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WHY DO WE BELIEVE IN SPONTANEOUS BREAKDOWN OF SYMMETRIES?*

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October 27, 1978

ABSTRACT

We construct solutions to free massless scalar field-theory, to the U(1) non-linear σ- model and to their corresponding gauge-theories in any number of dimensions. The global symmetry of these models is not broken. This contradicts the claims that it is impossible to construct a symmetric vacuum. Our treatment of the symmetry mode is identical to that of the gauge degrees of freedom in gauge-theories. It is shown that the U(1) non-linear σ-model is free in the continuum.

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I. INTRODUCTION

Consider a free, massless field theory in n spatial dimensions, with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 \tag{1}$$

where ϕ is a real scalar field. $\mathcal{L}$ is invariant under the symmetry

$$\phi(x) \rightarrow \phi(x) + \eta \tag{2}$$

where η is independent of space and time.

According to the existing literature,[1-6] this is the simplest model in which the symmetry is spontaneously broken. In this paper it is argued that the ground state of the system is symmetric. We are able to construct a solution and demonstrate that the symmetry is not spontaneously broken for the above mentioned free field theory, for the non linear σ-model with the U(1) symmetry (which is equivalent to the free field theory in the continuum) and for their corresponding gauge theories. However, it seems that our arguments apply to any theory of Bose fields with a continuous symmetry.

Let us emphasize that this paper deals with symmetries, and not with correlations. The symmetry of the Lagrangian is exact, and we are not interested in terms which break the symmetry. We do not view the system as a limit in which the strength of a symmetry breaking term goes to zero. We say that the symmetry (Eq. 2) is spontaneously broken, if the field ϕ has some preferred value. We are not interested here in correlations, which are the relations

between the values of the field at different points of space-time. We emphasize this point since some people understand the term "spontaneous breakdown of symmetry" as a synonym to "strong correlations". Note that a system can have strong correlations (imagine a system in which the dominating configurations are those which are very close to $\phi(x) = c$) but still be symmetric (namely c may assume any value with an equal probability). We shall not introduce the constraint that the field must assume some value c at infinity. By introducing this constraint one is studying correlations, and not the symmetry which is explicitly broken by the constraint.

The Traditional View:

Let us first describe the traditional view of the existing literature. We shall use as an illustration the free-field theory,[1-6] but very similar phenomena are believed to happen in any model in which the symmetry is claimed to be spontaneously broken.

According to the traditional view, there is an infinite set of possible solutions, parametrized by η. Each solution has its own vacuum state, $|0\rangle_\eta$, they all have the same energy, and satisfy

$$_\eta\langle 0|\phi(x)|0\rangle_\eta = \eta \qquad (3)$$

The current associated with the symmetry, $J_\mu = \partial_\mu\phi$, is conserved, but the charge (the generator of the symmetry) $Q = \int d^n\vec{x}\, J_o(\vec{x})$ is ill-defined. Instead, one should use a more careful definition

of Q,*

and get

$$\text{a.}\quad Q|0\rangle_\eta \neq 0 \qquad (4)$$

$$\text{b.}\quad e^{-i\eta' Q}|0\rangle_\eta = |0\rangle_{\eta+\eta'} .$$

This means that the vacuum is not symmetric. The claim in the literature is that it is impossible to construct a symmetric vacuum. It should be emphasized that the symmetry was treated as an exact symmetry, without using a symmetry-breaking term which goes to zero.

In our construction there is a unique vacuum, and it is symmetric. The generator is well-defined and it does annihilate the vacuum. However, the expectation value of the field (Eq. 3) is not defined, since the field is a quantity which is not invariant under the symmetry.

II. MASSLESS FREE FIELD

a. Solution of Free Massless Field Theory

We view the system as a quantum mechanical system with infinite number of degrees of freedom, and we use the Schrödinger representation. The degrees of freedom are $\phi(\vec{x})$, where $\vec{x}$ is viewed as a (continuous) index which runs over space only. The Fourier transformation is a linear change of variables (for simplicity we ignore factors of 2π):

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* A possible definition of Q is

$$Q = \lim_{R\to\infty} \int d^n\vec{x}\, f_R(\vec{x})\, J_o(\vec{x})$$

where

$$f_R(\vec{x}) = \begin{cases} 1 & |\vec{x}| \leq R \\ 0 & |x| > R + \varepsilon \end{cases}$$

$$\tilde{\phi}(\vec{k}) = \int d^n \vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\, \phi(\vec{x}). \tag{5}$$

In the new variables the Hamiltonian is

$$H = \int d^n\vec{k}\, \frac{1}{2}\, [\dot{\tilde{\phi}}^2(\vec{k}) + \omega_k^{\,2}\, \tilde{\phi}^2(\vec{k})] \tag{6}$$

$$\text{where} \qquad \omega_k^{\,2} = \vec{k}^2 .$$

(For simplicity we ignore the fact that $\tilde{\phi}(\vec{k})$ is complex. By taking linear combinations of $\tilde{\phi}(\vec{k})$ and $\tilde{\phi}(-\vec{k})$ one gets real oscillators.)

In order to work with discrete oscillators we use

$$\int d^n\vec{k} \rightarrow \frac{1}{V}\sum_{\vec{k}} \tag{7}$$

where V is the infinite volume of space (It is not necessary to use a box cut-off.). The volume factor is absorbed by defining normalized coordinates

$$q_{\vec{k}} = \frac{1}{\sqrt{V}}\, \tilde{\phi}(\vec{k}) , \quad p_{\vec{k}} = \dot{q}_{\vec{k}} , \tag{8}$$

and we get the following system

$$\left\{ \begin{array}{l} \text{An infinite set of independent oscillators labelled by } \vec{k} \\ \\ H_{\vec{k}} = \frac{1}{2}\, p_{\vec{k}}^2 + \frac{1}{2}\omega_k^{\,2}\, q_{\vec{k}}^2 \end{array} \right\} \tag{9}$$

The manipulations which led to Eq. (9) were formal. (There are configurations $\phi(\vec{x})$ for which the Fourier transform $\tilde{\phi}(\vec{k})$ does not exist; at Eqs. (7) and (8) we introduced an infinite constant). However, the system described by Eq. (9) is perfectly well-defined.

Each individual oscillator is well-defined and does not have any infinities. The coordinates are normalized such that the "mass" (the coefficient of the $\frac{1}{2}\, p_{\vec{k}}^2$ term) is one, and not infinite or zero. The commutation relation is $[p_{\vec{k}}, q_{\vec{k}}] = -i$ (and not $-i\delta(0)$). The facts that there are infinitely many oscillators, and that the $\vec{k}$ index is continuous, do not cause any problem, since in a free field theory the oscillators are independent of each other. Note that if we excite the $\vec{k}$ oscillator by $\Delta E_{\vec{k}}$, the system is excited by exactly the same amount. This means that no oscillator can be neglected as a "point in the continuum".

For each $\vec{k} \neq 0$ we define

$$a_{\vec{k}}^{\dagger} = 1/\sqrt{2\omega_k}\, (\omega_k\, q_{\vec{k}} - ip_{\vec{k}}) \tag{10}$$

and use the $(a_{\vec{k}}^{\dagger})^m |0\rangle_{\vec{k}}$ basis to the $\vec{k}$-Hilbert space.

For $\vec{k} = 0$

$$H_o = \frac{1}{2}\, p_o^2 \tag{11}$$

namely the potential term vanishes: $V(q_o) = 0$. Therefore, the $\vec{k} = 0$ oscillator has the symmetry

$$q_o \rightarrow q_o + a. \tag{12}$$

This corresponds to the symmetry (Eq. 2) of the original system. Since η in Eq. (2) is $\vec{x}$ independent, all the shift is in the $\tilde{\phi}(\vec{k} = 0)$ direction. The energy eigenstates of the $\vec{k} = 0$ oscillator are given by $\psi_p(q_o) = e^{-ipq_o}$ with $E_p = \frac{1}{2}\, p^2$, and the ground-state is

$$\psi_{g.s.}(q_o) = 1 \qquad \text{(for the } \vec{k} = 0 \text{ oscillator).} \quad (13)$$

This wave function is symmetric under Eq. (12), so that the symmetry is not spontaneously broken. However, the wave-function is not square-integrable, and this was probably the reason why it was rejected. This leads to the following apparent problem:

In quantum mechanics with one degree of freedom (our degree of freedom is q_o, and for simplicity we drop for a moment the index o) the transition amplitude from a state ψ at time t, to ψ' at t' is given by

$$\int dq'dq\ \psi'^*(q')\ \psi(q)\ G(q't';q,t) \qquad (14)$$

where $G = \langle q'|e^{-iH(t'-t)}|q\rangle$ is the Green-function. Let us choose the states ψ' and ψ to be the ground-state (Eq. (13)). Since the Hamiltonian (Eq. (11)) is invariant under Eq. (12) we get

$$G(q' + a,t';q + a,t) = G(q',t';q,t) \qquad (15)$$

and Eq. (14) factorizes as follows:

$$[\int dq'\, G(q',t';\bar{q},t)] \cdot [\int dq] \qquad (16)$$

where $\bar{q}$ is an arbitrarily fixed value of the coordinate q. The second factor in Eq. (16) is the infinite volume of the symmetry group, and the transition amplitude (Eq. (14)) is infinite.

A state ψ can be written as $\int dq\psi(q)|q\rangle$. Usually we assume that the states $|q\rangle$ and $|q'\rangle$ are distinct (if $q' \neq q$), and that a measurement can determine the value of the coordinate. In

that case $|\psi(q)|^2$ is the probability of finding the system in the state $|q\rangle$ and we must have $\int dq|\psi(q)|^2 = 1$. In gauge theories this approach is abandoned whenever we deal with the gauge degrees of freedom. Our basic assumption is that a degree of freedom under which the Hamiltonian is invariant should be treated in exactly the same way we treat the gauge degrees of freedom in gauge theories. We assume that the symmetry (Eq. (12)) is exact, and that no source term which breaks the symmetry can be introduced. Therefore, we can not measure the value of q_o. $|q_o\rangle$ and $|q_o'\rangle$ are not distinct states. They are two "copies" or "representatives" of the same physical state. Therefore, they cannot have different amplitudes $\psi(q_o)$ and $\psi(q_o')$. We must have $\psi(q_o)$ = const. as in Eq. (13). The q_o degree of freedom can never be excited; it is always in the ground state. This reflects the fact that q_o is not a physical degree of freedom. The infinity in Eq. (16) is due to overcounting. In the double integral of Eq. (14) the points (q',q) and $(q' + a,q + a)$ have equal contribution for any a. In the functional integral language, the trajectory $q(t)$ has the same contribution as $q(t) + a$. According to Fadeev and Popov, we have to eliminate this overcounting. This can be done by fixing some arbitrary value $\bar{q}$ at time t, and integrating over q' at time t'. This eliminates the $\int dq$ factor in Eq. (16). (To calculate the first factor in Eq. (16), we use the expression for the Green function of the free Hamiltonian of Eq. (11),

$$G = [2\pi i(t' - t)]^{-\frac{1}{2}} \exp[i(q' - q)^2/2(t' - t)].$$

The q' integration gives 1. This is the reason why we choose the constant in Eq. (13) to be 1.)

The symmetry we are dealing with is different from the rotational symmetry of an atom. We can measure the angular orientation of an atom by external sources or instruments and therefore we cannot conclude that the atom must always be in an s-state. Our symmetry (Eq. (12)) is a symmetry of the full "universe" described by the Lagrangian of Eq. (1), and it can be compared to the rotational invariance of the full universe (it does not make any sense to specify the angular orientation of the universe, and therefore these degrees of freedom are not physical).

To summarize:

(a) The degree of freedom in the symmetry direction (q_o) is not physical. It cannot be measured, since no symmetry breaking external source is allowed.

(b) Its wave function is always $\psi(q_o) = 1$.

(c) To avoid overcounting in calculations of transition amplitudes, we arbitrarily fix the value of q_o at the initial time.

The vacuum we have just constructed is unique. There is no meaning to the average value of the coordinate q_o, since it can have any value with equal probability. This means that $\langle 0|q_o|0\rangle$ is not well defined. In fact, the operator q_o or any non-trivial function of which (including $e^{-i\alpha q_o}$) is not defined. We assumed that the Hilbert space in the q_o coordinate includes only the $\psi(q_o) = 1$ state, and therefore any function of q_o will take us outside this

Hilbert space. This is just a reflection of the fact that q_o is not a physical degree of freedom. The set of operators in the problem is $\{q_k, p_k\}$. All of them, except for q_o, are invariant under the symmetry (Eq. (2)). We conclude that the only matrix elements we are allowed to calculate are those of operators which are invariant under the symmetry. The operator $\phi(\vec{x})$ has a q_o component in its expansion, and therefore it is not a legal operator. Only invariant combinations like $\partial_\mu\phi(\vec{x})$ or $\phi(\vec{x}) - \phi(\vec{y})$ are allowed. It should be emphasized that our solution violates cluster decomposition, only when ill-defined operators are used. Whenever we calculate matrix elements of invariant combinations of fields, the result is exactly the same as in any symmetry-breaking solution. Note that in order to have a meaning to $\langle 0|q_o|0\rangle = \bar{q}_o$ we have to construct a wave-packet $\psi(q_o)$ centered around $\bar{q}_o$ and with a width Δq_o. The spread of the conjugate momentum would then be $\Delta p_o \sim \frac{1}{\Delta q_o}$ and the cost in the kinetic energy would be $\sim (\Delta p_o)^2$ (q_o is normalized to have a unit mass). The kinetic energy term in the Hamiltonian, $p_o^2 = (-i\frac{\partial}{\partial q_o})^2$ tends to flatten $\psi(q_o)$, and when there is no potential-energy term $V(q_o)$ to fight this tendency (i.e., when there is a symmetry) the resulting wave-function of the ground-state is flat (Eq. (13)) .

The generator of the symmetry operation $q_o \to q_o + a$ is $p_o = -i\frac{\partial}{\partial q_o}$, and the unitary operator which corresponds to this operation is e^{-iap_o}. Since our Hilbert space in the q_o coordinate

contains only the state $\psi(q_o) = 1$, p_o is a well-defined operator, it does annihilate the vacuum, and the vacuum is invariant under e^{-iap_o}! (This is in contrast to the traditional literature which claims that it is impossible to construct a generator which annihilates the vacuum [Eq. (4)]).

b. The Normalization

Let us define the distance of a field configuration $\phi(\vec{x})$ from the origin $(\phi(\vec{x}) = 0)$ to be

$$\rho^2 \equiv \int d^n\vec{x}\, \phi^2(\vec{x}) = \int d^n\vec{k}\, \tilde{\phi}^2(\vec{k}) = \sum_{\vec{k}} q_{\vec{k}}^2 \qquad (17)$$

where the coordinates $\tilde{\phi}(\vec{k})$ and $q_{\vec{k}}$ are defined in Eqs. (5) and (8). The coordinates $q_{\vec{k}}$ are normalized in such a way that when we move from the origin in the $q_{\vec{k}}$ direction by an amount $\Delta q_{\vec{k}}$, the distance of the new configuration from the origin is exactly $\Delta q_{\vec{k}}$. In particular, when we move in the symmetry direction, $q_o \to q_o + a$, a is just the distance between the old and the new configurations. Note that our definition of distance (Eq. (17)) is such, that the normalized coordinates $q_{\vec{k}}$ have a unit mass (Eq. (9)). If instead of our $q_o = \frac{1}{\sqrt{V}}\int d^n\vec{x}\phi(\vec{x})$ we would use $q_o' = \frac{q_o}{\sqrt{V}}$ we would get a coordinate with an infinite mass $(m = V)$. In terms of this coordinate one can break the symmetry by constructing a wave function with a finite width $\Delta q_o'$. Its energy is still above the ground-state energy, but the difference goes like $\frac{1}{V}$ (I would like to thank Alan Guth for this observation). I prefer to avoid coordinates which have infinite (or zero) mass. In Section III, we will turn the symmetry (Eq. (2)) into a gauge symmetry and discover that for each value of $\vec{k}$ we have a

symmetry mode with exactly the same Hamiltonian as that of q_o (Eq. (11)). In that case, if we construct the symmetry-breaking wave-function for each such mode, the excitation energy $(\sim \frac{1}{V})$ is multiplied by the number of symmetry-modes $(N = V\, V_k$ where V_k is the volume in k space) and is infinite. Since there is no reason to treat the symmetry-modes in the gauge theory differently than we treat our global symmetry mode q_o, we have to reject the symmetry breaking wave-functions.

Note that a translation of the field $\phi(\vec{x})$ by a finite amount η (Eq. (2)) corresponds to a translation of q_o by $a = \sqrt{V}\,\eta$, which indicates that this translation corresponds to an infinite distance in configuration space. Since η can be any real number, the range of q_o is the real axis "expanded" by an infinite factor $\sqrt{V}$. Note that this abnormal range does not cause any difficulty, since we declared that q_o is not a physical degree of freedom, and our procedure for avoiding overcounting is such that the range of q_o is irrelevant.

In our normalization $p_o = \dot{q}_o = \frac{1}{\sqrt{V}}\int d^n\vec{x}\, \dot{\phi}(\vec{x})$ and therefore

$$p_o = \frac{1}{\sqrt{V}}\, Q \qquad (18)$$

where p_o is our generator, and Q is the generator in the literature. This means that the function $f_R(\vec{x})$ in the foot-note of Eq. (4) should be normalized to $\int f_R^2(\vec{x})\, d^n\vec{x} = 1$. The wrong normalization of Q explains why people had troubles with its infinite matrix elements. We note that the translation operator

can be written either as e^{-iap_o} or as $e^{-i\eta Q}$. The reason that p_o is the generator with the right normalization is that $a(= \sqrt{V}\,\eta)$ is the right measure of the translation distance.

c. A List of Pitfalls

1. The operator $a^{\dagger}_{\vec{k}}$ is ill-defined for $\vec{k} = 0$ ($\omega_{k=o} = 0$ in Eq. (10)). This operator is used in Ref. 3.
2. Using $[p_o, q_o] = -i$ it is concluded[3] that $\langle 0|[p_o,q_o]|0\rangle \neq 0$ and therefore $p_o|0\rangle \neq 0$. In our construction, it is clear that p_o is well defined and it does annihilate the vacuum. The troubles come from the fact that q_o is an illegal operator.
3. Assuming the existence of the operator $U(\eta) = e^{-iQ\eta}$, and using $U(\eta)\phi(\vec{x})U^{-1}(\eta) = \phi(\vec{x}) + \eta$ it is argued[4] that if $U(\eta)|0\rangle = |0\rangle$ we get $\langle 0|\phi(\vec{x})|0\rangle = \langle 0|\phi(\vec{x}) + \eta|0\rangle$. (In our notation we get $\langle 0|q_o|0\rangle = \langle 0|q_o + a|0\rangle$). This is considered to be a contradiction, since the two matrix elements are expected to differ by $\eta\,(a)$. At this point one may conclude that either the vacuum is not invariant under $U(\eta)$ or[4] the operator $U(\eta)$ does not exist. Our resolution on the other hand is that the two matrix elements are equally ill-defined.
4. The norm of $Q|0\rangle$ is sometimes used as a criterion for the breakdown of the symmetry. Using the $R \to \infty$ limit (foot-note above Eq. (4)), one gets $\langle 0|Q\,Q|0\rangle \sim R^{n-1}$, which indicates "troubles" for $n > 1$ but not for $n = 1$. Using the normalized generator $p_o = 1/\sqrt{V}\,Q$ where $V = R^n$ we get $\langle 0|p_o\,p_o|0\rangle \sim \frac{1}{R}$ and this goes to zero, independently of n.

d. Remarks on the Structure of the Hilbert Space

We have an infinite tensor product of Hilbert spaces (one for each $\vec{k}$). Let us look at the Hilbert space one gets by considering (the completion of) all vectors that differ from the ground-state (which is $|0\rangle_{\vec{k}}$ for each $\vec{k}$) only for a finite number of $\vec{k}$'s. For a denumerable set of indices, such a construction leads to a separable Hilbert space (namely, the set of basis vectors is denumerable). But since $\vec{k}$ is continuous, the Hilbert space we get is non-separable. For example, there are $\aleph$ one particle states $|1\rangle_{\vec{k}} = a^{\dagger}_{\vec{k}}|\,g.s.\rangle$. There is no problem in the normalization of these sharp-momentum states: they are normalized to one. We just do not have a denumerable basis for the Hilbert space. To construct a separable Hilbert space we may choose to work with states of the type

$$\sum_{\vec{k}} \frac{1}{\sqrt{V}}\, f(\vec{k})\, a^{\dagger}_{\vec{k}}|g.s.\rangle \equiv \int d^n\vec{k}\, f(\vec{k})\, a^{\dagger}(\vec{k})|g.s.\rangle \tag{19}$$

with $\int d^n\vec{k}\,|f(\vec{k})|^2 = 1$ and where

$$a^{\dagger}(\vec{k}) = \sqrt{V}\; a^{\dagger}_{\vec{k}} \tag{20}$$

is the usual operator-valued distribution, with the $\delta(\vec{k} - \vec{k}')$ normalization.

The reason we can define sums like the one in Eq. (19) is that although when we change $\vec{k}$, we change the Hilbert space, the new Hilbert space has exactly the same structure. In both we have, for example, the one particle state. This is not true, however,

for $\vec{k} = 0$. The operator $a^\dagger_{\vec{k}}$ (Eq. (10)) is not defined for $\vec{k} = 0$ and we cannot construct $a^\dagger_o|g.s.\rangle$. The corresponding Hilbert space has either a continuous energy spectrum, or, if one accepts the assumption that q_o is not a physical mode, it has a single state. Consequently, I suspect that there is no way to define $a^\dagger_o$ as a $\vec{k} \to 0$ limit, and Eq. (19) makes sense only if we impose[7]

$$f(\vec{k} = 0) = 0. \tag{21}$$

The conclusion is that the $\vec{k} = 0$ mode has nothing to do with the continuum. This is consistent with Lorentz-invariance: the $(\vec{k} = 0;\ E = 0)$ state is a single orbit, whereas all the $\vec{k} \neq 0$ one particle states form a different orbit (an orbit is a set of states which can be connected by a Lorentz transformation). Since the symmetry (Eq. (2)) involves only this isolated and unphysical mode, the considerations of this paper are independent of the way one chooses to treat the continuum.

e. Why is the Ferromagnet Different?

We shall discuss the problem on the level of one degree of freedom. For a spin system, the basis is discrete, say $|u\rangle$ and $|d\rangle$. Let us consider a diagonal Hamiltonian, $H = V_u|u\rangle\langle u| + V_d|d\rangle\langle d|$. If $V_u = V_d$, we have a symmetry. Both $|u\rangle$ and $|d\rangle$ (or any linear combination) are perfectly good ground states. In the $|u\rangle$ vacuum, for example, the symmetry is broken.

In this paper we are interested in a continuous degree of freedom q. We have a continuous set of basis vectors $|q\rangle$, and the

Hamiltonian contains a potential term $V(q)$ which can be written in the form $\sum_q V(q)|q\rangle\langle q|$. We have a symmetry if $V(q) = \text{const}$. The big difference is that now H contains the kinetic energy term $\frac{1}{2m}p^2 = -\frac{1}{2m}\frac{\partial^2}{\partial q^2}$. This term prefers a flat distribution $\psi(q) = 1$. which is symmetric, to a peaked distribution, which breaks the symmetry.

III. Free Gauge Theory

Let us turn the global symmetry (Eq. 2) into a local symmetry, and introduce a gauge field A_μ. The purpose of this exercise is to show that in this gauge theory, for each $\vec{k}$ there is one degree of freedom the Hamiltonian of which is identical to that of q_o (Eq. (11)). It would be clear that according to the current approach to gauge theories, one should treat these zero-modes in exactly the same way we treated q_o. We consider this to be a significant support for our treatment of the zero-mode in the case of the global symmetry.

The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi - g\,A_\mu)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{22}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and ϕ and A_μ are real fields. $\mathcal{L}$ is symmetric under

$$\begin{aligned} \phi(x) &\to \phi(x) + \eta(x) \\ A_\mu(x) &\to A_\mu(x) + \frac{1}{g}\partial_\mu\eta(x) \end{aligned} \tag{23}$$

Since we use the Schrödinger picture, we choose the $A_o = 0$ gauge. At each spatial point $\vec{x}$, we have $n + 1$ degrees of freedom:

$\phi(\vec{x})$ and $A_i(\vec{x})$ (n is the number of space dimensions). The symmetry is still given by Eq. (23), but now η is a function of spatial $\vec{x}$ only.

We define $q_{\vec{k}}$ and $p_{\vec{k}}$ as before (Eqs. (5) and (8)) and in a similar way we define

$$\vec{A}_{\vec{k}} = \frac{1}{\sqrt{V}} \tilde{\vec{A}}(\vec{k}) \quad , \quad \vec{E}_{\vec{k}} = \dot{\vec{A}}_{\vec{k}} \, . \tag{24}$$

(All the vectors refer to the real space.)

The Hamiltonian is $H = \sum_{\vec{k}} H_{\vec{k}}$, without any interaction between different $\vec{k}$ modes. For a given $\vec{k}$ we have

$$H_{\vec{k}} = \frac{1}{2}\dot{q}_{\vec{k}}^2 + \frac{1}{2}\dot{\vec{A}}_{\vec{k}}^2 + \frac{1}{2}(\vec{k}\, q_{\vec{k}} - g\vec{A}_{\vec{k}})^2 + \frac{1}{2}(\vec{k} \times \vec{A}_{\vec{k}})^2 \tag{25}$$

[to simplify the notation we ignored the fact that $q_{\vec{k}}$ and $\vec{A}_{\vec{k}}$ are complex, with $q_{-\vec{k}} = q_{\vec{k}}^*$ and $\vec{A}_{-\vec{k}} = \vec{A}_{\vec{k}}^*$. In fact, for each pair $(\vec{k},-\vec{k})$ we get two independent Hamiltonians. To get the first one we substitute in Eq. (25) $q_{\vec{k}} \to \sqrt{2}\,\mathrm{Re}\, q_{\vec{k}}$ and $\vec{A}_{\vec{k}} \to \sqrt{2}\,\mathrm{Im}\, \vec{A}_{\vec{k}}$. To get the second one we substitute $q_{\vec{k}} \to -\sqrt{2}\,\mathrm{Im}\, q_{\vec{k}}$ and $\vec{A}_{\vec{k}} \to \sqrt{2}\,\mathrm{Re}\vec{A}_{\vec{k}}$.] The transverse (relative to the $\vec{k}$ direction) component of $\vec{A}_{\vec{k}}$, is denoted by $A_{\vec{k}}^T$ (it is an n - 1 vector) and it decouples from $q_{\vec{k}}$. In order to diagonalize $H_{\vec{k}}$, we perform the following rotation in the plane of the longitudinal component $A_{\vec{k}}^L$ and $q_{\vec{k}}$:

$$x_{\vec{k}} = (kq_{\vec{k}} - gA_{\vec{k}}^L)/\omega_k$$

$$y_{\vec{k}} = (gq_{\vec{k}} + kA_{\vec{k}}^L)/\omega_k$$

where $k = |\vec{k}|$ and $\omega_k = \sqrt{g^2 + k^2}$

We now get

$$H_{\vec{k}} = \frac{1}{2}\dot{x}_{\vec{k}}^2 + \frac{1}{2}(\dot{A}_{\vec{k}}^T)^2 + \frac{1}{2}\dot{y}_{\vec{k}}^2 + \frac{1}{2}\omega_k^2\left[x_{\vec{k}}^2 + (A_{\vec{k}}^T)^2\right]$$

The coordinates $x_{\vec{k}}$ and $A_{\vec{k}}^T$ are n independent harmonic oscillators which represent the n degrees of freedom of a free vector particle with a mass $\mu^2 = g^2$. The Hamiltonian for $y_{\vec{k}}$ decouples from the other degrees of freedom. It does not have any potential term in $y_{\vec{k}}$. For each $\vec{k}$ we have a degree of freedom which has exactly the same Hamiltonian as in Eq. (11).

Obviously, the $y_{\vec{k}}$ are the coordinates in the gauge-symmetry direction. The $\vec{k}$ Fourier component of $\eta(\vec{x})$ in Eq. (23) induces a shift in $y_{\vec{k}}$, and leaves invariant all the other coordinates. The fact that the potential in this direction is flat corresponds to the statement that this is a symmetry of the Hamiltonian. The way to treat each one of these zero-modes, is identical to the way we treated the q_o mode (below Eq. (11)). The statement that q_o is not a physical degree of freedom corresponds in the gauge-theory to the statement that the gauge degrees of freedom are not physical and that we are not allowed to introduce gauge-variant sources. This treatment of the gauge degrees of freedom is identical to that of the current approach to gauge theories.

Note that in the gauge theory the Hamiltonian of the $\vec{k} = 0$ mode is

$$H_o = \frac{1}{2}\dot{q}_o^2 + \frac{1}{2}\dot{\vec{A}}_o^2 + \frac{1}{2}g^2\vec{A}_o^2 \qquad (28)$$

The zero mode is $y_o = q_o$ and it corresponds to the global symmetry of the gauge-theory (namely to the transformation of Eq. (23) with a constant η). We observe that the Hamiltonian of y_o is identical to that of any $y_{\vec{k}}$, and it should be treated in exactly the same way we treat the $\vec{k} \neq 0$ modes. The conclusion is that the global symmetry is not broken just as the gauge-symmetry is not broken. The expectation value of y_o in the vacuum is ill-defined just like the expectation value of any other $y_{\vec{k}}$. (The fact that the gauge-symmetry is not broken is generally accepted.[8])

IV. THE U(1) SYMMETRY

a. The non-linear σ-model

Let us consider the non-linear σ-model with the U(1) symmetry, in the continuum and in any number n of spatial dimensions. The system is defined by the following Lagrangian and constraint:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2, \qquad \vec{\phi}^2 = 1 \qquad (29)$$

where $\vec{\phi} = (\phi_1, \phi_2)$ (the vector here refers to the internal space). The constraint $\vec{\phi}^2 = 1$ describes a circle. Let us denote a point on this circle by z. The basic objects in the Schrödinger picture are the field configurations $z(\vec{x})$, which are mappings from the real n-dimensional space, to the circle. We assume these mappings to be

continuous (otherwise the potential-energy density $\frac{1}{2}(\partial_i \vec{\phi})^2$ is not defined). This assumption is crucial to our considerations. Note that it does not make any sense if $\vec{x}$ is on a lattice.

Let us now parametrize the points on the circle by a real angular variable θ. Given θ, the corresponding point z on the circle is determined uniquely. However, for any given z, θ is only determined modulu $2\pi m$ (with integer m).

Let us now parametrize a given configuration $z(\vec{x})$. We start from an arbitrary point in space, $\vec{x}_p$. We parametrize the corresponding point on the circle $z_p = z(\vec{x}_p)$ by θ_p. (Any $\theta_p + 2\pi m$ will do it, but we choose only one possibility. We will show that this is the only ambiguity in the description of $z(\vec{x})$!). Because of the continuity, we can proceed from $\vec{x}_p$ outwards, and parametrize $z(\vec{x})$ by a continuous function $\theta(\vec{x})$. Obviously, we could use parametrizations which have discontinuous jumps of $2\pi m$ whenever we wish. Our point, however, is that it is possible to use a parametrization $\theta(x)$ which is continuous all over space. Once we make this choice, the only ambiguity we have is the determination of θ_p at $\vec{x}_p$. Once this is done, $\theta(\vec{x})$ is uniquely determined everywhere else. (Note that on the lattice we have the $m2\pi$ ambiguity for any lattice point independently. In the continuum we have it only once!). Our argument is independent of the number of space-dimensions, n. It does, however, depend on the symmetry. (If z is a point on a sphere, the parametrization in terms of the θ and ϕ angles is not necessarily continuous, because of the singularity of the ϕ variable at $\theta = 0$).

We can now ignore the point $\vec{x}_p$, and conclude that for any continuous $z(\vec{x})$ there is a continuous parametrization $\theta(\vec{x})$ which is uniquely determined up to a global addition of $2\pi m$.

The system is symmetric under a global rotation. In the θ parametrization the symmetry transformation is

$$\theta(\vec{x}) \rightarrow \theta(\vec{x}) + \eta \tag{30}$$

where η is independent of $\vec{x}$.

This is very similar to Eq. (2) in the free field theory. The symmetry transformation carries us from one field configuration to a new field configuration which has the same potential energy. The only difference is that in Eq. (30), whenever $\eta = 2\pi m$, we get a new parametrization for the same field configuration.

We can now define the Fourier transform of $\theta(\vec{x})$ as in Eq. (5). Since the only ambiguity in $\theta(x)$ is the addition of a constant term, $2\pi m$, the only ambiguity in $\tilde{\theta}(\vec{k})$ is in the $\vec{k} = 0$ mode. The $\vec{k} \neq 0$ modes have nothing to do with the symmetry or with the ambiguity. For these modes we define

$$q_{\vec{k}} = \frac{1}{\sqrt{V}} \tilde{\theta}(\vec{k}) \tag{31}$$

just as we did in Eq. (8), and get exactly the same system of independent oscillators as in Eq. (9).

For the $\vec{k} = 0$ mode we have the ambiguity that $\theta(\vec{x})$ and $\theta(\vec{x}) + 2\pi m$ corresponds to the same configuration. When we Fourier transform (Eq. (5)) this means that $\tilde{\theta}(\vec{k} = 0)$ and $\tilde{\theta}(0) + 2\pi m V$, or (using Eq. (31) for $\vec{k} = 0$) that q_o and $q_o + 2\pi m\sqrt{V}$ correspond to the same point. This means that the normalized coordinate, q_o, describes a point on a circle of radius $\sqrt{V}$. The potential energy around this circle is constant (in fact, zero), which means that q_o is the symmetry direction. The fact that the circle is infinite is irrelevant, since q_o is not a physical degree of freedom (we forbid any source term which is not invariant under the rotational symmetry). The wave function in q_o is $\psi(q_o) = 1$, and we do not allow any excitation in this unphysical mode. The way to avoid the infinities due to the symmetry (the infinite circumference of the circle) is identical to that in the free field theory, which is just the analogue of the Faddeev-Popov procedure. The wave-function in q_o is symmetric (namely, each point on the circle has the same probability). We conclude that the global U(1) symmetry is not broken! The discussion of the local U(1) symmetry is identical to that of the free gauge theory in Section III, and the conclusionsare the same.

b. <u>The linear σ-model</u>

The Lagrangian of Eq. (29) is replaced by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{1}{2}\lambda(\vec{\phi}^2 - f^2)^2. \tag{32}$$

Since we do not know how to solve this field-theory, we cannot construct a solution and check whether it is symmetric or not. If we believe that the limit of the solutions of Eq. (32) as $\lambda \to \infty$ (and $f = 1$) is the solution of Eq. (29), then we have to conclude that the global $U(1)$ symmetry of Eq. (32) is not broken.

The $U(1)$ symmetry of Eq. (32) is as unphysical as the $U(1)$ symmetry of Eq. (29), and as before, we do not allow any symmetry breaking sources. The kinetic energy term in the Hamiltonian is still there and it tends to smear the wave-function ψ in the symmetry direction. Therefore we expect that the solutions of Eq. (32) are symmetric.

We would like to emphasize that we are not in conflict with any physical predictions of the usual approach. If we use the (ρ,θ) parametrization to $\vec{\phi}$ $(\phi_1 + i\phi_2 = \rho e^{i\theta})$, we agree with the statement that $\langle 0|\rho^2|0\rangle \sim f^2$. We disagree, however, with the statement that $\langle 0|\theta|0\rangle = \theta_o$, namely that there is some preferred but unmeasurable direction. We claim that the ground-state is an equal superposition of all possible directions: For any field configuration $[\rho(\vec{x}), \theta(\vec{x})]$ the wave-function ψ satisfies

$$\psi[\rho(\vec{x})e^{i\theta(\vec{x})}] = \psi[\rho(\vec{x})e^{i[\theta(\vec{x}) + \eta]}] \qquad (33)$$

with any value of the constant η.

Our considerations are not restricted to the $U(1)$ symmetry. The only requirement is that the symmetry will be continuous. To illustrate the meaning of a "continuous symmetry" let us consider a

field theory with the fields $\phi_a(\vec{x})$, where a is an internal degree of freedom. Formally, we can consider $|\phi_a(\vec{x})\rangle$ to be the basis of our Hilbert space (like the basis $|q\rangle$ in one-dimensional quantum mechanics). In the Schrödinger picture, the most general state is given by $\psi[\phi_a(\vec{x})]$. The symmetry is continuous if it induces in the space of all configurations $\phi_a(\vec{x})$, continuous surfaces along which the potential energy $V[\phi_a(\vec{x})]$ is a constant. The emphasis here is that the continuity is with respect to the $\phi_a(\vec{x})$ space and not with respect to $\psi[\phi_a(\vec{x})]$. The kinetic energy term in the Hamiltonian always tends to smear ψ along the equi-potential surfaces in the $\phi_a(\vec{x})$ space. This is in contrast to the spin system of Section II.e. There, the <u>basis</u> of the Hilbert space is discrete, and therefore the kinetic term is absent. The fact that the set of all possible states, $\psi = a|u\rangle + b|d\rangle$, is continuous in a and b, is not relevant. Another example in which the symmetry is not continuous is a scalar field with $V = \frac{1}{2}\lambda(\phi^2 - f^2)^2$, since the symmetry does not induce any equipotential trajectory which connects $\phi(\vec{x})$ to $-\phi(\vec{x})$.

When we couple gauge-bosons and fermions to the scalar field as in the Weinberg-Salam model, we agree with all the physical predictions such as the mass spectrum of the theory. We disagree, however, with the statement that the global symmetry is broken and that the scalar fields choose one particular direction.

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