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INSERTION OF SKEW QUADRUPOLES TO EXCHANGE $x - x'$ AND $y - y'$ PHASE SPACES*

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Some suggested HIF accelerators[1,2] require exchange of horizontal $(x - x')$ and vertical $(y - y')$ emittances in beam stacking and bunching. This can be accomplished by a 90° rotation in a solenoid. The purpose of this note is to demonstrate that this can also be accomplished in a skew quadrupole transport system, and to demonstrate necessary and sufficient conditions for such a system.

The skew quadrupole system considered is a normal quadrupole lattice rotated 45° in the transverse $(x - y)$ plane. The equations of motion in the usual quadrupole coordinates

$$\begin{aligned} x'' &= -Kx \\ y'' &= Ky \end{aligned} \qquad K = \frac{B'}{B\rho} \tag{1}$$

are rotated -45° to the lab coordinates to become:

$$\begin{aligned} x'' &= -Ky \\ y'' &= -Kx \end{aligned} \qquad \text{(lab)} \tag{2}$$

For our purposes it is simplest to express the problem in matrix terms. The effect of an arbitrary quadrupole transport system (not skew) can be represented by a 4 x 4 matrix Q for (x, x', y, y') as:

$$Q = \begin{pmatrix} A_x & B_x & 0 & 0 \\ C_x & D_x & 0 & 0 \\ 0 & 0 & A_y & B_y \\ 0 & 0 & C_y & D_y \end{pmatrix} \tag{3}$$

The effect of a skew quadrupole array can be represented by a matrix Q' where

$$Q' = R^{-1}\, Q\, R$$

and R is the matrix for a rotation of -45°.

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$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{4}$$

Then Q' becomes:

$$Q' = \frac{1}{2} \begin{pmatrix} A_x + A_y & B_x + B_y & A_y - A_x & B_y - B_x \\ C_x + C_y & D_x + D_y & C_y - C_x & D_y - D_x \\ A_y - A_x & B_y - B_x & A_x + A_y & B_x + B_y \\ C_y - C_x & D_y - D_x & C_x + C_y & D_x + D_y \end{pmatrix} \tag{5}$$

Complete exchange of horizontal and vertical phase space can be accomplished if

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} = - \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} \tag{6}$$

since then

$$Q' = \begin{pmatrix} 0 & 0 & A_y & B_y \\ 0 & 0 & C_y & D_y \\ A_y & B_y & 0 & 0 \\ C_y & D_y & 0 & 0 \end{pmatrix} \tag{7}$$

connects $(x, x')_{final}$ only with $(y, y')_{initial}$ and $(y, y')_{final}$ with $(x, x')_{initial}$.

If we represent (in betatron notion):

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} = I \cos \mu_x + J_x \sin \mu_x, \quad \text{where} \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \tag{8A}$$

and

$$\begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} = I \cos \mu_y + J_y \sin \mu_y \tag{8B}$$

and choose a FODO quadrupole lattice such that $\mu_y = 2n\pi$ and $\mu_x = (2n - 1)\pi$

then:

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

and:

$$\begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

and:

$$Q' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{11}$$

Which simply exchanges (x, x') with (y, y').

As an example, we note that this can be obtained by choosing a skew FODO lattice with 90° phase shift per period in the skew-y plane and 45° per period in the skew x-plane and following 4 periods to $\mu_y = 360°$, $\mu_x = 180°$. Such a lattice can be constructed easily using SYNCH and allowing the F quads to be set at a lower field gradient value than the D quads.

Necessary and Sufficient Conditions:

For a quadrupole lattice skewed at 45°, equation 5 gives the proper transfer matrix and the condition

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} = - \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} \tag{6}$$

must be satisfied if x and y phase spaces are to be completely exchanged. In place of (6), a lattice skewed at an angle θ other than 45° requires contradictory conditions of form

$$\begin{aligned} A_x \cos^2\theta + A_y \sin^2\theta &= 0 \\ A_x \sin^2\theta + A_y \cos^2\theta &= 0 \end{aligned} \tag{12}$$

for A, B, C, and D, which can be satisfied only if $\theta = 45°$ $(0° < \theta < 90°)$. Therefore, the 45° angle is necessary.

With condition (6) expressed in the form of equation 8, we are

immediately led to the requirement $\cos \mu_x = - \cos \mu_y$ which requires either $\mu_x + \mu_y = n\pi$ or $\mu_y = \mu_x + n\pi$ where n is an odd integer. If $\mu_x + \mu_y = n\pi$ (n odd), then $\sin \mu_x = \sin \mu_y$ and the condition $J_x \sin \mu_x = -J_y \sin \mu_y$ cannot be satisfied (since $\beta_x, \beta_y > 0$).

Thus, a necessary condition is $\mu_y = \mu_x + n\pi$ (n odd). Also, if $\sin \mu_x \neq 0$, then equation (6) requires $J_x = J_y$, since $\sin \mu_x = - \sin \mu_y$. Therefore, we have two choices of a second necessary condition: Either: 1) $\mu_x = m\pi$ (m any integer) as in the above example, or 2) μ_x can take any value but $J_x = J_y$ is required. We can demonstrate this possibility by generating a FODO lattice as above ($B_F \neq -B_D$) and requiring that at the center of the F (or D) quad $\beta_x = \beta_y$ instead of a requirement on μ_x. At this center point symmetry requires $\alpha_x = \alpha_y = 0$; also $\gamma = 1/\beta$. Then $J_x = J_y$. The skew insertion would begin and end at such a center point. Such a lattice is not difficult to generate. In both cases 1) and 2) the condition $\mu_y = \mu_x + n\pi$ (n odd) for the skew insertion must also be satisfied.

We thank Lloyd Smith for suggesting the problem and useful comments.

FOOTNOTES

1. R.C. Arnold, et.al., "Hearth Fire Reference Concept #3: A Rapid Cycling Synchrotron System", ANL Report ACC - 6 (June 16, 1978).

2. A. W. Maschke, "Conceptual Design of a Heavy Ion Fusion Energy Center", (March 13, 1978), BNL 50817.