(\underset{\sim}{x},\underset{\sim}{p},t)[\underset{\approx}{I}-\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)]$$

$$-\int d^6z\int_{-\infty}^{t} dt'\int d^3x'\ \eta_0(\underset{\sim}{x}|\underset{\sim}{z},t)M_0^{-1}(t)M_0(t')\{\eta(\underset{\sim}{x}',\underset{\sim}{z}),f_0(\underset{\sim}{z},t')\}$$

$$\underset{\sim}{A}_1(\underset{\sim}{x}',t') \quad . \tag{7.3.3}$$

To obtain the conductivity kernel from the above equation, we must insert into it the expression $\underset{\sim}{A}_1(\underset{\sim}{x},t) = -\int_{-\infty}^{t} dt''\underset{\sim}{E}_1(\underset{\sim}{x},t'')$. We find

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = e^2\int d^3p\ f_0(\underset{\sim}{x},\underset{\sim}{p},t)\gamma_0^{-1}(\underset{\sim}{x},\underset{\sim}{p},t)[\underset{\approx}{I}-\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)]\cdot$$

$$\int_{-\infty}^{t} dt'\underset{\sim}{E}_1(\underset{\sim}{x},t') + \int_{-\infty}^{t} dt'\int d^3x'\int d^6z\,\eta_0(\underset{\sim}{x}|\underset{\sim}{z},t)M_0^{-1}(t)M_0(t')$$

$$\{\eta_0(\underset{\sim}{x}'|\underset{\sim}{z},t),f_0(\underset{\sim}{z},t')\}\cdot\int_{-\infty}^{t'} dt''\underset{\sim}{E}_1(\underset{\sim}{x},t'') \quad . \tag{7.3.4}$$

Integrating the last term by parts we obtain

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = e^2\int d^3p\ f_0(\underset{\sim}{x},\underset{\sim}{p},t)\gamma_0^{-1}(\underset{\sim}{x},\underset{\sim}{p},t)[\underset{\approx}{I}-\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)]\cdot\int_{-\infty}^{t}\underset{\sim}{E}(\underset{\sim}{x},t')$$

$$+\int_{-\infty}^{t'} dt'\int d^3x'\int_{t'}^{t} dt''\int d^6z\ \eta_0(\underset{\sim}{x}|\underset{\sim}{z},t)M_0^{-1}(t)M_0(t'')$$

$$\left\{\eta_0(\underset{\sim}{x}'|\underset{\sim}{z},t),f_0(\underset{\sim}{z},t'')\right\}\cdot\underset{\sim}{E}_1(\underset{\sim}{x}',t') \quad . \tag{7.3.5}$$

From (7.3.5) we extract the conductivity[11,12]

$$\sigma(\underset{\sim}{x},\underset{\sim}{x}',t,t') = \bar{e}^2\int d^3p\; f_0(\underset{\sim}{x},\underset{\sim}{p},t)\gamma_0^{-1}(\underset{\sim}{x},\underset{\sim}{p},t)[\underset{\approx}{I}-\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p},t)]$$

$$\delta(\underset{\sim}{x}-\underset{\sim}{x}') + \int_{t'}^{t} dt''\int d^6z\; \underset{\sim}{\eta}_0(\underset{\sim}{x}|\underset{\sim}{z})M_0^{-1}(t)M_0(t'')$$

$$\left\{\underset{\sim}{\eta}_0(\underset{\sim}{x}'|\underset{\sim}{z},t),f_0(\underset{\sim}{z},t'')\right\} \tag{7.3.6}$$

which gives $\underset{\sim}{j}_1$ thru

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = \int d^3x'\int_{-\infty}^{t} dt'\; \underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',t,t')\cdot\underset{\sim}{E}_1(\underset{\sim}{x}',t) \quad . \tag{7.3.7}$$

We have thus generalized previous work[10-12] in that we have found the conductivity of a <u>time-dependent</u> relativistic Vlasov plasma. However, the practical use of Eq. (7.3.6) depends on one being able to solve for the time evolution operator $M_0(t)$. Although $M_0(t)$ cannot be found in the general case, there do exist a few time-dependent systems for which $M_0(t)$ is known, such as a uniform plasma in the presence of a spatially-uniform but time-dependent electric field.[13]

We can prove a fluctuation-dissipation theorem from Eq. (7.3.6), if f_0 is a function of $\underset{\sim}{q},\underset{\sim}{p}$ only through h_0: $f_0(h_0(\underset{\sim}{q},\underset{\sim}{p},t))$. Of course, this implies that h_0 (and therefore f_0) must be time independent, if we are to satisfy the unperturbed equation $\partial f_0/\partial t + \{f_0,h_0\} = 0$. In this case Eq. (7.3.6) reduces to

$$\sigma(\underset{\sim}{x},\underset{\sim}{x}',t,t') = e^2\delta(\underset{\sim}{x}-\underset{\sim}{x}')\int d^3p\ f_0(h_0(\underset{\sim}{x},\underset{\sim}{p}))\gamma_0^{-1}(\underset{\sim}{x},\underset{\sim}{p})[\underset{\approx}{I}-\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})]$$

$$+\int_{t'-t}^{0} dt''\int d^6z\ \frac{df_0}{dh_0}\ (h_0(\underset{\sim}{x},\underset{\sim}{p}))\eta_0(\underset{\sim}{x}|\underset{\sim}{z})M_0(t'')$$

$$\eta_0(\underset{\sim}{x}'|\underset{\sim}{z}),h_0(\underset{\sim}{z}) \quad . \tag{7.3.7}$$

In finding this expression we have used Eq. (2.2.19), which is applicable when h_0 is time-independent. We note that $\underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',t,t') = \underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',t-t',0)$ depends only on the time difference $t-t'$, as must he true in a time-independent system. How we use Eq. (2.2.8) to obtain

$$\underset{\approx}{\sigma}(x,x',\tau,0) = e^2\delta(\underset{\sim}{x}-\underset{\sim}{x}')\int d^3p\ f_0(h_0(\underset{\sim}{x},\underset{\sim}{p}))\gamma_0^{-1}(\underset{\sim}{x},\underset{\sim}{p})[\underset{\approx}{I}-\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})]$$

$$+\int_{-\tau}^{0} dt''\int d^6z\ \frac{df_0}{dh_0}\ (h_0(\underset{\sim}{z}))\underset{\sim}{\eta}_0(\underset{\sim}{x}|\underset{\sim}{z})\ \frac{\partial M_0}{\partial t}\ (t'')\underset{\sim}{\eta}_0(\underset{\sim}{x}'|z) \quad . \tag{7.3.8}$$

Upon computing the time integral in the above expression we find

$$\underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',\tau,0) = e^2\delta(\underset{\sim}{x}-\underset{\sim}{x}')\int d^3p\ f_0(h_0(\underset{\sim}{x},\underset{\sim}{p}))\gamma_0^{-1}(\underset{\sim}{x},\underset{\sim}{p})[\underset{\approx}{I}-\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})]$$

$$+ e^2\delta(\underset{\sim}{x}-\underset{\sim}{x}')\int d^3p\ \frac{df_0}{dh_0}\ (h_0(\underset{\sim}{x},\underset{\sim}{p}))\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})\underset{\sim}{v}_0(\underset{\sim}{x},\underset{\sim}{p})$$

$$-\int d^6z\ \frac{df_0}{dh_0}\ (h_0(\underset{\sim}{z}))\eta_0(\underset{\sim}{x}|\underset{\sim}{z})M_0(-\tau)\underset{\sim}{\eta}_0(\underset{\sim}{x}|\underset{\sim}{z}) \quad . \tag{7.3.9}$$

The first two terms in this expression cancel. To see this we integrate the second term by parts,

$$\int d^3p \frac{df_0}{dh_0} \underset{\sim}{v}_0 \underset{\sim}{v}_0 = \int d^3p \frac{\partial f_0}{\partial p} \underset{\sim}{v}_0$$

$$= -\int d^3p \, f_0 \frac{\partial \underset{\sim}{v}_0}{\partial \underset{\sim}{p}} \quad , \tag{7.3.10}$$

and we note (from Eq. (7.1.14) that

$$\frac{\partial \underset{\sim}{v}_0}{\partial \underset{\sim}{p}} = \gamma_0^{-1} [\underset{\approx}{I} - \underset{\sim}{v}_0 \underset{\sim}{v}_0] \tag{7.3.11}$$

The cancellation of the first two terms in Eq. (7.3.9) follows immediately from Eqs. (7.2.10-11). Thus Eq. (7.3.9) reduces to

$$\underset{\approx}{\sigma}(\underset{\sim}{x},\underset{\sim}{x}',\tau,0) = -\int d^6z \frac{df_0}{dh_0} (h_0(\underset{\sim}{z}))\underset{\sim}{\eta}_0(\underset{\sim}{x}|\underset{\sim}{z})M_0(-\tau)\underset{\sim}{\eta}_0(\underset{\sim}{x}'|\underset{\sim}{z}) \ . \tag{7.3.12}$$

This expression is a generalized Kubo relation. To obtain the usual form, one need only use thermal equilibrium: $df_0/dh_0 = -f_0/T$. We see that Eq. (7.3.12) relates $\underset{\approx}{\sigma}$ to the two point current correlation function weighted with the function $-df_0/dh_0$.

In conclusion, we have used Hamiltonian methods to derive the conductivity of a Vlasov plasma, a case where the standard Kubo procedure does not apply, since the perturbation changes the form of the current function. Even so, we are able to prove a fluctuation-dissipation theorem (7.3.12).

7.4 Relation between the Ponderomotive Hamiltonian and the Linear Susceptibility of Vlasov Plasma

The concept of ponderomotive potential has proven to be of great value in analyzing nonlinear wave propagation[14-19] and plasma confinement by rf oscillations.[7,20,21] The ponderomotive Hamiltonian is a generalized ponderomotive "potential," which allows one to include

kinetic effects in the slow evolution of f. In this section we show that the ponderomotive Hamiltonian $K_{2\nu 0}$ is simply related to the linear Vlasov susceptibility of a (possibly inhomogeneous and time-dependent) plasma. This relation is very powerful because it reduces the nonlinear calculation of the ponderomotive Hamiltonian to the linear calculation of the susceptibility.

A central assumption of this section is that we can transform away h_1 entirely when we compute the linear response of the plasma, i.e. $K_1 = 0$. In essence this means that there are no particles which are resonant with the linear perturbation under consideration. If we like, we may divide the particles into resonant particles and nonresonant particles. The remarks of this section apply only to the nonresonant particles.

We first discuss the calculation of the ponderomotive Hamiltonian. The second-order zeroth-harmonic Deprit equation is

$$K_{20} - \frac{1}{2}\left(\frac{\partial w_{20}}{\partial t} + \{w_{20}, h_0\}\right) = h_{20} + \frac{1}{2}\{w_1, h_1\} , \tag{7.4.1}$$

for $K_1 = 0$.

As we noted in section 7.2, we may split the quantities in the equation into second-order linear (λ) terms and second-order nonlinear (ν) terms, which satisfy

$$K_{2\nu 0} - \frac{1}{2}\left(\frac{\partial w_{2\nu 0}}{\partial t} + \{w_{2\nu 0}, h_0\}\right) = h_{2\nu 0} + \frac{1}{2}\{w_1, h_1\}_0 , \tag{7.4.2}$$

and

$$K_{2\lambda 0} - \frac{1}{2}\left(\frac{\partial w_{2\lambda 0}}{\partial t} + \{w_{2\lambda 0}, h_0\}\right) = h_{2\lambda 0} , \tag{7.4.3}$$

where $h_{2\nu}$ and $h_{2\lambda}$ are given by Eqs. (7.2.24) and (7.2.25).

In a similar fashion we find the linear charge-density response:

$$\rho_1(\underset{\sim}{x},t) = -\int d^6z\ (\underset{\sim}{x}|\underset{\sim}{z})\{w_1(\underset{\sim}{z},t),f_0(\underset{\sim}{z},t)\}\ . \tag{7.4.9}$$

Let us consider the bilinear quantity $\rho_1\Phi_1 - \underset{\sim}{j}_1\cdot\underset{\sim}{A}_1$. Integrating this quantity over all space gives

$$\int d^3x[\rho_1(\underset{\sim}{x},t)\Phi_1(\underset{\sim}{x},t) - \underset{\sim}{j}_1(\underset{\sim}{x},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)] =$$

$$-\int d^6z\left[\int d^3x[\Lambda(\underset{\sim}{x}|\underset{\sim}{z})\Phi_1(\underset{\sim}{x},t) - \underset{\sim}{\eta}_0(\underset{\sim}{x}|\underset{\sim}{z})\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)]\right]\{w_1(z,t),f_0(z,t)\}$$

$$-\int d^6z\int d^3x\ \underset{\sim}{\eta}_1(\underset{\sim}{x}|\underset{\sim}{z},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)\ f_0(\underset{\sim}{z},t)\quad . \tag{7.4.10}$$

By using Eqs. (7.1.21-27), we see that the spatial integrals on the right side of Eq. (7.4.10) are simply terms in the Hamiltonian:

$$\int d^3x[\Lambda(\underset{\sim}{x}|\underset{\sim}{z})\Phi_1(\underset{\sim}{x},t) - \underset{\sim}{\eta}_0(\underset{\sim}{x}|\underset{\sim}{z})\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)] = h_1(\underset{\sim}{z},t) \tag{7.4.11}$$

$$\int d^3x\ \underset{\sim}{\eta}_1(\underset{\sim}{x}|\underset{\sim}{z},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t) = -2h_{2\nu}(\underset{\sim}{z},t)\quad . \tag{7.4.12}$$

Thus we can write Eq. (7.4.10) in the shorter form:

$$\int d^3x[\rho_1(\underset{\sim}{x},t)\Phi_1(\underset{\sim}{x},t) - \underset{\sim}{j}_1(\underset{\sim}{x},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)] = \int d^6z\Big[-h_1(\underset{\sim}{z},t)$$

$$\{w_1(\underset{\sim}{z},t),f_0(\underset{\sim}{z},t)\} + 2h_{2\nu}(\underset{\sim}{z},t)f_0(\underset{\sim}{z},t)\Big] \tag{7.4.13}$$

The first term on the right hand side of the above equation can be integrated by parts.

$$-\int d^6z\ h_1(\underset{\sim}{z},t)\{w_1(\underset{\sim}{z},t),f_0(\underset{\sim}{z},t)\} = \int d^6z\ f_0(\underset{\sim}{z},t)\ \{w_1(\underset{\sim}{z},t),$$

$$h_1(\underset{\sim}{z},t)\} \tag{7.4.14}$$

Then the form of the entire integral on the right hand side of Eq. (7.4.10) may be changed by using the integration-variable transformation, $\underset{\sim}{z} \to M_0(t)\underset{\sim}{z}$.

$$\int d^6z\ f_0(\underset{\sim}{z},t)[\{w_1(\underset{\sim}{z},t),h_1(\underset{\sim}{z},t)\} + 2h_{2\nu}(\underset{\sim}{z},t)] =$$

$$\int d^6z\ M_0(t)f_0(\underset{\sim}{z},t)M_0(t)[\{w_1(\underset{\sim}{z},t),h_1(\underset{\sim}{z},t)\} + 2h_{2\nu}(\underset{\sim}{z},t)] \quad (7.4.15)$$

We next use the known evolution of f_o, $M_o(t)f_0(\underset{\sim}{z},t) = f_0(\underset{\sim}{z},0)$, and we use the expression (7.4.15) for the right side of Eq. (7.4.13) to obtain

$$\int d^3x[\rho_1(\underset{\sim}{x},t)\Phi_1(\underset{\sim}{x},t)-\underset{\sim}{j}_1(\underset{\sim}{x},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)] = \int d^6z\ f_0(\underset{\sim}{z},0)M_0(t)$$

$$\left[\{w_1(\underset{\sim}{z},t),h_1(\underset{\sim}{z},t)\} + 2h_{2\nu}(\underset{\sim}{z},t)\right] \quad (7.4.16)$$

Finally, we apply the averaging operator to the equation, and we use Eq. (7.4.7) to obtain

$$\int d^3x[\rho_1(\underset{\sim}{x},t)\Phi_1(\underset{\sim}{x},t)-\underset{\sim}{j}_1(\underset{\sim}{x},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)] = 2\int d^6z\ f_0(\underset{\sim}{z},0)M_0(t)K_{2\nu 0}(t), \quad (7.4.17)$$

and we again change the variable in the phase space integral, $\underset{\sim}{z} \to M_0^{-1}(t)\underset{\sim}{z}$. This gives[4]

$$\int d^3x[\rho_1(\underset{\sim}{x},t)\Phi_1(\underset{\sim}{x},t)-\underset{\sim}{j}_1(\underset{\sim}{x},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)] = 2\int d^6z\ f_0(\underset{\sim}{z},t)K_{2\nu 0}(t) \quad (7.4.18)$$

This relation is known as the K-χ theorem, since it relates the ponderomotive Hamiltonian $K_{2\nu 0}(t)$, on the right side of Eq. (7.4.18), to the linear response $\rho_1,\underset{\sim}{j}_1$, on the left side of Eq. (7.4.18); in a moment we will show how the linear response can be related to χ, the

plasma susceptibility. (We note that Johnston and Kaufman[22] have recently shown that this relation is one of a family of similar relations which connect K_n with the (n-1)th response.) This relation is very powerful, since if we know the linear response as a functional of the unperturbed distribution f_0, then we can functionally differentiate[23] both sides of Eq. (7.4.18) to obtain

$$K_{2\nu 0}(\underset{\sim}{z},t) = \frac{1}{2}\frac{\delta}{\delta f_0}\int d^3x[\rho_1(\underset{\sim}{x},t)\Phi_1(\underset{\sim}{x},t)-\underset{\sim}{j}_1(\underset{\sim}{x},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)] \ , \quad (7.4.19)$$

i.e., we can deduce the ponderomotive Hamiltonian from the linear response.

However, in order to compute the functional derivative of the right side of the above expression, one must know it for arbitrary f_0. (This is analogous to the fact that in order to compute the ordinary derivative of a function y(x) in some interval [a,b], one must know y(x) for all x in that interval.) Unfortunately, the linear response for arbitrary f_0 is not usually known. For example, consider the case $h_0 = \frac{1}{2}p^2$, i.e., unmagnetized plasma. The linear response for spatially homogeneous f_0 is published in many texts, e.g., ref. 24, but the class of arbitrary unperturbed distributions is much larger. This class contains all functions of the form

$$f_0(\underset{\sim}{q},\underset{\sim}{p},t) = g(\underset{\sim}{q}-\underset{\sim}{p}t,\underset{\sim}{p}) \ . \quad (7.4.20)$$

Still, we are able to exploit Eq. (7.4.18), if we also make some approximations. Since these approximations depend on the particular unperturbed state, we must defer discussing them to later chapters.

To prepare the way for later work, we specialize Eq. (7.4.18), first to time-independent systems, and then to systems which are also

spatially homogeneous. For convenience we use the radiation gauge, $\Phi_1 = 0$. In the time-independent case, the linear fields are taken to have a definite frequency.

$$\underset{\sim}{A}_1(\underset{\sim}{x},t) = -\frac{i}{\omega}\hat{\underset{\sim}{A}}_1(\underset{\sim}{x})e^{-i\omega t} + \text{c.c.} \tag{7.4.21}$$

$$\underset{\sim}{E}_1(\underset{\sim}{x},t) = \hat{\underset{\sim}{E}}_1(\underset{\sim}{x})e^{-i\omega t} + \text{c.c.} \tag{7.4.22}$$

The associated linear currents are found from the linear conductivity,

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = i\int d^3x'\ \underset{\approx}{\sigma}_a(\underset{\sim}{x},\underset{\sim}{x}',\omega)\cdot\hat{\underset{\sim}{E}}_1(\underset{\sim}{x}')e^{-i\omega t} + \text{c.c.} \tag{7.4.23}$$

We use only the antihermitian part of since our remarks are relevant to nonresonant particles only. We insert this expression into Eq. (7.4.18), and we use the averaging operator to eliminate the second-harmonic terms. The result is

$$-\frac{i}{\omega}\int d^3x d^3x'\hat{\underset{\sim}{E}}^*_1(\underset{\sim}{x})\cdot i\underset{\approx}{\sigma}_a(\underset{\sim}{x},\underset{\sim}{x}',\omega)\cdot\hat{\underset{\sim}{E}}_1(\underset{\sim}{x}') = \int d^6z\ f_0(\underset{\sim}{z})\ K_{2\nu 0}(\underset{\sim}{z})\ . \tag{7.4.24}$$

Of course, $\underset{\approx}{\sigma}$ is related to the susecptibility $\underset{\approx}{\chi}$ through $4\pi i\sigma/\omega = \underset{\approx}{\chi}$. Thus the above expression can be rewritten

$$-\frac{1}{4\pi}\int d^3x d^3x'\hat{\underset{\sim}{E}}^*_1(\underset{\sim}{x})\cdot\underset{\approx}{\chi}_{sh}(\underset{\sim}{x},\underset{\sim}{x}',\omega)\cdot\underset{\sim}{E}_1(\underset{\sim}{x}') = \int d^6z\ f_0(\underset{\sim}{z})\ K_{2\nu 0}(\underset{\sim}{z}). \tag{7.4.25}$$

Here we see that the ponderomotive Hamiltonian K is simply related to the linear susceptibility. Hence the name: K-χ theorem.

We further specialize to the case of a uniform background, $\underset{\approx}{\chi}(\underset{\sim}{x},\underset{\sim}{x}',\omega) = \underset{\approx}{\chi}(\underset{\sim}{x}-\underset{\sim}{x}',\omega)$, and nearly plane waves: $\hat{\underset{\sim}{E}}_1(\underset{\sim}{x}) = \hat{\underset{\sim}{\mathcal{E}}}_1(\underset{\sim}{x})\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})$, where $|\nabla\hat{\underset{\sim}{\mathcal{E}}}_1| \ll |\underset{\sim}{k}\hat{\underset{\sim}{\mathcal{E}}}_1|$. We find

$$-\frac{1}{4\pi}\int d^3x d^3x' \hat{\xi}_1^*(\underset{\sim}{x})\cdot\chi_h(\underset{\sim}{x}-\underset{\sim}{x}',\omega)\cdot\xi_1(\underset{\sim}{x}')e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}'-\underset{\sim}{x})}$$

$$=\int d^6z\ f_0(\underset{\sim}{z})\ K_{2\nu 0}(\underset{\sim}{z})\ . \tag{7.4.26}$$

Changing the integration variable on the left, $\underset{\sim}{x}' \to \underset{\sim}{s} \equiv \underset{\sim}{x}-\underset{\sim}{x}'$, gives

$$-\frac{1}{4\pi}\int d^3x \hat{\xi}_1^*(\underset{\sim}{x})\cdot\int d^3s\ \underset{\approx}{\chi}_h(\underset{\sim}{s},\omega)\cdot\xi_1(\underset{\sim}{x}-\underset{\sim}{s})e^{i\underset{\sim}{k}\cdot\underset{\sim}{s}}$$

$$=\int d^6z\ f_0(\underset{\sim}{z})K_{2\nu 0}(\underset{\sim}{z})\ . \tag{7.4.27}$$

We compute the left side of Eq. (7.4.27) by expanding in Fourier space (c.f. sec. 6.1). Keeping the first term we obtain

$$-\frac{1}{4\pi}\underset{\approx}{\chi}_h(\underset{\sim}{k},\omega):\int d^3x \hat{\xi}_1^*(\underset{\sim}{x})\hat{\xi}_1(\underset{\sim}{x}) = \int d^6z\ f_0(\underset{\sim}{z})K_{2\nu 0}(\underset{\sim}{z})\ , \tag{7.4.28}$$

which will be used in chapters 8 and 10 to determine $K_{2\nu 0}$.

In summary, we have introduced the concept of ponderomotive Hamiltonian, and we have proven a theorem relating it to the linear susceptibility of Vlasov plasma.

8. Nonlinear Wave Evolution in Unmagnetized Plasma

In this chapter we use the nonlinear wave formalism of section 6.3 and the Lie transform techniques outlined in section 7.2 to study the nonlinear evolution of high frequency waves in relativistic unmagnetized uniform plasma. (Sluijter and Montgomery[1] showed that a correct treatment must include relativistic effects.) Our primary goal is to derive the equation which describes the nonlinear evolution of the wave amplitude. Along the way we discuss many other nonlinear effects, such as quasistatic magnetic field generation, the self-consistent quasistatic density perburbation, and the meaning of wave energy and momentum. We follow the outlines of sections 6.3 and 7.2 and proceed order by order. In section 8.1 we discuss the first-order effect, the response of the plasma to a wave packet: $\underset{\sim}{E}_1(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t) + c.c.$ We calculate this response with Lie transforms; i.e., we use w_1 to transform away the first-order Hamiltonian. As a matter of course we determine how slowly $\underset{\sim}{\mathcal{E}}_1$ must vary in order to explicitly compute the integral (7.2.14) for w_1. The condition on $\underset{\sim}{\mathcal{E}}_1$ turns out to be that $\underset{\sim}{\mathcal{E}}_1$ must vary slowly along an orbit compared to the phase $(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)$, or $|(\underset{\sim}{v}_0 \cdot \nabla + \partial/\partial t)\underset{\sim}{\mathcal{E}}_1| \ll |(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)\underset{\sim}{\mathcal{E}}_1|$. This, of course, excludes resonant particles for which $\underset{\sim}{k} \cdot \underset{\sim}{p} - \omega$ vanishes. At the end of section 8.1 we let $\underset{\sim}{\mathcal{E}}_1$ be constant and we thereby obtain the usual expression for the linear susceptibility of relativistic unmagnetized Vlasov plasma.[2]

The second order calculation is split into two parts. The first part (section 8.2) is a discussion of the second-order zeroth-harmonic effects. We derive the ponderomotive Hamiltonian $K_{2\nu 0}$, and we discuss

its effects on the average particle motion. We also show that K_{2v0} reduces to the usual ponderomotive potential[3] upon taking appropriate limits. Then we determine the plasma response to the ponderomotive potential when the wave packet amplitude is nearly static. (The more general case does not appear to have simple analytical solutions.) Finally, we include the second-order zeroth-harmonic electric and magnetic fields. We include the effects of the second-order potential Φ_{20} in the calculation of the second-order density response, and we compute the wave generated magnetic field $\underset{\sim}{B}_{20}$.[4-11]

Since we spent some time in the earlier section 6.2 discussing wave energy and momentum of general media, we now investigate the meaning of these concepts in Vlasov plasma. Our method is to use Lie transforms to compute the second-order average momentum density. Our result is very simple. If there is no oscillation-center response, $F_{20} = 0$, then the wave momentum density equals the true momentum density. However, if the wave-packet is finite (so that $\underset{\sim}{\mathcal{E}}_1$ depends on $\underset{\sim}{x}$ and t), then the oscillation center response F_{20} to the ponderomotive Hamiltonian causes the true momentum density to differ from the wave momentum density.

The other part of the second-order calculation (section 8.3) is to compute the second-harmonic effects. This calculation proceeds in a straightforward manner. We compute the second-order nonlinear current, and use it to compute the second-order second-harmonic electromagnetic field. These second-order fields are important to the third-order calculation, since they appear in the third-order Hamiltonian.

Section 8.4 is a discussion of the third order effects. We compute the third-order nonlinear current $j_{3\nu 1}$, and we calculate (via the formulation of section 6.3) its effect on the nonlinear evolution of the first-harmonic amplitude $\underset{\sim}{\mathcal{E}}_1$. First we discuss Langmuir oscillations; we recover the usual nonlinear Schrödinger equation.[12] Then we discuss electromagnetic waves. Here we unify previous work in that our equation contains ponderomotive effects[13] and the nonlinearities[1,14-16] associated with plane waves. In addition, our equation includes the self-Faraday rotation caused by the self-generated magnetic field.

8.1 Linear Theory; The Lie Transform to First Order

Here we use Lie transforms to calculate the linear response of a relativistic unmagnetized plasma. In doing so we assume we can transform away the first-order Hamiltonian. As we shall see, this means that there are no resonant particles and that the wave amplitude varies slowly.

We begin by discussing the unperturbed system. Since the plasma is unmagnetized and uniform, we have $\underset{\sim}{A}_0 = 0$ and $\Phi_0 = 0$. We also choose units so that $m = c = 1$. Then Eq. (7.1.8) gives

$$h_0(\underset{\sim}{p}) = (1 + p^2)^{\frac{1}{2}} \quad . \tag{8.1.1}$$

We note that a uniform unperturbed distribution can be any function of the momenta, $f_0(\underset{\sim}{p})$. (Later we will require f_0 to isotropic.) The time development operator for h_0 is

$$M_0(t)\ g(\underset{\sim}{q},\underset{\sim}{p}) = (\underset{\sim}{q} + \underset{\sim}{v}t,\underset{\sim}{p}) \quad , \tag{8.1.2}$$

where, from Eq. (7.1.14),

$$\underset{\sim}{v}_0(\underset{\sim}{p}) = \underset{\sim}{p}/(1 + p^2)^{\frac{1}{2}} \quad . \tag{8.1.3}$$

We now impose a perturbation $\underset{\sim}{A}_1(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{A}}_1(\underset{\sim}{x},t)\exp(i\underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t) + c.c.$ (We use the radiation gauge: $\Phi_1 = 0$.) We find the response to $\underset{\sim}{A}_1$ by transforming away h_1, i.e., we use Eqs. (7.2.14-16). From (7.1.21) the first-order Hamiltonian is given by

$$h_1(\underset{\sim}{q},\underset{\sim}{p},t) = -\underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t)e^{i(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)} + c.c. \tag{8.1.4}$$

We insert this expression into Eq. (7.2.14) to find w_1:

$$w_1(\underset{\sim}{q},\underset{\sim}{p},t) = ee^{i(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)}\int^0 d\tau\, \underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q} + \underset{\sim}{v}_0\tau, t + \tau)$$

$$\times\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{v}_0-\omega)\tau} + \text{c.c.} \tag{8.1.5}$$

We calculate the above integral by repeated ingegrations by parts:

$$w_1(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)}}{i(\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega)} \sum_{n=0}^{\infty} \left[\frac{i}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega} \left(\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}}\right)\right]^n$$

$$\times\, \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t) + \text{c.c.} \tag{8.1.6}$$

The above expression is not very useful unless we can approximate the infinite sum by its first few terms. To do this, the approximation

$$\left|\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}}\right) \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1\right| (\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1| \tag{8.1.7}$$

must hold. (For the special case $\underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t) = \underset{\sim}{\mathcal{A}}_0 \exp(\nu t)$, the infinite sum in (8.1.6) converges only when the inequality $|\nu| < |\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0|$ is satisfied.) Thus, we see that in order to transform away h_1, $\underset{\sim}{\mathcal{A}}_1(\underset{\sim}{x},t)$ must be slowly varying, by which we mean (8.1.7) is satisfied. In particular we must exclude resonant particles, for which $\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0 \cong 0$. Keeping only the first term in Eq. (8.1.6) we obtain

$$w_1(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{-ie\underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t)e^{i(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)}}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0(\underset{\sim}{p}) - \omega} + \text{c.c.} \tag{8.1.8}$$

Suppose instead we consider plane waves, for which the expression (8.1.6) reduces to (8.1.8) exactly. Now we ask what conditions are required for the full Lie transform series (including all the w_n's) to converge. We expect convergence for untrapped particles, since Dewar[17] has shown that the Lie transform for untrapped particles in a longitudinal wave is analytic in e. A nonrelativistic particle is untrapped if the inequality,

$$\frac{1}{2}(\hat{k} \cdot \underset{\sim}{p} - \omega/k)^2 > \frac{1}{2} |e\underset{\sim}{\mathcal{A}}_1|/(\omega k) \quad , \tag{8.1.9}$$

holds, i.e., the kinetic energy of the particle in the wave frame must be greater than the wave amplitude. This condition also excludes resonant particles.

Since we want to be able to transform away the first-order Hamiltonian, we will assume throughout this chapter and the next that the approximations (8.1.7) and (8.1.9) hold. Thus, we are restricted to discussing electromagnetic waves and Langmuir waves with large phase velocity. (Ion sound waves have resonant electrons and so they cannot be analyzed by these methods. However, it may be possible to treat resonant particles by some other technique.) For electromagnetic wave packets and Langmuir wave packets with large plasma velocity the following separate inequalities hold: The nearly monochromatic assumption,

$$|\partial\underset{\sim}{\mathcal{A}}_1/\partial t| << |\omega\underset{\sim}{\mathcal{A}}_1| \quad , \tag{8.1.10}$$

and large phase velocity compared to particle velocity,

$$|\underset{\sim}{k} \cdot \underset{\sim}{v}_0/\omega| << 1 \tag{8.1.11}$$

and

$$|\underset{\sim}{v}_0 \cdot \nabla \underset{\sim}{\mathcal{A}}_1| \ll |\omega \underset{\sim}{\mathcal{A}}_1| \quad . \tag{8.1.12}$$

Together these inequalities imply (8.1.7).

We now use w_1 to calculate the linear response. First we find f_1 from Eq. (7.2.10).

$$f_1 = - \frac{e[(\underset{\sim}{k} - i\nabla)\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1]}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega} \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)} + \text{c.c.} \tag{8.1.13}$$

For shorthand we introduce the notation $\psi \equiv \underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t$ (or $\equiv \underset{\sim}{k} \cdot \underset{\sim}{q} - \omega t$ depending on context). Finally, we insert f_1 into Eq. (7.2.17) to find the linear current,

$$\underset{\sim}{j}_1 = \{[-e^2 \int d^3p f_0 \quad \frac{(I - \underset{\sim}{v}_0 \underset{\sim}{v}_0)}{(1+p^2)^{\frac{1}{2}}} - e^2 \int d^3p \quad \frac{\underset{\sim}{v}_0 \underset{\sim}{v}_0}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega} \quad \frac{\partial f_0}{\partial \underset{\sim}{p}}$$

$$\times (\underset{\sim}{k} - i\nabla)] \cdot \underset{\sim}{\mathcal{A}}_1\}e^{i\psi} + \text{c.c.} \tag{8.1.14}$$

To obtain the linear conductivity $\underset{\approx}{\sigma}(\underset{\sim}{k},\omega)$, we simply specialize to the case of plane waves. In this case $\underset{\sim}{\mathcal{A}}_1$ is independent of space and time and the electric field is

$$\underset{\sim}{E}_1(\underset{\sim}{x},t) \equiv \underset{\sim}{\mathcal{E}}_1 \; e^{i\psi} + \text{c.c.}$$

$$= i\omega \underset{\sim}{\mathcal{A}}_1 \; e^{i\psi} + \text{c.c.} \tag{8.1.15}$$

Hence, from (8.1.14) we deduce

$$\underset{\approx}{\sigma}(\underset{\sim}{k},\omega) = \frac{ie^2}{\omega}\int d^3p\left[\frac{(I - \underset{\sim}{v}_0\,\underset{\sim}{v}_0)f_0}{(1+p^2)^{\frac{1}{2}}} + \frac{\underset{\sim}{v}_0\,\underset{\sim}{v}_0}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega}\,\underset{\sim}{k}\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}}\right] \quad . \quad (8.1.16)$$

From σ we obtain the susceptibility $\underset{\approx}{\chi} \equiv 4\pi i\underset{\approx}{\sigma}/\omega$,

$$\underset{\approx}{\chi}(\underset{\sim}{k},\omega) = \frac{-4\pi e^2}{\omega^2}\int d^3p\left[\frac{(I - \underset{\sim}{v}_0\,\underset{\sim}{v}_0)f_0}{(1+p^2)^{\frac{1}{2}}} + \frac{\underset{\sim}{v}_0\,\underset{\sim}{v}_0}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega}\,\underset{\sim}{k}\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}}\right] \quad , \quad (8.1.17)$$

of a relativistic unmagnetized plasma.[2] Since we have assumed a lack of resonant particles, we can integrate (8.1.17) by parts:

$$\underset{\approx}{\chi}(\underset{\sim}{k},\omega) = \frac{-4\pi e^2}{\omega^2}\int d^3p\; f\,(\underset{\sim}{p})\left[\frac{I - \underset{\sim}{v}_0\,\underset{\sim}{v}_0}{(1+p^2)^{\frac{1}{2}}} - \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{p}}\,\frac{\underset{\sim}{v}_0\,\underset{\sim}{v}_0}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega}\right] \tag{8.1.18}$$

$$= \frac{-4\pi e^2}{\omega^2}\int d^3p\;\frac{f_0(\underset{\sim}{p})}{(1+p^2)^{\frac{1}{2}}}\left[I - \frac{\underset{\sim}{k}\,\underset{\sim}{v}_0 + \underset{\sim}{v}_0\,\underset{\sim}{k}}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega} + \frac{\underset{\sim}{v}_0\,\underset{\sim}{v}_0(k^2 - \omega^2)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega)^2}\right] .$$

Calculating the nonrelativistic limit ($v/c \to 0$) of (8.1.18) is a delicate process because the limiting value of kc/ω depends on the perturbation under consideration. For example, $kc/\omega = O(1)$ for electromagnetic waves satisfying $k \gtrsim \omega_e/c$ ($\omega_e^2 \equiv 4\pi n_0 e^2/m_e$ is the electron plasma frequency), but $kc/\omega \ll 1$ for electromagnetic waves satisfying $k \ll \omega_e/c$. To illustrate this point, we first restore c in the expression (8.1.18):

$$\underset{\approx}{\chi}(\underset{\sim}{k},\omega) = \frac{-4\pi e^2}{\omega^2}\int d^3p \; \frac{f_0}{(1 + p^2/c^2)^{\frac{1}{2}}}$$

$$\times \left[I - \frac{\underset{\sim}{k}\,\underset{\sim}{v}_0 + \underset{\sim}{v}_0\,\underset{\sim}{k}}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega} + \frac{\underset{\sim}{v}_0\,\underset{\sim}{v}_0(k^2 - \omega^2/c^2)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega)^2}\right] . \qquad (8.1.19)$$

One type of nonrelativistic limit we can take is to simply let c^{-1} vanish (and therefore $\underset{\sim}{v}_0 \to \underset{\sim}{p}$). This gives

$$\underset{\approx}{\chi}_1(\underset{\sim}{k},\omega) = \frac{-4\pi e^2}{\omega^2}\int d^3p \; f_0\left[I - \frac{\underset{\sim}{k}\,\underset{\sim}{p} + \underset{\sim}{p}\,\underset{\sim}{k}}{\underset{\sim}{k}\cdot\underset{\sim}{p} - \omega} + \frac{k^2\underset{\sim}{p}\,\underset{\sim}{p}}{(\underset{\sim}{k}\cdot\underset{\sim}{p} - \omega)^2}\right] . \qquad (8.1.20)$$

We call this the nonrelativistic limit of type 1. This is the nonrelativistic limit usually seen.[18] However, we have kept terms of order k^2p^2/ω^2 in the above expression. Thus, we should also keep terms of order p^2/c^2 when considering electromagnetic waves satisfying $k \gtrsim \omega_e/c$. This prescription gives

$$\underset{\approx}{\chi}_2 = \frac{-4\pi e^2}{\omega^2}\int d^3p \; f_0\left[\underset{\approx}{I}(1 - \tfrac{1}{2}\,p^2/c^2) + \frac{\underset{\sim}{k}\,\underset{\sim}{p} + \underset{\sim}{p}\,\underset{\sim}{k}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{p}}\right.$$

$$\left. + \frac{\underset{\sim}{p}\,\underset{\sim}{p}(k^2 - \omega^2/c^2)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{p})^2}\right] . \qquad (8.1.21)$$

We call this the nonrelativistic limit of type 2. For later reference we take the limit $\underset{\sim}{k}\cdot\underset{\sim}{p}/\omega \ll 1$ limit of Eq. (8.1.21) for isotropic f_0. We find

$$\underset{\approx}{\chi}_2 \approx -\frac{\omega_p^2}{\omega^2}\left[I\left(1 + k^2 v_T^2/\omega^2 - \frac{5}{2} v_T^2/c^2\right) + 2\,\underset{\approx}{k}\,\underset{\approx}{k}\, v_T^2/\omega^2\right] \quad , \qquad (8.1.22)$$

where

$$v_T^2 \equiv \frac{1}{3} n_0^{-1} \int d^3p\; f_0 p^2 \quad . \qquad (8.1.23)$$

In summary, we have used Lie transforms to find the linear response of a uniform relativistic unmagnetized Vlasov plasma.

8.2 Second-Order Zeroth-Harmonic Effects

The second-order zeroth-harmonic effects are those effects which vary on the temporal and spatial scales of the wave packet. These effects deserve study for two reasons. The first reason is that the second-order quantities must be known before the third-order calculation, which determines the nonlinear evolution of the wave packet, can be done. The other reason is that these effects are important by themselves. For example, the ponderomotive potential[3] may help confine plasma for fusion application. For another example, the wave generated quasistatic magnetic field[4-11] may affect particle transport[19] in the laser-fusion scheme.

We divide this section into three subsections. In the first we derive the equations which describe oscillation center evolution. This includes deriving and discussing the ponderomotive Hamiltonian for an unmagnetized particle. In the second subsection, we find the solution to the oscillation center equations in the case where the wave packet amplitude is nearly static. We find explicit expressions for the density perturbation and the quasistatic magnetic field produced by the wave. In the last subsection, 8.2c, we show that wave momentum has a simple interpretation in the oscillation-center formalism.

8.2a Second-Order Zeroth-Harmonic Equations: The Ponderomotive Hamiltonian and Oscillation-Center Evolution

In this subsection we derive a set of equations which govern the zeroth-harmonic quantities. We begin by deriving the ponderomotive Hamiltonian for an unmagnetized particle. The ponderomotive Hamintonian and the second-order zeroth-harmonic electromagnetic field determine the evolution of the oscillation-center distribution F via Liouville's equation. Then we transform F to obtain f, the true distribution. Finally, we show how to compute the second-order zeroth-harmonic current density and charge density, which determine the evolution of the electromagnetic field. Thus, we obtain a set of equations which, given the wave amplitude, determine the evolution of the second-order zeroth-harmonic quantities.

We derive the ponderomotive Hamiltonian K_{200} in two ways. First, we use the $K - \chi$ theorem of section 7.4. Unfortunately, the approximations involved in finding $K_{2\nu 0}$ are not apparent in this derivation. For this reason we give another derivation using ordinary Deprit perturbation theory.

For a wave packet,

$$\underset{\sim}{E}_1(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)e^{ix} + c.c. \quad , \tag{8.2.1}$$

Eqs. (7.4.28) and (8.1.18) combine to give the following relation,

$$\int d^3q\, d^3p\, f_0(\underset{\sim}{p}) K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2}{\omega^2}\int d^3x \int d^3p\, f_0(\underset{\sim}{p}) \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t)$$

$$\cdot \left[\frac{I - \underset{\sim}{v}_0\, \underset{\sim}{v}_0}{(1+p^2)^{\frac{1}{2}}} - \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}} \frac{\underset{\sim}{v}_0\, \underset{\sim}{v}_0}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega}\right] \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) + O(\nabla, \partial/\partial t) \quad , \quad (8.2.2)$$

involving the ponderomotive Hamiltonian for a relativistic unmagnetized particle. We functionally differentiate this equation with respect to $f_0(\underset{\sim}{p})$ to obtain

$$\int d^3q\, K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2}{\omega^2}\int d^3x\, \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t)$$

$$\times \left[\frac{I - \underset{\sim}{v}_0\, \underset{\sim}{v}_0}{(1+p^2)^{\frac{1}{2}}} - \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}} \frac{\underset{\sim}{v}_0\, \underset{\sim}{v}_0}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega}\right] \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) + O(\nabla, \partial/\partial t). \quad (8.2.3)$$

In the above expression we have two quantities with equal spatial integrals. The difference of these two quantities is therefore a function whose integral vanishes, i.e., a derivative. Thus, we deduce the formula,

$$K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2}{\omega^2}\left[\frac{|\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{q},t)|^2 - |\underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{q},t)|^2}{(1+p^2)^{\frac{1}{2}}}\right.$$

$$\left. - \underset{\sim}{k} \cdot \frac{\partial}{\partial p} \frac{|\underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{q},t)|^2}{k \cdot v\,(p) - \omega}\right] + O(\partial/\partial \underset{\sim}{q},\ \partial/\partial t) \quad , \quad (8.2.4)$$

for the ponderomotive Hamiltonian. We see that the K-X theorem provides a three-line derivation of the ponderomotive Hamiltonian. Unfortunately,

this derivation does not tell us what small parameter the error $O(\partial/\partial \underset{\sim}{q}, \partial/\partial t)$ represents.

To determine these errors, we must find $K_{2\nu 0}$ from $h_{2\nu 0}$, w_1, and h_1 using standard Deprit perturbation theory, i.e., Eq. (7.2.56). As usual, we analyze (7.2.56) by letting $K_{2\nu 0}$ equal the slowly varying terms on the right side of that equation, and we integrate the rapidly varying terms along a trajectory to find $w_{2\nu 0}$. In this case there are only zeroth-harmonic terms, which vary slowly, on the right side of Eq. (7.2.56). So we choose $w_{2\nu 0}$ to vanish, and $K_{2\nu 0}$ is given by

$$K_{2\nu 0} = h_{2\nu 0} + \frac{1}{2}\{w_1, h_1\}_0 \quad . \tag{8.2.5}$$

The first term in $K_{2\nu 0}$ is easily computed from Eq. (7.2.36):

$$h_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = e^2[|\underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t)|^2 - |\underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t)|^2]/(1 + p^2)^{\frac{1}{2}}. \tag{8.2.6}$$

To find the second term, we take w_1 from Eq. (8.1.6),

$$w_1 = \left[\frac{e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1}{i(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)} + \left(\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}}\right)\frac{e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1}{(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^2}\right]e^{i\psi} + \text{c.c.} \tag{8.2.7}$$

we take h_1 from (8.1.4), and we compute the zeroth harmonic terms in their Poisson bracket:

$$\{w_1,h_1\}_0 = -2e^2\, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}} \frac{|\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1|^2}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega}$$

$$+ e^2\{i \frac{\partial}{\partial \underset{\sim}{q}} \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1 \cdot [(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-1} \frac{\partial}{\partial \underset{\sim}{p}} \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1^*$$

$$+ \frac{\partial}{\partial \underset{\sim}{p}} (\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1^*(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-1})] + \text{c.c.}\}$$

$$+ e^2\{i(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-2} [(\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}})(\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1)]\underset{\sim}{k}$$

$$\cdot \frac{\partial}{\partial \underset{\sim}{p}}(\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1^*) - i\, \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1^*\, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}} [(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-2}$$

$$\times (\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}})\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1] + \text{c.c.}\} \quad . \qquad (8.2.8)$$

Inserting these results into Eq. (8.2.5) gives

$$K_{2v0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2(|\underset{\sim}{a}_1|^2 - |\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1|^2)}{(1+p^2)^{\frac{1}{2}}} - \underset{\sim}{e}^2 k \cdot \frac{\partial}{\partial \underset{\sim}{p}} \frac{|\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1|^2}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega}$$

$$+ \frac{e^2}{2} \{ i \frac{\partial}{\partial q} \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1 [(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-1} \frac{\partial}{\partial p} \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1^*$$

$$+ \frac{\partial}{\partial p}(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1^* (\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-1})] + c.c.\}$$

$$+ \frac{e^2}{2} \{ i(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-2} [(\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}}) \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1] \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}} \underset{\sim}{v}_0$$

$$\cdot \underset{\sim}{a}_1^* - i \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1^* k \cdot \frac{\partial}{\partial \underset{\sim}{p}} [(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^{-2}$$

$$\times (\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}}) \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1] + c.c.\} \quad . \tag{8.2.9}$$

We use $\underset{\sim}{\mathcal{E}}_1 = i\omega \underset{\sim}{a}_1 - \partial \underset{\sim}{a}_1/\partial t$ (or $\underset{\sim}{\mathcal{E}}_1 = \underset{\sim}{a}_1/i\omega + O(\partial/\partial\omega t)$) to compare (8.2.9) with (8.2.4). We see that the K - X relation does indeed give K_{2v0} correctly to $O(\nabla, \partial/\partial t)$. However, in (8.2.9) we have the correction terms and so we can estimate the errors. Examining (8.2.9) we see that neglecting the gradient and time derivative terms means we are approximating

$$|\frac{\partial}{\partial t} \underset{\sim}{a}_1| << |\omega \underset{\sim}{a}_1| \tag{8.2.10}$$

$$|(\frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}}) \underset{\sim}{a}_1| << |(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega) \underset{\sim}{a}_1| \tag{8.2.11}$$

and $$|\frac{\partial}{\partial \underset{\sim}{q}} \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1| << |\underset{\sim}{k} \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1| \quad . \tag{8.2.12}$$

The first approximation (8.2.10) is invoked when we use $\mathcal{E}_1 = \mathcal{A}_1/i\omega$, the second (8.2.11) in invoked when we neglect the last term in (8.2.9), and the last (8.2.12) is invoked when we neglect the next to last term in (8.2.9).

The ponderomotive Hamiltonian K_{2v0} is the kinetic generalization of ponderomotive potential. To obtain the ponderomotive potential Ψ from K_{2v0}, we simply take limit $\underset{\sim}{p} \to 0$ in the expression (8.2.9). This procedure gives

$$\Psi \equiv K_{2v0}(\underset{\sim}{q},\underset{\sim}{p}=0,t) = e^2|\underset{\sim}{\mathcal{A}}_1|^2 = \frac{e^2|\underset{\sim}{\mathcal{E}}_1|^2}{m\omega^2} \quad , \tag{8.2.13}$$

upon restoring ordinary units ($m \neq 1$). the above expression is the usual result for the ponderomotive potential of an unmagnetized particle. Thus, we see that the usual ponderomotive potential is correct for particles with small momentum, i.e., $\underset{\sim}{k} \cdot \underset{\sim}{v}_0(\underset{\sim}{p}) << \omega$, $|\underset{\sim}{v}_0| << c$, and $|\underset{\sim}{v}_0 \cdot \nabla \underset{\sim}{\mathcal{A}}_1| << |\omega \underset{\sim}{\mathcal{A}}_1|$.

To compare with previous work, we evaluate K_{2v0} to lowest order in $(\nabla,\partial/\partial t)$ in the nonrelativistic limit (of type 1, $\omega/kc \to 0$) when $\underset{\sim}{\mathcal{E}}_1$ is parallel to $\underset{\sim}{k}$. We find

$$K_{2v0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2|\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{q},t)|^2}{m(\omega - \underset{\sim}{k} \cdot \underset{\sim}{p}/m)^2} \tag{8.2.14}$$

upon restoring ordinary units ($m \neq 1$). An expression similar to (8.2.14) has previously appeared in the literature,[20] but in that work (8.2.14) was interpreted as a velocity dependent potential, which determined the average particle acceleration via

$$\dot{\underset{\sim}{v}} = -\nabla \frac{e^2|\mathcal{E}_1|^2}{m^2(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2} \quad . \tag{8.2.15}$$

However, this cannot be right since it violates Liouville's theorem in that the divergence of the flow,

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot (\dot{\underset{\sim}{x}}) + \frac{\partial}{\partial \underset{\sim}{v}} \cdot (\dot{\underset{\sim}{v}}) = 0 + 2\underset{\sim}{k} \cdot \nabla \frac{e^2|\mathcal{E}_1|^2}{m^2(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^3} \quad , \tag{8.2.16}$$

does not vanish.

We must also allow for second-order fields $\underset{\sim}{A}_{20}(\underset{\sim}{q},t)$ and $\Phi_{20}(\underset{\sim}{q},t)$, which appear in Eq. (7.2.52). To analyze that equation we note that the right-hand side is slowly varying. Thus, $w_{2\lambda 0} = 0$, and

$$K_{2\lambda 0} = h_{2\lambda 0} = -e\underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{A}_{20} + e\Phi_{20} \quad . \tag{8.2.17}$$

We combine $K_{2\lambda 0}$ and $K_{2\nu 0}$ to obtain the total second-order oscillation center Hamiltonian,

$$K_{20} = K_{2\lambda 0} + K_{2\nu 0} \quad . \tag{8.2.18}$$

By combining K_{20} with h_0, we obtain the total oscillation-center Hamiltonian

$$K = h_0 + K_{20} \quad . \tag{8.2.19}$$

The oscillation-center Hamiltonian determines the evolution of F through Liouville's equation,[21,22]

$$\frac{\partial F}{\partial t} + \{F,K\} = 0 \quad . \tag{8.2.20}$$

Actually, to preserve the ordering scheme, we should "linearize" this equation. That is, we should use

$$F = f_0 + F_{20} \quad , \tag{8.2.21}$$

where

$$\frac{\partial F_{20}}{\partial t} + \{F_{20}, h_0\} + \{f_0, K_{20}\} = 0 \quad . \tag{8.2.22}$$

Of course, it is always permissible to use the full nonlinear equation (8.2.20), but we must keep in mind that our result for F is valid only to second order, since K has been computed only to second order.

We digress for a moment to compare this formalism to the formalism of Aamodt and Vella.[23] Both formalisms must ultimately agree since they both use systematic perturbation theory. However, the ultimate equation to be solved in their formalism is a velocity space diffusion equation. In this formalism the equation we must solve is Liouville's equation.

It is interesting to note that the pondermotive Hamiltonian of a plane wave ($\partial \underset{\sim}{\mathcal{A}}_1/\partial \underset{\sim}{q} = 0$) does not affect F, since there is no ponderomotive force if the wave amplitude has no gradient. To prove this, we note that when K is independent of $\underset{\sim}{q}$, any time independent function $F(\underset{\sim}{p})$ of the momentum is a solution of (8.2.20). If the system evolves in time from zero wave amplitude and $F(t \to -\infty) = f_0(\underset{\sim}{p})$, then since F is time independent, $F = f_0$ for all time.

It is important to keep in mind that the oscillation-center distribution does <u>not</u> equal the time average of the distribution f. To compute f we must perform the inverse transform (7.2.3). Using the equations (4.1.15a-c) for T, the expression (8.2.22) for F, and the fact that $w_{20} = 0$, we obtain the zeroth-harmonic (or time-averaged) second order part of f:

$$f_{20} = \frac{1}{2}\{w_1,\{w_1,f_0\}\}_0 + F_{20} \quad . \tag{8.2.23}$$

Thus, we see that the oscillation-center distribution and the time averaged true distribution differ by the term $\frac{1}{2}\{w_1,\{w_1,f_0\}\}_0$. Using the expression (8.1.8) for w_1, we find that this term is given by

$$\frac{1}{2}\{w_1,\{w_1,f_0\}\}_0 = \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}}\left[\frac{e^2|\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1|^2}{(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^2}\,\underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}}\right] \tag{8.2.24}$$

to lowest order in $(\nabla,\partial/\partial t)$.

To complete this discussion we must show how to compute the second-order zeroth-harmonic charge density and current density. These quantities determine the second-order zeroth-harmonic electromagnetic fields via Maxwell's equations (6.3.42), and the electromagnetic fields, upon choosing a gauge, determine $\underset{\sim}{A}_{20}$ and Φ_{20}.

The second-order zeroth-harmonic charge density is determined from (7.1.31). Using (8.2.23) we find

$$\rho_{20}(\underset{\sim}{x},t) = e\int d^3p\ [\frac{1}{2}\{w_1,\{w_1,f_0\}\}_0|_{\underset{\sim}{x},\underset{\sim}{p},t} + F_{20}(\underset{\sim}{x},\underset{\sim}{p},t)] \quad . \tag{8.2.25}$$

We note that the first term (8.2.24) in the above integrand is a momentum derivative to lowest order in $(\nabla,\partial/\partial t)$, and so its integral vanishes. Thus we have

$$\rho_{20}(\underset{\sim}{x},t) = e\int d^3p\ F_{20}(\underset{\sim}{x},\underset{\sim}{p},t) \tag{8.2.26}$$

to lowest order in $(\nabla,\partial/\partial t)$.

To find the current density we extract the second-order zeroth-harmonic part of Eq. (7.1.30):

$$\underset{\sim}{j}_{20} = \int d^6z(\underset{\sim}{\eta}_0 f_{20} + (\underset{\sim}{\eta}_1 f_1)_0 + \underset{\sim}{\eta}_{20} f_0). \tag{8.2.27}$$

Using Eqs. (7.1.24), (7.1.25), (7.1.27), and (8.1.13), we reduce the above expression to

$$\begin{aligned} \underset{\sim}{j}_{20}(\underset{\sim}{x},t) = e\int d^3p\{&\underset{\sim}{v}_0(\underset{\sim}{p})f_{20}(\underset{\sim}{x},\underset{\sim}{p},t) + [e^2\gamma_0^{-1}(\underset{\sim}{\mathcal{A}}_1 - \underset{\sim}{v}_0\,\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1) \\ &\times \frac{(\underset{\sim}{k} + i\nabla)(\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1^*)}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega}\cdot\frac{\partial f_0}{\partial\underset{\sim}{p}} + \text{c.c.}] + [-e\gamma_0^{-1}(\underset{\sim}{A}_{20} - \underset{\sim}{v}_0\,\underset{\sim}{v}_0\cdot\underset{\sim}{A}_{20}) \\ &- \frac{1}{2}e^2\gamma_0^2(2\underset{\sim}{v}_0|\underset{\sim}{\mathcal{A}}_1|^2 + 2\underset{\sim}{\mathcal{A}}_1\,\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1^* + 2\underset{\sim}{\mathcal{A}}_1^*\,\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1 \\ &- 6\,\underset{\sim}{v}_0|\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1|^2)]f_0\} \quad , \end{aligned} \tag{8.2.28}$$

where f_{20} is given by (8.2.23).

Thus, given the wave amplitude $\underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t)$, we have a complete set of equations for the second-order zeroth-harmonic quantities. The wave amplitude $\underset{\sim}{\mathcal{A}}_1$ determines the ponderomotive Hamiltonian via Eq. (8.2.9). The ponderomotive Hamiltonian and the second-order fields determine the

oscillation center Hamiltonian via Eqs. (8.2.17-20). The oscillation center Hamiltonian determines the evolution of the oscillation center distribution F via Eq. (8.2.20). The oscillation-center distribution determines the second-order Zeroth-harmonic charge density and current density via Eqs. (8.2.23-27). Finally, the charge density and current density determine the second-order zeroth-harmonic fields via Maxwell's equations (6.3.42).

The importance of these equations is that we have removed the rapid oscillations from the Vlasov equation to second order. As we shall see in the next section, the fact that Eq. (8.2.20) has only slowly varying terms will allow us to find its solution in special cases. Even if we must resort to numerical integration of these equations, we have improved matters. Since the original equations contain rapidly varying terms, much more computer time would be required to integrate them.

8.2b Quasistatic Oscillation-Center Evolution

The equations of the previous section have analytic solutions when the wave-packet evolution is quasistatic, by which we mean that we can neglect the term $\partial F/\partial t$ in Eq. (8.2.20). This approximation is valid when the majority of the particles can cross the wave packet before it changes much. The quasistatic solutions are exact (within the framework of oscillation center theory) when the wave envelope is static, a case of wide applicability. For example, in most rf heating schemes the wave is present for a time comparable to the plasma lifetime.

If we can neglect the term $\partial F/\partial t$, Eq. (8.2.20) has the solution,

$$F(\underset{\sim}{z},t) = g[K(\underset{\sim}{z},t)] \quad , \tag{8.2.29}$$

where g is an arbitrary function. We set the wave amplitude equal to zero in this equation and find

$$f_0(\underset{\sim}{z},t) = g(h_0(\underset{\sim}{p})) \quad . \tag{8.2.30}$$

Thus, the solution (8.2.29) can be valid only when the unperturbed system is isotropic. Actually, we are justified in keeping our solution $F = f_0 + F_{20} + \ldots$ only up to second order. Expanding (8.2.29), we find

$$F_{20}(\underset{\sim}{z},t) = \frac{df_0}{dh_0}(h_0(\underset{\sim}{p}))K_{20}(\underset{\sim}{z},t) \quad . \tag{8.2.31}$$

When f_0 is Maxwellian, this expression reduces to

$$F_{20}(\underset{\sim}{z},t) = -f_0(h_0(\underset{\sim}{p}))K_{20}(\underset{\sim}{z},t)/T \quad . \tag{8.2.32}$$

To determine the condition for validity of this result, we examine the "linearized" equation (8.2.22). In finding the solution (8.2.31), we have neglected the first term in Eq. (8.2.22). For this term to be small we must have $\partial F_{20}/\partial t \ll \{F_{20}, b_0\}$ or, using Eq. (8.2.31),

$$\left|\frac{\partial K_{20}}{\partial t}\right| \ll |\underset{\sim}{v}_0 \cdot \nabla K_{20}| \quad . \tag{8.2.33}$$

Since K_{20} varies on the scale of the wave packet, this approximation means that particles can traverse the wave packet before it changes much.

Of course, not all particles satisfy Eq. (8.2.32) unless $\partial K_{20}/\partial t$ strictly vanishes. There are always some particles with zero unperturbed velocity. However, we reason that if the relative number of such particles is small, then our final results will be approximately correct. This condition is

$$\left|\frac{\partial K_{20}}{\partial t}\right| \ll v_T |\nabla K_{20}| \quad , \tag{8.2.34}$$

where v_T is the thermal speed. This reasoning is analogous to the case where one finds the linear response of a Vlasov species to a perturbation of frequency ω and wave vector k. When $\omega/k \ll v_T$, the Boltzmann response is valid.[24]

At this point we choose a gauge. Since $v_T < c$ must hold, quasi-static evolution (8.2.34) is slow compared to the speed of light, and so the displacement current $\partial \underset{\sim}{E}_{20}/\partial t$ may be neglected in Maxwell's equations (6.4.32). In this case the transverse gauge, $\nabla \cdot \underset{\sim}{A}_{20} = 0$, is convenient because it reduces Maxwell's equations (6.4.32) to

$$-\nabla^2 \Phi_{20} = 4\pi\rho_{20} \quad ,$$

$$\nabla \cdot \underset{\sim}{B}_{20} = 0 \quad ,$$

and $$\nabla \times \underset{\sim}{B}_{20} = \frac{4\pi}{c} \underset{\sim}{j}_{20} \quad , \tag{8.2.35}$$

where $\underset{\sim}{A}_{20}$, if needed, is determined from

$$\nabla \cdot \underset{\sim}{A}_{20} = 0$$

and $$\nabla \times \underset{\sim}{A}_{20} = \underset{\sim}{B}_{20} \quad . \tag{8.2.36}$$

This gauge differs from the (radiation) gauge that was used for the linear fields, but this difference in gauges does not cause any difficulties, since the first order Maxwell equations were separated from the second order equations in the last section.

We proceed to solve for the charge density perturbation in terms of the potential Φ_{20}. Inserting the expression (8.2.31) into (8.2.26), we find

$$\rho_{20} = e \int d^3p \, \frac{df_0}{dh_0} K_{20} = e \int d^3p \, \frac{df_0}{dh_0} [K_{2\upsilon 0} + e\Phi_{20} - e\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20}] \quad .$$

The last term inside the brackets does not contribute to the integral, since $\underset{\sim}{v}_0$ is an odd function of $\underset{\sim}{p}$, and f_0 is an even function. Thus, restoring species labels, we find

$$\rho^s_{20}(x,t) = \rho^s_{2\upsilon 0}(\underset{\sim}{x},t) + \rho^s_{2\lambda 0}(\underset{\sim}{x},t) \quad , \tag{8.2.37}$$

where

$$\rho^s_{2\lambda 0}(\underset{\sim}{x},t) = e_s \int d^3p \, \frac{df^s_0}{dh^s_0} (h^s_0(\underset{\sim}{p}))K^s_{2\nu 0}(\underset{\sim}{x},\underset{\sim}{p},t) \tag{8.2.38}$$

is the second-order nonlinear charge density, and

$$\rho^s_{2\lambda 0}(\underset{\sim}{x},t) = e_s^2 \, \Phi_{20}(\underset{\sim}{x},t) \int d^3p \, \frac{df^s_0}{dh^s_0} (h^s_0(\underset{\sim}{p})) \tag{8.2.39}$$

is the second-order linear charge density.

The last expression suggests that we define a generalized temperature,

$$T_s \equiv - n^s_0 / \int d^3p \, \frac{df^s_0}{dh^s_0} \quad , \tag{8.2.40}$$

which is seen to reduce to the usual temperature when f^s_0 is Maxwellian. Using this generalized temperature one can define a generalized Debye length for species s:

$$\lambda_s^{-2} \equiv 4\pi \, n^s_0 e_s^2 / T_s = -4\pi e_s^2 \int d^3p \, \frac{df^s_0}{dh^s_0} \quad . \tag{8.2.41}$$

The plasma Debye length λ_D is found in the usual way:

$$\lambda_D^{-2} \equiv \sum_s \lambda_s^{-2} \quad . \tag{8.2.42}$$

To find Φ_{20} we use Poisson's equation,

$$-\nabla^2 \Phi_{20} = 4\pi \sum_s \rho^s_{20} \quad .$$

Inserting (8.2.37) into the above equation and using the definitions (8.2.38-40), we obtain

$$(\nabla^2 - \lambda_D^{-2})\Phi_{20}(\underset{\sim}{x},t) = -4\pi \sum_s \rho^s_{2\nu 0}(\underset{\sim}{x},t) = -4\pi\rho_{2\nu 0}(\underset{\sim}{x},t) \quad . \qquad (8.2.43)$$

This equation is easily solved by using the Green's function for the Yukawa equation:

$$\Phi_{20}(\underset{\sim}{x},t) = \int d^3x' \; \rho_{2\nu 0}(\underset{\sim}{x}',t) \; \exp(-|\underset{\sim}{x} - \underset{\sim}{x}'|/\lambda_D)/|\underset{\sim}{x} - \underset{\sim}{x}'| \quad . \qquad (8.2.44)$$

Thus, we have obtained Φ_{20} in terms of the first-harmonic amplitude, which is in $\rho_{2\nu 0}$, and, by combining the equations (8.2.37) and (8.2.44), we therefore know the charge density ρ^s_{20} in terms of the first-harmonic amplitude.

With certain assumptions, these equations simplify dramatically. When the unperturbed distribution is Maxwellian, the nonlinear second-order charge density (8.2.38) becomes

$$\rho^s_{2\nu 0}(\underset{\sim}{x},t) = -\frac{e_s}{T_s}\int d^3p \; f^s_0(h^s_0(\underset{\sim}{p}))\bar{K}^s_{2\nu 0}(\underset{\sim}{x},\underset{\sim}{p},t)$$

$$= \frac{e_s}{4\pi T_s}\, \underset{\sim}{\mathcal{E}}^*_1(\underset{\sim}{x},t) \cdot \chi^s(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \quad , \qquad (8.2.45)$$

upon using the K-χ theorem (8.2.3-4). When the Debye length is small compared to the wave amplitude scale length, (8.2.43) becomes

$$\Phi_{20}(\underset{\sim}{x},t) = 4\pi\lambda_D^2\, \rho_{2v0}(\underset{\sim}{x},t) \quad . \tag{8.2.46}$$

Combining these equations we find

$$\Phi_{20}(\underset{\sim}{x},t) = \lambda_D^2 \sum_s \frac{e_s}{T_s}\, \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \chi^s(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \quad , \tag{8.2.47}$$

which, together with (8.2.39,40) implies

$$\rho^s_{2\lambda 0}(\underset{\sim}{x},t) = -\frac{n_0^s e_s^2}{T_s}\, \lambda_D^2 \sum_{s'} \frac{e_{s'}}{T_{s'}}\, \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \chi^{s'}(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \; . \tag{8.2.48}$$

Inserting (8.2.45) and (8.2.48) into (8.2.37) we obtain

$$\rho^s_{20}(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \left[\frac{e_s}{4\pi T_s}\, \chi^2(\underset{\sim}{k},\omega) - \frac{n_0^s e_s^2}{T_s}\, \lambda_D^2 \sum_{s'} \frac{e_{s'}}{T_{s'}}\, \chi^{s'}(\underset{\sim}{k},\omega)\right]$$

$$\cdot\, \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k},\omega) \quad . \tag{8.2.49}$$

Thus, we have expressions for ρ^s_{20} and Φ_{20} which depend on the wave amplitude $\underset{\sim}{\mathcal{E}}_1$ and the unperturbed system; the K-χ theorem allowed us to eliminate the ponderomotive Hamiltonian.

With one additional assumption, this equation becomes even simpler. We assume that our plasma has only one ion species which is singly charged, $e_i = -e_e \equiv e_0$. Then Eq. (8.2.49) reduces to

$$n^i_{20} = n^e_{20} = \rho^i_{20}/e_0 = \frac{\underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \chi(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k},\omega)}{4\pi(T_e + T_i)} \quad , \tag{8.2.50a}$$

where $\chi \equiv \chi^e + \chi^i$. Use of the linear propagation equation,

$D(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) = 0$, and Faraday's law, $\underset{\sim}{k} \times \underset{\sim}{\mathcal{E}}_1 = i\omega\underset{\sim}{\mathcal{B}}_1/c$, allows us to rewrite (8.2.51) in the form

$$n_{20} \equiv n_{20}^{i} = -\frac{|\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)|^2 - |\underset{\sim}{\mathcal{B}}_1(\underset{\sim}{x},t)|^2}{4\pi(T_e + T_i)} \quad . \tag{8.2.50b}$$

This formula generalizes the work of reference 25, where cold plasma waves were considered.

For future reference, we specialize to the case of cold plasma waves ($\chi^s \to -\omega_s^2 \underset{\approx}{I}/\omega^2$) in a plasma with one species of singly charged ions with $m_i >> m_e$. Equation (8.2.50b) becomes

$$n_{20} = -n_0\left(\frac{e_0^2|\mathcal{E}_1|^2}{m_e\omega^2}\right)/(T_e + T_i) \quad , \tag{8.2.51a}$$

and Eq. (8.2.47) becomes

$$\Phi_{20} = \frac{e_0}{1 + T_e/T_i}\,\frac{|\mathcal{E}_1|^2}{m_e\omega^2} \quad . \tag{8.2.51b}$$

The remaining part of this problem is to determine the second-order zeroth-harmonic current density and magnetic field. To determine the current density we use the expressions (8.2.23), (8.2.31), and the fact that f_0 is isotropic to reduce (8.2.28) to the form,

$$\Phi_{20}(\underset{\sim}{x},t) = 4\pi\lambda_D^2\, \rho_{2\nu 0}(\underset{\sim}{x},t) \quad . \tag{8.2.46}$$

Combining these equations we find

$$\Phi_{20}(\underset{\sim}{x},t) = \lambda_D^2 \sum_s \frac{e_s}{T_s}\, \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \chi^s(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \quad , \tag{8.2.47}$$

which, together with (8.2.39,40) implies

$$\rho^s_{2\lambda 0}(\underset{\sim}{x},t) = -\frac{n_0^s e_s^2}{T_s}\, \lambda_D^2 \sum_{s'} \frac{e_{s'}}{T_{s'}}\, \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \chi^{s'}(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \quad . \tag{8.2.48}$$

Inserting (8.2.45) and (8.2.48) into (8.2.37) we obtain

$$\rho^s_{20}(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \left[\frac{e_s}{4\pi T_s}\, \chi^2(\underset{\sim}{k},\omega) - \frac{n_0^s e_s^2}{T_s}\, \lambda_D^2 \sum_{s'} \frac{e_{s'}}{T_{s'}}\, \chi^{s'}(\underset{\sim}{k},\omega)\right]$$

$$\cdot\, \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k},\omega) \quad . \tag{8.2.49}$$

Thus, we have expressions for ρ^s_{20} and Φ_{20} which depend on the wave amplitude $\underset{\sim}{\mathcal{E}}_1$ and the unperturbed system; the K-χ theorem allowed us to eliminate the ponderomotive Hamiltonian.

With one additional assumption, this equation becomes even simpler. We assume that our plasma has only one ion species which is singly charged, $e_i = -e_e \equiv e_0$. Then Eq. (8.2.49) reduces to

$$n^i_{20} = n^e_{20} = \rho^i_{20}/e_0 = \frac{\underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \cdot \chi(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{k},\omega)}{4\pi(T_e + T_i)} \quad , \tag{8.2.50a}$$

where $\chi \equiv \chi^e + \chi^i$. Use of the linear propagation equation,

$D(\underset{\sim}{k},\omega) \cdot \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) = 0$, and Faraday's law, $\underset{\sim}{k} \times \underset{\sim}{\mathcal{E}}_1 = i\omega\underset{\sim}{\mathcal{B}}_1/c$, allows us to rewrite (8.2.51) in the form

$$n_{20} \equiv n_{20}^{i} = -\frac{|\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)|^2 - |\underset{\sim}{\mathcal{B}}_1(\underset{\sim}{x},t)|^2}{4\pi(T_e + T_i)} \quad . \tag{8.2.50b}$$

This formula generalizes the work of reference 25, where cold plasma waves were considered.

For future reference, we specialize to the case of cold plasma waves ($\chi^s \to -\omega_s^2 \underset{\approx}{I}/\omega^2$) in a plasma with one species of singly charged ions with $m_i >> m_e$. Equation (8.2.50b) becomes

$$n_{20} = -n_0\left(\frac{e_0^2|\mathcal{E}_1|^2}{m_e\omega^2}\right)/(T_e + T_i) \quad , \tag{8.2.51a}$$

and Eq. (8.2.47) becomes

$$\Phi_{20} = \frac{e_0}{1 + T_e/T_i}\,\frac{|\mathcal{E}_1|^2}{m_e\omega^2} \quad . \tag{8.2.51b}$$

The remaining part of this problem is to determine the second-order zeroth-harmonic current density and magnetic field. To determine the current density we use the expressions (8.2.23), (8.2.31), and the fact that f_0 is isotropic to reduce (8.2.28) to the form,

$$\underset{\sim}{j}_{20} = e\int d^3p\ \{\underset{\sim}{v}_0 \frac{1}{2}\{w_1,\{w_1,f_0\}\}_0 + \underset{\sim}{v}_0 \frac{df_0}{dh_0}[K_{2v0} - e\,\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20}]$$

$$+ [e^2\gamma_0^{-1}(\underset{\sim}{a}_1 - \underset{\sim}{v}_0\,\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1)\frac{(\underset{\sim}{k} + i\nabla)(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1^*)}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega}\frac{\partial f_0}{\partial \underset{\sim}{p}} + \text{c.c.}]$$

$$- e\,\gamma_0^{-1}(\underset{\sim}{A}_{20} - \underset{\sim}{v}_0\,\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20})f_0\} \quad . \tag{8.2.52}$$

The terms involving $\underset{\sim}{A}_{20}$ cancel, since integration by parts gives

$$-\int d^3p\ \underset{\sim}{v}_0 \frac{df_0}{dh_0}\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20} = -\int d^3p \frac{df_0}{\partial \underset{\sim}{p}}\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20}$$

$$= \int d^3p\ f_0 \frac{\partial}{\partial \underset{\sim}{p}}\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20}$$

$$= \int d^3p\ f_0\ \gamma_0^{-1}(\underset{\sim}{A}_{20} - \underset{\sim}{v}_0\,\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20}) \ . \tag{8.2.53}$$

Hence, the expression (8.2.52) for j_{20} reduces to

$$\underset{\sim}{j}_{20}(\underset{\sim}{x},t) = e\int d^3p\Bigg\{\underset{\sim}{v}_0 \frac{df_0}{dh_0}K_{2v0}(\underset{\sim}{x},\underset{\sim}{p},t) + \underset{\sim}{v}_0 \frac{1}{2}\{w_1,\{w_1,f_0\}\}\Big|_{\underset{\sim}{x},\underset{\sim}{p},t}$$

$$+ \Bigg[e^2\gamma_0^{-1}(\underset{\sim}{a}_1 - \underset{\sim}{v}_0\,\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1)$$

$$\times \frac{(\underset{\sim}{k} + i\nabla)(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1^*)}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega}\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}}\Bigg]\Bigg|_{\underset{\sim}{x},\underset{\sim}{p},t}\Bigg\} \tag{8.2.54}$$

Since $\underset{\sim}{A}_{20}$ does not appear in this expression, the remaining steps are straightforward. We evaluate the expression (8.2.54), we sum the

contributions from the various species, and we insert the total current into (8.2.35) and solve for $\underset{\sim}{B}_{20}$.

Let us evaluate the expression (8.2.54) to zeroth order in the gradients. Using (8.2.4) and (8.2.24) we find (after some algebra and integrations by parts)

$$\int d^3p \left\{ \underset{\sim}{v}_0 \frac{1}{2} \{w_1, \{w_1, f_0\}\}\Big|_{\underset{\sim}{x},\underset{\sim}{p},t} + \left[e^2 \gamma_0 (\underset{\sim}{\mathcal{A}}_1 - \underset{\sim}{v}_0 \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1) \frac{\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1^*}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega} \right.\right.$$

$$\left.\left. \times \underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}} + \text{c.c.} \right] \right\} = \int d^3p \; f_0 \frac{\partial K_{2v0}}{\partial \underset{\sim}{p}} \quad . \tag{8.2.55}$$

Inserting this result into (8.2.54) and integrating by parts, we obtain

$$\underset{\sim}{j}_{20}(\underset{\sim}{x},t) = 0 \quad . \tag{8.2.56}$$

That is, the quasistatic current vanishes to zeroth order in ∇.

At first glance this result seems surprising, but in fact it is required by charge conservation. The structure of the theory is such that the calculation of $\underset{\sim}{j}_{20}$ must result in an expression of the form

$$[\underset{\sim}{j}_{2v0}(\underset{\sim}{x},t)]_i = \sum_{\substack{j=1 \\ k=1}}^{3} \alpha_{ijk}(\underset{\sim}{k},\omega,f_0) \underset{\sim}{\mathcal{A}}_j(\underset{\sim}{x},t) \overset{*}{\underset{\sim}{\mathcal{A}}}_k(\underset{\sim}{x},t) \quad , \tag{8.2.57}$$

where the tensor α is a function of $\underset{\sim}{k},\omega$, and the background state. In the quasistatic limit, charge conservation requires $\nabla \cdot \underset{\sim}{j} = 0$. Requiring this in Eq. (8.2.44) gives

$$\sum_{ijk} \alpha_{ijk}\, \partial(\mathcal{A}_j \mathcal{A}_k^*)/\partial x_i = 0 \qquad (8.2.58)$$

At this point we note that $\mathcal{A}(\underset{\sim}{x},t)$ is an arbitrary function. We are simply using Lie transforms to find the second-order Vlasov response to the field $\underset{\sim}{A}(\underset{\sim}{x},t) = \mathcal{A}_1(\underset{\sim}{x},t)\exp\,(i\underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t) + c.c.$ Thus, the tensor $\partial(\mathcal{A}_j\mathcal{A}_k^*)/\partial x_i$ is arbitrary and the only way Eq. (8.2.58) can always bold is if $\alpha_{ijk} = 0$. We conclude $\underset{\sim}{j}_{20} = O(\nabla)$ for a quasistatic response. In fact, $\nabla \cdot \underset{\sim}{j}_{20} = 0$ implies that $\underset{\sim}{j}_{20}$ is the curl of some vector field.

This fact does not preclude a quaistatic magnetic field $\underset{\sim}{B}_{20}$ which is zeroth order in the amplitude gradient. For a zeroth order (in $\nabla\underset{\sim}{\mathcal{A}}_1$) $\underset{\sim}{B}_{20}$, from $\nabla \times \underset{\sim}{B}_{20} = 4\pi\underset{\sim}{j}_{20}$ we see that $\underset{\sim}{j}_{20}$ is first order in $\nabla\underset{\sim}{\mathcal{A}}_1$. Hence, to calculate $\underset{\sim}{B}_{20}$, we must calculate $\underset{\sim}{j}_{20}$ to first order in $\nabla_{\sim 1}$.

To determine $\underset{\sim}{B}_{20}$we must repeat the calculation without invoking the approximation (8.2.12). However, it is still permissible to use (8.2.11), which is necessary to transform away h_1. For simplicity we also approximate $\underset{\sim}{k} \cdot \underset{\sim}{v}_0 \ll \omega$ (large phase velocity) and $p \ll 1$ (non-relativistic thermal speed). With these approximations, Eq. (8.1.6) gives

$$w_1(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{ie}{\omega}\, \underset{\sim}{p} \cdot \underset{\sim}{\mathcal{A}}\,(\underset{\sim}{q},t)e^{i(\underset{\sim}{k}\cdot\underset{\sim}{q}-\omega t)} + c.c. \quad , \qquad (8.2.59)$$

and Eq. (8.2.9) reduces to

$$K_{2v0}(\underset{\sim}{q},\underset{\sim}{p},t) = e^2|\underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t)|^2 -$$

$$\times\, (i\underset{\sim}{\mathcal{A}}_1^*(\underset{\sim}{q},t) \cdot \frac{\partial}{\partial \underset{\sim}{q}}\, \underset{\sim}{p} \cdot \underset{\sim}{\mathcal{A}}_1(\underset{\sim}{q},t) + c.c.) \qquad (8.2.60)$$

We insert these results into Eq. (8.2.54). Assuming f_0 to be isotropic we find

$$\underset{\sim}{j}_{20} = \frac{-ie^2}{\omega}\int d^3p\,\frac{df_0}{\partial \underset{\sim}{p}}\,[\underset{\sim}{a}_1^* \cdot \nabla_{\underset{\sim}{p}} \cdot \underset{\sim}{a}_1 - \underset{\sim}{a}_1 \cdot \nabla_{\underset{\sim}{p}} \cdot \underset{\sim}{a}_1^*]$$

$$- \frac{ie^2}{\omega}\int d^3p\,[\underset{\sim}{a}_1^* \frac{\partial f_0}{\partial \underset{\sim}{p}} \cdot \nabla_{\underset{\sim}{p}} \cdot \underset{\sim}{a}_1 - \underset{\sim}{a}_1 \frac{\partial f_0}{\partial \underset{\sim}{p}} \cdot \nabla_{\underset{\sim}{p}} \cdot \underset{\sim}{a}_1^*] \qquad (8.2.61)$$

in the nonrelativistic limit. This expression integrates by parts yielding

$$\underset{\sim}{j}_{20} = \frac{ie^3 n_0}{\omega}\,[\underset{\sim}{a}_1^* \cdot \nabla \underset{\sim}{a}_1 - \underset{\sim}{a}_1 \cdot \nabla \underset{\sim}{a}_1^* + \underset{\sim}{a}_1^* \nabla \cdot \underset{\sim}{a}_1 - \underset{\sim}{a}_1 \nabla \cdot \underset{\sim}{a}_1^*]$$

$$= - \frac{ie^3 n_0}{\omega}\,\nabla \times (\underset{\sim}{a}_1^* \times \underset{\sim}{a}_1) \quad . \qquad (8.2.62)$$

Using $\underset{\sim}{a}_1 = c\underset{\sim}{\mathcal{E}}_1/i\omega$ and restoring ordinary units and species labels we find

$$\underset{\sim}{j}^2_{20}(\underset{\sim}{x},t) = \frac{n_0 e_s^3}{m_s^2\omega^3}\,\nabla \times [-i\underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \times \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)] \quad . \qquad (8.2.63)$$

We see that only the electron current is important.

Keeping only the elctron current in Ampere's law (8.2.35), we find

$$\nabla \times \underset{\sim}{B}_{20} = \frac{4\pi n_0 e_0^3}{m_e^2\omega^3 c}\,\nabla \times (i\,\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1) \quad , \qquad (8.2.64)$$

where $-e_0$ is the charge of an electron. From this we deduce

$$\underset{\sim}{B}_{20} = \frac{4\pi n_0 e_0^3 \, i \, \underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1}{m_e^2 \, \omega^3 c} - \nabla V \quad , \tag{8.2.65}$$

where V is determined by $\nabla \cdot \underset{\sim}{B}_{20} = 0$:

$$\nabla^2 V = \frac{4\pi n_0 e_0^3 \, i \, \nabla \cdot (\underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{\mathcal{E}}_1)}{m_e^2 \omega^3 c} \quad . \tag{8.2.66}$$

The equation is easily solved by using the Green's function $-(4\pi)^{-1}|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1}$ for Poisson's equation.

$$V(x) = \frac{-i n_0 e_0^3}{m_e^2 \omega^3 c} \int d^3x' \, \frac{\nabla' \cdot [\underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x}',t) \times \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x}',t)]}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \tag{8.2.67}$$

Inserting this expression into Eq. (8.2.65) gives our final result:

$$\underset{\sim}{B}_{20}(\underset{\sim}{x},t) = \frac{i e_0 \omega_e^2}{m_e \omega^3 c} [\underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x},t) \times \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) - \int d^3x' \, \frac{\underset{\sim}{x} - \underset{\sim}{x}'}{4\pi|\underset{\sim}{x} - \underset{\sim}{x}'|^3} \frac{\partial}{\partial x'}$$

$$\cdot (\underset{\sim}{\mathcal{E}}_1^*(\underset{\sim}{x}',t) \times \underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x}',t))] \quad . \tag{8.2.68}$$

This expression for $\underset{\sim}{B}_{20}$ has been noted previously by Bezzerides, et al[9,10] for one-dimensional variation of the envelope $\underset{\sim}{\mathcal{E}}_1$ and by Mora and Pellat[11] in the case where $\nabla \cdot (\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1) = 0$. Of course, in general $\nabla \cdot (\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1) \neq 0$ is true and we must include the second term in (8.2.57). This gives a nonlocal dependence of $\underset{\sim}{B}_{20}$ on $\underset{\sim}{\mathcal{E}}_1$.

Thus we see that we can solve the second-order zeroth-harmonic equations when the oscillation center evolution is quasistatic, i.e., when the particles have time to equilibrate with the oscillation-center

Hamiltonian. Equations (8.2.38-49) are explicit formulas for the electrostatic potential and the charge density perturbations associated with the wave packet. For a single-ion-species plasma with $e_i = -e_e$, we get the surprisingly simple result (8.2.51). Equations (8.2.63) and (8.2.68) are explicit formulas for the current density and magnetic field associated with the wave packet.

8.2c Interpretation of Wave Momentum

To calculate the time average kinetic (as opposed to canonical) momentum density $\underset{\sim}{G}_{20}$ of a Vlasov species to second order, we use the expression

$$\underset{\sim}{G}_{20}(\underset{\sim}{x},t) = \int d^3p\ \{(\underset{\sim}{p} - e\underset{\sim}{A}(\underset{\sim}{x},t))f(\underset{\sim}{x},\underset{\sim}{p},t)\}_{20}$$

$$= \int d^3p\ [\underset{\sim}{p}f_{20} - e(\underset{\sim}{A}_1 f_1)_{20} - e\underset{\sim}{A}_{20}f_0] \quad , \tag{8.2.69}$$

where f_{20} is determined by the theory of the preceding subsection. In section 6.2 we showed how to assign a "wave-momentum density" $\underset{\sim}{G}_{2w}$ to each species in a collisionless plasma. In this section we determine the relationship between $\underset{\sim}{G}_{20}$ and $\underset{\sim}{G}_{2w}$.

Let us use (8.2.23) in the above equation.

$$\underset{\sim}{G}_{20} = \int d^3p\ [\underset{\sim}{p}F_{20} - e\underset{\sim}{A}_{20}f_0 + \underset{\sim}{p}\ \frac{1}{2}\ \{w_1,\{w_1,f_0\}\}_0 - e(\underset{\sim}{A}_1 f_1)_0] \ . \tag{8.2.70}$$

The last two terms depend on only the amplitude of the wave packet, not its shape. In contrast, the first term vanishes for plane waves, but does not vanish otherwise. The second term depends on the second-order current response, which depends on F_{20}. Since the expression (6.2.17) for the wave momentum density is independent of the wave packet shape, we are motivated to guess the formula

$$\underset{\sim}{G}_{2w} = \int d^3p\ [\underset{\sim}{p}\ \frac{1}{2}\ \{w_1,\{w_1,f_0\}\}_0 - e\underset{\sim}{A}_1 f_1] \quad . \tag{8.2.71}$$

This equation is easily verified. Using (8.2.24) and (8.1.13) in Eq. (8.2.71) we find

$$\underset{\sim}{G}_{2w} = \int d^3p \left[\underset{\sim}{p}\, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}} \left(\frac{e^2 |\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1|^2}{(\underset{\sim}{k}\, \underset{\sim}{v}_0 - \omega)^2} \underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}} \right) \right.$$

$$\left. + \frac{e^2 \underset{\sim}{\mathcal{A}}_1\, \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1^* + e^2 \underset{\sim}{\mathcal{A}}_1^*\, \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega} \underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}} \right] \quad . \tag{8.2.72}$$

On the other hand, inserting (8.1.18) into (6.2.17) gives

$$\underset{\sim}{G}_{2w} = e^2 \int d^3p\, f_0\, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}}$$

$$\times \left[\frac{\underset{\sim}{k} |\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1|^2}{(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^2} - \frac{\underset{\sim}{\mathcal{A}}_1\, \underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1^* + \underset{\sim}{\mathcal{A}}_1^*\, v_0 \cdot \underset{\sim}{\mathcal{A}}_1}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega} \right] \quad . \tag{8.2.73}$$

Integrating by parts we see that (8.2.36) and (8.2.37) are equivalent.

To correlate these results with previous work,[26] we define the background momentum density:

$$\underset{\sim}{G}_{2B} \equiv \underset{\sim}{G}_{20} - \underset{\sim}{G}_{2w} = \int d^3p (\underset{\sim}{p}\, F_{20} - e\, \underset{\sim}{A}_{20}\, f_0) \quad . \tag{8.2.74}$$

The second term in the integrand can be ignored since it vanishes upon summing over species. To see this, we restore ordinary units ($m \neq 1$ and $c \neq 1$) and compute

$$\sum_s \int d^3p\, e_s\, \underset{\sim}{A}_{20}\, f_0/c = \underset{\sim}{A}_{20}/c \sum_s n_0^s e_s = 0 \quad .$$

The final sum vanishes because the unperturbed state is neutral. Thus, the important term is $\int d^3p\, \underset{\sim}{p}\, F_{20}$. In general, this term does not vanish. For the quasistatic response (8.2.31) one can explicitly show that $\int d^3p\, \underset{\sim}{p}\, F_{20}$ is nonzero.

This exercise demonstrates the meaning of wave momentum density in (uniform unmagnetized) Vlasov plasma. The wave momentum density is calculated by not allowing for the second-order field $\underset{\sim}{A}_{20}$, and by not allowing the oscillation centers to respond. In general, the wave momentum density is <u>not</u> the total momentum density. To find the total momentum density we must add the background momentum density, which is defined by (8.2.74). (Dewar[26] discussed the breakup of total momentum density into wave and background contributions for a Lagrangian system. The breakup presented here corresponds to Dewar's "canonical", meaning standard, not Hamiltonian, breakup.)

8.3 Second-Order Second-Harmonic Effects

The next step in the study of nonlinear wave evolution is the calculation of the second-order second-harmonic quantities. These quantities are of interest because their existence may change the character of the wave. For example, we find that waves which are transverse in the linear limit possess a longitudinal second-order second-harmonic electrid field.[16] Another reason for studying the second-order second-harmonic quantities is that they are needed in the calculation of the third-order first-harmonic currents, which alter the evolution of the wave envelope. The second-order electromagnetic field appears in the third-order Hamiltonian, and the second-order distribution function contributes to the third-order current.

The calculation of these quantities proceeds via the outline in section 7.2. We first calculate the second-order second-harmonic nonlinear quantities, including the current $\underset{\sim}{j}_{2\nu 2}$. Then we insert this current into Eq. (6.3.39) to find the electric field. Finally, we compute the second-order linear quantities.

We calculate the second-order second-harmonic quantities by transforming away the Hamiltonian. This choice, $K_{2\nu 2} = 0$, has been used to derive Eq. (7.2.47). The remaining analysis of that equation is to compute the right-hand side and then to integrate it along an orbit.

From Eq. (7.2.36) we obtain

$$h_{2\nu 2} = \frac{e^2[\underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{\mathcal{A}}_1 - (\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_1)^2]}{2(1 + p^2)^{\frac{1}{2}}} \; e^{2i\psi} + \text{c.c.} \tag{8.3.1}$$

Eqs. (8.1.4) and (8.1.8) combine to give

$$\{w_1,h_1\}_2 = -\frac{e^2(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1)^2[k^2 - (\underset{\sim}{k} \cdot \underset{\sim}{v}_0)^2]}{(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^2\,(1 + p^2)^{\frac{1}{2}}}\; e^{2i\psi} + \text{c.c.} \qquad (8.3.2)$$

to lowest order in $(\nabla,\partial/\partial t)$, i.e., we invoke the approximations (8.2.10-12) in this calculation. We insert these expressions into Eq. (7.2.47), and we integrate along an orbit, i.e., we divide by the resonant denominator $i(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)$, to find

$$-\frac{1}{2}\, w_{2\nu 2} = -\frac{ie^2}{4(1 + p^2)^{\frac{1}{2}}}\left[\frac{\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1}{k \cdot v - \omega}\right.$$

$$\left. - \frac{(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1)^2}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega}\left(\frac{k^2 - (\underset{\sim}{k} \cdot \underset{\sim}{v}_0)^2}{(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^2} + 1\right)\right] \cdot e^{2i\psi} + \text{c.c.} \qquad (8.3.3)$$

We are now able to compute $f_{2\nu 2}$ via Eq. (7.2.48). We use the expression (8.3.3) to compute the first term,

$$-\frac{1}{2}\,\{w_{2\nu 2},\, f_0\} = \frac{e^2}{2(1 + p^2)^{\frac{1}{2}}}\left[\frac{\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega}\right.$$

$$\left. - \frac{(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1)^2\,(k^2 + \omega^2 - 2\,\omega \underset{\sim}{k} \cdot \underset{\sim}{v}_0)}{(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^3}\right]$$

$$\times\; k \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}}\, e^{2i\psi} + \text{c.c.} \qquad (8.3.4)$$

and we use the expression (8.1.8) to compute the second term,

$$\frac{1}{2}\,\{w_1,\{w_1,f_0\}\}_2 = \frac{e^2(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1)^2}{2(\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega)^2}\left(\underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{p}}\right)^2 f_0 e^{2i\psi} + \text{c.c.} \qquad (8.3.5)$$

Thus, Eq. (7.2.48) gives

$$f_{2\nu 2} = \frac{e^2}{2}\left\{(1+p^2)^{\frac{1}{2}}\left[\frac{\mathcal{A}_1 \cdot \mathcal{A}_1}{\mathbf{k} \cdot \mathbf{v}_0 - \omega} - \frac{(\mathbf{v}_0 \cdot \mathcal{A}_1)^2(k^2+\omega^2-2\omega\mathbf{k}\cdot\mathbf{v}_0)}{(\mathbf{k}\cdot\mathbf{v}_0-\omega)^3}\right]\right.$$

$$\left. \times\, \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}} + \frac{(\mathbf{v}_0 \cdot \mathcal{A}_1)^2}{(\mathbf{k}\cdot\mathbf{v}_0-\omega)^2}(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}})^2 f_0\right\} e^{2i\psi} + \text{c.c.} \qquad (3.3.6)$$

Now we can use Eq. (7.2.49) to calculate $\mathbf{j}_{2\nu 2}$. Using (8.3.6) and (7.1.15) to find the first term,

$$\int d^6z\, \eta_0 f_{2\nu 2} = \frac{e^3}{2}\int d^3p\, v_0\left\{(1+p)^{-\frac{1}{2}}\left[\frac{\mathcal{A}_1 \cdot \mathcal{A}_1}{\mathbf{k}\cdot\mathbf{v}_0-\omega}\right.\right.$$

$$\left. - \frac{(\mathbf{v}_0 \cdot \mathcal{A}_1)^2(k^2+\omega^2-2\,\omega\mathbf{k}\cdot\mathbf{v}_0)}{(\mathbf{k}\cdot\mathbf{v}_0-\omega)^3}\right]\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{p}}$$

$$\left. + \frac{(\mathbf{v}_0 \cdot \mathcal{A}_1)^2}{(\mathbf{k}\cdot\mathbf{v}_0-\omega)^2}(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}})^2 f_0\right\} e^{2i\psi} + \text{c.c.} \qquad (8.3.7)$$

The second term in Eq. (7.2.49) is found by combining Eqs. (7.1.24), (7.1.27), and (8.1.13):

$$\int d^6z\, \eta_1 f_1 = e^3\int d^3p\, \frac{(\mathcal{A}_1 - \mathbf{v}_0\mathbf{v}_0\cdot\mathcal{A}_1)\mathbf{v}_0\cdot\mathcal{A}_1}{(1+p^2)^{\frac{1}{2}}(\mathbf{k}\cdot\mathbf{v}_0-\omega)}\,\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{p}}\, e^{i\psi}$$

$$+ \text{c.c.} \qquad (8.3.8)$$

and Eq. (7.2.39) allows us to compute the third term in Eq. (7.2.49):

$$\int d^6z\, \underset{\sim}{n}_{2\nu 2} f_0 = -\frac{e^3}{2}\int d^3p\, \frac{\underset{\sim}{v}_0 \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 + 2\underset{\sim}{a}_1 \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1 - 3\underset{\sim}{v}_0(\underset{\sim}{v}_0 \cdot \underset{\sim}{a}_1)^2}{(1 + p^2)}$$

$$\times\, f_0 e^{i\psi} + \text{c.c.} \qquad (8.3.9)$$

We sum these three expressions to find the total second-order second-harmonic nonlinear current $\underset{\sim}{j}_{2\nu 2}$. To second order in $\underset{\sim}{k} \cdot \underset{\sim}{v}_0/\omega$ and $p(p/mc)$ we obtain

$$\underset{\sim}{j}_{2\nu 2} = \frac{n_0 e^3}{2\omega}\, [\underset{\sim}{k}\, \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1\, (1 - 8v_T^2 + 6k^2 v_T^2/\omega^2) + \underset{\sim}{a}_1\, \underset{\sim}{k} \cdot \underset{\sim}{a}_1 (2$$

$$- 16v_T^2 + 12k^2 v_T^2/\omega^2) + 12\underset{\sim}{k}(\underset{\sim}{k} \cdot \underset{\sim}{a}_1)^2 v_T^2/\omega^2]\, e^{2i\psi} + \text{c.c.} \qquad (8.3.10)$$

where v_T^2 is defined by (8.1.23). We note that $j_{2\nu 2}$ is longitudinal for either transverse ($\underset{\sim}{k} \cdot \underset{\sim}{a}_1 = 0$) or longitudinal ($\underset{\sim}{k} \times \underset{\sim}{a}_1 = 0$) waves. In either case we have

$$\underset{\sim}{j}_{2\nu 2} = \frac{n_0 e^3}{2\omega}\, [\underset{\sim}{k}(\hat{k} \cdot \underset{\sim}{a}_1)^2 (2 - 16v_T^2 + 24k^2 v_T^2/\omega^2)$$

$$+ \underset{\sim}{k}\, \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 (1 - 8v_T^2 + 6k^2 v_T^2/\omega^2)]\, e^{2i\psi} + \text{c.c.} \qquad (8.3.11)$$

To find the corresponding density perturbation, we use continuity, $n_{2\nu 2} = \underset{\sim}{k} \cdot \underset{\sim}{j}_{2\nu 2}/e\omega$:

$$n_{2\nu 2} = \frac{n_0 e^3}{2\omega^2}\, [(\hat{k} \cdot \underset{\sim}{a}_1)^2 (2 - 16v_T^2 + 24k^2 v_T^2/\omega^2)$$

$$+ \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 (1 - 8v_T^2 + 6k^2 v_T^2/\omega^2)]\, e^{2i\psi} + \text{c.c.} \qquad (8.3.12)$$

To prepare for the self consistent calculation, we restore units and species labels in our expression for $\underset{\sim}{j}_{2\nu 2}$:

$$\underset{\sim}{j}^{s}_{2\nu 2} = \frac{n_0 e_s^3}{2m_s^2 \omega c^2} [\underset{\sim}{k}(\hat{k} \cdot \underset{\sim}{\mathcal{A}}_1)^2 (2 - 16v_s^2/c^2 + 24k^2 v_s^2/\omega^2)$$

$$+ \underset{\sim}{k}\, \underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{\mathcal{A}}_1 (1 - 8v_s^2/c^2 + 6k^2 v_s/\omega^2)]\, e^{2i\psi} + \text{c.c.} \quad , \qquad (8.3.13)$$

where $v_s^2 \equiv T_s/m_s$. The presence of the factor m_s^{-2} in this expression makes the ion current negligible. Thus the total nonlinear current is well approximated by the electron portion,

$$\underset{\sim}{j}^{e}_{2\nu 2} = \frac{n_0 e_0^3}{2m_e^2 \omega c^2} [\underset{\sim}{k}(\hat{k} \cdot \underset{\sim}{\mathcal{A}}_1)^2 (2 - 16v_e^2/c^2 + 24k^2 v_e^2/\omega^2)$$

$$+ \underset{\sim}{k}\, \underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{\mathcal{A}}_1 (1 - 8v_e^2/c^2 + 6k^2 v_e^2/\omega^2)]\, e^{2i\psi} + \text{c.c.} \qquad (8.3.14)$$

Inserting the electric field $\underset{\sim}{\mathcal{E}}_1 = i\omega \underset{\sim}{\mathcal{A}}_1/c$, we obtain

$$\underset{\sim}{j}^{e}_{2\nu 2} = \frac{n_0 e_0^3}{2m_e^2 \omega^3} [\underset{\sim}{k}(\hat{k} \cdot \underset{\sim}{\mathcal{E}}_1)^2 (2 - 16v_e^2/c^2 + 24k^2 v_e^2/\omega^2)$$

$$+ \underset{\sim}{k}\, \underset{\sim}{\mathcal{E}}_1 \cdot \underset{\sim}{\mathcal{E}}_1 (1 - 8v_e^2/c^2 + 6k^2 v_e^2/\omega^2)]\, e^{2i\psi} + \text{c.c.} \qquad (8.3.15)$$

To find the second-order second-harmonic electric field $\underset{\sim}{E}_{22}(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{x},t)\exp(2i\psi) + \text{c.c.}$, we use Eq. (6.3.52), for which we need the linear response tensor $\underset{\approx}{D} = \underset{\approx}{I}(1 - k^2c^2/\omega^2) + \underset{\sim}{k}\,\underset{\sim}{k}\,c^2/\omega^2 + \sum_s \chi_s$.

For high frequency waves we can neglect the ion currents. Hence we have

$$\underset{\approx}{D}(\underset{\sim}{k},\omega) = (\underset{\approx}{I} - \hat{k}\hat{k})\,[1 - k^2c^2/\omega^2 - \omega_e^2(1 + k^2v_e^2/\omega^2 - \tfrac{5}{2}v_e^2/c^2)/\omega^2]$$

$$+ \hat{k}\hat{k}\,[1 - \omega_e^2(1 + 3k^2v_e^2/\omega^2 - \tfrac{5}{2}v_e^2/c^2)/\omega^2] \quad , \qquad (8.3.16)$$

which gives

$$\underset{\approx}{D}^{-1}(\underset{\sim}{k},\omega) = \frac{\omega^2(\underset{\approx}{I} - \hat{k}\hat{k})}{\omega^2 - k^2c^2 - \omega_e^2(1 + k^2v_e^2/\omega^2 - \frac{5}{2}v_e^2/c^2)}$$

$$+ \frac{\omega^2\hat{k}\hat{k}}{\omega^2 - \omega_e^2(1 + 3k^2v_e^2/\omega^2 - \frac{5}{2}v_e^2/c^2)} \quad . \qquad (8.3.17)$$

Inserting (8.3.14) and (8.3.17) into (6.3.52), we obtain

$$\underset{\sim}{\mathcal{E}}_{22}(\underset{\sim}{x},t) = \frac{i\,e_0\,\omega_e^2\,\underset{\sim}{k}}{4m_e c^2[\omega^2 - \frac{1}{4}\omega_e^2(1 + 3k^2v_e^2/\omega^2 - \frac{5}{2}v_e^2/c^2)]}$$

$$\times\,[\underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{\mathcal{A}}_1(1 - 8v_e^2/c^2 + 6k^2v_e^2/\omega^2)$$

$$+ (\hat{k} \cdot \underset{\sim}{\mathcal{A}}_1)^2(2 - 16v_e^2/c^2 + 24k^2v_e^2/c^2)] \quad . \qquad (8.3.18)$$

From this we obtain the vector potential $\underset{\sim}{\mathcal{A}}_{22} = c\,\underset{\sim}{\mathcal{E}}_{22}/(2i\omega)$:

$$\underset{\sim}{\mathcal{A}}_{22}(\underset{\sim}{x},t) = \frac{e_0\,e^2\underset{\sim}{k}[\underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{\mathcal{A}}_1(1 - 8v_e^2/c^2 + 6k^2v_e^2/c^2) + (\hat{k} \cdot \underset{\sim}{\mathcal{A}}_1)(2 - 16v_e^2/c^2 + 24k^2v_e^2/c^2]}{8m_e c\omega[\omega^2 - \frac{1}{4}\omega_e^2(1 + 3k^2v_e^2/\omega^2 - \frac{5}{2}v_e^2/c^2)]} \quad . \qquad (8.3.19)$$

We digress momentarily to discuss the inversion of $\underset{\approx}{D}$ more thoroughly. If an eigenvalue of $\underset{\approx}{D}$ has a zero at the second harmonic, i.e. there exists a normal mode with wave vector equal to $2\underset{\sim}{k}$ and frequency equal to 2ω, then the inversion (8.3.17) is not possible, and so one must analyze the situation differently. However, in our specific case we do not have a problem. In unmagnetized plasma, high frequency waves satisfy $\omega > \omega_e$ and therefore greater than $2\omega_e$, but from Eq. (8.3.11) we see that there is not transverse current to drive a transverse wave to second order in $\underset{\sim}{k} \cdot \underset{\sim}{v}_0/\omega$ and p/mc. In fact, for isotropic f_0 one can show from Eqs. (8.3.7-9) that $\underset{\sim}{j}_{2\nu 2}$ is transverse to all orders in those parameters.

To complete our discussion of the second-order second-harmonic terms, we compute the second-order linear quantities. The current linear in $\underset{\sim}{A}_{22}$ is calculated using the linear conductivity:

$$\underset{\sim}{j}_{2\lambda 2} = \underset{\approx}{\sigma}(2\underset{\sim}{k},2\omega) \cdot \underset{\sim}{E}_{22} = \underset{\approx}{\sigma}(2\underset{\sim}{k},2\omega) \cdot (i2\omega \underset{\sim}{A}_{22})$$

$$= \frac{2\omega}{4\pi i} \cdot \underset{\approx}{\chi}(2\underset{\sim}{k},2\omega) \cdot (i2\omega \underset{\sim}{A}_{22})$$

$$= \frac{\omega^2}{\pi} \underset{\approx}{\chi}(2\underset{\sim}{k},2\omega) \cdot \underset{\sim}{A}_{22} \quad . \tag{8.3.20}$$

Inserting Eq. (8.1.22) into (8.3.20), and noting that Eq. (8.3.19) implies that $\underset{\sim}{A}_{22}$ is longitudinal, we find

$$\underset{\sim}{j}_{2\lambda 2} = -n_0 e^2 (1 + 3k^2 v_T^2/\omega^2 - \frac{5}{2} v_T^2) \underset{\sim}{\mathcal{A}}_{22} e^{2i\psi} + \text{c.c.} \tag{8.3.21}$$

through first order in v_T^2. To find the corresponding linear density perturbation, we use continuity, $n_{2\lambda 2} = \underset{\sim}{k} \cdot \underset{\sim}{j}_{2\lambda 2}/e\omega$:

$$n_{2\lambda 2} = -\frac{n_0 e}{\omega}\left(1 + 3k^2 v_T^2/\omega^2 - \frac{5}{2} v_T^2\right) \underset{\sim}{k} \cdot \underset{\sim}{\mathcal{A}}_{22} e^{2i\psi} + \text{c.c.} \tag{8.3.22}$$

For purposes of comparison, we restore units in Eqs. (8.3.12) and (8.3.22) and we sume to find the total second-order density perturbation. The result is

$$\eta_{22} = \frac{k^2\omega_e^2[\underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{\mathcal{A}}_1(1 - 8v_e^2/c^2 + 6k^2 v_e^2/c^2) + (\hat{k} \cdot \underset{\sim}{\mathcal{A}}_1)^2(2 - 16v_e^2/c^2 + 24k^2 v_e^2/c^2]}{8\pi m_e c^2[\omega^2 - \frac{1}{4}\omega_e^2(1 + 3k^2 v_e^2/\omega^2 - \frac{5}{2} v_e^2/c^2)]} \tag{8.3.23}$$

where η_{22} is the density amplitude. In the cold plasma limit this reduces to

$$\eta_{22} = \frac{k^2\omega_e^2[\underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{\mathcal{A}}_1 + 2(\hat{k} \cdot \underset{\sim}{\mathcal{A}}_1)^2]}{8\pi m_e c^2(\omega^2 - \frac{1}{4}\omega_e^2)} , \tag{8.3.24}$$

which was obtained by Arons and Max[16], their Eq. (14), in their discussion of nonlinear transverse plane-wave propagation.

This concludes the analysis of the second-order second-harmonic effects. By using Lie transforms to transform away the second-order second-harmonic Hamiltonian, we have obtained expressions for the electric field (8.3.18), the current (8.3.12) and (8.3.21), and the density (8.3.23).

8.4 Third-order Effects

As we noted in section 7.3, third-order quantities oscillate at the first harmonic or the third harmonic. The first step in calculating either type of quantity is to compute the third-order nonlinear current $\underset{\sim}{j}_{3v} = \underset{\sim}{j}_{3v1} + \underset{\sim}{j}_{3v3}$. The third-order first-harmonic nonlinear current $\underset{\sim}{j}_{3v1}$ modifies the evolution of the first-harmonic amplitude via Eqs. (6.3.46) and (6.3.51). The third-order third-harmonic current $\underset{\sim}{j}_{3v3}$ produces a third-order electric field $\underset{\sim}{E}_{33} = D^{-1}(3\underset{\sim}{k},3\omega) \cdot \underset{\sim}{j}_{3v3}$, just as the second-order second-harmonic current produced a second-order electric field, c.f. Eq. (8.3.14).

Here we are interested only in the nonlinear evolution of the first-harmonic amplitude $\underset{\sim}{\mathcal{E}}_1$. Hence, we ignore $\underset{\sim}{j}_{3v3}$, and we calculate only $\underset{\sim}{j}_{3v1}$ (in subsection 8.5a). Then (in subsection 8.5b) we use the theory of section 6.3 to determine how $\underset{\sim}{j}_{3v1}$ alters the evolution of $\underset{\sim}{\mathcal{E}}_1$. We find all previously discussed effects, such as the nonlinear change in the conductivity due to the density depression caused by the ponderomotive potential[13] and the self-precession effect previously calculated[16] for plane waves. In addition, we find a new, comparable effect: self Faraday rotation caused by self-generated quasistatic magnetic field. In the end, we unify previous work by calculating the third-order current including all effects to zeroth order in v_e^2/c^2 and $k^2v_e^2/\omega^2$ for the case of quasistatic evolution.

8.4a Calculation of the Third-Order First-Harmonic Current

The calculation of the third-order first-harmonic nonlinear current $\underset{\sim}{j}_{3\nu 1}$ follows the outline of section 7.2. We calculate the integral (7.2.76) to find $w'_{3\nu 1}$, which we use in Eq. (7.2.75) to find $f_{3\nu 1}$. Then we insert $f_{3\nu 1}$ into Eq. (7.2.77) to find $\underset{\sim}{j}_{3\nu 1}$.

We make a number of simplifying assumptions in this calculation. First, we treat the case of quasistatic evolution, for which we have an analytic expression (8.2.31) for the oscillation-center distribution. Secondly, we work to zeroth-order in $(\partial/\partial t, \nabla)\underset{\sim}{a}_1$, i.e., we invoke all of the approximations (8.2.10-12). Of course, as we noted in section 8.2b, to be consistent we must keep terms in the gradient of the vector potential, $\nabla \underset{\sim}{A}_{20}$. In addition, we study plasma with one species of singly charged ions, for which the quasistatic density change (8.2.51) is simple. Finally, we calculate $\underset{\sim}{j}_{3\nu 1}$ only to zeroth order in the temperature. From reference 1, we should expect a contribution to the current of size

$$j' \sim \delta n e^2 E/(m\omega) \approx e^2\, E|E|^2/m\omega T \quad ,$$

A kinetic correction to j' would be of magnitude $j'' \sim (e^2 E|E|^2/m\omega T) \times (k^2 T/m\omega^2)$, and so would be zeroth order in T. Thus, by calculating $\underset{\sim}{j}_{3\nu 1}$ to zeroth order in the temperature, we obtain the first kinetic correction to the work of reference 1.

We begin by combining Eqs. (7.2.75) and (7.2.77),

$$\underset{\sim}{j}_{3\nu 1} = e\int d^3p\ \underset{\sim}{v}_0\ \{-\tfrac{1}{3}\ w'_{3\nu 1}, f_0\} + e\int d^3p\ \underset{\sim}{v}_0\ \tfrac{1}{2}\ \{w_1,\{w_2,f_0\}\}$$

$$-\tfrac{1}{6}\ e\int d^3p\ \underset{\sim}{v}_0\{w_1,\{w_1,\{w_1,f_0\}\}\} - e\int d^3p\ \underset{\sim}{v}_0\ \{w_1,F_2\}$$

$$+ e\int d^3p(\underset{\sim}{v}_1 f_2) + e\int d^3p\ \underset{\sim}{v}_2 f_1 + e\int d^3p\ \underset{\sim}{v}_{3\nu 1}\ f_0 \quad , \qquad (8.4.1)$$

keeping only the first-harmonic terms. We must evaluate (8.4.1) term by term. Throughout this process, we will try to put these terms in the form, $\Delta j \sim \int d^3p\ f_0(p)\ g(\underset{\sim}{q},\underset{\sim}{p},t)$. In that form we need g only to zeroth order in $\underset{\sim}{p}$, since we want Δj to zeroth order in T.

The first term is

$$-\tfrac{1}{3}\ e\int d^3p\ \underset{\sim}{v}_0\{w_{3\nu 1},f_0\} = -\tfrac{1}{3}\ e\int d^3p\ \underset{\sim}{v}_0\ \frac{\partial w'_{3\nu 1}}{\partial \underset{\sim}{q}}\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}}\ .$$

$$= \tfrac{1}{3}\ e\int d^3p\ f_0\ \frac{\partial}{\partial \underset{\sim}{p}}\cdot\left(\frac{\partial w'_{3\nu 1}}{\partial \underset{\sim}{q}}\ \underset{\sim}{v}_0\right) \quad , \qquad (8.4.2)$$

where the last step follows from integration by parts. Since we need the quantity $(\partial/\partial\underset{\sim}{p})\cdot[(\partial w'_{3\nu 1}/\partial\underset{\sim}{q})\underset{\sim}{v}_0]$ to zeroth order in $\underset{\sim}{p}$, we need the quantity $(\partial w'_{3\nu 1}/\partial\underset{\sim}{q})\underset{\sim}{v}_0$ to first order in $\underset{\sim}{p}$. Thus, we use $\underset{\sim}{v}_0 \approx \underset{\sim}{p}$, and we compute $w'_{3\nu 1}$ to zeroth order in $\underset{\sim}{p}$. This reduces the first term to

$$-\frac{1}{3}\, e \int d^3p\; \underset{\sim}{v}_0 \{w'_{3\nu 1}, f_0\} = \frac{1}{3}\, e \int d^3p\; f_0\; \frac{\partial w'_{3\nu 1}}{\partial \underset{\sim}{q}}\, (\underset{\sim}{p} = 0)$$

$$= \frac{1}{3}\, n_0 e\, \frac{\partial w'_{3\nu 1}}{\partial \underset{\sim}{q}}\, (\underset{\sim}{p} = 0) \quad . \tag{8.4.3}$$

The quantity $w'_{3\nu 1}$ is found by computing the integral (7.2.76). Integration along a trajectory of a first-harmonic term does not change its leading order in $\underset{\sim}{p}$, since such an integration amounts to division by $-i(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_0)$. Therefore, to find $w'_{3\nu 1}$ to zeroth order in $\underset{\sim}{q}$, we need $h_{3\nu 1} + \{w_1, 3h_2 + \{w_1, h_1\}\}_1/3$ to zeroth order in $\underset{\sim}{p}$. This quantity can be written in the form,

$$h_{3\nu 1} + \frac{1}{3}\{w_1, 3h_2 + \{w_1, h_1\}\}_1 = h_{3\nu 1} + \frac{1}{3}\left[\frac{\partial w_1}{\partial \underset{\sim}{q}} \cdot \frac{\partial}{\partial \underset{\sim}{p}}\,(3h_2 + \{w_1, h_1\})\right]_1$$

$$- \frac{1}{3}\left[\frac{\partial w_1}{\partial \underset{\sim}{p}} \cdot \frac{\partial}{\partial \underset{\sim}{q}}\,(3h_2 + \{w_1, h_1\})\right]_1 \quad . \tag{8.4.4}$$

Using Eq. (7.2.73), we find the first term in this expression to be

$$h_{3\nu 1} = e^2 (\underset{\sim}{\mathcal{A}}_1^* \cdot \underset{\sim}{\mathcal{A}}_{22} + \underset{\sim}{\mathcal{A}}_1 \cdot \underset{\sim}{A}_{20}) e^{i\psi} + \text{c.c.} \tag{8.4.5}$$

to zeroth order in $\underset{\sim}{p}$. Since w_1 is first order in $\underset{\sim}{p}$, the second term in (8.4.4) must be at least first order in $\underset{\sim}{p}$, and so it is ignored. To compute the last term in (8.4.4), we need $3h_2 + \{w_1, h_1\}$ to zeroth order in $\underset{\sim}{p}$. Eqs. (7.1.22), (8.2.8), and (8.3.2) yield

$$3h_2 + \{w_1,h_1\} = 3e\,\Phi_{20} + 3e^2|\underset{\sim}{a}_1|^2$$

$$+ \frac{3}{2}\,e^2(\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1\, e^{2i\psi} + \text{c.c.}) + O(\underset{\sim}{p}) \quad . \tag{8.4.6}$$

This result, combined with (8.1.8), allows us to compute

$$\left[-\frac{1}{3}\,\frac{\partial w_1}{\partial \underset{\sim}{p}} \cdot \frac{\partial}{\partial \underset{\sim}{q}}\,(3h_2 + \{w_1,h_1\})\right]_1 = -\,\frac{e^3\underset{\sim}{k} \cdot \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1}{\omega}\, e^{i\psi}$$

$$+ \text{c.c.} + O(\underset{\sim}{p},\nabla) \quad . \tag{8.4.7}$$

Inserting (8.4.7) and (8.4.5) into (8.4.4) and integrating along a trajectory we find

$$-\frac{1}{3}\,w'_{3\nu 1} = [\,\frac{ie^2}{\omega}\,(\underset{\sim}{a}_1^* \cdot \underset{\sim}{a}_{22} + \underset{\sim}{a}_1 \cdot \underset{\sim}{A}_{20})$$

$$-\,\frac{ie^3}{\omega^2}\,\underset{\sim}{k} \cdot \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1]e^{i\psi} + \text{c.c.} \tag{8.4.8}$$

Finally, we insert (8.4.8) into (8.43) to obtain

$$-\frac{1}{3}\,e\int d^3p\;\underset{\sim}{v}_0\{w'_{3\nu 1},f_0\} = \left[\frac{n_0e^3\underset{\sim}{k}}{\omega}\,(\underset{\sim}{a}_1^* \cdot \underset{\sim}{a}_{22} - e\,\underset{\sim}{k} \cdot \underset{\sim}{a}_1^*\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1/\omega\right.$$

$$\left. + \underset{\sim}{a}_1 \cdot \underset{\sim}{A}_{20}) - \frac{in_0e^3}{\omega}\,(\nabla\,\underset{\sim}{A}_{20}) \cdot \underset{\sim}{a}_1\right]e^{i\psi} + \text{c.c.} \tag{8.4.9}$$

As always, we must remember that $\nabla\underset{\sim}{A}_{20}$ is zeroth order in $\nabla\underset{\sim}{a}_1$.

The second term on the right side of (8.4.1) is

$$\frac{1}{2}\, e \int d^3p\ \underset{\sim}{v}_0\{w_1,\{w_2,f_0\}\} = \frac{e}{2}\int d^3p\ \underset{\sim}{v}_0\left(\frac{\partial w_1}{\partial \underset{\sim}{q}}\cdot\frac{\partial}{\partial \underset{\sim}{p}}\frac{\partial w_2}{\partial \underset{\sim}{q}}\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}}\right.$$

$$\left. - \frac{\partial w_1}{\partial \underset{\sim}{p}}\cdot\frac{\partial}{\partial \underset{\sim}{q}}\frac{\partial w_2}{\partial \underset{\sim}{q}}\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}}\right) = \frac{e}{2}\int d^3p\ f_0\,\frac{\partial}{\partial \underset{\sim}{p}}\cdot\left[\frac{\partial w_2}{\partial \underset{\sim}{q}}\frac{\partial}{\partial \underset{\sim}{p}}\cdot\left(\frac{\partial w_1}{\partial \underset{\sim}{q}}\underset{\sim}{v}_0\right)\right.$$

$$\left. + \frac{\partial w_2}{\partial \underset{\sim}{q}\,\partial \underset{\sim}{q}}\cdot\frac{\partial w_1}{\partial \underset{\sim}{p}}\underset{\sim}{v}_0\right] \quad , \tag{8.4.10}$$

where the last step follows from integration by parts. Since w_1 and $\underset{\sim}{v}_0$ are both first order in $\underset{\sim}{p}$, we need w_2 only to zeroth order in $\underset{\sim}{p}$ for the evaluation of (8.4.10). Since $w_{2\lambda} \propto \underset{\sim}{p}\cdot\underset{\sim}{\mathcal{A}}_{22}$ we have $w_2 = w_{2v}$ + O(p) which from (8.3.3) is

$$w_2 = \frac{ie^2\underset{\sim}{\mathcal{A}}_1\cdot\underset{\sim}{\mathcal{A}}_1 e^{2i\psi}}{2\omega} + \text{c.c.} + O(p) \quad . \tag{8.4.11}$$

Inserting this expression for w_2 and w_1, (8.1.8), to lowest order in $\underset{\sim}{p}$ we find

$$\frac{1}{2}\, e \int d^3p\ \underset{\sim}{v}_0\{w_1,\{w_2,f_0\}\} = 0 + O(v_T) \quad . \tag{8.4.12}$$

The third term on the right side of (8.4.1) also vanishes to this order. To see this we note that $\underset{\sim}{v}_0 = O(p)$ and $w_1 = O(p)$. We apply the usual technique of integrating by parts to take the operator $\partial/\partial\underset{\sim}{p}$ off f_0 and onto the rest of the integrand. We see that we have three operators $\partial/\partial\underset{\sim}{p}$ but four functions $\underset{\sim}{v}_0$ and 3 w_1's of O(p). Hence the integrand is $O(\underset{\sim}{p})$.

The fourth term on the right side of (8.4.1) is found by first computing the Poisson bracket and then integrating by parts.

$$-e\int d^3p\ \underset{\sim}{v}_0\{w_1,F_2\} = -\ e^2\int d^3p\left[\frac{\underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{-1}}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0-\omega}\ k\cdot\frac{\partial}{\partial\underset{\sim}{p}}\ (-K_2f_0/T)\right.$$

$$\left.-\ v_0\ \frac{\partial}{\partial\underset{\sim}{p}}\left(\frac{-ie\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1}{\underset{\sim}{k}\cdot\underset{\sim}{v}\ -\omega}\right)\cdot\frac{\partial}{\partial\underset{\sim}{q}}\ (-K_2f_0/T)\right]e^{i\psi}+\text{c.c.}$$

$$=\frac{-e^2}{T}\int d^3p\ f_0K_2\ \underset{\sim}{k}\cdot\frac{\partial}{\partial\underset{\sim}{p}}\left(\frac{\underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0-\omega}\right)e^{i\psi}+\frac{ie^3}{T}\int d^3p$$

$$\times\ f_0\ \underset{\sim}{v}_0\ \frac{\partial K_2}{\partial\underset{\sim}{q}}\cdot\frac{\partial}{\partial\underset{\sim}{p}}\left(\frac{\underset{\sim}{v}_0\cdot\underset{\sim}{\mathcal{A}}_1}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0-\omega}\right)e^{i\psi}+\text{c.c.}\qquad(8.4.13)$$

In evaluating $\partial k_2/\partial\underset{\sim}{q}$ we keep only the term $\nabla\ \underset{\sim}{A}_{20}$. In evaluating the integrals we must keep in mind the T^{-1} factor, and hence we keep the integrands to $O(p^2)$. Carrying out the differentiations and integrations in (8.4.13) we find

$$-e\int d^3p\ \underset{\sim}{v}_0\{w_1,F_2\} = \left[\frac{n_0e^2\underset{\sim}{k}}{\omega}\left(\frac{\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1(3e^2|\underset{\sim}{\mathcal{A}}_1|^2+2e\Phi_{20})}{\omega}+\frac{e^2\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1^*\underset{\sim}{\mathcal{A}}_1\cdot\underset{\sim}{\mathcal{A}}_1}{\omega}\right.\right.$$

$$\left.-e\ \underset{\sim}{\mathcal{A}}_1\cdot\underset{\sim}{A}_{20}\right)+\frac{n_0e^2\underset{\sim}{\mathcal{A}}_1}{\omega^2}\left(e^2|\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1|^2+k^2(e^2|\underset{\sim}{\mathcal{A}}_1|^2+e\Phi_{20})\right)$$

$$+\frac{n_0e^4\underset{\sim}{\mathcal{A}}_1^*}{\omega^2}(\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1)^2-\frac{n_0e^3}{\omega}A_{20}\ \underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1$$

$$\left.+\frac{in_0e^3}{\omega}\underset{\sim}{\mathcal{A}}_1\cdot\nabla\ \underset{\sim}{A}_{20}\right]e^{i\psi}+\text{c.c.}\qquad(8.4.14)$$

The fifth integral on the right side of (8.4.1) is

$$e\int d^3p(\underset{\sim}{v}_1 f_2) = e\int d^3p\Big(\underset{\sim}{v}_1 F_2 + \frac{1}{2}\,\underset{\sim}{v}_1\{w_1,\{w_1,f_0\}\}$$

$$- \frac{1}{2}\,\underset{\sim}{v}_1\{w_2,f_0\}\Big)_1 \,. \tag{8.4.15}$$

To calculate this integral to zeroth order in T, we need $\underset{\sim}{v}_1$ in the last two terms in the integrand to $O(p^0)$, but we need $\underset{\sim}{v}_1$ to $O(p^2)$ in the first term. Thus, we find

$$e\int d^3p(\underset{\sim}{v}_1 f_2)_I = -e^2\underset{\sim}{A}_1 \cdot \int d^3p(\frac{1}{2}\,p^2\,\underset{\approx}{I} + \underset{\sim}{p}\,\underset{\sim}{p})f_0K_2/T$$

$$-e^2\Big[\underset{\sim}{A}_1 \cdot \int d^3p\Big(F_2 + \frac{1}{2}\,\{w_1,\{w_1,f_0\}\} - \frac{1}{2}\,\{w_2,f_0\}\Big)\Big]_1 \tag{8.4.16}$$

to zeroth order in T. We evaluate these expressions, noting that the last integral is simply n_2. We find

$$e\int d^3p(\underset{\sim}{v}_1 f_2) = -\frac{5}{2}\,n_0e^2\underset{\sim}{A}_1(e^2|\underset{\sim}{\mathcal{A}}_1|^2 + e\Phi_{20}) - e^2(\underset{\sim}{A}_1 n_2)_1$$

$$= \Big[-\frac{5}{2}\,n_0e^2\underset{\sim}{\mathcal{A}}_1(e^2|\underset{\sim}{\mathcal{A}}_1|^2 + e\Phi_{20}) - e^2\underset{\sim}{\mathcal{A}}_1 n_{20} - e^2\underset{\sim}{\mathcal{A}}_1^*\eta_{22}\Big]e^{i\psi} + \text{c.c.}\,, \tag{8.4.17}$$

where n_{20} is given by Eq. (8.2.49) and η_{22} is given by Eq. (8.3.23).

The sixth integral on the right side of (8.4.1) is

$$-\int d^3p(\underset{\sim}{v}_2 f_1)_1 = -e\int d^3p\left(\underset{\sim}{v}_2 \frac{\partial w_1}{\partial \underset{\sim}{q}} \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}}\right)_1$$

$$= e\int d^3p\ f_0 \frac{\partial}{\partial \underset{\sim}{p}} \cdot \left(\frac{\partial w_1}{\partial \underset{\sim}{q}} \underset{\sim}{v}_2\right)_1 \quad . \tag{8.4.18}$$

We see that we need w_1 to $O(p)$ and $\underset{\sim}{v}_2$ to $O(p^0)$. Thus,

$$e\int d^3p(\underset{\sim}{v}_2 f_1)_1 = -\left(e^2\underset{\sim}{A}_2\int d^3p\ f_0 \frac{\partial^2 w_1}{\partial \underset{\sim}{p} \cdot \partial \underset{\sim}{q}}\right)_1 = e^2\left(\underset{\sim}{A}_2\int d^3p\{w_1, f_0\}\right)_1$$

$$= -e^2(\underset{\sim}{A}_2 n_1) \quad . \tag{8.4.19}$$

To find n_1 we use continuity, $n_1 = \underset{\sim}{k} \cdot \underset{\sim}{j}_1/e\omega$:

$$n_1 = (\underset{\sim}{k} \cdot \underset{\approx}{\sigma} \cdot \underset{\sim}{\mathcal{E}}_1/e\omega)e^{i\psi} + \text{c.c.} = \omega(4\pi e)^{-1}\ \underset{\sim}{k} \cdot \underset{\approx}{\chi} \cdot \underset{\sim}{\mathcal{A}}_1 e^{i\psi} + \text{c.c.}$$

$$= -(n_0 e\ \underset{\sim}{k} \cdot \underset{\sim}{\mathcal{A}}_1/\omega)e^{i\psi} + \text{c.c.} \tag{8.4.20}$$

The last step follows from the use of (8.1.22). Inserting (8.4.20) into (8.4.19) we obtain

$$e\int d^3p(\underset{\sim}{v}_2 f_1) = [n_0 e^3 \underset{\sim}{A}_{20}\ \underset{\sim}{k} \cdot \underset{\sim}{\mathcal{A}}_1/\omega$$

$$+ n_0 e^3\ \underset{\sim}{\mathcal{A}}_{22}\ \underset{\sim}{k} \cdot \underset{\sim}{\mathcal{A}}_1^*/\omega]e^{i\psi} + \text{c.c.} \tag{8.4.21}$$

For the last integral in (8.4.1) we need $\underset{\sim}{v}_{3v1}$ only to zeroth order in p. Using (7.1.26) we find

$$e\int d^3p\ \underset{\sim}{v}_{3\upsilon 1}\ f_0 = n_0 e^4(\underset{\sim}{a}_1|\underset{\sim}{a}_1|^2 + \tfrac{1}{2}\underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1)e^{i\psi} + \text{c.c.} \qquad (8.4.22)$$

We can now sum these integrals to find the total current (8.4.1). First we calculate the partial sum $j'_{3\upsilon 1}$, which is defined to be the total current neglecting ponderomotive effects, i.e., to calculate $j'_{3\upsilon 1}$ we neglect terms containing F_2, n_2, Φ_{20}, and A_{20}. Thus, we keep the first two terms in (8.4.9), the last term in (8.4.17), the last term in (8.4.21), and all of (8.4.22). The result is

$$\begin{aligned} j'_{3\upsilon 1} = {} & [n_0 e^3 \underset{\sim}{k}(\underset{\sim}{a}_1^* \cdot \underset{\sim}{a}_{22}/\omega - e\ \underset{\sim}{k} \cdot \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1/\omega^2) - n_0 e^4 \underset{\sim}{a}_1|\underset{\sim}{a}_1|^2 \\ & + \tfrac{1}{2}\, n_0 e^4\, \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 + e^2 \underset{\sim}{a}_1^* \eta_{22} \\ & + n_0 e^3 \underset{\sim}{a}_{22}\ \underset{\sim}{k} \cdot \underset{\sim}{a}_1^*/\omega]e^{i\psi} + \text{c.c.} \end{aligned} \qquad (8.4.23)$$

Restoring units in this expression we find the amplitude,

$$\begin{aligned} \underset{\sim}{\mathcal{J}}'_{3\upsilon 1} = {} & n_0 e_s^3\ \underset{\sim}{k}\ \underset{\sim}{a}_1^* \cdot \underset{\sim}{a}_{22}/m_s^2 c^2 \omega - n_0 e_s^4 \underset{\sim}{k}\ \underset{\sim}{k} \cdot \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1/m_s^3 c^3 \omega^2 \\ & + n_0 e_s^4 \underset{\sim}{a}_1|\underset{\sim}{a}_1|^2/m_s^3 c^5 + \tfrac{1}{2}\, n_0 e_s^4\, \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1/m_s^3 c^5 \\ & - \eta_{22}\ e_s^2\ \underset{\sim}{a}_1^*/m_s c + n_0 e_s^3 \underset{\sim}{a}_{22}\ \underset{\sim}{k} \cdot \underset{\sim}{a}_1^*/m_s^2 c^2 \omega \quad . \end{aligned} \qquad (8.4.24)$$

We note that only the electron currents are significant because of the factors of m_s^{-1}. We now use Eqs. (8.3.19) and (8.3.24) in Eq. (8.4.24) and $e = -e_0$, and so obtain

$$\mathcal{J}'_{3v1} = -\frac{n_0 e_0^4}{m_e^3 \omega^2 c^3} \underset{\sim}{k}\,\underset{\sim}{k} \cdot \underset{\sim}{a}_1^* \left[\frac{\omega^2 \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1}{\omega^2 - \omega_e^2/4} + \frac{\omega_e^2 (\hat{k} \cdot \underset{\sim}{a}_1)^2}{4\omega^2 - \omega_e^2} \right]$$

$$+ \frac{n_0 e_0^4}{2 m_e^3 c^5} \underset{\sim}{a}_1^* \left[\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 \left(1 - \frac{k^2 c^2}{\omega^2 - \omega_e^2/4} \right) - \frac{2k^2 c^2 (\hat{k} \cdot \underset{\sim}{a}_1)^2}{\omega^2 - \omega_e^2/4} \right]$$

$$+ \frac{n_0 e_0^4 \underset{\sim}{a}_1 |\underset{\sim}{a}_1|^2}{m_e^3 c^5} \quad . \tag{8.4.25}$$

This result is correct when the particles do not have time to respond to ponderomotive effects, such as the case of a fast wave packet on the case of the early development of a laser pulse. As discussed by Arons and Max,[16] fast means that the wave packets changes before a sound wave has time to cross it. For comparison we transverse waves. Then (8.4.25) reduces to

$$\mathcal{J}'_{3v1} = \frac{n_0 e_0^4}{2 m_e^3 c^5} \left[2 \underset{\sim}{a}_1 |\underset{\sim}{a}_1|^2 + \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 \left(\frac{3\omega_e^2}{4\omega^2 - \omega_e^2} \right) \right] \quad , \tag{8.4.26}$$

which agrees with Eq. (15) of Arons and Max.[16]

Now we sum the remaining terms, the ponderomotive terms. From Eqs. (8.4.9, 14, 17, and 21) we find

$$\mathcal{J}_{3v1} - \mathcal{J}'_{3v1} = -e^2 \underset{\sim}{a}_1 \Big[n_{20} + n_0 e^2 |\underset{\sim}{a}_1|^2 (5/2 - k^2/\omega^2) - n_0 e^2 |\hat{k} \cdot \underset{\sim}{a}_1|^2 k^2/\omega^2$$

$$+ n_0 e \Phi_{20} (5/2 - k^2/\omega^2) \Big] + n_0 e^2 (\underset{\sim}{k}/\omega^2) \Big[\underset{\sim}{k} \cdot \underset{\sim}{a}_1 \; (3e^2 |\underset{\sim}{a}_1|^2 + 2e\Phi_{20})$$

$$+ e^2 \underset{\sim}{k} \cdot \underset{\sim}{a}_1^* \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_1 \Big] + n_0 e^4 \underset{\sim}{a}_1^* (\underset{\sim}{k} \cdot \underset{\sim}{a}_1)^2/\omega^2 - i n_0 e^3 \underset{\sim}{a}_1 \times \underset{\sim}{B}_{20}/\omega \quad . \tag{8.4.27}$$

We restore units in this equation, and we use $e = -e_0$ to obtain

$$\mathcal{J}_{\sim 3v1} - \mathcal{J}'_{\sim 3v1} = - (e_0^2 \mathcal{A}_{\sim 1}/m_e c)\Big[n_{20} + n_0 e_0^2 |\mathcal{A}_{\sim 1}|^2 (5/2 - k^2c^2/\omega^2)/m_e^2 c^4$$

$$- n_0 e_0^2 (\hat{k} \cdot \mathcal{A}_{\sim 1}|^2 \; k^2/m_e^2\omega^2 c^2 - n_0 e_0 \Phi_{20}(5/2 - k^2c^2/\omega^2)/m_e c^2\Big]$$

$$+ n_0 e_0^2 (\underset{\sim}{k}c/m_e\omega^2)\Big[\underset{\sim}{k} \cdot \mathcal{A}_{\sim 1}(3e_0^2 |\mathcal{A}_{\sim 1}|^2/m_e^2 c^4 - 2e_0\Phi_{20}/m_e c\;)$$

$$+ e_0^2 \underset{\sim}{k} \cdot \mathcal{A}_{\sim 1}^* \mathcal{A}_{\sim 1} \cdot \mathcal{A}_{\sim 1}/m_e^2 c^4\Big] + n_0 e_0^4 \mathcal{A}_{\sim 1}^* (\underset{\sim}{k} \cdot \mathcal{A}_{\sim 1})^2/m_e^3\omega^2 c^3$$

$$+ i\, n_0 e_0^3 \mathcal{A}_{\sim 1} \times \underset{\sim}{B}_{20}/m_e^2 \omega c^2 \quad , \qquad (8.4.28)$$

where n_{20}, Φ_{20}, and $\underset{\sim}{B}_{20}$ are given by Eqs. (8.2.49, 47, and 68). We see that the ponderomotive terms contribute substantially to the third-order nonlinear current.

Finally we sum Eqs. (8.4.25) and (8.4.28) to obtain the total third-order first-harmonic nonlinear current. We find

$$\underset{\sim}{\mathcal{J}}_{3v1} = -\frac{e_0 \underset{\sim}{1}}{m\,c}\Big[n_{20} + n_0 e_0^2 |\underset{\sim}{\mathcal{A}}_1|^2 (3/2 - k^2c^2/\omega^2)/m_e^2 c^4$$

$$- n_0 e_0^2 |\hat{k}\cdot\underset{\sim}{\mathcal{A}}_1|^2 k^2/m_e^2\omega^2 c^2 - n_0 e_0 \Phi_{20}(5/2 - k^2c^2/\omega^2)/m_e c^2\Big]$$

$$- \frac{n_0 e_0^2\, \underset{\sim}{\mathcal{A}}_1^*}{m_e c}\left[\frac{e_0^2\, \underset{\sim}{\mathcal{A}}_1\cdot\underset{\sim}{\mathcal{A}}_1\, \omega_e^2}{m_e^2 c^4(4\omega^2 - \omega_e^2)} + \frac{e_0^2(\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1)^2\, \omega_e^2}{m_e^2 c^2\omega^2(4\omega^2 - \omega_e^2)}\right]$$

$$+ \frac{n_0 e_0^2\, \underset{\sim}{k}c}{m_e\omega^2}\left[\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1\left(\frac{3e_0^2|\underset{\sim}{\mathcal{A}}_1|^2}{m_e^2 c^4} - \frac{2e_0\Phi_{20}}{m_e c^2}\right) - \underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1^*\,\frac{e_0^2(\hat{k}\cdot\underset{\sim}{\mathcal{A}}_1)^2\omega_e^2}{m_e^2 c^4(4\omega^2 - \omega_e^2)}\right.$$

$$\left. - \underset{\sim}{k}\cdot\underset{\sim}{\mathcal{A}}_1^*\left(\frac{e_0^2\, \underset{\sim}{\mathcal{A}}_1\cdot\underset{\sim}{\mathcal{A}}_1\, \omega^2}{m_e^2 c^4(4\omega^2 - \omega_e^2)}\right)\right] + i n_0 e_0^3 \underset{\sim}{\mathcal{A}}_1 \times \underset{\sim}{B}_{20}/m_e^2\omega c^2 \quad . \qquad (8.4.29)$$

Putting this in terms of the electric field $\underset{\sim}{\mathcal{E}}_1 = i\omega\underset{\sim}{\mathcal{A}}_1/c$, we obtain

$$\underset{\sim}{\mathcal{J}}_{3v1} = \frac{i e_0\, \underset{\sim}{\mathcal{E}}_1}{m_e\omega}\Big[n_{20} + n_0 e_0^2 |\underset{\sim}{\mathcal{E}}_1|^2 (3/2 - k^2c^2/\omega^2)/m_e^2\omega^2 c^2$$

$$- n_0 e_0^2 |\hat{k}\cdot\underset{\sim}{\mathcal{E}}_1|^2 k^2/m_e^2\omega^4 - n_0 e_0 \Phi_{20}(5/2 - k^2c^2/\omega^2)/m_e c^2\Big]$$

$$- \frac{i n_0 e_0\, \underset{\sim}{\mathcal{E}}_1^*}{m_e\omega}\left[\frac{3e_0^2\, \underset{\sim}{\mathcal{E}}_1\cdot\underset{\sim}{\mathcal{E}}_1\, \omega_e^2}{2m_e^2\omega^2 c^2(4\omega^2 - \omega_e^2)} - \frac{e_0^2(\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{E}}_1)^2\, \omega_e^2}{m_e^2\omega^4(4\omega^2 - \omega_e^2)}\right]$$

$$+ \frac{i n_0 e_0^2 \underset{\sim}{k}c^2}{m_e\omega^3}\left[\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{E}}_1^*\left(\frac{e_0^2\, \underset{\sim}{\mathcal{E}}_1\cdot\underset{\sim}{\mathcal{E}}_1\, \omega_e^2}{m_e^2\omega^2 c^2(4\omega^2 - \omega_e^2)} + \frac{e_0^2(\hat{k}\cdot\underset{\sim}{\mathcal{E}}_1)^2\, \omega_e^2}{m_e^2\omega^2 c^2(4\omega^2 - \omega_e^2)}\right)\right.$$

$$\left. - \underset{\sim}{k}\cdot\underset{\sim}{\mathcal{E}}_1\left(\frac{3e_0|\underset{\sim}{\mathcal{E}}_1|^2}{m_e^2\omega^2 c^2} - \frac{2e_0\Phi_{20}}{m_e c^2}\right)\right] + n_0 e_0\, \underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20}/m_e\omega c \quad , \qquad (8.4.30)$$

where n_{20}, Φ_{20}, and $\underset{\sim}{B}_{20}$ are given by Eqs. (8.2.49, 47, and 68). Two of the terms in this equation have simple interpretations. The term

$in_{20}e_0^2\,\underset{\sim}{\mathcal{E}}_1/m_e\omega$ is the change in the electron current response due to the change n_{20} in the average density. The term $n_0e_0^3\,\underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20}/m_e^2\omega^2c$ is the change in the response due to the laser generated magnetic field. Both terms can be obtained by the heuristic method of section 6.4. Furthermore, the first term, $in_{20}e_0^2\,\underset{\sim}{\mathcal{E}}_1/m_e\omega$, is the largest term; all others are smaller by the factor $k^2v_e^2/\omega^2$ or v_e^2/c^2. The fact that the heuristic technique gives the dominant term leads one to trust it. However, we also note that the heuristic technique gives the term $n_0e_0^3\,\underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20}/m_e^2\omega^2c^2$ with no hint of the many other terms in Eq. (8.4.26) of the same magnitude.

For the final result of this chapter, we insert the formulas (8.2.47) and (8.2.50b) into Eq. (8.4.30). We find

$$
\begin{aligned}
\underset{\sim}{\mathcal{J}}_{3v1} = {} & \frac{e_0^2\,\underset{\sim}{\mathcal{E}}_1}{4\pi i m_e\omega}\left\{\frac{|\underset{\sim}{\mathcal{E}}_1|^2 - |\underset{\sim}{\mathcal{B}}_1|^2}{T}\right. \\
& \left. + \frac{\omega_e^2}{\omega^2}\left[\frac{|\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{E}}_1|^2}{m_e\omega^2} + \frac{|\underset{\sim}{\mathcal{E}}_1|^2}{m_ec^2}\left(\frac{2T_i + T_e(k^2c^2/\omega^2 - 3)}{T}\right)\right]\right\} \\
& + \frac{e_0^2\,\underset{\sim}{\mathcal{E}}_1^*}{4\pi i m_e\omega}\left[\frac{3e_0^2\,\underset{\sim}{\mathcal{E}}_1\cdot\underset{\sim}{\mathcal{E}}_1\,\omega_e^2}{2m_e^2\omega^2c^2(4\omega^2 - \omega_e^2)} - \frac{e_0^2|\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{E}}_1|^2\omega_e^2}{m_e^2\omega^4(4\omega^2 - \omega_e^2)}\right] \\
& + \frac{e_0\,\hat{k}\,\hat{k}\cdot\underset{\sim}{\mathcal{E}}_1}{4\pi i m_e\omega}\left(\frac{k^2\omega_e^2|\underset{\sim}{\mathcal{E}}_1|^2}{m_e^2\,\omega^4}\right)\left(\frac{3T_e + T_i}{T}\right) + n_0e_0^3\,\underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20}/m_e^2\omega^2c \\
& - \frac{e_0^2\,\hat{k}\,\hat{k}\cdot\underset{\sim}{\mathcal{E}}_1^*}{4\pi i m_e\omega}\left(\frac{\omega_e^4\,k^2[\underset{\sim}{\mathcal{E}}_1\cdot\underset{\sim}{\mathcal{E}}_1 + (\hat{k}\cdot\underset{\sim}{\mathcal{E}}_1)^2]}{\omega^4 m_e(4\omega^2 - \omega_e^2)}\right) ,
\end{aligned}
\tag{8.4.31}
$$

where $T \equiv T_e + T_i$.

In conclusion, we have derived the nonlinear conductivity of the plasma to second order in $\underset{\sim}{\mathcal{E}}_1$ and to zeroth order in v_e^2, keeping both ponderomotive effects and cold plasma effects, thereby extending and unifying previous work.

8.4b Nonlinear Evolution of the Wave Packet

Finally we have the ingredients needed for deriving the nonlinear evolution of the wave packet amplitude. We assemble these ingredients here in a discussion of the nonlinear wave-packet evolution of longitudinal waves and of transverse waves. In both cases we concentrate our discussion on the derivation and form of the nonlinear evolution equations. We point out the new terms in the nonlinear evolution equation, and we briefly discuss their effects. However, for detailed discussions of the solutions of these equations we refer to the literature.

We begin with a discussion of linear waves in order to calculate the quantities needed for the left side of Eq. (6.3.65). In section 8.1 we noted that this work is restricted to Langmuir waves and electromagnetic waves, since we ignore resonant particle effects. For such waves $\omega \gg kv_e$ is satisfied, which implies that $\underset{\approx}{D}(\underset{\sim}{k},\omega)$ can be written in the (diagonalized) form, c.f.(8.3.16),

$$\underset{\approx}{D}(\underset{\sim}{k},\omega) = D_{\parallel}(\underset{\sim}{k},\omega)\hat{k}\hat{k} + D_{\perp}(\underset{\sim}{k},\omega)(\underset{\approx}{I} - \hat{k}\hat{k}) \quad , \tag{8.4.32}$$

where

$$D_{\parallel}(\underset{\sim}{k},\omega) = 1 - (\omega_e^2/\omega^2)[1 + 3k^2v_e^2/\omega^2 - \frac{5}{2}(v_e^2/c^2)] \quad , \tag{8.4.33}$$

and

$$D_{\perp}(k,\omega) = 1 - k^2c^2\omega^2 - (\omega_e^2/\omega^2)[1 + k^2v_e^2/\omega^2 - \frac{5}{2}(v_e^2/c^2)] . \tag{8.4.34}$$

(For Langmuir waves $D_{\parallel}(\underset{\sim}{k},\omega) = 0$ and for transverse waves $D_{\perp}(\underset{\sim}{k},\omega) = 0$.) Eq. (8.3.16) implies that the natural polarization unit vectors of $\underset{\approx}{D}$ are $\hat{k}$, $\hat{u}_{\perp 1}(\underset{\sim}{k})$, and $\hat{u}_{\perp 2}(\underset{\sim}{k})$, where $\hat{u}_{\perp 1}$ and $\hat{u}_{\perp 2}$ form an orthonormal basis for the plane perpendicular to $\hat{k}$. We define the corresponding $\mathcal{E}_1^{\alpha}(\underset{\sim}{x},t)$'s of section 6.3 to be $\mathcal{E}_1^{\parallel}$, $\mathcal{E}_1^{\perp 1}$, and $\mathcal{E}_1^{\perp 2}$.

To find the electric field amplitude $\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t)$ from the quantities $\mathcal{E}_1^{\alpha}(\underset{\sim}{x},t)$, we use Eq. (6.3.42), which, in this case, is

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\nabla\cdot\frac{\partial}{\partial \underset{\sim}{k}}\right)^n \left[\mathcal{E}_1^{\parallel}(\underset{\sim}{x},t)\hat{k} + \mathcal{E}_1^{\perp 1}(\underset{\sim}{x},t)\hat{u}_{\perp 1}(\underset{\sim}{k}) + \mathcal{E}_1^{\perp 2}(\underset{\sim}{x},t)u_{\perp 2}(\underset{\sim}{k})\right] . \tag{8.4.35}$$

Note that we now use $\underset{\sim}{k}$, rather than $\underset{\sim}{k}_0$, to denote the central wave vector. To lowest order in ∇, this formula yields

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \approx \mathcal{E}_1^{\parallel}(\underset{\sim}{x},t) + \mathcal{E}_1^{\perp 1}(\underset{\sim}{x},t)\hat{u}_{\perp 1}(\underset{\sim}{k}) + \mathcal{E}_1^{\perp 2}(\underset{\sim}{x},t)\hat{u}_{\perp 2}(\underset{\sim}{k}) \quad , \tag{8.4.36}$$

the usual result from lowest order WKB theory. It is important to remember that (8.4.36) is valid only to lowest order in $\nabla\cdot\partial/\partial\underset{\sim}{k}$, otherwise one encounters what seem to be contradictions, as the following example shows. Consider a longitudinal wave, for which (8.4.35-36) reduce to

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\nabla\cdot\frac{\partial}{\partial \underset{\sim}{k}}\right) [\hat{k}\,\mathcal{E}_1^{\parallel}(\underset{\sim}{x},t)] \tag{8.4.37}$$

and

$$\underset{\sim}{\mathcal{E}}_1(\underset{\sim}{x},t) \approx \hat{k}\,\mathcal{E}_1^{\parallel}(\underset{\sim}{x},t) \quad . \tag{8.4.38}$$

The curl of the electric field found by using the second expression does not vanish:

$$\nabla \times \underset{\sim}{\mathcal{E}}_1 \approx - (\hat{k} \times \nabla \mathcal{E}_1^{\parallel}) e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)} + c.c. \qquad (8.4.39)$$

and so it seems our longitudinal wave has nonzero curl. However, one must remember that we have restricted ourselves to working to zeroth order in $\nabla \mathcal{E}_1^{\parallel}$ by using (8.4.38). To that order, the expression (8.4.39) does vanish. If we wish to go to higher order in ∇, we must use (8.4.37). In fact, it is easy to show that $\nabla \times \underset{\sim}{\mathcal{E}}_1$ does vanish if one uses (8.4.37). We have

$$\nabla \times \underset{\sim}{E}_1 = e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)} (i\underset{\sim}{k} \times \underset{\sim}{\mathcal{E}}_1 + \nabla \times \underset{\sim}{\mathcal{E}}_1) + c.c.$$

$$= e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)} (i\underset{\sim}{k} + \nabla) \times \sum_{n=0}^{\infty} \frac{1}{n!} (-i \nabla \cdot \frac{\partial}{\partial k})^n [\hat{k} \mathcal{E}_1^{\parallel}(\underset{\sim}{x}, t)] + c.c. \quad , \qquad (8.4.40)$$

which is seen to vanish upon noting the identity,

$$i\underset{\sim}{k} \times (-i \nabla \cdot \frac{\partial}{\partial \underset{\sim}{k}})^n \underset{\sim}{g}(\underset{\sim}{x}, \underset{\sim}{k}) = - n \nabla \times (-i \nabla \cdot \frac{\partial}{\partial \underset{\sim}{k}})^{n-1} \underset{\sim}{g}(\underset{\sim}{x}, \underset{\sim}{k})$$

$$+ (-i \nabla \cdot \frac{\partial}{\partial \underset{\sim}{k}})^n i\underset{\sim}{k} \times \underset{\sim}{g}(\underset{\sim}{x}, k) \quad . \qquad (8.4.41)$$

Analogous statements hold for transverse waves.

We now discuss the nonlinear propagation of Langmuir waves, i.e. $\underset{\sim}{k},\omega$ is a root of $D_{\|}$ but not a root of $D_{\perp}$. (We do not discuss $\underset{\sim}{k} \to 0$, where $D_{\|} = D_{\perp}$.) In this case Eq. (8.4.34) applies, and Eqs. (6.3.61) and (6.3.65) become

$$\underset{\sim}{\mathcal{E}}^{\perp}_{31}(\underset{\sim}{x},t) = -4\pi i(\underset{\approx}{I} - \hat{k}\hat{k}) \cdot \underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)/[\omega D_{\perp}(\underset{\sim}{k},\omega)] \tag{8.4.42}$$

and

$$[i\frac{\partial}{\partial t} + i\underset{\sim}{v}_{\|} \cdot \nabla + \frac{1}{2}\underset{\sim}{d}_{\|} : \nabla\nabla]\mathcal{E}^{\|}_{1}(\underset{\sim}{x},t) = -4\pi i\hat{k} \cdot \underset{\sim}{\mathcal{J}}_{3\nu 1}(\underset{\sim}{x},t)/\overline{\omega D_{\|}}(k) \quad . \tag{8.4.43}$$

From Eq. (8.4.28) we obtain the quantities:

$$\omega_{\|}(\underset{\sim}{k}) = \omega_e\left[1 - \frac{5}{2}(v_e^2/c^2) + \frac{3}{2}(k^2\lambda_e^2)\right] \quad , \tag{8.4.44}$$

$$\underset{\sim}{v}_{\|}(k) = 3\underset{\sim}{k}v_e^2/\omega_e \quad , \tag{8.4.45}$$

$$\underset{\approx}{d}_{\|}(\underset{\sim}{k}) = \underset{\approx}{I}\,3v_e^2/\omega_e \quad , \tag{8.4.46}$$

and

$$\overline{\omega D_{\|}}(\underset{\sim}{k}) = 2(\omega_e^2/\omega^2)(1 + 6k^2v_e^2/\omega^2) \quad . \tag{8.4.47}$$

In deriving these quantities we have used $kv_e \ll \omega_e$. Inserting the expression (8.4.38) into (8.4.30), and using Eqs. (8.2.51a,b), we find

$$\mathcal{J}_{3v1} = -\hat{k}\left(\frac{in_0 e_0^2 \mathcal{E}_1^{\parallel}}{m_e\omega}\right)\left(\frac{e_0^2|\mathcal{E}_1^{\parallel}|^2}{m_e\omega^2(T_e + T_i)}\right)\left\{1 - \frac{T_e + T_i}{m_e c^2}\left[\frac{3(2\omega^2 - \omega_e^2)}{4\omega^2 - \omega_e^2} - \frac{5T_i}{2(T_e + T_i)}\right.\right.$$

$$\left.\left. + \frac{k^2c^2}{\omega^2}\left(\frac{9\omega_e^2 - 20\omega^2}{4\omega^2 - \omega_e^2} + \frac{3T_i}{2(T_e + T_i)}\right)\right]\right\} \qquad (8.4.48)$$

This expression and Eq. (8.4.42) yield $\mathcal{E}_{31}^{\perp} = 0$, i.e. no transverse electric field at the fir harmonic is generated since $\mathcal{J}_{3v1}$ is purely longitudinal. Inserting (8.4.44-48) into (8.4.43) we find the evolution equation of $\mathcal{E}_1^{\parallel}(\underset{\sim}{x},t)$:

$$[i\frac{\partial}{\partial t} + 3i(v_e^2/\omega_e)\underset{\sim}{k}\cdot\nabla + 3(v_e^2/\omega_e)\nabla^2]\mathcal{E}_1^{\parallel} = -\alpha_{\parallel}\mathcal{E}_1^{\parallel}|\mathcal{E}_1^{\parallel}|^2 \quad , \quad (8.4.49)$$

where

$$\alpha_{\parallel} \equiv \frac{\omega\left\{1 - \frac{(T_e + T_i)}{m_e c^2}\left[\frac{3(2\omega^2 - \omega_e^2)}{4\omega^2 - \omega_e^2} - \frac{5T_i}{2(T_e + T_i)} + \frac{k^2c^2}{\omega^2}\left(\frac{9\omega_e^2 - 20\omega^2}{4\omega^2 - \omega_e^2} + \frac{3T_i}{2(T_e + T_i)}\right)\right]\right\}}{8\pi n_0(T_e + T_i)(1 + 6k^2v_e^2/\omega^2)}$$

(8.4.50)

Thus we have obtained the nonlinear evolution equation for the Langmuir wave amplitude. Taking the limit $kv_e/\omega \to 0$, we find agreement with Zakharov's[12] Eq. (1.8). Our calculation complements Zakharov's. We have systematically derived (8.4.50) including kinetic corrections, but with the approximation $|k\underset{\sim}{\mathcal{E}}_1| \gg |\nabla\underset{\sim}{\mathcal{E}}_1|$. Zakharov presented a heuristic derivation of his Eq. (1.8), such that he could not include kinetic corrections. However, he did not require $|k\underset{\sim}{\mathcal{E}}_1| \gg |\nabla\underset{\sim}{\mathcal{E}}_1|$. Of course, both

derivations require quasistatic evolution. Since the solutions to (8.4.50) have been discussed elsewhere,[12,27] we do not further discuss nonlinear Langmuir wave propagation.

We now discuss the nonlinear evolution of electromagnetic waves, i.e. we assume $D_\perp(\underset{\sim}{k},\omega) = 0$. In this case Eqs. (8.4.36), (6.3.61), and (6.3.65) become

$$\underset{\sim}{\mathcal{E}}_\perp(\underset{\sim}{x},t) \approx \underset{\sim}{\mathcal{E}}_\perp^\perp(\underset{\sim}{x},t) \equiv \mathcal{E}_\perp^{\perp 1}(\underset{\sim}{x},t)\hat{u}_{\perp 1}(\underset{\sim}{k}) + \mathcal{E}_\perp^{\perp 2}(\underset{\sim}{x},t)\hat{u}_{\perp 2}(\underset{\sim}{k}) \qquad (8.4.51)$$

$$\mathcal{E}_{31}^{\parallel}(\underset{\sim}{x},t) = -4\pi i\,\hat{k}\cdot\underset{\sim}{\mathcal{J}}_{3v1}(\underset{\sim}{x},t)/[\omega D_\parallel(\underset{\sim}{k},\omega)] \quad , \qquad (8.4.52)$$

and

$$\left(i\frac{\partial}{\partial t} + i\underset{\sim}{v}_\perp\cdot\nabla + \frac{1}{2}\underset{\approx}{d}_\perp : \nabla\nabla\right)\underset{\sim}{\mathcal{E}}_\perp^\perp(\underset{\sim}{x},t) = -4\pi i(\underset{\approx}{I} - \hat{k}\hat{k})\cdot\underset{\sim}{\mathcal{J}}_{3v1}(\underset{\sim}{x},t)/\sqrt{\omega D_\perp}(\underset{\sim}{k}) \quad . \qquad (8.4.53)$$

From Eq. (8.4.34) we find quantities needed for the left side of Eq. (3.4.53):

$$\omega_\perp(\underset{\sim}{k}) = (\omega_e^2 + k^2c^2)^{\frac{1}{2}} \quad , \qquad (8.4.54)$$

$$\underset{\sim}{v}_\perp(\underset{\sim}{k}) = \underset{\sim}{k}c^2/\omega_\perp(\underset{\sim}{k}) \quad , \qquad (8.4.55)$$

and

$$\underset{\approx}{d}_\perp(\underset{\sim}{k}) = (\underset{\approx}{I} - \underset{\sim}{k}\,\underset{\sim}{k}\,c^2/\omega_\perp^2)(c^2/\omega_\perp^2) \quad , \qquad (8.4.56)$$

to zeroth order in v_e^2. The thermal corrections to these quantities do not change the physics, $\underset{\sim}{v}_\perp$ remains parallel to $\underset{\sim}{k}$ and $\underset{\approx}{d}_\perp(k)$ can still be written as the sum of $\underset{\approx}{I}$ times one factor and $\underset{\sim}{k}\underset{\sim}{k}$ times another. Therefore we neglect these thermal corrections. On the other hand, we want the RHS to first order in v_e^2, so we need $\overline{\omega D_\perp}^{-1}$ to first order. From Eq. (8.4.34) we find

$$\overline{\omega D_\perp}(\underset{\sim}{k}) = 2(1 + k^2 v_e^2 \omega_e^2/\omega^4) \quad . \tag{8.4.57}$$

We also need the nonlinear current at the first harmonic. Using $\hat{k} \cdot \underset{\sim}{\mathcal{E}}_1 = 0$ in the expression (8.4.30), we find

$$-4\pi i \underset{\sim}{\mathcal{J}}_{3\nu 1} = -\frac{\omega_e^2 \underset{\sim}{\mathcal{E}}_1}{\omega} \frac{e_0^2 |\underset{\sim}{\mathcal{E}}_1|^2}{m_e \omega^2 (T_e + T_i)} \left[1 + \frac{T_i}{m_e c^2} - \frac{T_e}{m_e c^2}\left(\frac{3}{2} - \frac{k^2 c^2}{\omega^2}\right)\right]$$

$$- \frac{i\omega_e^2 e_0}{m_e \omega^2 c} \underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20} + \left(\frac{\omega_e^2 \underset{\sim}{\mathcal{E}}_1^*}{\omega}\right)\left(\frac{e_0^2 \underset{\sim}{\mathcal{E}}_1 \cdot \underset{\sim}{\mathcal{E}}_1}{m_e^2 \omega^2 c^2}\right)\left(\frac{3\omega_e^2}{2(4\omega^2 - \omega_e^2)}\right) \quad . \tag{8.4.58}$$

To obtain the longitudinal part of the first harmonic amplitude we insert (8.4.58) and (8.4.30) into (8.4.52). We find

$$\underset{\sim}{\mathcal{E}}_{31}^{\parallel} = -\frac{i\omega_e^2 e_0 \hat{k}\hat{k} \cdot \underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20}}{m_e \omega c[\omega^2 - \omega_e^2(1 - 3k^2 v_e^2/\omega^2 - \frac{5}{2} v_e^2/c^2)]} \quad . \tag{8.4.59}$$

We see that only the self generated magnetic field term contributes $\underset{\sim}{\mathcal{E}}_{31}^{\parallel}$. Thus the longitudinal part of the first harmonic amplitude vanishes when $\underset{\sim}{B}_{20}$ vanishes.

To derive the nonlinear evolution equation for $\underset{\sim}{\mathcal{E}}_1$, we insert Eqs. (8.4.57) and (8.5.58) into (8.4.53) and obtain

$$[i(\frac{\partial}{\partial t} + v_1 \cdot \nabla) + \frac{1}{2}\, \underset{\approx}{d}_1 : \nabla\nabla]\underset{\sim}{\mathcal{E}}_1 = -\alpha \underset{\sim}{\mathcal{E}}_1 |\underset{\sim}{\mathcal{E}}_1|^2 + \beta \underset{\sim}{\mathcal{E}}_1^* \underset{\sim}{\mathcal{E}}_1 \cdot \underset{\sim}{\mathcal{E}}_1 - i\gamma(\underset{\approx}{I} - \hat{k}\hat{k}) \cdot \underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20} \quad , \tag{8.4.60}$$

where

$$\alpha = \frac{\omega_e^2[1 + T_i/m_e c^2 - T_e(\frac{3}{2} - k^2c^2/\omega^2)/m_e c^2]}{\omega\, 8\pi\, n_0(T_e + T_i)(1 + 3k^2 v_e^2 \omega_e^2/\omega^2)} \tag{8.4.61}$$

$$\beta = \frac{3e_0^2 \omega_e^4}{2m_e^2 \omega^3 c^2 (4\omega^2 - \omega_e^2)} \tag{8.4.62}$$

$$\gamma = \frac{e_0 \omega_e^2}{2m_e \omega^2 c} \quad . \tag{8.4.63}$$

Since $\hat{k} \cdot \underset{\sim}{\mathcal{E}}_1 = 0$, the last term in (8.4.60) can be simplified:

$$-\, i\gamma(\underset{\approx}{I} - \hat{k}\hat{k}) \cdot \underset{\sim}{\mathcal{E}}_1 \times \underset{\sim}{B}_{20} = -\, i\gamma \hat{k} \cdot \underset{\sim}{B}_{20} \underset{\sim}{\mathcal{E}}_1 \times \hat{k} \quad . \tag{8.4.64}$$

In addition, the second term may be written

$$\beta\, \underset{\sim}{\mathcal{E}}_1^*\, \underset{\sim}{\mathcal{E}}_1 \cdot \underset{\sim}{\mathcal{E}}_1 = \beta\, \underset{\sim}{\mathcal{E}}_1 \times (\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1) + \beta\, \underset{\sim}{\mathcal{E}}_1 |\underset{\sim}{\mathcal{E}}_1|^2$$

$$= \beta\, \hat{k} \cdot (\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1) \underset{\sim}{\mathcal{E}}_1 \times \hat{k} + \beta\, \underset{\sim}{\mathcal{E}}_1 |\underset{\sim}{\mathcal{E}}_1|^2 \quad . \tag{8.4.65}$$

Thus Eq. (8.4.60) becomes

$$[i(\frac{\partial}{\partial t} + \underset{\sim}{v}_1 \cdot \nabla) + \frac{1}{2} d_1 : \nabla\nabla]\underset{\sim}{\mathcal{E}}_1 = -(\alpha + \beta)\underset{\sim}{\mathcal{E}}_1 |\underset{\sim}{\mathcal{E}}_1|^2 - i\underset{\sim}{\mathcal{E}}_1 \times \hat{k}\hat{k} \cdot (i\beta\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1)$$

$$- i\underset{\sim}{\mathcal{E}}_1 \times \hat{k}\hat{k} \cdot (\gamma \underset{\sim}{B}_{20}) \quad . \qquad (8.4.66)$$

Two of the nonlinear terms in (8.4.66) have appeared previously in the literature. The first nonlinear term $-(\alpha + \beta)\underset{\sim}{\mathcal{E}}_1 |\underset{\sim}{\mathcal{E}}_1|^2$ was obtained in Ref. 13 by heuristic means. Our work is a slight improvement in that we have obtained the corrections of orders $k^2 v_e^2/\omega^2$ and v_e^2/c^2 to the coefficient $(\alpha + \beta)$. The second nonlinear term $-i\underset{\sim}{\mathcal{E}}_1 \times \hat{k}\hat{k} \cdot (i\beta\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1)$, appears in References 1,14-16. Arons and Max[16] showed that this term causes the polarization of the wave to precess, as one might infer from the presence of the cross product $\underset{\sim}{\mathcal{E}}_1 \times \hat{k}$ in this term. The last term $-i\underset{\sim}{\mathcal{E}}_1 \times \hat{k}\hat{k} \cdot (\gamma \underset{\sim}{B}_{20})$ has not appeared previously in the literature. The similarity between the last term and the second term allows us to deduce that the last term also contributes to precession of the wave polarization. In fact, this term causes self Faraday rotation by the self generated magnetic field $\underset{\sim}{B}_{20}$.

Since these last two terms are so similar, let us compare their magnitudes. Neglecting ∇ in Eq. (8.2.65) we find

$$\frac{\hat{k} \cdot i\beta\underset{\sim}{\mathcal{E}}_1^* \times \underset{\sim}{\mathcal{E}}_1}{\hat{k} \cdot \gamma \underset{\sim}{B}_{20}} = \frac{3\omega^2}{4\omega^2 - \omega_e^2} \quad . \qquad (8.4.67)$$

Thus the self precession effect of Arons and Max[16] is comparable to the self Faraday rotation effect found here. This illustrates how the heuristic method(c.f. sec. 6.4) for including nonlinearity can err. Had we used that method, we would have simply inserted the change in the conductivity due to $\underset{\sim}{B}_{20}$ into Eq. (6.4.6), thereby obtaining the last term in Eq. (8.4.66). However, we would have neglected the second term, which is of the same magnitude, and which contributes to the same effect, polarization precession. Thus the heuristic method leaves out terms which are comparable to the terms it includes.

In conclusion, we have used the systematic method of section 6.3 to derive the equations which describe the nonlinear evolution of Langmuir waves and of electromagnetic waves. In the case of Langmuir waves we verified previous heuristic work and make a slight improvement by finding corrections to the coefficient $\alpha_{\parallel}$ of the nonlinearity. In electromagnetic waves we also verified previous work, but in addition we found that the previously neglected self Faraday rotation effect is comparable to the self precession effect discussed previously.[16]

8.5 Summary of Chapter 8

We have presented a systematic discussion of nonlinear wave evolution. Our discussion is founded on the systematic perturbation scheme of section 6.3 and on the Lie transform perturbation theory outlined in section 7.2. Our results include a self consistent discussion of the fields and particle response through third order in the wave amplitude.

In first order (section 8.1) we showed how Lie transforms are used to derive the linear response of a relativistic unmagnetized plasma. We pointed out that there exist two nonrelativistic limits of the linear response which differ in the order assigned to ω/kc.

In second order we found responses at the zeroth harmonic (section 8.2) and the second harmonic (section 8.3). Central to the zeroth-harmonic discussion is the concept of pondermotive Hamiltonian, a generalization of the pondermotive potential. The pondermotive Hamiltonian, together with the zeroth-harmonic fields Φ_{20} and $\underset{\sim}{A}_{20}$, determine the evolution of the oscillation-center distribution. We showed how the pondermotive Hamiltonian can be obtained very simply from the K-X theorem of section 7.4. In the case of quasistatic evolution, where the oscillation centers have time to equilibrate with the wavepacket, we found simple expressions for the nonlinear oscillation center response (8.2.31), the nonlinear density response (8.2.50b), and the self generated magnetic field (8.2.68). In addition we found, in section 8.2c, that the oscillation center formalism leads to a simple interpretation of wave momentum.

The calculation of the second-order second-harmonic response was very short. First we used Lie transforms to calculate the second-order second-harmonic nonlinear current. Then we used the formalism of section 6.3 to find the associated electric field.

Finally (in section 8.4) we computed the third-order nonlinear current $j_{3\nu 1}$ and determined its effect on the evolution of the first-harmonic amplitude. In this calculation we unified and added to previous work by including all of the previously know effects[1,13-16] and a previously neglected effect, self Faraday rotation, in a single equation for the nonlinear evolution of an electromagnetic wave in an unmagnetized plasma. An important conclusion of this study is that one must use a systematic perturbation theory to discuss nonlinear wave evolution, since the heuristic method commonly in use may leave out effects comparable to the effects it includes.

9. Propagation of Test Waves in the Laser Plasma System

In the last section we studied the laser plasma system. We calculated the nonlinear density change and the self generated magnetic field, and we derived the equation which governs the evolution of the laser amplitude. In this section we complicate matters by adding in a small amplitude "test wave". Our goal is to understand the linear propagation of this test wave in the laser plasma system.

We are motivated by recent experiments 1-3, which use the Faraday rotation effect to measure the laser-generated magnetic field.[4-6] In a typical experiment a laser propagates through the plasma until it reaches the critical layer (defined to be where $\omega_p(\underset{\sim}{x})=\omega$), where it is reflected. In this process several mechanisms are present which lead to magnetic field generation (see ref. 7 and references therein). We are concerned with the so called resonance-absorption-generated magnetic field[9-12], the field produced mainly at the resonance layer, or critical layer. According to Eq. (8.2.68), the self-generated magnetic field is large whenever the electric field amplitude is large, such as at the resonance point.

One way to measure magnetic fields is to use magnetic probes.[4-6] Unfortunately, magnetic probes are much larger than the laser-plasma interaction region, and so they measure a spatially averaged magnetic field which is much smaller than the peak magetic field. Magnetic probes are small enough to measure the resonance-absorption-generated magnetic field in microwave simulations [13,14] of the laser-plasma interaction, but the relation of such simulations to the laser-plasma interaction is not entirely clear.

Thus the more direct method of using the Faraday rotation effect to measure the laser-generated magnetic field was introduced[1]. This method

employs a small amplitude test wave, which propagates through the laser irradiated region, (although both waves are generated by lasers, to distinguish them we call the large amplitude wave the laser or laser beam, and we call the small amplitude wave the test wave.) As the test wave propagates, the quasistatic magnetic field causes its polarization to rotate; this is the Faraday rotation effect. Therefore, one can deduce the magnetic field by measuring the angle through which the polarization rotates upon traversing the magnetized region. Typically, such measurements[1-3] yield magnetic field magnitudes of 0.1-2.0 megagauss.

At present, Faraday rotation experiments do not measure the resonance-absorption-generated magnetic field, because the test-wave frequency is too low to allow the test-wave to propagate near the critical layer. However, if the test wave could propagate near the critical layer while the laser was present, then one might expect to see other effects besides Faraday rotation, since the self-generated magnetic field is not the only quantity which is driven by the laser. For example, the laser generates second-harmonic electromagnetic fields, and it changes the particle motion via the ponderomotive Hamiltonians.

In this chapter we determine these effects systematically for the following situation. We study a uniform plasma, through which a large amplitude laser beam and a small aplitude test wave propagate. Thus, the first-order electric field has the form $\underset{\sim}{E}_1(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}_L(\underset{\sim}{x},t)\exp(i\psi) + \underset{\sim}{\mathcal{E}}_t \exp(i\psi_t) + \text{c.c.}$, where $\underset{\sim}{\mathcal{E}}_L$ is the laser amplitude, $\underset{\sim}{\mathcal{E}}_t$ is the test wave amplitude, $\psi \equiv \underset{\sim}{k}.\underset{\sim}{x} - \omega t$ is the rapidly varying phase of the laser, and $\psi_t \equiv \underset{\sim}{k}_t.\underset{\sim}{x} - \omega_t t$ is the rapidly varying phase of the test wave.

Our method for determining these effects is very similar to the method of chapter 6, which was used in chapter 8 to derive the nonlinear evolution equations governing waves in unmagnetized plasma. This method can be summarized as follows:

From the first order fields we calculate the second-order nonlinear currents. We separate these currents according to their rapidly varying phase, i.e. harmonic number. In second order (it turns out that) the phases do not correspond to waves, and so we find the associated electromagnetic fields by inverting $\underset{\sim}{D}$, as in Eqs. (6.3.50). Having solved the second order problem, we can compute the third-order nonlinear current.

The third-order current can be divided into term according to their phase. The phases of some of these terms do not correspond to waves, and so the associated electromagnetic field can be found using $\underset{\sim}{D}^{-1}$. Other terms exist with phases exp $(\pm i\psi_t)$, i.e. they are in resonance with the test wave. Such terms modify the evolution of the test-wave amplitude as in Eq. (6.3.65).

We divide this procedure into two parts. In the first-part, sec. 9.1, we calculate the third-order nonlinear current at the first-harmonic of the test wave. In this calculation we keep only those terms that are linear in the test wave amplitude, since the test wave is assumed small. As a result we obtain the laser-modified test-wave conductivity, the tensor which gives the current at the first harmonic of the test wave when multiplied by the test-wave electric field.

In the second part of this procedure, sec. 9.2, we determine the effect of the laser-induced conductivity on the evolution of the test-wave. We find a number of new effects. One of these effects is that an electromagnetic test wave acquires a longitudinal part. In addition, the test-wave polarization changes as the test wave propagates through the laser-irradiated plasma. In part this polarization change is due to Faraday rotation by the laser-generated magnetic field, but there are also other effects which are not simply a rotation of the test-wave polarization. For example, an initially plane-polarized wave can become circularly polarized. Thus we

conclude that the inference of the resonance-absorption-generated quasistatic magnetic field from measurements of the polarization change cannot be accomplished by using only Faraday rotation theory.

9.1 Conductivity of the Laser-Plasma System

9.1a First Order

The first part of the problem is to calculate the currents produced by the test wave in the laser plasma system. This calculation is similar to the calculation of section 8.4a, so the reader may wish to skip to the result, Eq. (9.1.67-68).

The first-order vector potential is assumed to have the form,

$$\underset{\sim}{A}_1(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{A}}_L(\underset{\sim}{x},t)e^{i\psi_L} + \underset{\sim}{\mathcal{A}}_t(\underset{\sim}{x},t)e^{i\psi_t} + \text{c.c.} \quad , \tag{9.1.1}$$

where $\psi_L \equiv \underset{\sim}{k}.\underset{\sim}{x} - \omega t$ and $\psi_t \equiv \underset{\sim}{k}_t \cdot \underset{\sim}{x} - \omega_t t$. Therefore, the first-order Hamiltonian is

$$h_1(\underset{\sim}{q},\underset{\sim}{p},t) = h_{L1}(\underset{\sim}{q},\underset{\sim}{p},t) + h_{t1}(\underset{\sim}{q},\underset{\sim}{p},t) \quad , \tag{9.1.2}$$

where

$$h_{L1}(\underset{\sim}{q},\underset{\sim}{p},t) = -e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_L(\underset{\sim}{q},t)e^{i\psi_L} + \text{c.c.} \tag{9.1.3}$$

and

$$h_{t1}(\underset{\sim}{q},\underset{\sim}{p},t) = -e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_t(\underset{\sim}{q},t)e^{i\psi_t} + \text{c.c.} \tag{9.1.4}$$

In these equations we have introduced notation that we will use throughout this chapter. The subscript L denotes quantities that are independent of the test wave amplitude, i.e. quantities associated with only the laser. The quantities of Chapter 8 are all of this sort. The subscript t denotes quantities that are linearly dependent on the test-wave amplitude. A quantity with neither the subscript L nor the

subscript t is the sum of the laser quantity and the test wave quantity. The rest of the notation is identical to that of previous chapters. The first numerical subscript refers to order, and the Greek letters λ and ν refer to linear and nonlinear.

Using Eqs. (9.1.2-4) we calculate the remaining first-order quantities. Integration along an orbit gives w_1:

$$w_1 = w_{L1} + w_{t1} \quad , \tag{9.1.5}$$

where

$$w_{L1} = -\frac{ie\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_L e^{i\psi_L}}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega} + \text{c.c.} \tag{9.1.6}$$

and

$$w_{t1} = -\frac{ie\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_t e^{i\psi_t}}{\underset{\sim}{k}_t \cdot \underset{\sim}{v}_0 - \omega_t} + \text{c.c.} \tag{9.1.7}$$

Next we compute $f_1 = -\{w_1, f_0\}$:

$$f_1 = f_{L1} + f_{t1} \quad , \tag{9.1.8}$$

where

$$f_{L1} = -\frac{e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_L e^{i\psi_L}}{\underset{\sim}{k}_t \cdot \underset{\sim}{v}_0 - \omega} \underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{p}} + \text{c.c.} \tag{9.1.9}$$

and

$$f_{t1} = -\frac{e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_t e^{i\psi_t}}{\underset{\sim}{k}_t \cdot \underset{\sim}{v}_0 - \omega_t} k_t \cdot \frac{\partial f_0}{\partial p} + \text{c.c.} \tag{9.1.10}$$

From f_1 we obtain the density:

$$n_1 = n_{L1} + n_{t1} \quad , \tag{9.1.11}$$

where

$$n_{L1} = - \frac{n_0 e \underset{\sim}{k} \cdot \underset{\sim}{\mathcal{A}}_L e^{i\psi_L}}{\omega} + c.c. + O(k^2 v_T^2/\omega^2) \tag{9.1.12}$$

and

$$n_{t1} = - \frac{n_0 e \underset{\sim}{k} \cdot \underset{\sim}{\mathcal{A}}_t e^{i\psi_t}}{\omega_t} + c.c. + O(k_t^2 v_T^2/\omega_t^2) \quad , \tag{9.1.13}$$

and the current $\underset{\sim}{j}_1 = \int d^3p \; (\underset{\sim}{\eta}_0 f_1 + \underset{\sim}{\eta}_1 f_0)$:

$$\underset{\sim}{j}_1 = \underset{\sim}{j}_{L1} + \underset{\sim}{j}_{t1} \quad , \tag{9.1.14}$$

where

$$\underset{\sim}{j}_{L1} = - n_0 e^2 \underset{\sim}{\mathcal{A}}_L e^{i\psi_L} + c.c. + O(k^2 v_T^2/\omega^2) \quad , \tag{9.1.15}$$

and

$$\underset{\sim}{j}_{t1} = - n_0 e^2 \underset{\sim}{\mathcal{A}}_t e^{i\psi_t} + c.c. + O(k_t^2 v_T^2/\omega_t^2) \quad . \tag{9.1.16}$$

The laser density n_{1L} vanishes since $\underset{\sim}{k} \cdot \underset{\sim}{\mathcal{A}}_L = 0$. However, we do not require $\underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t = 0$ at this time.

9.1b Second Order

The second-order calculation proceeds in the usual manner. First we compute the second-order nonlinear current. Then we use $\underset{\approx}{D}^{-1}$ to find the second-order electric field $\underset{\sim}{E}_2$. Finally, we use $\underset{\sim}{E}_2$ to calculate the second-order linear quantities.

The second-order quantities are of three types. There are terms quadratic in $\underset{\sim}{a}_L$ (denoted by subscript L2) with phase 0 or phase $2\psi_L$. These terms were calculated in sections 8.2 and 8.3. There are terms quadratic in $\underset{\sim}{a}_t$, which we ignore, since $\underset{\sim}{a}_t$ is small. Finally, there are terms bilinear in $\underset{\sim}{a}_L$ and $\underset{\sim}{a}_t$ (denoted by the subscript t2), which we calculate here. For example, from Eqs. (7.2.25) and (9.1.1) we obtain

$$h_{2\nu} = h_{L2\nu 0} + h_{L2\nu 2} + h_{t2\nu +} + h_{t2\nu -} \quad , \tag{9.1.17}$$

where $h_{L2\nu 0}$ and $h_{L2\nu 2}$ are given by Eq. (8.2.6) and Eq. (8.3.1), and where

$$h_{t2\nu +} = e^2 \gamma_0^{-1} [\underset{\sim}{a}_L \cdot \underset{\sim}{a}_t - \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_L \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_t] \, e^{i(\psi_t + \psi_L)} + \text{c.c.} \tag{9.1.18}$$

and

$$h_{t2\nu -} = c^2 \gamma_0^{-1} [\underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_t - \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_L^* \underset{\sim}{v}_0 \cdot \underset{\sim}{a}_t] \, e^{i(\psi_t - \psi_L)} + \text{c.c.} \tag{9.1.19}$$

(Note that the final subscripts ± refer to the phases $\psi_\pm \equiv \psi_t \pm \psi_L$. For convenience we also define $\underset{\sim}{k}_\pm \equiv \underset{\sim}{k}_t \pm \underset{\sim}{k}$ and $\omega_\pm = \omega_t \pm \omega$.)

To calculate the effects of the second-order test wave quantities we must first examine the second-order nonlinear Lie transform equations,

$$K_{t2\nu\pm} - \frac{1}{2}\left(\frac{\partial w_{t2\nu\pm}}{\partial t} + \{W_{t2\nu\pm}, h_0\}\right) = h_{t2\nu\pm} + \frac{1}{2}\{w_1, h_1\}_{t\pm} \ , \quad (9.1.20)$$

which are obtained by extracting the test wave part of Eq. (7.2.34). The terms on the right side of (9.1.20) are rapidly varying if $k_\pm v_T \ll \omega_\pm$ is satisfied. The "+" inequality is easily satisfied since

$$k_+ v_T = |\underset{\sim}{k}_t + \underset{\sim}{k}|\ v_T < (k_t + k)v_T = \frac{v_T}{c}(k_+ c + kc) < \frac{v_T}{c}(\omega_t + \omega)$$

$$= \frac{v_T}{c}\omega_+ \quad (9.1.21)$$

holds and $v_T/c \ll 1$ has been assumed. The "−" inequality holds whenever $(v_T/c)(\omega_+/|\omega_-|) \ll 1$ is satisfied as we see from the relation,

$$\frac{k_- v_T}{|\omega_-|} < \frac{v_T}{c}\frac{(k_t + k)c}{|\omega_-|} < \frac{v_T}{c}\frac{\omega_+}{|\omega_-|} \quad . \quad (9.1.22)$$

Therefore, as long as ω_- is not too small (i.e. the test wave frequency is not too close to the laser frequency), the right hand side of (9.1.20) is rapidly varying, and we may choose $K_{t2\nu\pm} = 0$ and compute $w_{t2\nu\pm}$ by integrating the right side of (9.1.20) along a trajectory.

Before computing $w_{t2\nu\pm}$, let us look ahead to the calculation of $\underset{\sim}{j}_{t2\nu\pm}$, which, from Eq. (7.2.35), is given by

$$\underset{\sim}{j}_{t2\nu\pm} = e\int d^3p\ \underset{\sim}{v}_0\ f_{t2\nu\pm} + e\int d^3p(\underset{\sim}{v}_1 f_1)_{t\pm} \quad . \quad (9.1.23)$$

For simplicity we calculate $j_{t2\nu\pm}$ only to zeroth-order in the quantities $k^2 v_T^2/\omega^2$, $k_t^2 v_T^2/\omega_t^2$, $k_\pm^2 v_T^2/\omega_\pm^2$, and v_T^2/c^2, all of which have been assumed to

be small. Essentially this means we are keeping only those terms which are zeroth order in v_T^2. To calculate the first integral in (9.1.23) we need $f_{t2\nu\pm}$. From Eq. (7.2.32) this is given by

$$f_{t2\nu\pm} = -\frac{1}{2}\{w_{t2\nu\pm},f_0\} + \frac{1}{2}\{w_1,\{w_1,f_0\}\}_{t\pm} \quad . \tag{9.1.24}$$

($F_{t2\nu\pm}$ vanishes since $K_{t2\nu\pm}$ vanishes.) Therefore, the first integral of (9.1.23) is

$$e\int d^3p\ \underset{\sim}{v}_0\ f_{t2\nu\pm} = e\int d^3p[\underset{\sim}{v}_0 - \frac{1}{2}\{w_{t2\nu\pm},f_0\} + \frac{1}{2}\ v_0\{w_1,\{w_1,f_0\}\}_{t\pm}] \quad . \tag{9.1.25}$$

The last term in Eq. (9.1.25) gives no contribution to zeroth order in v_T^2. To see this we note that w_1 and $\underset{\sim}{v}_0$ are first order (and higher) in $\underset{\sim}{p}$, and that the two Poisson brackets can introduce at most two $\underset{\sim}{p}$ derivatives. Thus the quantity $\underset{\sim}{v}_0\{w_1,\{w_1,f_0\}\}$ is at least first order in $\underset{\sim}{p}$, and so its integral is at least of order v_T^2 (since f_0 is isotropic, odd moments of p vanish). Hence we have,

$$e\int d^3p\ \underset{\sim}{v}_0\ f_{t2\nu\pm} = -\frac{e}{2}\int d^3p\ \underset{\sim}{v}_0\ \frac{\partial w_{t2\nu\pm}}{\partial \underset{\sim}{q}}\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}} \tag{9.1.26}$$

to zeroth order in v_T^2. Integrating this term by parts,

$$e\int d^3p\ \underset{\sim}{v}_0\ f_{t2\nu\pm} = \frac{1}{2}\ e\int d^3p\ f_0\ \frac{\partial}{\partial \underset{\sim}{p}}\cdot\left(\frac{\partial w_{t2\nu\pm}}{\partial \underset{\sim}{q}}\ \underset{\sim}{v}_0\right) \quad , \tag{9.1.27}$$

we see that we need $w_{t2\nu\pm}$ only to zeroth order in $\underset{\sim}{p}$ in order to find $\underset{\sim}{j}_{t2\nu\pm}$ to zeroth order in v_T^2.

Now we return to the calculation of $w_{t2\nu\pm}$. Since $K_{t2\nu\pm} = 0$, we find $w_{t2\nu\pm}$ by integrating the right side of (9.1.20) along a trajectory. This amounts to division by $i(\underset{\sim}{k}_{\pm} \cdot \underset{\sim}{p} - \omega_{\pm})$, and so the leading order in $\underset{\sim}{p}$ of $w_{t2\nu\pm}$ equals the leading order in $\underset{\sim}{p}$ of the right side of (9.1.20). Thus we need to calculate $h_{t2\nu\pm} + \frac{1}{2}\{w_1,h_1\}_{t\pm}$ only to zeroth order in $\underset{\sim}{p}$. From Eqs. (9.1.18) and (9.1.19) we obtain

$$h_{t2\nu\pm} = e^2 \underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t e^{i\psi_+} + \text{c.c.} \tag{9.1.28}$$

and

$$h_{t2} = e^2 \underset{\sim}{\mathcal{A}}_L^* \cdot \underset{\sim}{\mathcal{A}}_t e^{i\psi_-} + \text{c.c.} \tag{9.1.29}$$

and from Eqs. (9.1.2-7) we find that $\{w_1,h_1\}_t$ vanishes to zeroth order in $\underset{\sim}{p}$. Thus Eq. (9.1.20) yields

$$-\frac{1}{2} w_{t2\nu+} = i e^2 (\underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t/\omega_+) e^{i\psi_+} + \text{c.c.} \tag{9.1.30}$$

and

$$-\frac{1}{2} w_{t2\nu-} = i e^2 (\underset{\sim}{\mathcal{A}}_L^* \cdot \underset{\sim}{\mathcal{A}}_t/\omega_-) e^{i\psi_-} + \text{c.c.} \tag{9.1.31}$$

to zeroth order in $\underset{\sim}{p}$.

At this point we have everything needed to calculate the nonlinear current density and charge density. From Eqs. (9.1.27) and (9.1.30-31) we find

$$e \int d^3p\, \underset{\sim}{v}_0\, f_{t2\nu+} = e^3 n_0 \underset{\sim}{k}_+ (\underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t/\omega_+) e^{i\psi_+} + \text{c.c.} \tag{9.1.32}$$

and

$$e\int d^3p\ \underset{\sim}{v}_0\ f_{t2\nu-} = e^3 n_0 \underset{\sim}{k}_- (\underset{\sim}{\mathcal{A}}_L^* \cdot \underset{\sim}{\mathcal{A}}_t/\omega_-) e^{i\psi_-} + \text{c.c.} \tag{9.1.33}$$

From (7.1.24) and (9.1.23) we find

$$e\int d^3p\ (\underset{\sim}{v}_1 f_1)_+ = n_0 e^3 \underset{\sim}{\mathcal{A}}_L (\underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/\omega_t) e^{i\psi_+} + \text{c.c.} \tag{9.1.34}$$

and

$$e\int d^3p\ (\underset{\sim}{v}_1 f_1)_- = n_0 e^3 \underset{\sim}{\mathcal{A}}_L^* (\underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/\omega_t) e^{i\psi_-} + \text{c.c.} \tag{9.1.35}$$

Combining these results in Eq. (9.1.23) we obtain

$$\underset{\sim}{j}_{t2\nu+} = n_0 e^3 [\underset{\sim}{k}_+ \underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t/\omega_+ + \underset{\sim}{\mathcal{A}}_L \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/\omega_t] e^{i\psi_+} + \text{c.c.} \tag{9.1.36}$$

and

$$\underset{\sim}{j}_{t2\nu-} = n_0 e^3 [\underset{\sim}{k}_- \underset{\sim}{\mathcal{A}}_L^* \cdot \underset{\sim}{\mathcal{A}}_t/\omega_- + \underset{\sim}{\mathcal{A}}_L^* \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/\omega_t] e^{i\psi_-} + \text{c.c.} \tag{9.1.37}$$

Using continuity $[n_{t2\nu\pm} = \underset{\sim}{k}_+ \cdot \underset{\sim}{j}_{t2\nu\pm}/(e\omega_\pm)]$ we determine the second-order nonlinear test wave density perturbation:

$$n_{t2\nu+} = n_0 e^2 [k_+^2 \underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t/\omega_+^2 + \underset{\sim}{k}_+ \cdot \underset{\sim}{\mathcal{A}}_L \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/(\omega_+ \omega_t)] e^{i\psi_+} + \text{c.c.} \tag{9.1.38}$$

and

$$n_{t2\nu-} = n_0 e^2 [k_-^2 \underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t/\omega_-^2 + \underset{\sim}{k}_- \cdot \underset{\sim}{\mathcal{A}}_L \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/(\omega_- \omega_t)] e^{i\psi_-} + \text{c.c.} \tag{9.1.39}$$

Finally, we restore ordinary units ($m \neq 1$ and $c \neq 1$) in Eqs. (9.1.36) and (9.1.37):

$$\underset{\sim}{j}_{t2\nu+} = n_0 e^3 m^{-2} c^{-2} [\underset{\sim}{k}_+ \underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t/\omega_+ + \underset{\sim}{\mathcal{A}}_L \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/\omega_t] e^{i\psi_+} + \text{c.c.} \quad (9.1.40a)$$

$$\underset{\sim}{j}_{t2\nu-} = n_0 e^3 m^{-2} c^{-2} [\underset{\sim}{k}_- \underset{\sim}{\mathcal{A}}_L^* \cdot \underset{\sim}{\mathcal{A}}_t/\omega_- + \underset{\sim}{\mathcal{A}}_L^* \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t/\omega_t] e^{i\psi_-} + \text{c.c.} \quad (9.1.40b)$$

We note that the factor m^{-2} in these expressions causes the ion current to be insignificant. Thus we are justified in using the above expressions (with $e = -e_0$) for the total nonlinear second-order test-wave current.

The next step in the second-order calculation is the determination of $\underset{\sim}{E}_2$ from $\underset{\sim}{j}_2$. The electric field $\underset{\sim}{E}_2$ produced by $\underset{\sim}{j}_{t2\nu}$ has the form

$$\underset{\sim}{E}_{t2} = \underset{\sim}{\mathcal{E}}_{t2+}(\underset{\sim}{x},t) e^{i\psi_+} + \underset{\sim}{\mathcal{E}}_{t2-}(\underset{\sim}{x},t) e^{i\psi_-} + \text{c.c.} \quad , \qquad (9.1.41)$$

where, in analogy to (6.3.52), $\underset{\sim}{\mathcal{E}}_{t2\pm}$ is given by

$$\underset{\sim}{\mathcal{E}}_{t2\pm} = -\frac{4\pi i}{\omega_+} \underset{\approx}{D}^{-1}(\underset{\sim}{k}_+, \omega_+) \cdot \underset{\sim}{\mathcal{J}}_{t2\nu\pm} \quad . \qquad (9.1.42)$$

Using Eqs. (8.3.16) and (9.1.40a–40b) with $e = -e_0$, we obtain

$$\underset{\sim}{\mathcal{E}}_{t2+} = \frac{i\omega_e^2 e_0 \underset{\sim}{k}_+ \underset{\sim}{\mathcal{A}}_L \cdot \underset{\sim}{\mathcal{A}}_t}{(\omega_+^2 - \omega_e^2) mc^2} + \frac{i\omega_e^2 e_0 \omega_+ \hat{k}_+ \hat{k}_+ \cdot \underset{\sim}{\mathcal{A}}_L \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t}{\omega_t (\omega_+^2 - \omega_e^2) mc^2}$$

$$+ \frac{i\omega_e^2 e_0 \omega_+ (\underset{\sim}{\mathcal{A}}_L - \hat{k}_+ \hat{k}_+ \cdot \underset{\sim}{\mathcal{A}}_L) \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t}{\omega_t (\omega_+^2 - \omega_e^2 - k_+^2 c^2) mc^2} \qquad (9.1.43)$$

and

$$\mathcal{E}_{t2-} = \frac{i\omega_e^2 e_0 \underset{\sim}{k}_- \underset{\sim}{\mathcal{A}}_L^* \cdot \underset{\sim}{\mathcal{A}}_t}{(\omega_-^2 - \omega_e^2)mc^2} + \frac{i\omega_e^2 e_0 \omega_- \hat{\underset{\sim}{k}}_- \hat{\underset{\sim}{k}}_- \cdot \underset{\sim}{\mathcal{A}}_L^* \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t}{\omega_t(\omega_-^2 - \omega_e^2)mc^2}$$

$$+ \frac{i\omega_e^2 e_0 \omega_- (\underset{\sim}{\mathcal{A}}_L^* - \hat{\underset{\sim}{k}}_- \hat{\underset{\sim}{k}}_- \cdot \underset{\sim}{\mathcal{A}}_L^*) \underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{A}}_t}{\omega_t(\omega_-^2 - \omega_e^2 - \hat{k}_-^2 c^2)mc^2} \quad . \tag{9.1.44}$$

In these formulas we see another restriction. The resonance denominator $\omega_\pm^2 - \omega_e^2(- k_\pm^2 c^2)$ causes our result to blow up for $\omega_\pm^2 \approx \omega_e^2(+ k_\pm^2 c^2)$. This corresponds to the three wave interaction of the laser and the test wave decaying into a wav of frequency $\omega_\pm$ and wave vector $\underset{\sim}{k}_\pm$. The treatment of this case requires different techniques. For our purposes we simply assume that the three wave resonance condition is not met.

To complete the second-order calculation we must find the second-order linear quantities, for which we use Eqs. (7.2.28-31). We first use (7.1.22) to find $h_{t2\lambda}$:

$$h_{t2\lambda} = - e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_{t2+} e^{i\psi_+} - e\underset{\sim}{v}_0 \cdot \underset{\sim}{\mathcal{A}}_{t2-} e^{i\psi_-} + \text{c.c.} \tag{9.1.45}$$

where

$$\underset{\sim}{\mathcal{A}}_{t2\pm} = \underset{\sim}{\mathcal{E}}_{t2\pm}/(i\omega_\pm c) \quad . \tag{9.1.46}$$

The second-order linear calculation parallels the first-order linear calculation. In analogy to (9.1.5-16) we find

$$w_{t2\lambda\pm} = \frac{ie\underset{\sim}{p} \cdot \underset{\sim}{\mathcal{A}}_{t2\pm} e^{i\psi_\pm}}{\omega_\pm} + \text{c.c.} \tag{9.1.47}$$

$$n_{t2\lambda\pm} = -\frac{n_0 e \underset{\sim}{k}_{\pm} \cdot \underset{\sim}{\mathcal{A}}_{t2\pm} e^{i\psi_{\pm}}}{\omega_+} + \text{c.c.} \tag{9.1.48}$$

$$\underset{\sim}{j}_{t2\lambda\pm} = -n_0 e^2 \underset{\sim}{\mathcal{A}}_{t2_{\pm}} e^{i\psi_{\pm}} + \text{c.c.} \tag{9.1.49}$$

to the lowest order in $\underset{\sim}{p}$ and v_T^2.

9.1c Third Order

Finally we are prepared to calculate the third-order nonlinear test-wave current. Since this current is linear in $\mathcal{A}_{\underset{\sim}{t}}$ and quadratic in $\mathcal{A}_{\underset{\sim}{L}}$, it contains terms with phases $\psi_t \pm \psi_L \pm \psi_L$. Here we keep only in those terms with phase ψ_t, i.e. the terms in resonance with the first-order test-wave electromagnetic field. These terms modify the test-wave evolution in third order. The remaining terms simply give rise to self consistent third-order fields with phases $\psi_t \pm 2\psi_L$. For simplicity, we calculate $\underset{\sim}{j}_{t3\nu}$ only to zeroth-order in v_T^2.

Combining Eqs. (7.2.75) and (7.2.77) and selecting the test-wave terms, we obtain

$$\underset{\sim}{j}_{t3\nu} = e\int d^3p\, \underset{\sim}{v}_0[-\tfrac{1}{3}\{w'_{t3\nu},f_0\} + \tfrac{1}{2}\{w_1,\{w_2,f_0\}\}_t - \tfrac{1}{6}\{w_1,\{w_1,\{w_1,f_0\}\}\}_t$$

$$- \{w_{t1},F_{L2}\}] + e\int d^3p[(\underset{\sim}{v}_1 f_2 + \underset{\sim}{v}_2 f_1)_t + \underset{\sim}{v}_{t3\nu} f_0] \quad .(9.1.50)$$

All of the quantities in this expression are known except for $w'_{t3\nu}$, which is calculated via

$$-\frac{1}{3}\, w'_{t3\nu} = \int^0 d\tau\, M_0(\tau)[h_{t3\nu} + \tfrac{1}{3}\{w_1,3h_2+\{w_1,h_1\}\}_t] \quad . (9.1.51)$$

The above equation is obtained by extracting the test-wave terms from Eq. (7.2.26). We now proceed to evaluate $\underset{\sim}{j}_{t3\nu}$ term by term.

Following the development from Eq. (9.1.26) to Eq. (9.1.27), we find the first term in (9.1.50) to be given by

$$-\frac{1}{3}\, e\int d^3p\, \underset{\sim}{v}_0\{w'_{3\nu 1},f_0\} = \frac{1}{3}\, e\int d^3p\, f_0\, \frac{\partial w'_{t3\nu}}{\partial \underset{\sim}{q}} \qquad (9.1.52)$$

to zeroth order in v_T^2, where we need $w'_{t3\nu}$ only to zeroth order in $\underset{\sim}{p}$. Thus we need the quantity in bracket in (9.1.51) only to zeroth order in $\underset{\sim}{p}$. From (7.1.21-23) we find

$$h_{t3\nu} = e^2(\underset{\sim}{a}_L \cdot \underset{\sim}{a}_{t2-} + \underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_{t2+} + \underset{\sim}{a}_t \cdot \underset{\sim}{A}_{L20})e^{i\psi_t} + c.c. \qquad (9.1.53)$$

to zeroth order in p. The next term is obtained by using Eqs. (9.1.5-7) and (7.1.22). The result is

$$\{w_1,h_2\}_t = \frac{e^3}{\omega}(\underset{\sim}{k}_- \cdot \underset{\sim}{a}_L \underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_t - \underset{\sim}{k}_+ \cdot \underset{\sim}{a}_L^* \underset{\sim}{a}_L \cdot \underset{\sim}{a}_t)e^{i\psi_t} + c.c. \qquad (9.1.54)$$

to zeroth order in $\underset{\sim}{p}$. The last term $\{w_1,\{w_1,h_1\}\}$ vanishes to zeroth order in $\underset{\sim}{p}$. Inserting these results into (9.1.51), we find

$$\frac{1}{3} w'_{t3\nu} = -\frac{ie^2}{\omega_t}[\underset{\sim}{a}_L \cdot \underset{\sim}{a}_{t2-} + \underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_{t2+} + \underset{\sim}{a}_t \cdot \underset{\sim}{A}_{L20} + \frac{e}{\omega}(\underset{\sim}{k}_- \cdot \underset{\sim}{a}_L \underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_t$$

$$- \underset{\sim}{k}_+ \cdot \underset{\sim}{a}_L^* \underset{\sim}{a}_L \cdot \underset{\sim}{a}_t)]e^{i\psi_t} + c.c. \qquad (9.1.55)$$

This expression can be inserted into (9.1.52) to obtain

$$-\frac{1}{3} e \int d^3p\, \underset{\sim}{v}_0\{w'_{t3\nu},f_0\} = \frac{n_0 e^3 \underset{\sim}{k}_t}{\omega_t}[\underset{\sim}{a}_L \cdot \underset{\sim}{a}_{t2-} + \underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_{t2+} + \underset{\sim}{a}_t \cdot \underset{\sim}{A}_{L20}$$

$$+ \frac{e}{\omega}(\underset{\sim}{k}_t \cdot \underset{\sim}{a}_L \underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_t - \underset{\sim}{k}_t \cdot \underset{\sim}{a}_L^* \underset{\sim}{a}_L \cdot \underset{\sim}{a}_t)]e^{i\psi_t}$$

$$- \frac{in_0 e^3}{\omega_t}(\nabla \underset{\sim}{A}_{L20}) \cdot \underset{\sim}{a}_t e^{i\psi_t} + c.c. \qquad (9.1.56)$$

Note that we have kept the ∇A_{L20} term since it is zeroth in $k^{-1}\nabla \ln|\underset{\sim}{a}|^2$.

Integrating by parts, we find the second term on the right side of (9.1.50):

$$\frac{1}{2} e \int d^3p \; \underset{\sim}{v}_0\{w_1,\{w_2,f_0\}\}_t = \frac{1}{2} e \int d^3p \; f_0 \frac{\partial}{\partial \underset{\sim}{p}} \cdot [\frac{\partial w_2}{\partial \underset{\sim}{q}} \frac{\partial}{\partial \underset{\sim}{p}} \cdot (\frac{\partial w_1}{\partial \underset{\sim}{q}} \underset{\sim}{v}_0)$$

$$+ \frac{\partial^2 w_2}{\partial \underset{\sim}{q} \, \partial \underset{\sim}{q}} \cdot \frac{\partial w_1}{\partial \underset{\sim}{p}} \underset{\sim}{v}_0]_t \quad . \qquad (9.1.57)$$

By straightforward calculation we obtain

$$\frac{1}{2} e \int d^3p \; \underset{\sim}{v}_0\{w_1,\{w_2,f_0\}\} = \frac{n_0 e^4 \underset{\sim}{k}_t}{\omega} [\frac{\underset{\sim}{a}_L \cdot \underset{\sim}{a}_t \underset{\sim}{k}_t \cdot \underset{\sim}{a}_L^*}{\omega_+}$$

$$- \frac{\underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_t \underset{\sim}{k}_+ \cdot \underset{\sim}{a}_L}{\omega_-}] e^{i\psi_t} + c.c. \qquad (9.1.58)$$

The third term $-\frac{1}{6} e \int d^3p \; \underset{\sim}{v}_0 L_1^3 f_0$ vanishes to zeroth order in $\underset{\sim}{p}$ as one can see by using the same reasoning as was used in the paragraph following Eq. (8.4.12).

To calculate the fourth term in (9.1.50) we evaluate the Poisson bracket, integrate by parts, and so obtain

$$-e \int d^3p \; \underset{\sim}{v}_0\{w_{t1},F_{L2}\} = -\frac{e}{T} \int d^3p \; f_0[K_{L2} \frac{\partial}{\partial p} \cdot (\frac{\partial w_{t1}}{\partial q} \underset{\sim}{v}_0)$$

$$+ \underset{\sim}{v}_0 \frac{\partial w_{t1}}{\partial p} \cdot \frac{\partial K_2}{\partial q}] \quad , \qquad (9.1.59)$$

where we have used $F_{L2} = -K_{L2} f_0/T$. Since the factor $(1/T)$ is present in (9.1.59), we must expand the quantity in brackets to $O(p^2)$ to obtain a

result valid to zeroth order in T (i.e. v_T^2). In addition we allow the gradient in the quantity $\partial K_{L2}/\partial \underset{\sim}{q}$ to act only on $\underset{\sim}{A}_{L20}$ for reasons we have discussed previously. Evaluation of (9.1.59) gives

$$-e\int d^3p\ \underset{\sim}{v}_0\{w_{t1},F_{L2}\} = n_0e^2[\underset{\sim}{a}_t(k_t^2/\omega_t^2)(e^2|\underset{\sim}{a}_L|^2 + e\Phi_{L20})$$

$$+ ie\underset{\sim}{a}_t\cdot\nabla\underset{\sim}{A}_{L20}/\omega_t]e^{i\psi_t} + 2n_0e^2\underset{\sim}{k}_t(e^2|\underset{\sim}{a}_L|^2 + e\Phi_{L20})$$

$$\times\ (\underset{\sim}{k}_t\cdot\underset{\sim}{a}_t/\omega_t^2)e^{i\psi_t} - n_0e^3(\underset{\sim}{k}_t\underset{\sim}{a}_t\cdot\underset{\sim}{A}_{L20}/\omega_t$$

$$+ \underset{\sim}{A}_{L20}\underset{\sim}{k}_t\cdot\underset{\sim}{a}_t/\omega_t)e^{i\psi_t} + \text{c.c.} \qquad (9.1.60)$$

Following the development from Eq. (8.4.15) to Eq. (8.4.17), we find the fifth term in (9.1.50):

$$e\int d^3p(\underset{\sim}{v}_1f_2)_t = -\frac{5}{2}\,n_0e^2\underset{\sim}{A}_{t1}(e^2|\underset{\sim}{a}_L|^2 + e\Phi_{L20}) - e^2(\underset{\sim}{A}_1n_2)_t$$

$$= [-\frac{5}{2}\,n_0e^2\underset{\sim}{a}_t(e^2|\underset{\sim}{a}_L|^2 + e\Phi_{L20}) - n_{L20}e^2\underset{\sim}{a}_t$$

$$-\mathcal{n}_{t2+}e^2\underset{\sim}{a}_L^* - \mathcal{n}_{t2-}e^2\underset{\sim}{a}_L]e^{i\psi_t} + \text{c.c.} \qquad (9.1.61)$$

Here, $_{t2\pm}$ is the amplitude of $n_{t2\pm}$. Using Eqs. (9.1.39,40, and 47), we can rewrite (9.1.61) in the form

$$e\int d^3p\,(\underset{\sim}{v}_1 f_2)_t = -e^2\underset{\sim}{\mathcal{A}}_t[n_{L20} + \tfrac{5}{2}\,n_0(e^2|\underset{\sim}{\mathcal{A}}_L|^2 + e\Phi_{20})]e^{i\psi_t}$$

$$- n_0 e^2\underset{\sim}{\mathcal{A}}_L[e^2k^2\underset{\sim}{\mathcal{A}}_L^*\cdot\underset{\sim}{\mathcal{A}}_t/\omega_-^2 + e^2\underset{\sim}{k}_t\cdot\underset{\sim}{\mathcal{A}}_L^*\underset{\sim}{k}_t\cdot\underset{\sim}{\mathcal{A}}_t/(\omega_-\omega_t)$$

$$- e\underset{\sim}{k}_-\cdot\underset{\sim}{\mathcal{A}}_{t2-}/\omega_-]e^{i\psi_t} - n_0e^2\underset{\sim}{\mathcal{A}}_L^*[e^2k_+^2\underset{\sim}{\mathcal{A}}_L\cdot\underset{\sim}{\mathcal{A}}_t/\omega_+^2$$

$$+ e^2\underset{\sim}{k}_t\cdot\underset{\sim}{\mathcal{A}}_L\underset{\sim}{k}_t\cdot\underset{\sim}{\mathcal{A}}_t/(\omega_+\omega_t) - e\underset{\sim}{k}_+\cdot\underset{\sim}{\mathcal{A}}_{t2+}/\omega_+]e^{i\psi_t} + \text{c.c.}$$

(9.1.62)

Following the development from Eq. (8.4.19) to Eq. (8.4.20), we find the sixth term in (9.1.50) to be given by

$$e\int d^3p\,(\underset{\sim}{v}_2 f_1)_t = -e^2(\underset{\sim}{A}_{20}n_1)_t = n_0e^3\underset{\sim}{A}_{20}(\underset{\sim}{k}_t\cdot\underset{\sim}{\mathcal{A}}_{t1}/\omega_t)e^{i\psi_t} + \text{c.c.}$$

(9.1.63)

For the last term in (9.1.50) we need $\underset{\sim}{v}_{t3\nu}$ only to zeroth order in p. Using (7.1.26) we find

$$e\int d^3p\;\underset{\sim}{v}_{t3\nu}f_0 = n_0e^4[\underset{\sim}{\mathcal{A}}_t\cdot\underset{\sim}{\mathcal{A}}_L\underset{\sim}{\mathcal{A}}_L^* + \underset{\sim}{\mathcal{A}}_t\cdot\underset{\sim}{\mathcal{A}}_L^*\underset{\sim}{\mathcal{A}}_L + \underset{\sim}{\mathcal{A}}_t|\underset{\sim}{\mathcal{A}}_L|^2]e^{i\psi_t} + \text{c.c.}$$

(9.1.64)

We now collect the formulas (9.1.56,58,60,62-64) to obtain the total third-order nonlinear test-wave current amplitude:

$$\mathcal{J}_{t3\nu} = - e^2 \mathcal{A}_t [n_{L20} + n_0 e^2 |\mathcal{A}_L|^2 (\tfrac{3}{2} - k_t^2/\omega_t^2) + n_0 e\Phi_{L20} (\tfrac{5}{2} - k_t^2/\omega_t^2)]$$

$$+ n_0 e^2 \mathcal{A}_L [e^2 \mathcal{A}_L^* \cdot \mathcal{A}_t (1 - k_-^2/\omega_-^2) - e^2 \mathbf{k}_t \cdot \mathcal{A}_L^* \mathbf{k}_t \cdot \mathcal{A}_t / (\omega_- \omega_t) + e\mathbf{k}_- \cdot \mathcal{A}_{t2-} / \omega_-]$$

$$+ n_0 e^2 \mathcal{A}_L^* [e^2 \mathcal{A}_L^* \cdot \mathcal{A}_t (1 - k_+^2/\omega_+^2) - e^2 \mathbf{k}_t \cdot \mathcal{A}_L \mathbf{k}_t \cdot \mathcal{A}_t / (\omega_+ \omega_t) + e\mathbf{k}_+ \cdot \mathcal{A}_{t2+} / \omega_+]$$

$$+ (n_0 e^2 \mathbf{k}_t \omega_t) [2(e^2 |\mathcal{A}_L|^2 + e\Phi_{L20}) \mathbf{k}_t \cdot \mathcal{A}_t / \omega_t + e(\mathcal{A}_L \cdot \mathcal{A}_{t2-} + \mathcal{A}_L^* \cdot \mathcal{A}_{t2+})$$

$$- e^2 (\mathbf{k}_t \cdot \mathcal{A}_L \mathcal{A}_L^* \cdot \mathcal{A}_t / \omega_- + \mathbf{k}_t \cdot \mathcal{A}_L^* \mathcal{A}_L \cdot \mathcal{A}_t / \omega_+)]$$

$$+ i n_0 e^3 \mathbf{B}_{L20} \times \mathcal{A}_t / \omega_t \quad . \qquad (9.1.65)$$

At this point we restore ordinary units ($m \neq 1$ and $c \neq 1$):

$$\mathcal{J}_{t3\nu} = - (e^2 \mathcal{A}_t / mc) [n_{L20} + n_0 (e^2 |\mathcal{A}_L|^2 / m^2 c^4) (\tfrac{3}{2} - k_t^2 c^2 / \omega_t^2)$$

$$- n_0 (e\Phi_{L20} / mc^2) (k_t^2 c^2 / \omega_t^2 - \tfrac{5}{2})]$$

$$+ n_0 (e^2 \mathcal{A}_L / mc) [e^2 \mathcal{A}_L^* \cdot \mathcal{A}_t (1 - k^2 c^2 / \omega_-^2) / (m^2 c^4) + e\mathbf{k}_- \cdot \mathcal{A}_{t2-} / (mc\omega_-)$$

$$- e^2 \mathbf{k}_t \cdot \mathcal{A}_L^* \mathbf{k}_t \cdot \mathcal{A}_t / (m^2 c^2 \omega_- \omega_t)] +$$

$$+ n_0(e^2\underset{\sim}{a}_L^*/mc)[e^2\underset{\sim}{a}_L \cdot \underset{\sim}{a}_t(1 - k_+^2c^2/\omega_+^2)/(m^2c^4) + e\underset{\sim}{k}_+ \cdot \underset{\sim}{a}_{t2+}/(mc\omega_+)$$

$$- e^2\underset{\sim}{k}_t \cdot \underset{\sim}{a}_L\underset{\sim}{k}_t \cdot \underset{\sim}{a}_t/(m^2c^2\omega_+\omega_t)]$$

$$+ n_0(ec^2\underset{\sim}{k}_t/\omega_t)[e^2\underset{\sim}{a}_L \cdot \underset{\sim}{a}_{t2-} + \underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_{t2+})/(m^2c^4) +$$

$$+ 2e\underset{\sim}{k}_t \cdot \underset{\sim}{a}_t(e^2|\underset{\sim}{a}_L|^2/mc^2 + e\Phi_{L20})/(m^2\omega_t c^3)$$

$$- e^3(\underset{\sim}{k}_t \cdot \underset{\sim}{a}_L\underset{\sim}{a}_L^* \cdot \underset{\sim}{a}_t/\omega_- + \underset{\sim}{k}_t \cdot \underset{\sim}{a}_L^*\underset{\sim}{a}_L \cdot \underset{\sim}{a}_t/\omega_+)/(m^3c^5)]$$

$$+ i n_0 e^3\underset{\sim}{B}_{L20} \times \underset{\sim}{a}_t/(m^2\omega_t c^2) \quad . \qquad (9.1.66)$$

We see that the factor m^{-1} in each of these terms causes the ion current to be insignificant. Hence Eq. (9.1.66) with $e = -e_0$ gives essentially the total current. We now make this substitution ($e = -e_0$), use Eqs. (9.1.40a-40b), and $\underset{\sim}{\mathcal{E}} = i\omega\underset{\sim}{a}/c$ in the above formula. The result is

$$\underset{\sim}{j}_{t3\nu} = \underset{\approx}{\sigma}_2 \cdot \underset{\sim}{\mathcal{E}}_t \qquad (9.1.67)$$

where $\underset{\approx}{\sigma}_2$, the laser induced conductivity is given by

$$\underset{\approx}{\sigma}_2 = i\underset{\approx}{I}(e_0^2/m\omega_t)[n_{L20} + n_0(e_0^2|\mathcal{E}_L|^2/m^2\omega^2c^2)(\tfrac{3}{2} - k_t^2c^2/\omega_t^2)$$

$$- n_0(e_0\Phi_{L20}/mc^2)(\tfrac{5}{2} - k_t^2c^2/\omega_t^2)] -$$

$$- i\,\hat{u}_L\,\hat{u}_L^*(n_0e_0^2/m\omega_t)\{e_0^2|\underset{\sim}{\mathcal{E}}_L|^2[1 - k_-^2c^2/(\omega_-^2 - \omega_e^2)]/m^2\omega^2c^2\}$$

$$- i\,\hat{u}_L^*\,\hat{u}_L(n_0e_0^2/m\omega_t)\{e_0^2|\underset{\sim}{\mathcal{E}}_L|^2[1 - k_+^2c^2/(\omega_+^2 - \omega_e^2)]/m^2\omega^2c^2\}$$

$$- \underset{\sim}{B}_{L20} \times \underset{\approx}{I}\; n_0e_0^3/m^2\omega^2c$$

$$+ i\,\hat{k}_t\,\hat{u}_L^*(n_0e_0^2/m\omega_t)[e_0^2k_t^2|\underset{\sim}{\mathcal{E}}_L|\hat{k}_t\cdot\underset{\sim}{\mathcal{E}}_L\omega_-/(\omega_-^2 - \omega_e^2)m^2\omega^2\omega_t]$$

$$+ i\,\hat{u}_L\,\hat{k}_t(n_0e_0^2/m\omega_t)[e_0^2k_t^2|\underset{\sim}{\mathcal{E}}_L|\hat{k}_t\cdot\underset{\sim}{\mathcal{E}}_L^*\omega_-/(\omega_-^2 - \omega_e^2)m^2\omega^2\omega_t]$$

$$+ i\,\hat{k}_t\,\hat{u}_L(n_0e_0^2/m\omega_t)[e_0^2k_t^2|\underset{\sim}{\mathcal{E}}_L|\hat{k}_t\cdot\underset{\sim}{\mathcal{E}}_L^*\omega_+/(\omega_+^2 - \omega_e^2)m^2\omega^2\omega_t]$$

$$+ i\,\hat{u}_L^*\,\hat{k}_t(n_0e_0^2/m\omega_t)[e_0^2k_t^2|\underset{\sim}{\mathcal{E}}_L|\hat{k}_t\cdot\underset{\sim}{\mathcal{E}}_L\omega_+/(\omega_+^2 - \omega_e^2)m^2\omega^2\omega_t]$$

$$+ i\,\hat{k}_t\,\hat{k}_t(n_0e_0^2/m\omega_t)\{- 2k_t^2(e_0^2|\underset{\sim}{\mathcal{E}}_L|^2/m^2\omega^2 - e_0\Phi_{L20}/m)/\omega_t^2$$

$$+ (\omega_e^2e_0^2k_t^2/m^2\omega^2\omega_t^2)[|\hat{k}_-\cdot\underset{\sim}{\mathcal{E}}_L|^2/(\omega_-^2 - \omega_e^2) + |\hat{k}_+\cdot\underset{\sim}{\mathcal{E}}_L|^2/(\omega_+^2 - \omega_e^2)$$

$$+ |\hat{k}_-\times\underset{\sim}{\mathcal{E}}_L|^2/(\omega_-^2 - \omega_e^2 - k_-^2c^2) + |\hat{k}_+\times\underset{\sim}{\mathcal{E}}_L|^2/(\omega_+^2 - \omega_e^2 - k_+^2c^2)]\Big\} \quad , \tag{9.1.68}$$

and where the quantities n_{L20}, Φ_{L20}, and B_{L20} are given by Eqs. (8.2.47, 49, and 68).

To conclude this section, we note a few properties of $\underset{\approx}{\sigma}_2$. Two of the terms in $\underset{\approx}{\sigma}_2$ have simple interpretations. The first part of the first term, $i\,\underset{\approx}{I}\,n_{L20}e_0^2/m\omega_t$, is the change in the conductivity due to the quasistatic density perturbation caused by the laser. The fourth term, $-\underset{\sim}{B}_{L20} \times \underset{\approx}{I}\,n_0 e_0^3/(m^2\omega_t^2c^2)$, is the change in the conductivity due to the quasistatic magnetic field caused by the laser. However, the remaining terms cannot be simply interpreted. Secondly, we note that all of the terms except for $i\,\underset{\approx}{I}\,n_{L20}e_0^2/m\omega_t$ are of zeroth order in v_T^2. So if one term is kept, such as the magnetic field term in a discussion of Faraday rotation, then all must be kept. Finally, we note thate the laser completely destroys the symmetry of the conductivity.

Whereas the unperturbed conductivity can be diagonalized by transforming to a coordinate system where $\underset{\sim}{k}_t$ is along an axis, the laser induced conductivity cannot be diagonalized by any trivial transformation.

9.2 Test Wave Propagation

Motivated by the Faraday rotation experiments,[1-3] we wish to analyze the situation shown in Fig. 1. A large amplitude laser propagates through a plasma. A test wave propagates across the laser irradiated region. While propagating, the test wave polarization changes from its initial value $\hat{e}_i$ to its final value $\hat{e}_f$. Our goal is to calculate this polarization change. Our calculation closely parallels the development of sec. 8.4b. The test-wave evolution is taken to be governed basically by the homogeneous plasma equations. The effect of the laser induced current $\underset{\sim}{j}_{t3\nu}$ is calculated perturbatively by using the formalism of sec. 6.3.

The first step in this formalism is the projection of $\underset{\sim}{\mathcal{E}}_t$ along the eigenvectors of $\underset{\approx}{D}(\underset{\sim}{k}_t,\omega_t)$. We are interested in the case where the unperturbed test wave is transversely polarized as in the Faraday rotation experiments. Thus we assume that $D_\perp(\underset{\sim}{k}_t,\omega_t) = 0$, c.f. Eq. (8.4.29), but ω_t is far from being a root of $D_\parallel(\underset{\sim}{k},\omega_t)$. In addition we assume $\underset{\sim}{\mathcal{E}}_t = \underset{\sim}{\mathcal{E}}_t^\perp$ to first order, where $\underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{E}}_t^\perp = 0$, i.e. $\underset{\sim}{\mathcal{E}}_t$ is a two dimensional vector in the plane perpendicular to $\underset{\sim}{k}_t$. The next step is the calculation of the effect of $\underset{\sim}{j}_{t3\nu}$ on $\underset{\sim}{\mathcal{E}}_t$. From Eq. (6.3.59) we find the third-order longitudinal current driven by $\underset{\sim}{j}_{t3\nu}$:

$$\underset{\sim}{\mathcal{E}}_{t3}^\parallel = -\frac{4\pi i\hat{k}_t \cdot \underset{\sim}{\mathcal{J}}_{t3}}{\omega_t D_\parallel(\underset{\sim}{k}_t,\omega_t)} = -\frac{4\pi i\hat{k}_t \cdot \underset{\approx}{\sigma}_2 \cdot \underset{\sim}{\mathcal{E}}_t^\perp}{\omega_t D_\parallel(\underset{\sim}{k}_t,\omega_t)} \quad , \tag{9.2.1}$$

and from Eq. (6.3.65) we determine the evolution equation for the test-wave amplitude in the presence of the laser:

$$\{i[\frac{\partial}{\partial t} + \underset{\sim}{v}(\underset{\sim}{k}_t)\cdot\nabla] + \frac{1}{2}\underset{\approx}{d}(\underset{\sim}{k}_t):\nabla\nabla\}\underset{\sim}{\mathcal{E}}_t^{\perp} = -\frac{4\pi i(\underset{\approx}{I} - \hat{k}_t\hat{k}_t)\cdot\underset{\sim}{\mathcal{J}}_{t3v}}{\overline{\omega D_{\perp}}(\underset{\sim}{k}_t)}$$

$$= -\frac{4\pi i(\underset{\approx}{I} - \hat{k}_t\hat{k}_t)\cdot\underset{\approx}{\sigma}_2\cdot\underset{\sim}{\mathcal{E}}_t^{\perp}}{\overline{\omega D_{\perp}}(\underset{\sim}{k}_t)} \quad . \tag{9.2.2}$$

The functions $\underset{\sim}{v}_{\perp}$, $\underset{\approx}{d}_{\perp}$, and $\overline{\omega D_{\perp}}$, which appear in these equations, are given by Eqs. (8.4.51-53), and $\underset{\approx}{\sigma}_2$ is given by Eq. (9.1.68).

Eq. (9.2.1) states that the test wave acquires a longitudinal part while it is in the laser irradiated region. This effect is of minor importance and will not be discussed further.

Eq. (9.2.2) shows how the test wave propagates in the presence of the test wave. For our purposes, we may neglect the dispersion term $\underset{\approx}{d}:\nabla\nabla_{\sim t}$, since the test wave beam typically has a very narrow spectrum in k space. In addition we restrict ourselves to the steady state problem. As a result we obtain the equation,

$$i\,v_{\perp}(\underset{\sim}{k}_t)\cdot\nabla\underset{\sim}{\mathcal{E}}_t^{\perp} = -\frac{4\pi i}{\overline{\omega D_{\perp}}(\underset{\sim}{k}_t)}\underset{\approx}{I}_t^{\perp}\cdot\underset{\approx}{\sigma}_2\cdot\underset{\sim}{\mathcal{E}}_t^{\perp} \quad , \tag{9.2.3}$$

where $\underset{\approx}{I}_t \equiv \underset{\approx}{I} - \hat{k}_t\hat{k}_t$. This equation can be solved by the method of characteristics. We define s to be a coordinate along $\underset{\sim}{v}(\underset{\sim}{k}_t)$ and so obtain

$$i\frac{\partial}{\partial s}\underset{\sim}{\mathcal{E}}_t^{\perp} = \underset{\approx}{H}(s)\cdot\underset{\sim}{\mathcal{E}}_t^{\perp} \tag{9.2.4}$$

where $\underset{\approx}{H}$, a two by two matrix in the plane perpendicular to $\hat{k}_t$, is given by

$$\underset{\approx}{H} = -4i\pi \underset{\approx}{I}_t^{\perp} \cdot \underset{\approx}{\sigma}_2 \cdot \underset{\approx}{I}_t^{\perp} / [v_{\perp}(\underset{\sim}{k}_t)\overline{\omega D_{\perp}}(\underset{\sim}{k}_t)] \quad . \tag{9.2.5}$$

(We are justified in right multiplying $\underset{\approx}{\sigma}_2$ by $\underset{\approx}{I}_t^{\perp}$, since $\underset{\sim}{k}_t \cdot \underset{\sim}{\mathcal{E}}_t^{\perp} = 0$.) Inserting the formula (9.1.68) into (9.2.5) we obtain

$$\underset{\approx}{H} = \underset{\approx}{I}_t^{\perp} \cdot (\underset{\approx}{I}a + i\hat{b} \times \underset{\approx}{I}b + \hat{u}_L^* \hat{u}_L \alpha + \hat{u}_L \hat{u}_L^* \beta) \cdot \underset{\approx}{I}_t^{\perp} \quad , \tag{9.2.6}$$

where

$$a = -\frac{\omega_e^2}{2k_t c^2} [n_{L20} - n_0(e_0^2 |\underset{\sim}{\mathcal{E}}_L|^2 / m^2\omega^2 c^2)(k_t^2 c^2/\omega_t^2 - \tfrac{7}{2})$$

$$+ n_0(e_0 \Phi_{L20}/mc^2)(k_t^2 c^2/\omega_t^2 - \tfrac{5}{2})] \tag{9.2.7}$$

$$\hat{b} = \underset{\sim}{B}_{L20}/|B_{L20}| \tag{9.2.8}$$

$$b = \frac{1}{2} e_0 B_{L20} \omega_e^2 / (k_t m c^3 \omega_t) \tag{9.2.9}$$

$$\alpha = -[\omega_e^2 e_0^2 |\underset{\sim}{\mathcal{E}}_L|^2 / (2k_t c^4 m^2 \omega^2)][1 - k_+^2 c^2/(\omega_+^2 - \omega_e^2)] \tag{9.2.10}$$

$$\beta = -[\omega_e^2 e_0^2 |\underset{\sim}{\mathcal{E}}_L|^2 / (2k_t c^4 m^2 \omega^2)][1 - k_-^2 c^2/(\omega_-^2 - \omega_e^2)] \quad , \tag{9.2.11}$$

Formulas for the quantities n_{L20}, Φ_{L20}, and $\underset{\sim}{B}_{L20}$ are in the preceding chapter, Eqs. (8.2.47, 49, and 68).

We note that the matrix $\underset{\approx}{H}$ is manifestly hermitian. Therefore $\underset{\approx}{H}$ can be diagonalized,

$$\underset{\approx}{H}(s) = \sum_{\alpha=1}^{2} H_\alpha(s)\, \hat{u}_\alpha(s)\, \hat{u}^*(s) \quad , \tag{9.2.12}$$

in terms of real eigenvalue $H_\alpha(s)$ and orthomounal eigenvectors $\hat{u}_\alpha(s)$. The hermiticity also implies that the solution $\underset{\sim}{\mathcal{E}}_t^\perp(s)$ of Eq. (9.2.4) has constant magnitude: $(\partial/\partial s)|\underset{\sim}{\mathcal{E}}_t^\perp(s)| = 0$.

In general, Eq. (9.2.4) is not amenable to an analytic solution. Of course, one can always integrate (9.2.4) numerically. However, to gain physical insight we study a special case which is analytically tractable. We assume that the eigenvectors $\hat{u}_\alpha$ of $\underset{\approx}{H}$ are independent of s, although the eigenvalues do depend on s. This condition is not unphysical. It holds whenever the directions of $\underset{\sim}{\mathcal{E}}_L$ and $\underset{\sim}{B}_{L20}$ are uniform, even though their magnitudes vary from place to place.

Invoking this condition, we find that the solution to (9.2.4) is

$$\underset{\sim}{\mathcal{E}}_t^\perp(s) = \hat{u}_1 \xi_1 e^{-i\phi_1(s)} + \hat{u}_2 \xi_2 e^{-i\phi_2(s)} \quad , \tag{9.2.13}$$

where ξ_1 and ξ_2 are constants, and $\phi_1(s)$ and $\phi_2(s)$ are given by

$$\phi_\alpha(s) = \int^{s} ds'\, H_\alpha(s') \quad . \tag{9.2.14}$$

The verification of this solution is trivial. Now we require the test wave to have polarization $\hat{e}_i$ at s_i, $\underset{\sim}{\mathcal{E}}_t^\perp(s_i) = |\underset{\sim}{\mathcal{E}}_t|\hat{e}_i$. Then we find that

the polarization of $\underset{\sim}{\xi}_t^1$ at s_f is given by

$$\hat{e}_f = \hat{u}_1\hat{u}_1^* \cdot \hat{e}_i e^{-i\phi_1} + \hat{u}_2\hat{u}_2^* \cdot \hat{e}_i e^{-i\phi_2} \quad , \tag{9.2.15}$$

where the phases ϕ_1 and ϕ_2 (no agrument attached) are given by

$$\phi_\alpha = \int_{s_i}^{s_f} ds' \, H_\alpha(s') \quad . \tag{9.2.16}$$

We interpret this solution as follows. Outside the irradiated region ($s < s_i$ or $s > s_f$) the electromagnetic test wave is degenerate; there are two polarizations with the same dispersion relation. Inside the irradiated region ($s_i < s < s_f$) the laser induced conductivity $\underset{\approx}{\sigma}_2$ breaks this degeneracy. So the test wave splits into the two polarizations $\hat{u}_1$ and $\hat{u}_2$, the two polarizations propagate with different wavenumbers, $i(\partial/\partial s) = H_1$ or H_2, and as a result, the two polarizations obtain different overall phase shifts ϕ_1 and ϕ_2.

Regarding this result we note that the identity portion of $\underset{\approx}{H}$ does not affect the final polarization. To see this we rewrite (9.2.15) in the form

$$\hat{e}_f = e^{-i(\phi_1+\phi_2)/2} [\hat{u}_1\hat{u}_1^* \cdot \hat{e}_i e^{-i\phi} + \hat{u}_2\hat{u}_2^* \cdot \hat{e}_i e^{i\phi}] \quad , \tag{9.2.17}$$

where

$$\phi = (\phi_1 - \phi_2)/2 = \frac{1}{2} \int_{s_i}^{s_f} ds' [H_1(s') - H_2(s')] \quad . \tag{9.2.18}$$

If we change $\underset{\approx}{H}$ to $\underset{\approx}{H} + \kappa \underset{\approx}{I}$, we do not change the eigenvectors $\hat{u}_1$ and $\hat{u}_2$, and we change the eigenvalues by adding κ to each: $H_\alpha \to H_\alpha + \kappa$. This change does not affect the relative phase ϕ. It only affects the overall phase $\phi_1 + \phi_2$, which is not observable in a polarization measurement. Thus we may change Eq. (9.2.4) to

$$i \frac{\partial}{\partial s} \underset{\sim}{\mathcal{E}}_t^\perp = \underset{\approx}{H}' \cdot \underset{\sim}{\mathcal{E}}_t^\perp \quad , \tag{9.2.19}$$

where

$$\underset{\approx}{H}' = \underset{\approx t}{I}^\perp \cdot (i\hat{b} \times \underset{\approx}{I}b + \hat{u}_L^* \hat{u}_L \alpha + \hat{u}_L \hat{u}_L^* \beta) \tag{9.2.20}$$

without affecting our result for $\hat{e}_f$ in a measurable way. For future reference we define the transfer matrix,

$$\underset{\approx}{u} \equiv \hat{u}_1 \hat{u}_1^* e^{-i\phi} + \hat{u}_2 \hat{u}_2^* e^{i\phi} \quad , \tag{9.2.21}$$

which gives the final polarization via

$$\hat{e}_f = \underset{\approx}{U} \cdot \hat{e}_i \quad . \tag{9.2.22}$$

We note that $\underset{\approx}{U}$ is unitary, as it should be since $\underset{\approx}{H}$ is hermitian.

Let us now apply this formalism to a few examples. First we treat the familiar case of Faraday rotation, i.e. we arbitrarily set $\alpha = \beta = 0$ in the expression (9.2.19) for $\underset{\approx}{H}'$:

$$\underset{\approx}{H}' = \underset{\approx t}{I}^\perp \cdot i\hat{b} \times \underset{\approx}{I}b \cdot \underset{\approx t}{I}^\perp = b\hat{k}_t \cdot \hat{b}i\hat{k}_t \times \underset{\approx}{I} \tag{9.2.23}$$

The second equality in the above relation follows from the use of vector identities. At this point we choose our coordinate system such that $\hat{k}_t = \hat{z}$, and therefore $\underset{\approx}{H}'$ is the following two by two x-y matrix:

$$\underset{\approx}{H}' = b\hat{k}_t \cdot \hat{b} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad . \tag{9.2.24}$$

This matrix has eigenvectors,

$$\hat{e}_{\pm} \equiv \sqrt{\frac{1}{2}} \; (\hat{x} \pm i\hat{y}) \quad , \tag{9.2.25}$$

and eigenvalues

$$H'_{\pm} = \pm \, b\hat{k}_t \cdot \hat{b} \quad . \tag{9.2.26}$$

Thus the transfer matrix $\underset{\approx}{U}$ is given by

$$\underset{\approx}{U} = (\hat{e}_+\hat{e}_- e^{-i\phi} + \hat{e}_-\hat{e}_+ e^{i\phi})$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad , \tag{9.2.27}$$

where

$$\phi = \int_{s_i}^{s_f} ds' \; b\hat{b} \cdot \hat{k}_t = \frac{1}{2} \frac{e_0\omega^2 e}{mc^2\omega_t k_t} \int_{s_i}^{s_f} ds' \; \hat{k}_t \cdot \underset{\sim}{B}_{L20} \quad . \tag{9.2.28}$$

The last equality follows from use of Eqs. (9.2.8) and (9.2.9). We see, from (9.2.27), that the polarization undergoes a rotation by angle ϕ, which is determined by (9.2.28). This result agrees with standard theory.[15]

Having verified the familiar case of Faraday rotation, we move on to another example. This time we assume that the laser is circularly polarized, and that $\underset{\sim}{k}_t$ and $\underset{\sim}{k}$ are nearly parallel. Let us write $\underset{\approx}{H}'$ in the form

$$\underset{\approx}{H}' = \underset{\approx}{I}_t^{\perp} \cdot [\tfrac{1}{2}(\alpha + \beta)(\hat{u}_L \hat{u}_L^* + \hat{u}_L^* \hat{u}_L)] \cdot \underset{\approx}{I}_t^{\perp}$$

$$+ \underset{\approx}{I}_t^{\perp} \cdot \{i[\tfrac{1}{2}(\alpha - \beta) i\hat{u}_L^* \times \hat{u}_L + b\hat{b}] \times \underset{\approx}{I}\} \cdot \underset{\approx}{I}_t^{\perp} \quad . \tag{9.2.29}$$

Since the laser is circularly polarized, we have

$$\hat{u}_L \hat{u}_L^* + \hat{u}_L^* \hat{u}_L = (\underset{\approx}{I} - \hat{k}\hat{k}) \quad , \tag{9.2.30}$$

and since $\underset{\sim}{k}_t$ is nearly parallel to $\underset{\sim}{k}$, we have

$$\underset{\approx}{I}_t^{\perp} \cdot (\underset{\approx}{I} - \hat{k}\hat{k}) \cdot \underset{\approx}{I}_t^{\perp} \approx \underset{\approx}{I}_t^{\perp} \quad . \tag{9.2.31}$$

Thus the first term in (9.2.29) can be ignored since it is proportional to the identity matrix for the plane perpendicular to $\hat{k}_t$. The second term in (9.2.29) is identical in form to the expression (9.2.23). Hence the results are similar. The polarization rotates by angle ϕ, but now ϕ is given by

$$\phi = \int_{s_i}^{s_f} ds' [b\hat{b} \cdot k_t + \tfrac{1}{2}(\alpha - \beta) i\hat{u}_L^* \times \hat{u}_L \cdot \hat{k}_t] \quad . \tag{9.2.32}$$

The additional rotation can be accounted for by invoking the presence of a "pseudomagnetic field" $\underset{\sim}{B}'$ given by

$$\underset{\sim}{B}' = \frac{mc^3\omega_t k_t}{e_0\omega_e^2}(\alpha - \beta)i\hat{u}_L^* \times \hat{u}_L$$

$$= \frac{e_0\omega_t i(\underset{\sim}{\mathcal{E}}_L^* \times \underset{\sim}{\mathcal{E}}_L)}{2mc\omega^2}\left[\frac{k_+^2c^2}{\omega_+^2 - \omega_e^2} - \frac{k_-^2c^2}{\omega_-^2 - \omega_e^2}\right] \quad . \tag{9.2.33}$$

Of course, $\underset{\sim}{B}'$ is strikingly different from a true magnetic field in that its magnitude depends on the frequency and wavevector of the test wave used to measure it. For order of magnitude purposes we neglect v in Eq. (8.2.65) and so obtain

$$\frac{|\underset{\sim}{B}'|}{|\underset{\sim}{B}_{L20}|} = \frac{\omega_t\omega}{2\omega_e^2}\left[\frac{k_+^2c^2}{\omega_+^2 - \omega_e^2} - \frac{k_-^2c^2}{\omega_-^2 - \omega_e^2}\right] \quad . \tag{9.2.34}$$

We see that by proper choice of $\underset{\sim}{k}_t$ and ω_t, this ratio can take on any value from $-\infty$ to $+\infty$. (Actually, we have made approximations such as (9.1.22) which restrict the variation of this ratio to some several orders of magnitude.) Hence Faraday rotation can be completely masked by pseudo Faraday rotation, i.e. the contribution to ϕ of $\underset{\sim}{B}'$.

Finally, we treat a third example to illustrate other possible effects. We examine the case where $\hat{k}_t = \hat{z}$, $\hat{k} = \hat{x}$, and the laser is plane polarized in the y direction, i.e. $\hat{u}_L = \hat{u}_L^* = \hat{y}$. In this case Eq. (8.2.65) implies that $\underset{\sim}{B}_{L20} = 0$, so that $\underset{\approx}{H}'$, from (9.2.29) is the following two by two

matrix in the x-y plane:

$$\underset{\approx}{H}' = (\alpha + \beta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad . \tag{9.2.35}$$

Obviously, the eigenvectors of $\underset{\approx}{H}'$ are $\hat{x}$ and $\hat{y}$, and so, from (9.2.29), the transfer matrix is

$$\underset{\approx}{U} = \hat{x}\hat{x}e^{-i\phi} + \hat{y}\hat{y}e^{i\phi} \quad , \tag{9.2.36}$$

where, from (9.2.18) ϕ is given by

$$\phi = \frac{1}{2} \int_{s_i}^{s_f} ds'[\alpha(s') + \beta(s')]$$

$$= -\frac{\omega_e^2 e_0^2}{4k_t c^4 m^2 \omega^2} [2 - \frac{k_+^2 c^2}{(\omega_+^2 - \omega_e^2)} - \frac{k_-^2 c^2}{(\omega_-^2 - \omega_e^2)}] \int_{s_i}^{s_f} ds' \, |\mathcal{E}_L(s')|^2 \quad . \tag{9.2.37}$$

Now let us suppose that $\underset{\sim}{\mathcal{E}}_t^{\perp}$ is initially plane polarized at 45° to the x axis:

$$\hat{e}_i = (\hat{x} + \hat{y})/\sqrt{2} \quad . \tag{9.2.38}$$

Then the final polarization is given by

$$\hat{e}_f = (\hat{x}e^{-i\phi} + \hat{y}e^{i\phi})/\sqrt{2} \quad . \tag{9.2.39}$$

We see that $\hat{e}_f$ is plane polarized for $\phi = m\ /2$, circularly polarized for $\phi = (2m + 1)\pi/2$, and elliptically polarized for other values of ϕ. This is in contrast to the Faraday rotation effect where the final polarization is always planar, though rotated.

9.3 Summary and Conclusions

By systematic perturbation theory we have obtained the conductivity of the laser-plasma system (9.1.68) to second order in $|\mathcal{E}_L|$ and zeroth order in v_T^2, and we have used this conductivity to derive the equations of evolution (9.2.1) and (9.2.2) for a test wave in the laser plasma system. The laser induced conductivity causes two basic effects. First, an electromagnetic test wave acquires a longitudinal portion. Secondly, the laser-induced conductivity breaks the degeneracy of transverse modes in an unmagnetized plasma.

Then we used our formalism to calculate the change in the polarization of the test wave as it propagates through the laser irradiated region. We noted that the polarization of the test wave can change in a number of ways depending on the characteristics of the laser $(\hat{u}_L, \underset{\sim}{k}, \omega)$ and the characteristics of the test wave $(\hat{e}_t, \underset{\sim}{k}_t, \omega_t)$. For example, the polarization may be rotated by the laser generated magnetic field $\underset{\sim}{B}_{L20}$ and the pseudomagnetic field $\underset{\sim}{B}'$, or the polarization may change its character from planar to elliptical as in Eq. (9.2.35). These effects are all comparable. Thus, a Faraday rotation measurement of the resonance-absorption-generated magnetic field does not just measure $\underset{\sim}{B}_{L20}$, and the analysis of Faraday rotation measurements of laser-generated magnetic fields is more complicated than previously believed.

10. Nonlinear Wave Evolution in Magnetized Plasma

A thorough discussion of nonlinear wave evolution in magnetized plasma requires much more labor than does a similar unmagnetized-plasma discussion. In magnetized plasma there are more waves: MHD, cyclotron, ordinary, extraordinary...; and the description of a problem involves more parameters: gyrofrequency, gyroradius, $k_\perp$, $k_{||}$, ... To keep this chapter reasonably short we do not fully explore nonlinear wave evolution in magnetized plasma, but we do discuss the basics. We construct the machinery needed for the magnetized-plasma discussion, and we obtain a few simple results.

We begin by discussing the unperturbed problem. We introduce convenient variables and we construct the time development operator. We discuss the Hamiltonian formulation of magnetized-plasma Vlasov theory. Next we discuss linear theory; we find the response to wave-packet perturbation. Here we see what approximations are required by the formalism, for example: How slow is slowly varying? We also correlate our results to the more common velocity-space Vlasov theory. Here and throughout this chapter we use nonrelativistic theory.

The third and final section is a discussion of some of the nonlinear effects that occur in magnetized plasma. Using the K-χ theorem we derive the ponderomotive Hamiltonian and we discuss some of its effects such as: ponderomotive particle drifts and the nonlinear gyrofrequency shift. We find the second-order quasistatic density perturbation caused by the ponderomotive effects. Finally, we use the heuristic method for including nonlinearity to derive the evolution equation for magnetized plasma waves including the nonlinear quasistatic density depression.

10.1 The Unperturbed System and Convenient Variables

When calculating the generating functions w_n, we must integrate along the unperturbed trajectories. To facilitate this process we should choose variables for which the time development operator has a simple form. In the unmagnetized case the variables $\underset{\sim}{q}$, $\underset{\sim}{p}$ are well suited since the time development operation is simple translation in $\underset{\sim}{q}$, but in the magnetized case the time development operator involves gyration, and other variables are more convenient.

The unperturbed nonrelativistic Hamiltonian for a uniform magnetic field is

$$h_0 = \frac{1}{2}\,(\underset{\sim}{p} - e\underset{\sim}{A}_0(\underset{\sim}{q}))^2 \tag{10.1.1}$$

where the vector potential is given by

$$\underset{\sim}{A}_0(\underset{\sim}{q}) = q_x B_0 \hat{y} \tag{10.2.1}$$

for the magnetic field $\underset{\sim}{B}_0 = \hat{z}\, B_0$. For this system convenient canonical variables are (Y,P), (ϕ,J), and (q,p), defined by the relations

$$\begin{aligned}
q_x &= P/\Omega + \sqrt{2J/\Omega}\,\sin\phi \\
p_x &= \sqrt{2\Omega J}\,\cos\phi \\
q_y &= Y + \sqrt{2J/\Omega}\,\cos\phi \\
p_y &= P \\
q_z &= q \\
p_z &= p \quad ,
\end{aligned} \tag{10.1.3}$$

where $\Omega \equiv eB_0$ is the signed gyrofrequency. We see that $\Omega \lessgtr 0$ implies $J \lessgtr 0$; if $B_0 > 0$, negatively charged particles have negative J, and

positively charged particles have positive J. In this system the unperturbed Hamiltonian is

$$h_0 = \Omega J + \frac{1}{2} p^2 \quad . \tag{10.1.4}$$

The physical interpretation of these variables follows from (10.1.3) and (10.1.4). The x guiding center position is P/Ω, the y guiding center position is Y, ΩJ is the perpendicular energy, and ϕ is of course the gyrophase.

Sometimes we use the q_x gyrocenter $X = P/\Omega$ and the gyrospeed $v = \sqrt{2\Omega J}$ instead of the variables P and J. With these new variables the transformation (10.1.3) becomes

$$\begin{aligned} q_x &= X + (v/\Omega) \sin \phi \\ p_x &= v \cos \phi \\ q_y &= Y + (v/\Omega) \cos \phi \\ p_y &= \Omega X \\ q_z &= q \\ p_z &= p \ . \end{aligned} \tag{10.1.5}$$

However, the variable pairs (Y,X) and (ϕ,v) are not canonical; instead we have $\{Y,X\} = 1/\Omega$ and $\{\phi,v\} = \Omega/v$. Hence the Poisson bracket of two functions $f(X,Y,\phi,v,q,p)$ and $g(X,Y,\phi,v,q,p)$ is given by

$$\{f,g\} = \frac{1}{\Omega}\left(\frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y}\right) + \frac{\Omega}{v}\left(\frac{\partial f}{\partial \phi}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial \phi}\right)$$

$$+ \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right) . \tag{10.1.6}$$

On occasion we will refer to the gyrocenter as a vector $\underset{\sim}{R} = (X,Y,q)$, and we denote the gyrating part of $\underset{\sim}{q}$ by $\underset{\sim}{\xi} \equiv \underset{\sim}{q} - \underset{\sim}{R}$.

In addition we sometim_s use a third set of variables for which the transformation involves complex exponentials instead of sines and cosines as in (10.1.5). We introduce the orthonomial basis

$$\hat{e}_+ = (\hat{x}+i\hat{y})/\sqrt{2}$$

$$\hat{e}_- = (\hat{x}-i\hat{y})/\sqrt{2} \tag{10.1.7}$$

$$\hat{e}_z = \hat{z} \ ,$$

and then define

$$\xi_\pm = \hat{e}^*_\pm \cdot \underset{\sim}{\xi} = \hat{e}_\mp \cdot \underset{\sim}{\xi} \tag{10.1.8}$$

and

$$R_\pm = \hat{e}^*_\pm \cdot \underset{\sim}{R} = \hat{e}_\mp \cdot \underset{\sim}{R} = (X \mp iY)/\sqrt{2} \ , \tag{10.1.9a}$$

which satisfy

$$\{R_+,R_-\} = -i \ , \tag{10.1.9b}$$

so that the gyrovector is given by

$$\underset{\sim}{R} = R_+\hat{e}_+ + R_-\hat{e}_- + R_z \ , \tag{10.1.10}$$

and similarly for $\underset{\sim}{\xi}$ and $\underset{\sim}{q}$. Then we note that

$$\xi_\pm = \mp \frac{iv}{\Omega\sqrt{2}} e^{\pm i\phi} \ , \tag{10.1.11}$$

and hence

$$\underset{\sim}{g} = \hat{e}_+\left(R_+ - \frac{iv}{\Omega\sqrt{2}}\, e^{i\phi}\right) + \hat{e}_-\left(R_- + \frac{iv}{\Omega\sqrt{2}}\, e^{-i\phi}\right) + \hat{e}_z q \ . \qquad (10.1.12)$$

The Poisson bracket of f and g in the variables (R_+,R_-,ϕ,v,q,p) is

$$\{f,g\} = \frac{-i}{\Omega}\left(\frac{\partial f}{\partial R_+}\frac{\partial g}{\partial R_-} - \frac{\partial f}{\partial R_-}\frac{\partial g}{\partial R_+}\right) + \frac{\Omega}{v}\left(\frac{\partial f}{\partial \phi}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial v}\right) + \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right) \qquad (10.1.13)$$

With any of these set of variables the time development operator has a simple action. The variable X,P,Y,J,v,R_+,R_-, and p remain constant in time, while ϕ and q evolve by simple translation: $M_o(t)(\phi,q) = (\phi+\Omega t, q+pt)$.

The reader may now he worrying ahout the symbols we shall use to denote oscillation center variables, since we are using capital letters (X,Y,J) to denote ordinary variables. We note, however, that in the last chapter we never used the oscillation center variables $\underset{\sim}{Q},\underset{\sim}{P}$, or $\underset{\sim}{Z}$. At each stage of the calculation we dealt with functions such ar F,K, or w; all variables were dummy variables. The same holds true in this chapter.

The unperturbed distribution function $f_o(z)$ can be any function of the invariants of h_o, i.e. $f_o(X,Y,\phi-\Omega t,v,q-pt,p)$. Here we require, for simplicity, that f_o be time independent and spatially homogeneous. Hence the unperturbed distribution can only be a function of J and p, $f_o(J,p)$, or equivalently v and p.

Additional ingredients needed in the calculation are the current density function and the charge density function. The charge density function is simply

$$\Omega(\underset{\sim}{x}|\underset{\sim}{Z}) = e\delta(\underset{\sim}{x}-\underset{\sim}{q}) = e\delta(\underset{\sim}{x}-\underset{\sim}{R}-\underset{\sim}{\xi})$$

$$= e\delta(x-X-r\sin\phi)\,\delta(y-Y-r\cos\phi)\,\delta(z-q)\ . \qquad (10.1.14)$$

In this expression we have introduced the gyroradius $r \equiv v/\Omega$. The current density function is

$$\underset{\sim}{\eta}\,(\underset{\sim}{x},t|\underset{\sim}{z}) = e\dot{\underset{\sim}{q}}\,\delta(\underset{\sim}{x}-\underset{\sim}{q})\ . \qquad (10.1.15)$$

Of course the velocity $\dot{\underset{\sim}{q}} = \partial h/\partial \underset{\sim}{p}$ has terms of all orders. For the nonrelativistic Hamiltonian

$$h = \frac{1}{2}\left(\underset{\sim}{p}-e\sum_{n=0}^{\infty}\underset{\sim}{A}_n\right)^2 + e\sum_{n=0}^{\infty}\phi_n\ , \qquad (10.1.16)$$

we have

$$\dot{\underset{\sim}{q}} = \sum_{n=0}^{\infty}\underset{\sim}{v}_n\ , \qquad (10.1.17)$$

where

$$\underset{\sim}{v}_o = \underset{\sim}{p} - e\underset{\sim}{A}_0\ . \qquad (10.1.18)$$

and

$$\underset{\sim}{v}_n = -e\underset{\sim}{A}_n \quad \text{for} \quad n > 0\ . \qquad (10.1.19)$$

In terms of the more convenient variables of (10.1.5) we have

$$v_{ox} = v\cos\phi$$

$$v_{oy} = -v\sin\phi$$

$$v_{oz} = p\ , \qquad (10.1.20)$$

or $\underset{\sim}{v}_o = \sqrt{\frac{1}{2}}\, v e^{i\phi}\, \hat{e}_+ + \sqrt{\frac{1}{2}}\, v e^{-i\phi}\, \hat{e}_- + p\hat{e}_z$

and

$$\underset{\sim}{v}_n = -e\underset{\sim}{A}_n(X + r \sin \phi\ ,\ Y + r \cos \phi\ ,\ q,t)\ . \tag{10.1.21}$$

To compute integrals over the new variables we need the Jacobian. The Jacobian for the canonical transformation (10.1.3) is of course unity, but the Jacobian for the transformation (10.1.5) is not. A simple calculation gives

$$\begin{aligned} d^3q d^3p &= dYdPdqdpd\phi dJ \\ &= dXdYdqd\phi d\Omega Jdp \\ &= dXdYdqd\phi vdvdp\ . \end{aligned} \tag{10.1.22}$$

As an example we calculate the unperturbed density.

$$n_o = \int dXdYdqd\phi vdvdp\ \delta(x-X-r \sin \phi)\ \delta(y-Y-r \cos \phi)\ .$$

$$\delta(z-q)\ f_o(v,p)$$

$$= \int_0^{2\pi} d\phi \int_0^{\infty} vdv \int_{-\infty}^{\infty} dp\ f_o(v,p) = 2\pi \int_0^{\infty} vdv \int_{-\infty}^{\infty} dp\ f_o(v,p) \tag{10.1.23}$$

From this example we see that the quantity $d\phi\ vdvdp\ f_o(v,p)$ equals the quantity $d^3v\ f(v\ ,v_{||})$ of the usual non-Hamiltonian formulation of magnetized Vlasov plasma theory.

We have just seen the main variables and quantities which will be used in discussing nonrelativistic magnetized plasma theory. We have seen that the time development transformation is particularly

simple for the variables X, Y, ϕ, v, q, and p, and we have noted that f_o can only be a function of v and p for a time-independent spatially-homogeneous unperturbed system. Furthermore, we have expressed the charge density and current density in terms of these variables.

10.2. Linear Theory

To compute the magnetized-particle Lie transform in first order, i.e. W_1, we must assume that resonances are unimportant, and that the amplitude of the wave packet is slowly varying. In this section we compute W_1 and see what these assumptions mean specifically.

Without loss of generality we take the wavevector of the perturbing field to be in the x-z plane.

$$\underset{\sim}{A}_1(x,t) = \underset{\sim}{\mathcal{A}}(x,t)\, e^{i(k\,x - k_{||} z - \omega t)} + \text{c.c.} \tag{10.2.1}$$

Inserting $\underset{\sim}{A}_1$ into the first order Hamiltonian and using the variables introduced in the last section we find

$$h_1 = [-ev \cos\phi\, \mathcal{A}_x(X + r\sin\phi,\ Y + r\cos\phi, q) + ev\cos\phi$$

$$\mathcal{A}_y(X + r\sin\phi, Y + r\cos\phi, q) - ep\, \mathcal{A}_z(X + r\sin\phi, Y + r\cos\phi, q)]\cdot$$

$$e^{i(k_\perp X + k_\perp r\sin\phi + k_{||} q - \omega t)} + \text{c.c.}\ . \tag{10.2.2}$$

To put this expression in a form which can be easily integrated along a trajectory, we first expand $\underset{\sim}{\mathcal{A}}$ in the basis (10.1.7):

$$\underset{\sim}{\mathcal{A}} = \mathcal{A}_+ \hat{e}_+ + \mathcal{A}_- \hat{e}_- + \mathcal{A}_z \hat{e}_z\ , \tag{10.2.3}$$

where

$$\mathcal{A}_\pm = \hat{e}_\mp \cdot \underset{\sim}{\mathcal{A}}\ . \tag{10.2.4}$$

Now Eq. (10.2.2) becomes

$$h_1 = -\frac{ev}{\sqrt{2}} e^{-i\phi} \mathcal{A}_+(\underset{\sim}{R}+\underset{\sim}{\xi}) - \frac{ev}{\sqrt{2}} e^{i\phi} \mathcal{A}_-(\underset{\sim}{R}+\underset{\sim}{\xi}) - ep\mathcal{A}_z(\underset{\sim}{R}+\underset{\sim}{\xi}) \ .$$

$$e^{i(k_\perp X + k_\perp r \sin\phi + k_{||} q - \omega t)} + \text{c.c.} \ . \qquad (10.2.5)$$

The next step is the expansion of the complex exponential in a power series.

$$e^{ik_\perp r \sin\phi} = \sum_{\ell=-\infty}^{\infty} J_\ell(k_\perp r) e^{i\ell p} \qquad (10.2.6)$$

This puts Eq. (10.2.5) into the form

$$h_1 = e^{ik_\perp X} \sum_\ell J_\ell(k_\perp r) \left\{ -\frac{ev}{\sqrt{2}} \mathcal{A}_+(\underset{\sim}{R}+\underset{\sim}{\xi}) e^{i[(\ell-1)\phi + k_{||} q - \omega t]} \right.$$

$$\left. - \frac{ev}{\sqrt{2}} \mathcal{A}_-(\underset{\sim}{R}+\underset{\sim}{\xi}) e^{i[(\ell+1)\phi + k_{||} q - \omega t]} - ep\mathcal{A}_z(\underset{\sim}{R}+\underset{\sim}{\xi}) e^{i(\ell\phi + k_{||} q - \omega t)} \right\}$$

$$+ \text{c.c.} \ . \qquad (10.2.7)$$

The last simplication we would like to achieve is the removal of the implicit appearance of ϕ through $\underset{\sim}{\xi}$ in the argument of $\underset{\sim}{\mathcal{A}}(\underset{\sim}{R}+\underset{\sim}{\xi})$. If $\underset{\sim}{\mathcal{A}}$ varies slowly over a typical gyroradius, we can achieve this simplification by the Taylor expansion

$$\underset{\sim}{\mathcal{A}}(\underset{\sim}{R}+\underset{\sim}{\xi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\underset{\sim}{\xi} \cdot \frac{\partial}{\partial \underset{\sim}{R}}\right)^n \underset{\sim}{\mathcal{A}}(\underset{\sim}{R})$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{ir}{\sqrt{2}} e^{i\phi} \frac{\partial}{\partial R_+} + \frac{ir}{\sqrt{2}} e^{-i\phi} \frac{\partial}{\partial R_-} \right)^n \underset{\sim}{\mathcal{A}}(R_+, R_-, q) \ . \qquad (10.2.8)$$

In fact we keep only the first few terms in this expansion. Thus we are making the approximation

$$|r\nabla_{\perp}\underset{\sim}{\mathcal{A}}| << |\underset{\sim}{\mathcal{A}}| \ . \tag{10.2.9}$$

The ϕ dependence of h_1 now being explicit, we proceed with the integration

$$w_1 = - \int d\tau \, M_o(\tau) h_1(t+\tau) \tag{10.2.10}$$

to find w_1. If we substitute Eq. (10.2.8) into (10.2.7), the terms in the above integral all have the form

$$\int^0 d\tau g(X,Y,q+p\tau,p,J) e^{i[\ell(\phi+\Omega\tau)+k_{||}(q+p\tau)-\omega(t+\tau)]}$$

$$= - \sum_{m=0}^{\infty} \left(\frac{i}{\ell+k_{||}p-\omega}\right)^{m+1} e^{i\psi_\ell} \left(p \frac{\partial}{\partial q} + \frac{\partial}{\partial \tau}\right)^m g(X,Y,q,p,J) \ ,$$

where $\psi_\ell \equiv \ell\phi+k_{||}q-\omega t$, and the above integration follows from repeated integrations by parts. Again, in practice we only keep the first few terms in the above series and so we must assume

$$\left|\left(p \frac{\partial}{\partial q} + \frac{\partial}{\partial \tau}\right)\underset{\sim}{\mathcal{A}}\right| << \left|(\ell\Omega+k_{||}p-\omega)\underset{\sim}{\mathcal{A}}\right| \tag{10.2.12}$$

for all values of ℓ for which the gyroharmonic is significant. This condition eliminates the possibility of heating resonant particles.

We not invoke the two approximation (10.2.9) and (10.2.12) for the integration of h_1. Keeping only the first term in the double expansion we obtain

$$w_1 = e^{ik_\perp X}\sum_\ell J_\ell(k_\perp r)\left\{-\frac{iev\mathcal{A}_+(\underset{\sim}{R})e^{i\psi_{\ell-1}}}{\sqrt{2}[(\ell-1)\Omega+k_{||}p-\omega]} - \frac{iev\mathcal{A}_-(\underset{\sim}{R})e^{i\psi_{\ell+1}}}{\sqrt{2}[(\ell+1)\Omega+k_{||}p-\omega]}\right.$$

$$\left. - \frac{iep\mathcal{A}_z(\underset{\sim}{R})e^{i\psi_\ell}}{(\ell\Omega+k_{||}p-\omega)}\right\} + \text{c.c.} \quad . \tag{10.2.13}$$

From w_1 we compute $f_1 = -\{w_1, f_o\}$.

$$f_1 = -e^{ik_\perp X}\sum_\ell J_\ell(k_\perp r)\ \frac{ev\mathcal{A}_+(\underset{\sim}{R})e^{i\psi_{\ell-1}}}{\sqrt{2}[(\ell-1)\Omega+k_{||}p-\omega]}\left[(\ell-1)\frac{\partial f_o}{\partial J} + k_{||}\frac{\partial f_o}{\partial p}\right]$$

$$+ \frac{ev\mathcal{A}_-(\underset{\sim}{R})e^{i\psi_{\ell+1}}}{\sqrt{2}[(\ell+1)\Omega+k_{||}p-\omega]}\left[(\ell+1)\frac{\partial f_o}{\partial J} + k_{||}\frac{\partial f_o}{\partial p}\right]$$

$$+ \frac{ep\mathcal{A}_z(\underset{\sim}{R})e^{i\psi_\ell}}{[\ell\Omega+k_{||}p-\omega]}\left[\ell\frac{\partial f_o}{\partial J} + k_{||}\frac{\partial f_o}{\partial p}\right] + \text{c.c.} \tag{10.2.14}$$

Using f_1 we can compute the linear current. From Eqs. (10.1.15), (10.1.18), and (10.1.19) we find

$$\underset{\sim}{j}_1(x,t) = \int d^6z\ \underset{\sim}{\eta}_o(\underset{\sim}{x}|\underset{\sim}{z})\ f_1(\underset{\sim}{z},t) + \int d^6z\ \underset{\sim}{\eta}_1(\underset{\sim}{x}|\underset{\sim}{z})f_o$$

$$= e\int dXdYdqd\phi\Omega dJdp\ \underset{\sim}{v}_o(p,\phi,J)\ \delta(\underset{\sim}{x}-\underset{\sim}{R}-\underset{\sim}{\xi})\ f_1(\underset{\sim}{R},p,\phi,J)$$

$$- n_o e^2\ \underset{\sim}{A}_1(x,t) \quad . \tag{10.2.15}$$

Using the δ function we integrate over dXdYdq.

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = e\int d\phi\Omega dJdp\ \underset{\sim}{v}_o(p,\phi,J)\ f_1(\underset{\sim}{x}-\underset{\sim}{\xi},p,\phi,J)$$

$$- n_o e^2\ \underset{\sim}{A}_1(\underset{\sim}{x},t) \tag{10.2.16}$$

Inserting f_1 from Eq. (10.2.14) into the above expression we find

$$\underset{\sim}{j}_1(x,t) = -e^2\int d\phi\Omega dJdp\ e^{ik_1x+ik_1r\sin\phi}\ \underset{\sim}{v}_o(p,\phi,J)\ \sum_{\ell} J_\ell(k_1 r)\ .$$

$$\left\{\frac{ev\mathcal{A}_+(\underset{\sim}{x}-\underset{\sim}{\xi})e^{i\psi_{\ell-1}}}{\sqrt{2}[(\ell-1)\Omega+k_{||}p-\omega]}\left[(\ell-1)\frac{\partial f_o}{\partial J}+k_{||}\frac{\partial f_o}{\partial p}\right]+\frac{ev\mathcal{A}_-(\underset{\sim}{x}-\underset{\sim}{\xi})e^{i\psi_{\ell+1}}}{\sqrt{2}[(\ell+1)\Omega+k_{||}p-\omega]}\right.$$

$$\left.\left[(\ell+1)\frac{\partial f_o}{\partial J}+k_{||}\frac{\partial f_o}{\partial p}\right]+\frac{ep\mathcal{A}_z(\underset{\sim}{x}-\underset{\sim}{\xi})e^{i\psi_\ell}}{(\ell\Omega+k_{||}p-\omega)}\left[\ell\frac{\partial f_o}{\partial J}+k_{||}\frac{\partial f_o}{\partial p}\right]\right\}$$

$$+\ \text{c.c.} - n_o e^2\ \underset{\sim}{A}_1(\underset{\sim}{x},t)\ . \tag{10.2.17}$$

To calculate this integral we again invoke the small gyroradius approximation allowing us to replace $\underset{\sim}{\mathcal{A}}(\underset{\sim}{x}-\underset{\sim}{\xi})$ in the above expression by $\underset{\sim}{\mathcal{A}}(\underset{\sim}{x})$. We also expand the exponential of $k_1 r\sin\phi$ using (10.2.26) and we use (10.1.20) for $\underset{\sim}{v}_o$. We find

$$\underset{\sim}{j}_1(x,t) = -e^2\ e^{i(k_1x+k_{||}z-\omega t)}\int d\phi\Omega dJdp\ \sum_{\ell,m} J_\ell(k_1 r)\ J_m(k_1 r)\ .$$

$$\left[\sqrt{\tfrac{1}{2}}\ ve^{i\phi}\hat{e}_+ + \sqrt{\tfrac{1}{2}}\ ve^{-i\phi}\hat{e}_- + p\hat{e}_z\right]\cdot\left\{\frac{ev\mathcal{A}_+(\underset{\sim}{x})e^{i\psi_{m+\ell-1}}}{\sqrt{2}[(\ell-1)\Omega+k_{||}p-\omega]}\right.\ .$$

$$\left[(\ell-1)\frac{\partial f_o}{\partial J} + k_{||}\frac{\partial f_o}{\partial p}\right] + \frac{ev_{-}(\underset{\sim}{x})e^{i\psi_{m+\ell+1}}}{\sqrt{2}[(\ell+1)\Omega+k_{||}p-\omega]}\left[(\ell+1)\frac{\partial f_o}{\partial J} + k_{||}\frac{\partial f_o}{\partial p}\right]$$

$$\left. + \frac{ep_{2}(\underset{\sim}{x})e^{i\psi_{\ell+m}}}{\ell\Omega+k_{||}p-\omega}\left[\ell\frac{\partial f_o}{\partial J} + k_{||}\frac{\partial f_o}{\partial p}\right]\right\} + \text{c.c.} - n_o e^2 \underset{\sim}{A}_1(\underset{\sim}{x},t) \ . \tag{10.2.18}$$

The remaining steps in finding the linear susceptibility, integrating over ϕ and returning to a Cartesian coordinate system, are straightforward, so we simply quote the result from Ichimaru[1] (Eq. 3.71):

$$\underset{\sim}{\chi}(\underset{\sim}{k},\omega) = -\frac{\omega_p^2}{\omega^2} + 4\pi e^2 \sum_{\ell=-\infty}^{\infty} \int 2\pi v_\perp dv_\perp dp \ \frac{\ell\Omega}{v_\perp}\frac{\partial f_o}{\partial v_\perp} + k_{||}\frac{\partial f}{\partial p} \cdot \frac{\underset{\sim}{\Pi}_\ell}{\ell\Omega+k_{||}p-\omega}$$

where

$\underset{\sim}{\Pi}_\ell =$

$$\begin{bmatrix} \dfrac{\ell^2\omega^2 J_\ell^2(k_\perp r)}{k_\perp^2} & iv_\perp \dfrac{\ell\Omega}{k_\perp} J_\ell(k_\perp r)J_\ell'(k_\perp r) & p\dfrac{\ell\Omega}{k_\perp} J_\ell^2(k_\perp r) \\ -iv_\perp \dfrac{\ell\Omega}{k_\perp} J_\ell(k_\perp r)\ J_\ell'(k_\perp r) & v_\perp^2(J_\ell'(k_\perp r))^2 & -ipv_\perp\ J_\ell(k_\perp r)\ J_\ell'(k_\perp r) \\ p\dfrac{\ell\Omega}{k_\perp} J_\ell^2(k_\perp r) & ipv_\perp J_\ell(k_\perp r)\ J_\ell'(k_\perp r) & p^2 J_\ell^2(k_\perp r) \end{bmatrix} \tag{10.2.20}$$

The longitudinal part of $\underset{\approx}{\chi}$ is given by

$$\chi \equiv \hat{k}\cdot\underset{\sim}{\chi}\cdot\hat{k} = -\frac{4\pi e^2}{\omega^2}\sum_{\ell=-\infty}^{\infty}\int 2\pi v\, dv\, dp\left(\frac{\ell\Omega}{v_\perp}\frac{\partial f_o}{\partial v_\perp} + k_{||}\frac{\partial f}{\partial v_{||}}\right)\frac{J_n^2(k_\perp r)}{\ell\Omega + k_{||}v_{||} - \omega}\ , \tag{10.2.21}$$

again from Ichimaru.

In summary, we have calculated the linear part of the Lie transform and shown how it can be used to compute the linear susceptibility of Vlasov plasma.

Thus the calculation of the linear response is straightforward. We integrate h_1 to find w_1, and then we use w_1 to find f_1. However, to make progress we had to use two approximations. The first (10.2.9) states that the amplitude must vary slowly in a gyroradius. The second (10.2.13) states that the variation of the amplitude along a trajectory must be slow compared to the variation of the phase.

10.3 Nonlinear Evolution of Magnetoplasma Waves

As we noted in the introduction to this chapter, magnetoplasma problems tend to involve much labor. Here we try to avoid as much of this labor as possible. We first discuss a special case to display the theory, but then we use the K-χ theorem to derive the general form for the ponderomotive Hamiltonian. We discuss some of the consequences of K (for example the ponderomotive drifts, the nonlinear gyrofrequency shift, and the self consistent density perturbation), but we leave others (such as the self generated magnetic field) for future research. In discussing the nonlinear evolution of a wave packet we do not use the systematic method, but instead use the heuristic method for including nonlinearity.

We first consider the special case $k_\perp = 0$ and $\mathcal{A}_z = 0$, i.e. $\underset{\sim}{A}_1(\underset{\sim}{x},t) = \mathcal{A}_x(\underset{\sim}{x},t)\exp(ik_{||}z-i\omega t) + \mathcal{A}_y(\underset{\sim}{x},t)\exp(ik_{||}z-i\omega t) + \text{c.c.}$ Invoking the approximations (10.2.9) and (10.2.12) we find from Eqs. (10.2.7) and (10.2.13) that

$$h_1 = -\frac{1}{\sqrt{2}}\, ev\mathcal{A}_+(\underset{\sim}{R})\, e^{i(-\phi+k_{||}q-\omega t)} - \frac{1}{\sqrt{2}}\, ev\mathcal{A}_-(\underset{\sim}{R})\, e^{i(\phi+k_{||}q-\omega t)} + \text{c.c.} \tag{10.3.1}$$

and

$$w_1 = \frac{iev\mathcal{A}_+(\underset{\sim}{R})e^{i(-\phi+k_{||}q-\omega t)}}{(\omega+\Omega-k_{||}p)\sqrt{2}} + \frac{iev\mathcal{A}_-(\underset{\sim}{R})e^{i(\phi+k_{||}q-\omega t)}}{(\omega-\Omega-k_{||}p)\sqrt{2}} + \text{c.c.} \tag{10.3.2}$$

To find the ponderomotive Hamiltonian we need to know $\{w_1,h_1\}$, the zeroth harmonic (in ωt) part of which is

$$\{w_1,h_1\} = \frac{-e^2|\mathcal{A}_+|^2}{(\Omega+\omega-k_{||}p)}\left(\Omega - \frac{k_{||}^2v^2}{2(\Omega+\omega-k_{||}p)}\right)$$

$$- \frac{e^2|\mathcal{A}_-|^2}{(\Omega-\omega+k_{||}p)}\left(\Omega - \frac{k_{||}^2v^2}{2(\Omega-\omega+k_{||}p)}\right)$$

$$+ \frac{1}{2}\, e^2k_{||}^2v^2\mathcal{A}_-^*\mathcal{A}_+\left[(\omega+\Omega-k_{||}p)^{-2} + (\omega-\Omega-k_{||}p)^{-2}\right]e^{-2i\phi}$$

$$+ \text{c.c.} \quad . \tag{10.3.3}$$

This quantity and the quantity

$$h_{2\nu o} = e^2|\underset{\sim}{\mathcal{A}}|^2 \tag{10.3.4}$$

are inserted into the second-order Deprit equation (7.2.52), which is then analyzed to find $K_{2\nu o}$.

We note that $\{w_1,h_1\}_o$ in this magnetized case differs from $\{w_1,h_1\}_o$ in Eq. (8.2.7) in a very important respect. Due to the phase $\exp(2i\phi)$, the last term in Eq. (10.3.3) oscillates along a particle orbit, and hence can be put in $w_{2\nu o}$, in contrast to the unmagnetized case where $w_{2\nu o} = 0$. Therefore, the analysis of Eq. (8.2.7) gives

$$K_{2\nu o} = e^2|\mathcal{A}_+|^2\left(\frac{\omega-k_{||}p}{\Omega+\omega-k_{||}p} + \frac{k_{||}^2v^2}{2(\Omega+\omega-k_{||}p)^2}\right)$$

$$+ e^2|\mathcal{A}_-|^2\left(\frac{\omega-k_{||}p}{\omega-\Omega-k_{||}p} + \frac{k_{||}^2v^2}{2(\omega-\Omega-k_{||}p)^2}\right) \qquad (10.3.5)$$

and

$$w_{2\nu o} = -\frac{i}{4}e^2k_{||}^2v^2\mathcal{A}_-^*\mathcal{A}_+\left[(\omega+\Omega-k_{||}p)^{-2} + (\omega-\Omega-k_{||}p)^{-2}\right]e^{-2i\phi} + \text{c.c.} \ . \qquad (10.3.6)$$

We now use the K - χ theorem to derive the general expression for the ponderomotive Hamiltonian. To do so we insert $\underset{\approx}{\chi}$ from linear theory, Eq. (10.2.19), into Eq. (7.4.25):

$$\int dXdYdqd\phi vdvdp\ f_o(v,p)\ K_{2\nu o}(X,Y,q,v,p) =$$

$$\int d^3x\int 2\pi vdvdp\ f_o(v,p)\ \underset{\sim}{\mathcal{E}}^*(\underset{\sim}{x},t)\cdot\underset{\approx}{\mathcal{K}}(v,p)\cdot\underset{\sim}{\mathcal{E}}(x,t)\ , \qquad (10.3.7)$$

where the tensor $\underset{\approx}{\ }$ is given by

$$\underset{\approx}{\mathcal{K}} = \frac{e^2}{\omega^2}\underset{\approx}{I} + e^2\sum_{\ell=-\infty}^{\infty}\left(\frac{\ell\Omega}{v}\frac{\partial}{\partial v} + k_{||}\frac{\partial}{\partial p}\right)\frac{\underset{\approx}{\Pi}_\ell(v,p)}{\ell\Omega+k_{||}p-\omega}\ , \qquad (10.3.8)$$

and Π_ℓ is given by Eq. (10.2.20). We noted in the previous example that $K_{2\nu o}$ is not a function of ϕ, terms in Eq. (8.2.4) which depend on ϕ are absorbed in $w_{2\nu o}$.

Functional differentiation of Eq. (10.3.7) gives

$$\int d^3R\ K_{2\nu o}(\underset{\sim}{R},v,p) = \int d^3x\ \underset{\sim}{\mathcal{E}}^*(\underset{\sim}{x},t)\cdot\underset{\approx}{\mathcal{K}}(v,p)\cdot\underset{\sim}{\mathcal{E}}(\underset{\sim}{x},t)\ . \qquad (10.3.9)$$

Since the two integrals are equal, we conclude (as in section 8.2a) that the two integrands are equal except for a derivative.

$$K_{2\nu o}(\underset{\sim}{R},r,p) = \underset{\sim}{\mathcal{E}}^{*}(\underset{\sim}{R},t)\cdot\underset{\approx}{\mathcal{H}}(v,p)\cdot\underset{\sim}{\mathcal{E}}(\underset{\sim}{R},t) + O(\nabla) \tag{10.3.10a}$$

The approximations involved in obtaining this expression are simply those used in the sample calculation (10.2.9) and (10.2.12) plus the wave packet assumptions

$$|\nabla\mathcal{E}| << |\underset{\sim}{k}\underset{\sim}{\mathcal{E}}|$$

and

$$\left|\frac{\partial\underset{\sim}{\mathcal{E}}}{\partial t}\right| << \left|\omega\underset{\sim}{\mathcal{E}}\right| . \tag{10.3.10b}$$

Noting that the expressions (10.3.10), (10.3.8) and (10.2.20) give a messy result for $K_{2\nu o}$ we infer that the calculation of $K_{2\nu o}$ without the K - χ theorem would involve much tedious algebra.

For purposes of comparison we take the limit $v \to 0$ and $p \to 0$ in Eq. (10.3.10). We find

$$K_{2\nu o}(\underset{\sim}{R},v=0,p=0) = \frac{e^2}{m}\left[\frac{|\mathcal{E}_z|^2}{\omega^2} + \frac{|\mathcal{E}_x|^2+|\mathcal{E}_y|^2}{\omega^2-\Omega^2} + \frac{i\Omega(\mathcal{E}_x^{*}\mathcal{E}_y - \mathcal{E}_x\mathcal{E}_y^{*})}{\omega(\omega^2-\Omega^2)}\right], \tag{10.3.11}$$

the standard result[2] for the ponderomotive potential.

Let us discuss oscillation center motion. For this discussion only we consider X,Y,q,p,ϕ,J as being oscillation center variables. The total oscillation center Hamiltonian is

$$= \Omega J + \tfrac{1}{2}p^2 + K_{2\nu o}(\underset{\sim}{R},J,p) \ . \tag{10.3.12}$$

Hence we have the equations of motion

$$\dot{X} = \{X,K\} = -\frac{1}{\Omega}\frac{\partial K_{2\nu o}}{\partial Y} \tag{10.3.13a}$$

$$\dot{Y} = \{Y,K\} = \frac{1}{\Omega}\frac{\partial K_{2\nu o}}{\partial X} \tag{10.3.13b}$$

$$\dot{\phi} = \{\phi,K\} = \Omega + \frac{\partial K_{2\nu o}}{\partial J} \tag{10.3.13d}$$

$$\dot{J} = \{J,K\} = 0 \tag{10.3.13d}$$

$$\dot{q} = \{q,K\} = p + \frac{\partial K_{2\nu o}}{\partial p} \tag{10.3.13e}$$

$$\dot{p} = \{p,K\} = -\frac{\partial K_{2\nu o}}{\partial q} \ . \tag{10.3.13f}$$

We see that oscillation center motion is much like guiding center motion. The X-Y motion is given by a perpendicular drift

$$\dot{\underset{\sim}{R}} = -\frac{\partial K_{2\nu o}}{\partial \underset{\sim}{R}} \times \frac{\hat{B}_o}{\Omega} \tag{10.3.14}$$

corresponding to a perpendicular force $-\partial K_{2\nu o}/\partial \underset{\sim}{R}$, but here the force depends on p and J. From Eq. (10.3.13c) we see that the gyrofrequency shifts in the presence of the wave by the amount

$$\Delta\Omega = \frac{\partial K_{2\nu o}}{\partial J} \ . \tag{10.3.15}$$

From Eq. (10.3.13d) we see that the oscillation center magnetic moment J is conserved. And we see that parallel motion, governed by Eqs. (10.3.13e) and (10.3.13f), is similar to the motion of unmagnetized particles. A nice aspect of this Hamiltonian formalism is that <u>all</u> of these effects (drifts, gyrofrequency shift, and parallel motion) are described by one function, the ponderomotive Hamiltonian $K_{2\nu o}$.

Having noted that the oscillation center magnetic moment is conserved, it is appropriate to emphasize its relation to the usual magnetic moment. Temporarily denoting oscillation center variables by the superscript osc., the Lie transform equations (4.1.15-16) give

$$J^{osc} = J - \{w_1(z),J\} - \tfrac{1}{2}\{w_2(z),J\} + \tfrac{1}{2}\{w_1(z),\{w_1(z),J\}\}$$

and

$$J = J^{osc} + \{w_1(z^{osc}),J^{osc}\} + \tfrac{1}{2}\{w_2(z^{osc}),J^{osc}\} + \tfrac{1}{2}\{w_1(z^{osc}),\{w_1(z^{osc}),J^{osc}\}\}$$

to second order. Thus, the fact that J^{osc} is conserved implies that the right side of the first of these equations is conserved. It does not imply that J is conserved.

For electrostatic waves, rather than work with Eq. (10.3.10), it is easier to derive K from $\hat{k}\cdot\chi\cdot\hat{k}$ of Eq. (10.2.21). Following the same procedure as before we obtain

$$K_{2\nu o}(\underset{\sim}{R},v,p,t) = -\frac{e^2|\underset{\sim}{E}|^2(\underset{\sim}{R},t)}{k^2}\sum_{\ell}\left(\ell\frac{\partial}{\partial J} + k_{||}\frac{\partial}{\partial p}\right)\frac{J_\ell^2(k_\perp r)}{\ell\Omega+k_{||}p-\omega}\,. \tag{10.3.17}$$

Thus, the gyrofrequency shift for an electrostatic wave with $k_{||} = 0$ is given by

$$\Delta\Omega = \frac{\partial K_{2\nu o}}{\partial J} = -\frac{e^2|\underset{\sim}{\mathcal{E}}|^2}{k^2} \sum_{\ell=-\infty}^{\infty} \ell \frac{\partial^2}{\partial J^2} \frac{J_\ell^2(k_\perp r)}{\ell\Omega - \omega} . \tag{10.3.18}$$

This formula was first obtained by Aamodt et al[3] (their Eq. (2)). However, by the Lie transform technique we have obtained this gyrofrequency shift for <u>electromagnetic</u> waves with $k_{||} \neq 0$. Since Aamodt et al state that the gyrofrequency shift (10.3.18) plays an important role in the nonlinear dynamics of the drift-cyclotron instability, we infer that the more general formula given by (10.3.15) may be important in the nonlinear dynamics of the Alfven-ion-cyclotron instability.[4,5] We leave further investigation of this point for future research.

At this point we interject a warning that concerns the use of Eq. (10.3.17). That equation was derived using $\underset{\sim}{k}\times\underset{\sim}{\mathcal{E}} = 0$. This condition is not identical to the electrostatic condition, $(\underset{\sim}{k}-i\nabla) \times \underset{\sim}{\mathcal{E}} = 0$, but it is equivalent if the approximation $k^{-1} \nabla \ln \mathcal{E} << 1$ holds. Hence one should not use the formula (10.3.17) for electrostatic waves unless it is also true that $k^{-1} \nabla \ln \mathcal{E} << 1$ holds.

Let us allow for self consistent fields Φ_{2o} and $\underset{\sim}{A}_{20}$ in the second-order Hamiltonian

$$h_{2\lambda o} = -e\underset{\sim}{v}_o \cdot \underset{\sim}{A}_{20}(\underset{\sim}{q},t) + e\Phi_{20}(\underset{\sim}{q},t) . \tag{10.3.19}$$

Using the small gyroradius expansion (10.2.8) and keeping the leading term we find

$$h_{2\lambda o} = -\frac{1}{\sqrt{2}} ev\, e^{-i\phi} A_{20+}(\underset{\sim}{R},t) - \frac{1}{\sqrt{2}} ev\, e^{i\phi} A_{20-}(\underset{\sim}{R},t)$$

$$- ep\, A_{20z}(\underset{\sim}{R},t) + e\Phi_{20} \; . \qquad (10.3.20)$$

We analyze these terms in the second-order linear Eq. (7.2.48) by putting the slowly varying terms in $K_{2\lambda o}$:

$$K_{2\lambda o} = -ep\, A_{20z}(\underset{\sim}{R},t) + e\Phi_{20}(\underset{\sim}{R},t) \qquad (10.3.21)$$

and the oscillating terms in $w_{2\lambda o}$:

$$\frac{1}{2} w_{2\lambda o} = \frac{i}{\Omega\sqrt{2}} ev\, A_{20+}\, e^{-i\phi} - \frac{i}{\Omega\sqrt{2}} ev\, A_{20-} e^{i\phi} \; . \qquad (10.3.22)$$

We now discuss the quasistatic evolution of the oscillation-center distribution F. Including K_{20}, the evolution of F obeys

$$\frac{\partial F}{\partial t} + \{F, h_o + K_{20}\} = 0 \qquad (10.3.23)$$

In the unmagnetized case we noted that F could be any function of the only (in general) invariant K. Here there is another invariant, J, since $\partial K/\partial\phi = 0$. Thus F can be any function of the two invariants, $\frac{1}{2}p^2 + K_{20}(X,Y,q,p,J)$ and J.

$$F = g[\tfrac{1}{2}p^2 + K_{20}(X,Y,q,p,J), J] \qquad (10.3.24)$$

Setting the wave amplitude (and therefore K_2) equal to zero, and $F = f_o$, we find that $g = f_o$. Inserting this (10.3.24) we obtain

$$F(\underset{\sim}{z}) = f_o(\tfrac{1}{2}p^2 + K_{20}(\underset{\sim}{z}), J) \; . \qquad (10.3.25)$$

This is just

$$\left|\frac{\partial K_{20}}{\partial t}\right| << \left|p\,\frac{\partial K_{20}}{\partial q}\right|$$

or

$$\left|\frac{\partial|\underset{\sim}{\mathcal{E}}|^2}{\partial t}\right| << \left|p\,\frac{\partial|\underset{\sim}{\mathcal{E}}|^2}{\partial z}\right| . \qquad (10.3.29b)$$

It means that the wave envelope changes so slowly that the particles have enough time to traverse the envelope by moving along field lines. Suppose instead that the opposite were true; that the wave envelope extent was infinite in the z direction. This would be true for a radiation filament in the z direction. In this case we would have $K_{20}(X,Y,J,p,t)$ independent of q and the solution is simply

$$F = f_o(\tfrac{1}{2}p^2,J) , \qquad (10.3.30)$$

i.e. there is no gradient $\partial K_{20}/\partial q$ to derive the oscillation center evolution.

To find the second order average distribution in particle phase space we must perform the inverse transformation

$$f_{20} = F_2 - \tfrac{1}{2}\{w_{20},f_o\} + \tfrac{1}{2}\{w_1,\{w_1,f_o\}\}_o . \qquad (10.3.31)$$

The density perturbation found from f_{20} is

$$n_{20}(\underset{\sim}{x},t) = \int d^6z\ \delta(\underset{\sim}{x}-\underset{\sim}{R}-\underset{\sim}{\xi})\ f_{20}(\underset{\sim}{R},p,J,\phi)$$

$$= \int d\phi\Omega dJdp\ f_{20}(\underset{\sim}{x}-\underset{\sim}{\xi},p,\phi,J) . \qquad (10.3.32)$$

We can expand this expression as in Eq. (9.2.8). Assuming (9.2.9) we keep only the first term in this expansion:

$$n_{20}(\underset{\sim}{x},t) = \int d\phi\Omega dJdp\ f_{20}(\underset{\sim}{R}=\underset{\sim}{x},p,\phi,J)$$
$$= \int d\phi\Omega dJdp \left[F_2(\underset{\sim}{R}=\underset{\sim}{x},p,J) - \tfrac{1}{2}\{w_{20},f_o\}\Big|_{\underset{\sim}{R}=\underset{\sim}{x}} + \tfrac{1}{2}\{w_1,\{w_1,f_o\}\}\Big|_{\underset{\sim}{R}=\underset{\sim}{x}}\right], \tag{10.3.33}$$

We evaluate this quantity to lowest order in the gradients, and so we keep only the ϕ-J part of the Poisson brackets. The second term

$$-\tfrac{1}{2}\int d\phi\Omega dJdp\ \frac{\partial w_{20}}{\partial\phi}\frac{\partial f_o}{\partial J} = 0 \tag{10.3.34}$$

must vanish since it is the integral of a periodic function in ϕ. Similarly the third term must vanish

$$\tfrac{1}{2}\int d\phi\Omega dJdp\left[\frac{\partial w_1}{\partial\phi}\frac{\partial}{\partial J}\left(\frac{\partial w}{\partial\phi}\frac{\partial f_o}{\partial J}\right) - \frac{\partial w_1}{\partial J}\frac{\partial}{\partial\phi}\left(\frac{\partial w_1}{\partial\phi}\frac{\partial f_o}{\partial J}\right)\right]$$
$$= -\tfrac{1}{2}\int d\phi\Omega dJdp\ \frac{\partial}{\partial\phi}\left(\frac{\partial w_1}{\partial J}\frac{\partial w_1}{\partial\phi}\right)\frac{\partial f_o}{\partial J} = 0 \tag{10.3.35}$$

(the first equality follows from integration by parts in J), since it is the integral over ϕ of a ϕ derivative. We are left with

$$n_{20}(\underset{\sim}{x},t) = \int d\phi\Omega dJdp\ \frac{\partial f_o}{\partial_2 p^2}\ K_{20}(\underset{\sim}{R}=\underset{\sim}{x},p,J,t)\ . \tag{10.3.36}$$

When f_o is gaussian in p, we can reduce this expression to a particularly simple form. We have

$$n_2(\underset{\sim}{x},t) = -\frac{1}{T_{||}}\int d\phi\lambda dJdp\ f_o\left[K_{2\nu o}(\underset{\sim}{R}=\underset{\sim}{x},p,J,t) + K_{2\lambda o}(\underset{\sim}{R}=\underset{\sim}{x},p,J,t)\right] . \tag{10.3.37}$$

The first term in $K_{2\lambda o}$, c.f. Eq. (10.3.21), is odd in p and gives no contribution. We know the contribution of $K_{2\nu o}$ from the K-X theorem. Thus

$$n_2^s(\underset{\sim}{x},t) = -\frac{1}{T_{||_s}}\left[-\frac{1}{4\pi}\underset{\sim}{\mathcal{E}}^{*}(\underset{\sim}{x},t)\cdot\underset{\approx}{\chi}_s(\underset{\sim}{k},w)\cdot\underset{\sim}{\mathcal{E}}(\underset{\sim}{x},t) + n_o\ e_s\ \phi_{20}\right] .$$

At this point the magnetized plasma calculation is identical to the unmagnetized plasma (c.f. sec. 8.2), and so Eqs. (8.2.47-50b) hold with the substitution $T_s \to T_{||s}$. In particular, for a plasma with one species of singly charged ions we have

$$n_{20} = -\frac{|\underset{\sim}{\mathcal{E}}_1|^2-|\underset{\sim}{\mathcal{B}}_1|^2}{4\pi(T_{e||}+T_{i||})} . \tag{10.3.39}$$

The next quantity to calculate is the quasistatic second-order self-consistent current perturbation. From our experience in dealing with unmagnetized plasma (see section 8.2c) we know that the current vanishes to zeroth order in the gradient of the amplitude because of charge conservation. Hence to correctly treat this problem we must return to linear theory and calculate $\dot{a}_1$ and w_1 to O(Va). Then we find K_2 and w_2 to O(Va). We will then know f_2 from which we can find $\underset{\sim}{j}_{20}$. Finally we must include $\underset{\sim}{A}_{20}$ self-consistently. The multiple expansions (10.2.8) and (10.2.11) make this calculation quite tedious, and so it is left for future research.[6]

We also leave the calculation of the second-harmonic terms and the systematic calculation of the third-order current for later research. However, we do give the general nonlinear evolution equation for a wave packet including the second-order density perturbation by the heuristic method. First we let $\underset{\sim}{\lambda} = n$ in Eq. (6.4.2):

$$\left[i\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_\alpha\cdot\nabla\right) + \tfrac{1}{2}\underset{\approx}{d}_\alpha:\nabla\nabla\right]\mathcal{E}_\alpha(\underset{\sim}{x},t) - n_{20}\frac{\partial\omega_\alpha}{\partial n_o}\mathcal{E}_\alpha(\underset{\sim}{x},t) = 0 \ . \tag{10.3.40}$$

Next we use (10.3.39), noting that $\mathcal{B}_{\underset{\sim}{\alpha}} = c\underset{\sim}{k} \times \hat{u}_\alpha\mathcal{E}_\alpha/\omega$ ($\hat{u}_\alpha$ is the polarization of the mode), to obtain

$$\left[\left(i\frac{\partial}{\partial t} + \underset{\sim}{v}_\alpha\cdot\nabla\right) + \tfrac{1}{2}\underset{\approx}{d}_\alpha:\nabla\nabla\right]\mathcal{E}_\alpha + \frac{1-|c\underset{\sim}{k}\times\hat{u}_\alpha/\omega|^2}{4\pi(T_{e||}+T_{i||})}\frac{\partial\omega_\alpha}{\partial n_o}\mathcal{E}_\alpha|\mathcal{E}_\alpha|^2 = 0 \ . \tag{10.3.41}$$

This is a very general evolution equation. No statement has been made about the medium; this equation applies to both magnetized and unmagnetized plasma. Furthermore we note that all the parameters in this equation ($\hat{u}_\alpha, \underset{\sim}{v}_\alpha, \underset{\approx}{d}$ ...) can be obtained from linear theory. Of course we must keep in mind that this equation is valid only for quasistatic wave-packet perturbations; i.e. Eq. (10.3.29) must be true. In addition we must keep in mind that this equation may be incomplete (and therefore wrong) since it was derived using the heuristic method for including nonlinearity.

We have seen a number of differences between the magnetized-plasma discussion and the unmagnetized plasma discussion. In magnetized plasma w_{20} does not vanish; there are zero-harmonic (in ωt) terms which do not vary slowly along a trajectory because of the

$\exp(i\phi)$ factor. The convenient variables for magnetized-plasma calculations (10.1.5) are not as convenient as the unmagnetized plasma variables $\underline{q},\underline{p}$, which can be easily manipulated as vectors. In general, magnetized plasma calculations involve more algebra. Yet even though these two systems are very different, the use of general Lie transform techniques has allowed us to pr[illegible] the K-X theorem in both cases, to derive a general formula for t[illegible] [illegible]cond-order quasistatic density perturbation Eq. (10.3.36), and [illegible]erive the general nonlinear wave equation (10.3.41) that includes this density perturbation.

Much work remains to be done on the nonlinear evolution of waves in magnetized plasma. For motivation we point to the proposals for using large rf power sources to heat tokamaks[7] and mirrors.[8] In addition, linear instability may cause wave amplitudes to grow until nonlinear effects become important. So far, nonlinear calculations have used the heuristic method, but as we showed in Chapter 8, the heuristic method gives an incomplete answer. For a complete answer one must attack these problems systematically. In this chapter we have taken the first few steps toward this systematic discussion.

11. Summary and Conclusions

In three steps we have discussed nonlinear wave evolution in Vlasov plasma. In the first step, chapters 2-5, we reviewed the Lie transform and its use in constructing a Hamiltonian perturbation theory. In the second step, chapter 6, we developed a formalism for deriving nonlinear wave equations given the nonlinear current. In the third step, chapters 7-10, we combined the first two steps; we used Lie transforms to derive the nonlinear currents, and we used the formalism of chapter 6 to determine their effect on the nonlinear evolution of wave packets.

In the first part of this thesis we saw that Lie transforms are a concise and flexible tool for analyzing Hamiltonian systems. Lie transforms are concise because all operations are expressed in terms of Poisson brackets; Taylor expansions in the separate variables, as in the older Poincaré-Von Zeipel technique, are avoided. Lie transforms are flexible because one is not committed to transforming away all terms in every order: slowly varying terms, which would produce resonant denominators upon integration, can be kept in the new system and analyzed by another technique.

In chapter 6 we discussed general aspects of the evolution of wave packets. We clarified the concepts of wave energy and wave momentum. We generalized these concepts to apply to perturbations which are not normal modes. This generalization allowed us to write the wave energy and wave momentum as sums of contributions from the electromagnetic field and the various species in a Vlasov plasma, where the wave energy (momentum) of species s is the total energy (momentum) transferred to species s by the linear fields. We also constructed a general formalism

for deriving nonlinear wave equations in terms of the nonlinear currents, and we compared this formalism to the often used heuristic method for including nonlinearity.

In chapter 7 we discussed general aspects of the application of Lie transforms to the Vlasov system of equations. We gave the Hamiltonian formalism for the Vlasov function, and we showed how self consistency is included via the current operators. In addition we proved two theorems. The first was a generalization of Kubo's fluctuation dissipation theorem: if the unperturbed Vlasov distribution is a function of only the Hamiltonian, then the linear conductivity is the weighted average of the current correlation function, where the weighting function is the derivative of the unperturbed distribution with respect to the energy. The second theorem was a simple relation between the linear susceptibility and the ponderomotive Hamiltonian. This theorem proved to be very useful throughout this thesis, since it allowed us to deduce a nonlinear quantity, the ponderomotive Hamiltonian, from a previously calculated linear quantity, the susceptibility, with very little effort.

In chapter 8 we applied these ideas to nonlinear wave evolution in unmagnetized plasma. We discussed the ponderomotive Hamiltonian $K_{2\nu 0}$ and showed how it causes a second-order quasistatic density depression and a self generated quasistatic magnetic field. In third order we derived the equation that governs the nonlinear evolution of waves. This equation, which was derived by using the systematic formalism of section 6.3, unified much previous work in that it contains the change in the conductivity due to the second-order quasistatic density perturbation, the self precession effect of cold plasma theory, and the self Faraday

rotation effect. On the other hand, the heuristic method yields the density perturbation effect and the Faraday rotation effect, but it leaves out the self precession effect, which is comparable to the Faraday rotation effect. We conclude that the heuristic method gives incomplete answers.

In chapter 9 these same ideas were applied to the problem of test wave propagation in a laser-plasma system. Motivated by experiment we wanted to know how the test wave polarization changes as the test wave propagates through the irradiated region. We found a number of comparable effects. In addition to the expected Faraday rotation of the polarization by the quasistatic magnetic field, we found that the laser induced conductivity can change the nature of the test wave polarization, for example the test wave polarization may change from plane to elliptical. In light of these findings we conclude that a more thorough analysis of Faraday rotation experiments is in order. By varying parameters, such as the test wave frequency and direction of propagation, one can isolate the various effects and obtain a more certain measurement of the quasistatic magnetic field.

Finally, we have applied these ideas to magnetized plasma. We derived the ponderomotive Hamiltonian from the linear susceptibility, and we discussed the effects of the ponderomotive Hamiltonian on particle motion: the parallel force, drifts, and the nonlinear gyrofrequency shift. We derived the universal formula for the second-order quasistatic density depression, and we included this effect in the nonlinear evolution of the wave amplitude using the heuristic method.

As we have seen, the Lie transform is a useful mathematical tool for

studying Hamiltonian systems. It allows simple proof of general theorems, and it diminishes the computational effort necessary to analyze specific systems.

APPENDIX A

Notations for Derivatives

The derivative notation used in this paper has been chosen to be very explicit, and as a result it is detailed and cumbersome. In the usual physics use of derivatives, such a detailed notation is not needed, and one may wonder if such a cumbersome notation is needed. In this appendix we show how confusion can arise if one uses a less explicit notation.

We first consider two functions, f and F, which obey the following relation,

$$f(\underset{\sim}{z},t,\theta) = F(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t). \tag{A1}$$

In our discussion of Lie transforms, we need to differentiate equations such as (A1) with respect to z_m. In the usual physics notation this would be written

$$\frac{\partial f}{\partial z_m} = \sum_n \frac{\partial F}{\partial Z_n} \frac{\partial Z_n}{\partial z_m} . \tag{A2}$$

We know what $\frac{\partial F}{\partial \underset{\sim}{Z}}$ means in this case, but we need an interpretation of this symbol for other cases.

The two most common interpretations of the symbol $\frac{\partial F}{\partial \underset{\sim}{Z}}$ are:

A. F is always a function of Z, so $\frac{\partial F}{\partial \underset{\sim}{Z}}$ is simply the derivative of F with respect to its argument.

B. Given a symbol such as $\frac{\partial F}{\partial \underset{\sim}{Z}}$, the object below the line (in this case $\underset{\sim}{Z}$) denotes both the argument of the function and the variable with respect to which the derivative is taken.

Let us first consider A. This interpretation cannot be used in Lie transform discussions, since in these discussions we approximate (A1) by

$$f(\underset{\sim}{z},t,\theta) \doteq F(\underset{\sim}{z},t) + \theta\,\{w_1, F\}$$

$$\doteq F(\underset{\sim}{z},t) + \theta \left[\frac{\partial w_1}{\partial \underset{\sim}{q}} \cdot \frac{\partial F}{\partial \underset{\sim}{p}} \quad \frac{\partial w_1}{\partial \underset{\sim}{p}} \cdot \frac{\partial F}{\partial \underset{\sim}{q}}\right]. \tag{A3}$$

The derivatives which appear in (A3) mean

$$\frac{\partial F}{\partial \underset{\sim}{Z}} = \frac{\partial}{\partial \underset{\sim}{z}} \left[F(\underset{\sim}{z},t)\right] , \tag{A4}$$

whereas, using interpretation A, we would assign them the meaning

$$\frac{\partial F}{\partial \underset{\sim}{z}} \overset{?}{=} \frac{\partial}{\partial \underset{\sim}{z}} \left[F\big(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta), t\big)\right] . \tag{A5}$$

Thus we cannot use interpretation A in Lie transform discussions.

Now we consider interpretation B. This interpretation is ambiguous when we consider derivatives of the form

$$\frac{\partial F}{\partial t} . \tag{A6}$$

Since t (the object below the line) doesn't indicate $\underset{\sim}{Z}$ or $\underset{\sim}{z}$, we don't know whether (A7) means

$$\frac{\partial}{\partial t} \big(F(\underset{\sim}{z},t)\big) \tag{A7}$$

or

$$\frac{\partial}{\partial t} \big(F(Z(\underset{\sim}{z},t,\theta), t)\big) \tag{A8}$$

or even

$$\frac{\partial}{\partial t} \big(F(\underset{\sim}{z},t)\big) \quad \text{evaluated at} \quad \underset{\sim}{z} = \underset{\sim}{Z}(\underset{\sim}{z},t,\theta) . \tag{A9}$$

It is important to distinguish between these three objects in the expansion of (A1) using the chain rule, which might be written in the following way:

$$\frac{\partial}{\partial t} \big(F(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t)\big) = \sum_k \frac{\partial F}{\partial Z_k} \cdot \frac{\partial Z_k}{\partial t}$$

$$+ \frac{\partial}{\partial t} \big(F(\underset{\sim}{z},t)\big) \quad \text{evaluated at} \quad z = \underset{\sim}{Z}(\underset{\sim}{z},t,\theta) . \tag{A10}$$

Though the notation used in (A10) is clear in this instance, it is cumbersome.

So we [illegible] that the standard notation is ambiguous when using Lie transforms, and we do not use it. Instead we use the notation introduced in Chapter 2 which is unambiguous though cumbersome.

APPENDIX B

Further Notation and List of Symbols

B1. Choice of Symbols

We use script letters to denote the amplitudes of oscillating quantities, e.g. $\underset{\sim}{E}(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}(\underset{\sim}{x},t)\exp(i\psi)+c.c.$ As much as possible we use Roman capitals to denote variables and functions which pertain to oscillation center phase space and lower case Roman letters to denote variables and functions which pertain to the usual phase space. The exceptions are the use of $\underset{\sim}{n}$ for the current operator, and the use of X,Y,I, etc. in the magnetized plasma discussion.

B2. Superscripts and Subscripts

Superscripts refer to species.

A single numerical subscript (e.g. $\underset{\sim}{\mathcal{J}}_3^s$) refers to the order in $\underset{\sim}{\varepsilon}_1$. A second numerical subscript or a subscripted + or - after a numerical subscript (e.g. $\underset{\sim}{\mathcal{J}}_{31}^s$ or $\underset{\sim}{\mathcal{J}}_{3+}^s$) refers to the harmonic number of the term. A subscripted Greek letter, which always appears immediately after the first numerical subscript (e.g. $\underset{\sim}{\mathcal{J}}_{3\lambda 1}$ or $\underset{\sim}{\mathcal{J}}_{2\nu}$), denotes whether this quantity is the linear response (λ) to the same order electric fields, or the nonlinear response (ν) to lower order electric fields. A subscripted Roman letter (L,t,x,y,z) or + or - before the first numerical subscript either distinguishes between laser (L) or test wave (t) fields, or when that distinction is not needed, it denotes the component (x,y,z,+,or-) of a vector.

Examples of subscript conventions

$\underset{\sim}{\mathcal{J}}_{3\lambda 1}$ - The linear (λ) response to the third-order (3) electric field $\underset{\sim}{E}_{31}$ at the first (1) harmonic.

$\underset{\sim}{\mathcal{J}}_{t2\nu+}$ - The test-wave (t) second order (2) nonlinear (ν) response (to the lower order fields $\underset{\sim}{E}_t$ and $\underset{\sim}{E}_L$) at the "plus" (+) harmonic ($\propto \exp\,[i(\omega_r+\omega)t]$.

$\underset{\sim}{A}_2$ - The second order (2) vector potential.

A_{+2} - The "plus" (+) component of the second order vector potential ($\hat{e}_+^* \cdot \underset{\sim}{A}_2$).

B3. List of Symbols

Symbol	Meaning (chapters in which first introduced, special symbol)
A	vector potential (7)
a	
B	magnetic field (6)
b	
C	
c	speed of light (7)
D	linear response tenor (7)
d	dispersion tenor (6)
E	electric field
e	Naperian base, charge of an unspecified particle (7), electronic charge (8, e_o)
F	oscillation center distribution (3)
f	usual distribution in phase space, or Liouville function (3)
G	wave momentum density (6)
g	
H	transformed old Hamiltonian (3)
h	old Hamiltonian (2)
I	unit tensor
i	$\sqrt{-1}$ or an index
J	action (4), magnetic moment (10)
j	current density (6) or an index
K	new Hamiltonian (3)
k	new Hamiltonian in the old phase space (3 only), wave vector (6)
L	Lie operator (2)
l	index
M	time development transformation (2)
m	mass (3)
N	number of degrees of freedom (2)
n	density (7)
O	order of
o	
P	new momentum (3)
p	old momentum (2)
Q	new coordinate (3)
q	old coordinate (2)
R	transformed remainder function (3 only), gyrocenter (10, $\underline{R}$)
r	untransformed remainder function (3 only), gyroradius (10)
S	
s	species (6)
T	Lie transform (3)
t	time (2)

$\mathbb{U}$ unitary operator (2)
u unit vector (6,$\hat{u}$)
V
v velocity (7,$\underset{\sim}{v}$), thermal velocity [8,$v_\beta \equiv (T_\beta/m_\beta)^{1/2}$], gyrovelocity [10,$v \equiv (2\Omega J/m)^{1/2}$]
W wave energy density (6)
w Lie generating function (3)
X gyrocenter (10)
x position (6)
Y gyrocenter (10)
y
Z oscillation center (3)
z phase space variables (2)
α index (6)
β index (6)
Γ
γ relativistic energy (7)
δ functional derivative (7)
∂ partial derivative
ϵ perturbation parameter (4)
ζ
η current at $\underset{\sim}{x}$ due to a particle at $\underset{\sim}{z}$ at time t [7, $\underset{\sim}{\eta}(\underset{\sim}{x}|\underset{\sim}{z},t)$]
θ Lie parameter (3)

κ Fourier transform variable (6)
Λ
λ denotes nonlinear terms that are linear in the electric field of the same order (6)
μ
ν denotes nonlinear terms that are nonlinear in lower order fields (6)
Ξ
ξ gyrovector (10)
$\underset{\sim}{\sigma}$
Π a tensor which appears in the linear susceptibility of magnetized plasma (10)
π 3.14159...
ρ charge density (6)
Σ
σ conductivity (6)
τ

Φ electrostatic potential (7)

ϕ gyrophase (10)
χ linear susceptibility (6)
ψ wave phase (8)
Ψ ponderomotive potential (8)
ω frequency (4), plasma frequency of species s (7, ω_s)
Ω Fourier transform variable (6 only), gyrofrequency (10)

$\approx$ approximately equal to
$\sim$ equal in order of magnitude
$\propto$ proportional to
$\mathcal{A}$ vector potential amplitude (8)
$\mathcal{B}$ magnetic field amplitude (6)
$\mathcal{E}$ electric field amplitude (6)
$\mathcal{J}$ current amplitude (6)
$\mathcal{O}$ order of
ρ charge density at $\underset{\sim}{x}$ for a particle at $\underset{\sim}{z}$ at time t [7, $\rho(\underset{\sim}{x}|\underset{\sim}{z},t)$]

Figure 1 Caption: Sketch showing a test wave entering the laser beam with polarization $\hat{u}_i$ and leaving the laser beam with polarization $\hat{u}_f$.

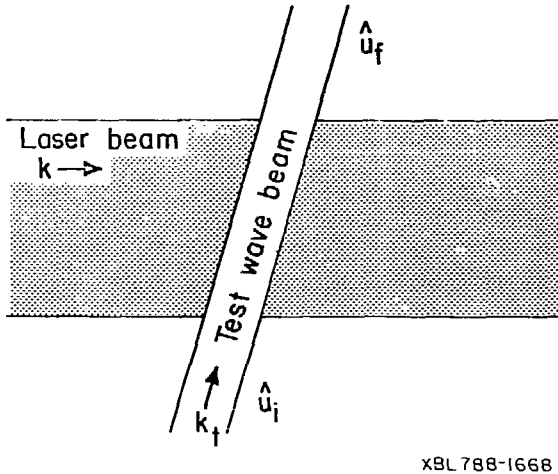

Fig. 1

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