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EQUILIBRIUM AND STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS

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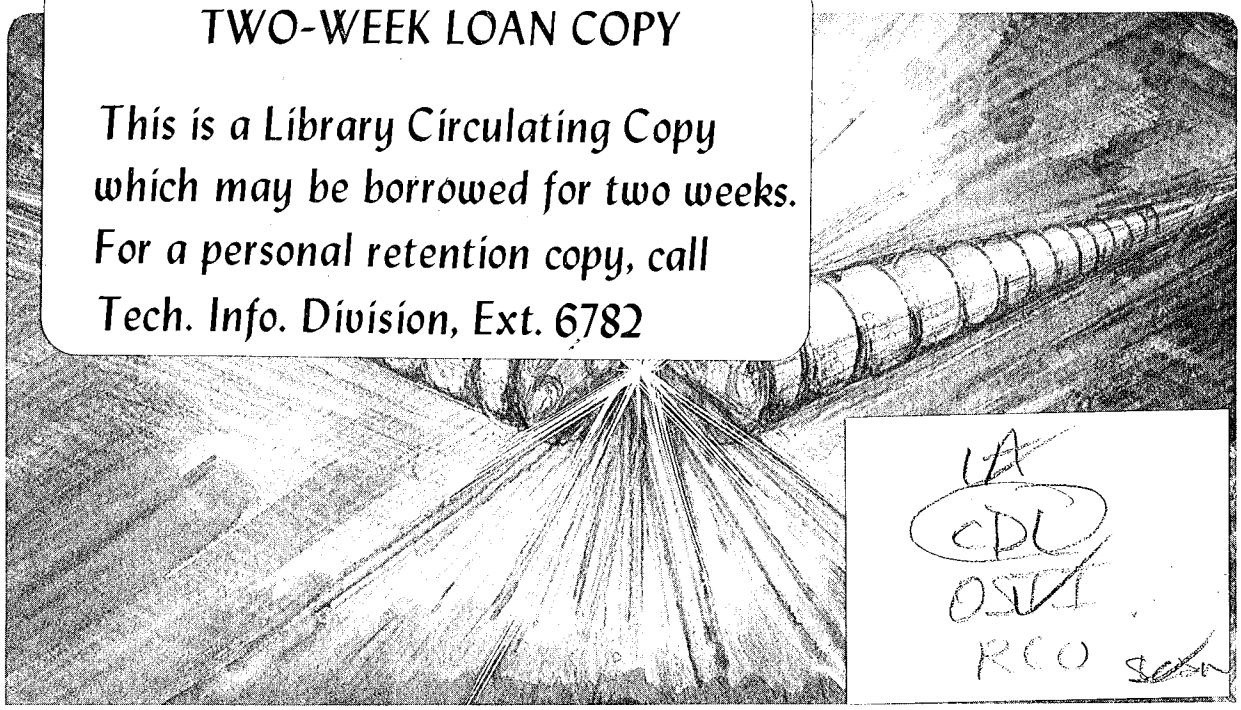

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EQUILIBRIUM AND STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS

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EQUILIBRIUM AND STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS

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ABSTRACT

In some confinement schemes of current interest, for example in conventional and field-reversed mirrors, and in the Tormac sheath, the parameter η, defined as the ratio of ion gyroradius to the plasma scale length $L_\perp$ perpendicular to the magnetic field, may take on appreciable values, as large as 1/3. In this work, formalisms appropriate to the study of such strongly inhomogeneous plasmas are developed and applied, for three general aspects of plasma behavior. Each of these three topics is studied using Hamiltonian formalism, which greatly aids in expressing results in a general and concise manner.

In Part I we study the equilibrium of strongly inhomogeneous, collisionless, slab plasmas, using a generalized version of a formalism previously developed, which permits the generation of self-consistent equilibria, for plasmas with arbitrary magnetic shear, and variation of magnetic field strength. The principal accomplishment of Part I is the development of an interpretive method, which connects the canonical variables, in terms of which the formalism is expressed, to the physical quantities needed to model a given physical system, and which is both

valid and practical, even for cases of strong magnetic inhomogeneity, and in particular for magnetic shear. The method provides a concrete physical content for the canonical formalism, permitting application to problems of interest. We apply the formalism to the particular problem of the Tormac sheath, modeling both the magnetic field and the loss-cone form of the ion distribution function in that region. The interpretive method of Part I is also of considerable use in the formalisms of Parts II and III.

Part II is concerned with the development of a systematic procedure for deriving the form of the guiding-center Hamiltonian K, for finite η, in an axisymmetric geometry. In the process of obtaining K, an expression for the first adiabatic invariant (the gyroaction) is obtained, which generalizes the usual expression $\frac{1}{2}mv_\perp^2/\Omega_c$ ($\Omega_c \equiv eB/mc$), to finite η and magnetic shear.

In Part III a formalism is developed for the study of the stability of strongly-inhomogeneous, magnetized slab plasmas; it is then applied to the ion-drift-cyclotron instability. We restrict our study to longitudinal perturbations. Use of Hamiltonian formalism and operator techniques permits a very concise expression of the full nonlocal mode equation, in a form valid for arbitrary inhomogeneity, but which transparently reduces in the appropriate limit to the weakly inhomogeneous expressions usually used to study stability. Certain new effects of potential significance are found, arising naturally from the mathematics. In particular, we find a generalized expression for the diamagnetic drift

frequency ω_*, of central importance in many plasma instabilities. For nonzero $v_\parallel/(\Omega_c L_s)$ (where $v_\parallel$ is the parallel velocity of a particle at its guiding center, and L_s is the shear scale length), ω_* becomes a function of position in phase space, with a shift from its unsheared value ω_*^o given by $\omega_* = \omega_*^o [1 + (v_\parallel/\Omega_c L_s)]^{-1}$.

The formalism is phrased in terms of a variational principle which, in addition to serving as a labor-saving device in parameter regimes where use of a finite-order differential equation is an adequate approximation to the full nonlocal mode equation, also permits the derivation of valid dispersion equations when the problem is fully nonlocal in character.

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INTRODUCTION

In plasmas of current interest, for example in the Tormak sheath region, in mirror machines, and in some Tokamak experiments, the ratio η of the ion gyroradius to plasma scale lengths perpendicular to the magnetic field, may be an appreciably large parameter. The meaning of "appreciably large" will be clarified in each of the three parts of this thesis, but generally we mean by this that the inhomogeneity is strong enough to modify the actual form of the equations governing a given aspect of the plasma behavior, rather than entering the equations in only a parametric fashion. In the latter situation, we shall say that the plasma is "weakly inhomogeneous."

Since in going from weak to strong inhomogeneity one removes a simplifying approximation from the theory, formalisms applicable to strong inhomogeneity will tend to be complicated and cumbersome. Therefore any formal means which makes the formalism more succinct and elegant is desireable. All three areas of study here are accordingly treated making maximal use of Hamiltonian formalism, in terms of which the equations governing different plasma geometries tend to look the same, and those governing the strongly inhomogeneous case have a transparent formal resemblance to their weakly inhomogeneous limits.

In the first part of this thesis we study the equilibrium of strongly inhomogeneous Vlasov slab plasmas, extending and clarifying the work of P. Channell.[1] In Part II we develop a canonical formulation of guiding center motion valid for particles having appreciable η

("finite gyroradius"), moving in axisymmetric geometries. Part III is concerned with the development and application of a formalism appropriate to the systematic study of the effects on plasma stability of strong inhomogeneity.

Each of these three parts has its own introductory section, in which the work of that section is described in greater detail. The parts are presented in an order which allows for a natural progression in the development of ideas and formalism, and later parts do make some reference to the preceding ones. Nevertheless the areas of study are distinct, and any of the three parts should be comprehensible without having to read the other two.

PART I. VLASOV SLAB EQUILIBRIA: INTERPRETATION AND PRACTICAL APPLICATION OF THE CHANNEL FORMALISM

I.A. Introduction To Part I

The work of P. Channell[1] enables one to generate self-consistent, exact Vlasov equilibria in a slab geometry (i.e., a geometry with a single direction of inhomogeneity, which we take to be the x direction), allowing for arbitrary magnetic shear, as well as arbitrary variation in magnetic field strength $B \equiv |\underline{B}|$ and in plasma density, the scale lengths of these quantities (L_s, L_B, and L_n, respectively) being limited only by the self-consistency of the equilibria generated. The possibility of generating such equilibria is particularly interesting for plasmas in which the ratio η of the ion gyroradius ρ_i to the magnetic scale length $L_{\underline{B}} \equiv \min(L_B, L_s)$ is not a negligibly small parameter. For such plasmas, the ion orbits become complicated, distorted from the helical form valid in the small η regime. Accordingly, calculations of plasma quantities such as density, pressure, and magnetic field variation, using the more elementary approaches adequate for small η, become of questionable validity.

In the Channell approach, from writing out the x-component of the pressure balance equation in terms of the constants of the motion (H, P_y, P_z)(the single-particle energy, and the y and z components of canonical momentum), one obtains an equation for the development in x of the vector potential $\underline{A} = (o, A_y, A_z)$, and the scalar potential Φ, formally the same as the equation of motion of a particle in a three

dimensional space $(\underline{A},\Phi)$, propagating in pseudo-time x, in an effective potential $U(\underline{A},\Phi)$. (U turns out to be a constant times the xx-component Π_{xx} of the pressure tensor $\underline{\underline{\Pi}}$.) Imposing quasineutrality on this system enables one to eliminate one degree of freedom (say Φ), and so have a two dimensional particle motion problem.

Channell's work did not allow for a scalar potential, and his formalism was derived only for a highly restricted class of distribution functions $f_s(H,P_y,P_z)$. H. Grad showed[2] that, for the one dimensional pseudo-motion problem of an unsheared magnetic field, the same formalism holds, for arbitrary choice of f_s. Recently, A. N. Kaufman showed[3] that one can choose f_s arbitrarily for the sheared (2-D) problem Channell studied.

For the Channell formalism to be useful, one must be able to design the canonical form $f_s(H,P_y,P_z)$ of the distribution function to model systems of physical interest. Physically, one may be given a specification of the desired magnetic field $\underline{\tilde{B}}(x)$, and so the reference trajectory $\underline{\tilde{A}}(x)$ which yields that $\underline{\tilde{B}}$, and a general description of the <u>physical</u> form $f_s(X_o,v_\perp^2,v_\parallel)$ ($X_o \equiv$ guiding center position) of the distribution function (e.g., loss cone, with a prescribed density variation with X_o) to be modeled. The present work establishes the connections between these physical quantities, and the canonical quantities (H,P_y,P_z,f_s,U) of the Channell formalism.

We begin in Sec. I.B by deriving the Channell formalism in its generalized form. In Sec. I.C, methods of interpreting the formalism are developed and applied. Section I.C.1 discusses the connection

between the canonical variables (H,P_y,P_z) and the physical ones $(X_o,v_\perp,$ $v_\parallel)$. Graphical methods are developed which facilitate making this connection, and which enable one to easily and rapidly extract a large amount of information, even in situations where the corresponding algebraic expressions become analytically intractable. These methods are extended in Secs. I.C.2 to I.C.4. In Sec. I.C.2 we develop a prescription for determining f_s, given f_s and $\underline{\tilde{A}}$. Section I.C.3 presents an explicit application of this procedure, modeling the Tormac sheath region, in which $\tilde{f}_s$ has a loss-cone form, and where $\underline{\tilde{A}}(x)$ gives a sheared magnetic field.

Having determined f_s, in Sec. I.C.4 we establish the connection between the topologies of f_s and of U. The discussion is largely qualitative, again aided by graphical visualization of the algebraic statements. This sort of analysis is continued in Sec. I.C.5, in which a remaining question of self-consistency is answered, thereby showing the validity of the prescription for choosing f_s, laid down in Sec. I.C.2.

Quantitative support for the qualitatively-determined conclusions reached in Secs. I.C.2 to I.C.5 is given in Sec. I.C.6, and in Appendix A. Appendix A carries out in an analytic fashion an evaluation of U, given f_s, and arrives at a form having the same features as expected from the more qualitative analysis of Sec. I.C.4. Section I.C.6 displays the results of numerical adaptation by W. M. Sharp of the methods developed in the foregoing sections. The results bear out the conclusions and expectations developed in those sections.

I.B. Generalized Channell Formalism

As described in the introduction to Part I, we consider a slab geometry, with x the sole inhomogeneity direction. The magnetic field $\underline{B}(x) = (o, B_y(x), B_z(x))$, obtained from the vector potential $\underline{A}(x) \equiv (o, A_y(x), A_z(x))$ through the relation

$$\underline{B} = \nabla \times \underline{A} = \hat{x} \times d\underline{A}/dx, \tag{I-1}$$

has at this point arbitrary variation with x, as does the electrostatic potential $\Phi(x)$. (Their x dependence will be determined self-consistently from the Channell formalism.)

The pressure balance equation for species s is

$$\underline{\nabla} \cdot \underline{\underline{\Pi}}^s = c^{-1} \underline{j}_s \times \underline{B} - \rho_s \underline{\nabla} \Phi . \tag{I-2}$$

In slab geometry, its only nontrivial component is the x-component:

$$\partial \Pi^s_{xx}/\partial x = c^{-1} \hat{x} \cdot (j_s \times \underline{B}) - \rho_s \ \partial \Phi/\partial x. \tag{I-3}$$

Writing out Π^s_{xx} in terms of canonical variables, we will see shortly [cf. Eq. (I-8)] that it is a functional of $\underline{A}$ and Φ, its x dependence entering only through these functions of x. The canonical momentum $\underline{p} \equiv (p_x, p_y, p_z)$ has y and z components which are constants of the motion, $(o, P_y, P_z) \equiv (o, p_y, p_z) \equiv \underline{P}$ = constant. A third constant of the motion is the single-particle Hamiltonian H_s, given by

$$H_s = p_x^2/2m_s + V_s(x|\underline{P}), \tag{I-4}$$

where

$$V_s(x|\underline{P}) \equiv [\underline{P} - (e_s/c)\ \underline{A}(x)]^2/2m_s + e_s\ \Phi(x) \tag{I-5}$$

is the effective 1-D potential in which a particle with momentum $\underline{P}$ oscillates.

With the distribution function f_s written in terms of these three constants, Π^s_{xx} appears as

$$\Pi^s_{xx} \equiv m_s^{-1} \int d^3 \underline{p}\, p_x^2 f_s(H_s, \underline{P}). \tag{I-6}$$

We now define h_s by

$$2m_s h_s \equiv p_x^2 = 2m_s(H_s - V_s) = 2m_s(H_s - e_s\Phi) - (\underline{P} - (e_s/c)\underline{A})^2 \tag{I-7}$$

and change the variables of integration in Eq. (I-6) from $\underline{p}$ to $(h_s, \underline{P})$, obtaining

$$\Pi^s_{xx}(\underline{A},\Phi) = 2\int d\underline{P} \int_0^\infty dh_s (2m_s h_s)^{1/2} f_s\left\{H_s = h_s + e_s\phi + [\underline{P} - (e_s/c)\underline{A}]^2/2m_s, \underline{P}\right\}. \tag{I-8}$$

From this form of Π^s_{xx}, we see that we can write

$$\begin{aligned} \partial_x \Pi^s_{xx}(\underline{A},\Phi) &= (\partial\underline{A}/\partial x)\cdot(\partial\Pi^s_{xx}/\partial\underline{A}) + (\partial\phi/\partial x)(\partial\Pi^s_{xx}/\partial\Phi) \\ &= \hat{x}\cdot[(\partial\Pi^s_{xx}/\partial\underline{A})\times\underline{B}] + (\partial\Phi/\partial x)(\partial\Pi^s_{xx}/\partial\Phi). \end{aligned} \tag{I-9}$$

Putting this into (I-3) and identifying the functions accompanying the $\underline{B}$ and Φ terms on the right and left hand sides, we find

$$\partial\Pi^s_{xx}/\partial\underline{A} = c^{-1}\, \underline{j}_s \tag{I-10}$$

and

$$\partial\Pi^s_{xx}/\partial\Phi = -\rho_s\,. \tag{I-11}$$

Using Eqs. (I-10) and (I-11) in Ampere's and Poisson's equations, respectively, gives

$$d^2 \underline{A}/dx^2 = - 4\pi c^{-1} \underline{j} = - (\partial/\partial \underline{A})\, U(\underline{A},\Phi) \tag{I-12}$$

and

$$d^2\Phi/dx^2 = - 4\pi\rho = - (\partial/\partial\Phi)\, U(\underline{A},\Phi) \tag{I-13}$$

where

$$U(A,\Phi) \equiv 4\pi \sum_s \Pi^s_{xx}\, (A,\Phi). \tag{I-14}$$

Equations (I-12) and (I-13) are a closed set of equations for exact, self-consistent equilibria, i.e., they embody the generalized Channell formalism. We may obtain a somewhat simpler version, if we replace (I-13) by the quasineutrality condition,

$$0 = - 4\pi\rho = (\partial/\partial\Phi)\, U(\underline{A},\Phi). \tag{I-15}$$

One may solve this for $\Phi(A)$, and combine this with (I-12) to obtain

$$d^2\underline{A}/dx^2 = - (d/d\underline{A})\, U[\underline{A},\Phi(\underline{A})]. \tag{I-16}$$

Equation (I-16) describes the motion of a pseudoparticle moving in a 2-dimensional pseudospace $\underline{A}$, evolving in pseudotime x. It is the form of the Channell formalism we shall work with here.

I.C. Interpretive Methods

We want to develop an understanding of the connections between the canonical variables (e.g., $\underline{P}, H_s, f_s, U$) in terms of which the Channell formalism is couched, and the physical variables (e.g., $X_o, v_\perp^2(X_o), v_\parallel(X_o)$, $\tilde{f}_s$) in terms of which a system to be modeled is specified. Here we describe a simple graphical method to aid one in making these connections.

Using this approach, much of the relevant canonical and physical information can be visually ascertained at a glance. To each graphically obtained piece of information, there is a corresponding algebraic statement. The graphical results are in principle as precise as the algebraic ones, and enable one to make qualitative statements about the relations among the various quantities in the theory, even in situations where the algebraic counterparts become mathematically intractable. The technique will also be useful in Parts II and III.

I.C.1. Connecting the Physical and Canonical Variables

We begin by noting that $\underline{A}$ makes its appearance in the theory only in the combination $(\underline{P} - (e_s/c)\underline{A})$. Because of this, it will prove convenient to visualize all quantities over a single $\underline{P}$ or $\underline{A}$ plane, where $\underline{A} \equiv (e_s/c)\,\underline{A}$. The Channell formalism becomes useful in the regime of appreciable gyroradius to scale length, where insuring self-consistency of an equilibrium becomes difficult. Therefore, in the following discussion, we shall be considering the ion species, $e_s = e_i = e$. Though less useful, the formalism is equally valid for the electrons, however, and everything that will be said here concerning the ions can be applied (with appropriate changes of signs and of mass) to the electrons. This is elaborated upon at the end of Sec. I.C.4.

Suppose we are given the physical information to be modeled, $\tilde{f}_s(X_o, v_\perp^2, v_\parallel)$ and the desired magnetic field $\tilde{\underline{B}}(x)$, through the reference trajectory $\tilde{\underline{A}}(x)$. We want to design a canonical distribution $f_s(H, \underline{P})$ which conforms to these physical requirements as closely as possible. We first draw in $\tilde{\underline{A}}(x)$, as in Fig. 1. Using Eq. (I-1), we can then draw in the corresponding magnetic field at various points $x_1, x_2, x_3 \ldots$

along the trajectory. We have chosen $\underline{\tilde{A}}$ (x) to model Tormac, the elbow in $\underline{\tilde{A}}$ yielding the shear in $\underline{\tilde{B}}$ which occurs across the sheath region.

Now suppose we have found the self-consistent magnetic field $\underline{B}^{SC}(x)$ and its $\underline{A}^{SC}(x)$. (We are designing things so that $\underline{A}^{SC}$ lies as near $\underline{\tilde{A}}$ as possible). We can make a plot analogous to Fig. 1 for $\underline{A}^{SC}$. Given $\underline{A}^{SC}$, the canonical variables (H, $\underline{P}$) acquire a definite physical significance. A natural definition of a particle's guiding center position X_o is the bottom of the well $V(x|\underline{P})$ in which it is sloshing, i.e., we define X_o by

$$0 = (\partial/\partial x)\ V(x|\underline{P})\Big|_{x = X_o} . \qquad \text{(I-17)}$$

One may in principle solve this for $X_o(\underline{P})$. In practice, this will in general be analytically intractable, so one may have to solve numerically to obtain explicit results. A graphical visualization of (I-17) is, however, much simpler. We refer to Fig. 2, which neglects $e\Phi$ in Eq.(I-5). Neglect of $e\Phi$ gives a V which is simpler to analyse graphically (setting $c = m_i = 1$),

$$V(x|\underline{P}) = \frac{1}{2}[\underline{P} - \underline{A}^{SC}(x)]^2 . \qquad \text{(I-18)}$$

In Tormac or mirror machines, where Φ comes from the escape of electrons along field lines, $e\Phi \sim T_e$, and so for a hot-ion plasma, $e\Phi \sim T_e << T_i$ $\sim \frac{1}{2}(P-\underline{A})^2$. Neglect of $e\Phi$ is thus a physically valid approximation in these cases, in addition to aiding pedagogic clarity.

From (I-18), we see that $V(x|\underline{P})$ is just half the square of the distance, in the $\underline{P}$ - $\underline{A}$ plane, from the point $\underline{P}$ which characterize a

given particle, to the point $\underline{A}^{SC}(x)$ on the trajectory. Extrema of this distance, i.e., points $\underline{A}(x)$ at which Eq. (I-17) is satisfied, occur when the line passing from $\underline{P}$ to $\underline{A}^{SC}(x)$ is normal to the trajectory $\underline{A}^{SC}$ at $\underline{A}^{SC}(x)$. Therefore all points $\underline{P}$ lying along this line have the same guiding center position. This is illustrated in Fig. 2. All points $\underline{P}$ on the lines labeled 1, 2, and 3 have as their guiding-center position x_1, x_2, and x_3, respectively.

We have drawn Fig. 2 so that a certain ambiguity becomes apparent; some values of $\underline{P}$ lie on more than one X_o line, such as point a. This corresponds to the potential $V(x)$ having more than one extremum. For the trajectory $\underline{A}^{SC}(x)$ considered here, a little thought shows that $V(x|\underline{P})$ has either two minima and one maximum, as for point a, or a single minimum and no maxima, as for points b and c. The former situation is illustrated in Fig. 3. Particles with energy $H = H_2$ may be trapped in either of the two wells, and so the canonical distribution $f_s(H, \underline{P})$ must in general be double-valued for such $(H, \underline{P})$.

Given $X_0(\underline{P})$, we are in a position to use Fig. 2 to read off $v_{\parallel}(X_o)$ for particles having a given $\underline{P}$. We introduce the x-dependent coordinate system with right handed triad $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$, where $\hat{b}_1 \equiv \hat{x}$, $\hat{b}_3 \equiv \hat{B}(x)$ points along the magnetic field at x, and $\hat{b}_2 \equiv \hat{b}_3 \times \hat{b}_1$ forms the third orthogonal unit vector. At a given point $x = X_o$, Hamilton's equations tell us that the parallel velocity is

$$v_{\parallel}(X_o, \underline{P}) = b_3(X_o) \cdot [\underline{P} - \underline{A}(X_o)] \equiv (P_3 - A_3)(X_o). \qquad \text{(I-19)}$$

Graphically, $v_{\parallel}$ is just the distance along the constant X_o line from the trajectory $\underline{A}^{sc}$ to the point $\underline{P}$ in question, as shown for point c in Fig.2. Since $X_o = X_o(\underline{P})$, we may write $v_{\parallel} = v_{\parallel}(\underline{P})$.

The final piece of physical information, $v_{\perp}^2(X_o)$, may now be obtained using conservation of energy:

$$H = \frac{1}{2}[v_{\perp}^2(X_o) + v_{\parallel}^2(X_o)] + e\Phi(X_o),$$

and thus

$$v_{\perp}^2(X_o) \equiv v_1^2(X_o) + v_2^2(X_o) = 2[H - e\Phi(X_o)] - v_{\parallel}^2(X_o). \qquad \text{(I-20)}$$

We see that $v_{\perp}^2$ is a function of $\underline{P}$ and H.

To clarify these ideas, we consider a simple illustration, viz., a magnetic field $\underline{B}(x)$ with constant modulus B_o and shear length $L_s \equiv \kappa_s^{-1}$; $\underline{B}(x) = B_o(\hat{y}\sin\kappa_s x + \hat{z}\cos\kappa_s x)$. This field is given by the circular pseudotrajectory with radius $A_o \equiv eB_oL_s$,

$$\underline{A}(x) = A_o(\hat{y}\sin\kappa_s x + \hat{z}\cos\kappa_s x) = eL_s\underline{B}(x),$$

as shown in Fig. 4. For any point $\underline{P} = P(\hat{y}\sin\theta_p + \hat{z}\cos\theta_p)$, $(P \geq 0)$, the effective potential $V(x|\underline{P}) = \frac{1}{2}[\underline{P} - \underline{A}(x)]^2$ is then given by

$$V(x|P) = \frac{1}{2}(P^2 + A_o^2) - P A_o \cos(\kappa_s x - \theta_p).$$

This has an infinite number of evenly spaced minima X_{on}, satisfying $\kappa_s X_{on} = \theta_p + 2\pi n$ $(n = 0, \pm 1, \pm 2, \ldots)$. The parallel velocity at any of these guiding center positions X_{on} is $v_{\parallel}(X_o) = P - A_o$. The X_o = constant lines in the $\underline{P}$ plane project radially outward from the origin $\underline{P} = 0$.

The point $\underline{P} = 0$ has $P = 0$, and so $V(x|\underline{P}) = \frac{1}{2} A_o^2$, independent of x. All other points $\underline{P}$ have a sinusoidally varying $V(x)$.

I.C.2. Obtaining f_s from $\tilde{f}_s$

Now that we know how to move between the canonical variables $(H, \underline{P})$ and the physical ones $[X_o, v_\perp^2(X_o), v_\parallel(X_o)]$, we are in a position to consider what $f_s(H, P)$ to write down to give the desired physical distribution $\tilde{f}_s$. We visualize f_s over the same $\underline{P}$ - plane as in Figs. 1 and 2, with the direction normal to this plane serving as the H - axis. Given $\underline{A}^{sc}(x)$, the physical distribution $\tilde{f}_s$ can be determined from f_s, using the interpretive methods just described. However, one is not given $\underline{A}^{sc}$ from the outset (this is determined self-consistently from Eq. (I-16), which requires a knowledge of f_s to determine U), but rather the reference trajectory $\tilde{\underline{A}}$. We thus proceed as follows. Treating $\underline{A}$ as though it were the real trajectory $\underline{A}^{sc}$, we determine the functions $\tilde{X}_o(\underline{P})$, $\tilde{v}_\parallel(P)$, and $\tilde{v}_\perp^2(H, \underline{P})$ in the same manner as X_o, $v_\parallel$ and $v_\perp$ are determined if $\underline{A}^{sc}$ is a known trajectory in the $\underline{P}$ plane. The functions $\tilde{X}_o$ and $\tilde{v}_\parallel$ provide a new coordinate system (in certain regions double-valued) of the $\underline{P}$ plane. The basic prescription for writing down f_s from $\tilde{f}_s$ is then to replace the arguments X_o, $v_\perp^2$, $v_\parallel$ in $\tilde{f}_s$ with these new functions $\tilde{X}_o$, $\tilde{v}_\perp^2$, $\tilde{v}_\parallel$ of the canonical variables H, $\underline{P}$:

$$f_s(H, \underline{P}) \equiv |\Omega I^{-1}| \tilde{f}_s[\tilde{X}_o(P), \tilde{v}_\perp^2(H, \underline{P}), \tilde{v}_\parallel(\underline{P})]. \qquad \text{(I-21)}$$

Two points in Eq. (I-21) require elaboration. The first is the presence of the factor $|\Omega I^{-1}|$, which adjusts between the normalizations of f_s and $\tilde{f}_s$. Here $\Omega \equiv \Theta \equiv \partial H/\partial J$ is the gyrofrequency of a particle

with momenta $\underline{P}$ and gyroaction J, and I is the Jacobian $I \equiv \partial(P_y, P_z)/\partial(\tilde{X}_o, \tilde{v}_\parallel)$ of our two parametrizations of the $\underline{P}$ plane. The distribution $\tilde{f}_s$ is normalized so that

$$n_s(X_o) = \int\int dv_\parallel \; d^2\underline{v}_\perp \; \tilde{f}_s = \int\int dv_\parallel dv_\perp^2 \; \tilde{f}_s(X_o, v_\perp^2, v_\parallel), \qquad \text{(I-22)}$$

where n_s is the density of guiding centers at X_o.

Distribution f_s is normalized over the canonical phase space $\underline{Z} \equiv (\Theta, J; Y, P_y; Z, P_z)$. (Here J and Θ are the canonical gyroaction and gyrophase, and Y and Z, the variables conjugate to P_y and P_z, are the guiding center y and z coordinates.)

Thus

$$d\underline{Z} = d\Theta \, dJ \, d\underline{P} \, dY \, dZ = (\Omega^{-1} I) \, d\Theta \, dH \, d\tilde{v}_\parallel \, d^3\underline{X},$$

where $d^3\underline{X} \equiv d\tilde{X}_o \, dY \, dZ$ is a volume element. Using (I-20), we have

$$dH \, d\tilde{v}_\parallel = d\tilde{v}_\perp^2 \, d\tilde{v}_\parallel \, \partial(H, \tilde{v}_\parallel)/\partial(\tilde{v}_\perp^2, \tilde{v}_\parallel) = \frac{1}{2} \, d\tilde{v}_\perp^2 \, d\tilde{v}_\parallel,$$

and so the integral over velocity space analogous to (I-22) gives

$$n_s(\tilde{X}_o) = \pi \int d\tilde{v}_\perp d\tilde{v}_\parallel |\Omega^{-1} I| f_s, \qquad \text{(I-23)}$$

and Eq. (I-21) follows.

The second point concerning Eq. (I-21) which requires clarification is that, for a given $\underline{P}$, H is always greater than or equal to the value $H_{min}(\underline{P}) \equiv V(X_o|\underline{P})$, and so it is only over this restricted, "physical region" of the $(H, \underline{P})$ - space that f_s has any physical meaning. Inside the physical region, we must have

$$f_s \geq 0, \qquad \text{(I-24)}$$

and we take f_s there to be given by Eq. (I-21), unless (I-21) violates condition (I-24). In this case we may take $f_s = 0$.

Outside the physical region the value of f_s is in principle irrelevant, and so one may assign f_s any convenient value there. However, the physical region is determined from the self-consistent trajectory $\underline{A}^{sc}(x)$ or $\underline{A}^{sc}(x)$, which is not known when writing down the distribution f_s. What is known is the reference trajectory $\underline{\tilde{A}}(x)$, which hopefully lies close to $\underline{A}^{sc}(x)$, but in general not precisely on it. Therefore we require condition (I-24) to hold not only in the region which would be physical if $\underline{A}^{sc} = \underline{\tilde{A}}$, but also in a sufficiently large neighborhood about this "quasi-physical" region that (I-24) will hold in the true physical region as well. A convenient choice which satisfies these criteria is to take f_s from (I-21) in the quasi-physical region, to have it go rapidly to zero at this region's boundaries, and remain zero elsewhere in the (H, $\underline{P}$) space.

I.C.3. Explicit Example: Loss Cone Distribution

We now illustrate these ideas more concretely. We denote by H' the particle kinetic energy at X_o, $H' \equiv H - e\Phi(X_o) = \frac{1}{2}[v_\perp^2(X_o) + v_\parallel^2(X_o)]$ $= h(X_o) + \frac{1}{2}[\underline{P} - \underline{A}(X_o)]^2$. Analogously, we denote by $\tilde{H}'$ the kinetic energy the particle would have if $\underline{A}^{sc}$ lay precisely on $\underline{\tilde{A}}$, $\tilde{H}' \equiv H - e\Phi(X_o)$ $\equiv \frac{1}{2}[\tilde{v}_\perp^2(X_o) + \tilde{v}_\parallel^2(X_o)]$. Since the reference potentials $\underline{\tilde{A}}$, $\tilde{\Phi}$ are known functions, $\tilde{H}'$ is a function of only the canonical variables (H, $\underline{P}$).

We begin by visualizing the locus of the physical and quasiphysical domains in the (H, $\underline{P}$) space. As in Sec. I.C.1, we neglect $e\Phi$ in comparison with H in this visualization, for simplicity. Including the

effect of $e\Phi$ would distort the sketches somewhat from those shown (for example, the line X_o = constant in Fig. 4a would no longer be exactly straight, and the domain boundaries of Fig. 4b would no longer be exactly parabolic), but the basic features would remain.

In Fig. 5a we draw the $\underline{P}$ plane of the $(H,\underline{P})$ space, showing the self-consistent and nearby reference trajectories $\underline{A}^{SC}$ and $\tilde{\underline{A}}$, a single X_o = constant line, and a number of sample points $\underline{A}_j$ on and off the trajectories.

The pseudopotential $U(\underline{A})$ defined by Eqs. (I-8) and (I-14) is valid for all $\underline{A}$ (or A). The lower limit $\tilde{H}'_{min} \simeq H_{min} \simeq \frac{1}{2}(\underline{P} - \underline{A})^2$. In Fig. 5b we draw the integration domains for each of the $\underline{A}_j$ of Fig. 5a, in the slice of the $(H, \underline{P})$ space having X_o = constant. The boundaries are labeled by the corresponding j. The area above the 0-boundary is the physical region, and above the 1-boundary is the quasiphysical region.

Now we specify a distribution function. In the quasiphysical region, we shall take f_s to be of the general form

$$f(H, \underline{P}) = \sum_{j=0}^{N} g_j(\underline{P}) \exp(-\gamma_j H). \qquad \text{(I-25)}$$

In his work[1], Channell assumed a special case of the form (I-25) throughout the $(H, \underline{P})$ space, namely $f = e^{-\gamma H} g(\underline{P})$. Here we will be modeling a loss-cone type distribution, such as exists in the Tormac sheath. We assume a substracted Maxwellian form. We shall thus take N = 1 in (I-25), with g_o positive, and g_1 negative. As a result, f will be positive in the quasiphysical region, going to zero at the quasiphysical boundary. Use of form (I-25) would make f negative outside the quasiphysical domain,

in violation of (I-24). We accordingly take $f = 0$ outside the quasiphysical region.

For the physical form, $\tilde{f}_{LC}$, of our distribution function, we write

$$f_{LC}(X_o, v_\perp^2, v_\parallel) = n\, e^{-\gamma_\parallel v_\parallel^2/2} \left[\bar{g}_o\, e^{-\gamma_\perp v_\perp^2/2} - \bar{g}_1\, e^{-R\gamma_\perp v_\perp^2/2}\right]. \qquad \text{(I-26)}$$

Here $n = n(X_o)$ is the guiding center density, and $R = R(X_o) \geq 1$ gives the loss cone width at X_o, $R \to \infty$ corresponding to a vanishingly narrow loss cone. In Tormac, we expect R to vary from ∞ in the interior, dropping as one moves across the sheath to some value $R_o \gtrsim 1$, given by the sheath mirror ratio. The normalization condition (I-22) requires

$$1 = (2\pi/\gamma_\parallel)^{1/2}\,(2\pi/\gamma_\perp)\,[\bar{g}_o - \bar{g}_1/R]\,. \qquad \text{(I-27)}$$

For simplicity we shall take $\gamma_\parallel = \gamma_\perp \equiv \gamma$, and $\bar{g}_o = \bar{g}_1$. In preparation for writing down $f_{LC}(H, \underline{P})$, we write (I-26) in the more nearly canonical form

$$\tilde{f}_{LC} = n\, \bar{g}_o\, (e^{-\gamma H'} - e^{-R\gamma H'}\, e^{(R-1)\gamma v_\parallel^2/2}). \qquad \text{(I-28)}$$

Finally, we apply Eq. (I-21) and the guidelines discussed subsequently, to define f_{LC}:

$$f_{LC}(H,\underline{P}) = \max\left\{ \begin{matrix} |\Omega I^{-1}|\, n\, \bar{g}_o (e^{-\tilde{\gamma} H'} - e^{-R\gamma \tilde{H}'}\, e^{(R-1)\gamma \tilde{v}_\parallel^2/2}) \\ 0 \end{matrix} \right. . \qquad \text{(I-29)}$$

Here n, R, and $\tilde{v}_\parallel^2$ are all evaluated at $\tilde{X}_o(\underline{P})$. f_{LC} is of the form of (I-25) in the quasiphysical region, with $\gamma_o = \gamma$, $\gamma_1 = R\gamma$, and g_o and g_1 given by

$$g_o(\underline{P}) = |\Omega I^{-1}|\ n\ \bar{g}_o\ e^{\gamma e\Phi(\tilde{X}_o)}$$

$$g_1(\underline{P}) = -|\Omega I^{-1}|\ n\ \bar{g}_o\ e^{R\gamma e\Phi(\tilde{X}_o)}\ e^{(R-1)\gamma\tilde{v}_{\parallel}^2/2})$$

in the region where $f_{LC} > 0$, and $g_o = g_1 = 0$ elsewhere.

Using the relation $\tilde{H}' = \frac{1}{2}(\tilde{v}_{\parallel}^2 + \tilde{v}_{\perp}^2)$, from (I-26) or (I-29) one sees that f_{LC} passes through zero on the curve $\tilde{H}' = \tilde{v}_{\parallel}^2/2$. The loss-cone boundary lies, roughly at $\tilde{v}_{\perp}^2 = (R-1)^{-1}\ \tilde{v}_{\parallel}^2$ or $\tilde{H}' = (1-R^{-1})^{-1}\ \tilde{v}_{\parallel}^2$. In Fig. 6 we sketch the contours of f_{LC} = constant, in an $\tilde{X}_o$=constant slice of the $(\tilde{H}', \underline{P})$ space.

I.C.4. Obtaining U from f_s

From Eqs. (I-8) and (I-14), we write $U_s(\underline{A})$, the contribution to $U(\underline{A})$ from species s.

$$U_s(\underline{A}) = \int d\underline{P}\ F_s(\underline{P},\underline{A}), \qquad \text{where}$$

$$F_s(\underline{P},\underline{A}) \equiv 8\pi \int_0^\infty dh\,(2h)^{1/2}\ f_s[H = h + \tfrac{1}{2}(\underline{P}-\underline{A})^2 + e\Phi,\ \underline{P}]. \qquad \text{(I-30)}$$

Here we are again setting $m_s = c = 1$, and suppressing the Φ dependence in $F_s(\underline{P}, \underline{A})$, it being understood that we will let $\Phi = \Phi(\underline{A})$ afterwards.) In the form written, U_s appears as a simple integral transform of the h-integrated distribution function. Using (I-25) for f_s, we have

$$F_s(\underline{P},\underline{A}) = C_1 \sum_j \gamma_j^{-3/2}\ e^{-\gamma_j e\Phi}\ e^{-\gamma_j(\underline{P}-\underline{A})^2/2}\ g_j\ (\underline{P}), \qquad \text{(I-31)}$$

where $\qquad C_1 \equiv 8\pi\sqrt{2}\int_0^\infty dy\,\sqrt{y}\,e^{-y} = 2(2\pi)^{3/2}.$

The factor $\exp-\gamma_j\ (\underline{P} - A)^2/2$ in (I-31) is of the same form as the propagator for a diffusion equation, and accordingly doing the $\underline{P}$-integration on F_s to get U_s will smear out the form of g_j over a region of radius $v_j \equiv \gamma_j^{-1/2}$. The topologies of U_s and the g_j will thus be related in this simple way; U_s will tend to have peaks where the g_j have peaks and troughs where the g_j have troughs. (Of course, if the differing g_j have their peaks and troughs in different places, one must look more closely at Eqs. (I-30) and (I-31) to develop valid qualitative expectations.) In particular, since the distribution f_{LC} we have chosen in (I-29) is large only around $\underline{A} = \underline{A}^{sc}$, with a height varying as $n(\tilde{X}_o)$, we expect U to have this same topology, i.e., a ridge centered about $\underline{A}^{sc}$, and decreasing with $\tilde{X}_o$ as n does.

We also observe that, since F_s has integrated over the h-dependence of f_s, U_s should be insensitive to the particular h-structure of f_s. Thus, whether f_s is of the loss cone type (I-29) or just a Maxwellian with a similar $n(\tilde{X}_o)$ - variation, the qualitative features of U_s should be the same. Physically, this is reasonable, since it is only very gross properties of the distribution function, e.g., its pressure and parallel currents, which determine the form $\underline{B}(x)$ of the magnetic field.

All the features discussed so far for the forms of f_s, F_s, and U_s are in principle applicable to the electrons as well as the ions. Retaining the original definition for $\underline{A}$, $\underline{A} \equiv (e_i/c)\ \underline{A} = (e/c)\ \underline{A}$, we see that the electron quasiphysical region lies around the trajectory $\underline{A} = -\tilde{\underline{A}}$, and so the part of f_e of principal physical relevance should be centered around this trajectory. The width of this region about $\tilde{\underline{A}}$ is typically much narrower for electrons than for ions, however, the ratio of thermal spreads in this momentum space being $(p_e/p_i) = (m_e T_e/m_i T_i)^{1/2} << 1$.

In moving from f_e to U_e, the fact that $e_e = -e_i$ introduces another reflection through the $\underline{A}$ origin, causing U_e to be centered along $\tilde{\underline{A}}$, just as is U_i. The mass ratio causes U_e to be narrower than U_i by the same factor (p_e/p_i), provided we choose f_e according to the same principles as used for f_i (namely, modeling the physical situation in the quasiphysical region, and falling rapidly to zero outside this region).

Having f_e fall to zero outside the quasiphysical region is not necessary, however, nor is it always desirable. For comparable pressures, and so comparable heights of the U_s ridges, since the electron ridge designed in this way is much narrower than the ion ridge, $\partial U_e/\partial \underline{A} >> \partial U_i/\partial \underline{A}$, and so the electrons will dominate in causing any lateral accelerations in the trajectory $\underline{A}^{sc}(x)$, i.e., electron parallel current will be principally responsible for any magnetic shear. This may not be called for, by a given physical model. One may instead prescribe an f_e which falls off much more slowly than f_i as the two move away from their quasiphysical domains. This will cause the ion pseudopotential U_i to be narrower than U_e, and so allow the ion parallel current to induce the shear. However, the electron <u>physical</u> region will still be much narrower than the ion physical region; this is independent of the choice of f_s or the resultant U_s. In Sec. I.C.6 we discuss a numerical application in which this latter method for choosing f_e has been employed.

I.C.5. Self-Consistency of $\underline{A}^{sc}$ and $\underline{\tilde{A}}$

In the following section and Appendix A, we will present more quantitative substantiation of the qualitative features of the pseudopotential U. However we are already in a position to answer a final important question concerning the Channell formalism's practical application. This is a question of self-consistency. We have chosen our distributions f_s so that, if the self-consistent trajectory $\underline{A}^{sc}$ lies close to the reference trajectory $\underline{\tilde{A}}$, then the distribution function will have the physical features prescribed by f_s. The question is, then, given the f_s we have chosen, is the form of U capable of yielding an $\underline{A}^{sc}$ lying close to $\underline{\tilde{A}}$? The answer to this question is "yes", as we now explain.

From the preceding subsection, we see that U_s for both species, and hence U, have the form of a ridge centered on $\underline{\tilde{A}}$ $(\tilde{X}_o)$, and with height varying as $n(\tilde{X}_o)$. On a pseudopotential of this form, we want to be able to shoot in our pseudoparticle, so that its trajectory $\underline{A}^{sc}(x)$ follows the ridge crest. Clearly, for the incoming pseudo-kinetic energy $\frac{1}{2}|d\underline{A}/dx|^2 = \frac{1}{2}\underline{B}^2$ not too large compared to the height of the pseudopotential ridge $U = 4\pi\,\Pi_{xx}$, one can do this, injecting the particle far enough on the lower downhill side of the ridge that it is accelerated around the elbow in $\underline{\tilde{A}}$. The ratio (pseudopotential/pseudo kinetic energy) $= 8\pi\,\Pi_{xx}/\underline{B}^2$ is just the plasma β, so the constraint just described says that the amount of shear a plasma can support becomes smaller with the plasma β.

Since the optimal trajectory $\underline{A}^{sc}$ must lie somewhat downhill from the ridge crest (which lies at $\underline{\tilde{A}}$), we see that, for a nonzero shear,

$\underline{\tilde{A}}$ cannot lie exactly on $\underline{\tilde{A}}$, for the f_s we have chosen. This disparity points to an inappropriate feature of the model (I-26) we chose for f_s; as written, f_{LC} there has no parallel current, which is necessary for the shear stipulated by $\underline{\tilde{A}}$. From Fig. 6, we see that it is exactly the effect of $\underline{A}^{sc}$ lying somewhat downhill from $\underline{\tilde{A}}$ to make $v_{\parallel} \neq \tilde{v}_{\parallel}$, and thereby to give the true distribution the parallel current necessary to produce the shear in $\underline{A}^{sc}$. If desired, one could now adjust the choice of $\tilde{f}_s$, for example introducing a parallel drift velocity $w_{\parallel}(\tilde{X}_o)$, and adjusting the $\tilde{X}_o$ dependence in order to make $\underline{A}^{sc}$ more nearly equal $\underline{\tilde{A}}$. We shall use this refined form for f_s in choosing our distribution in Part III.

Support for the validity of these qualitative statements is given in the next section, where we present results of numerical application of the interpretive methods thus far presented, and in Appendix A, where we make a more explicit estimate of the size and shape of the potential ridge U.

I.C.6. Numerical application

In the foregoing sections we have seen how to apply and extract information from the Channell formalism, using graphical methods, from which qualitative conclusions may be relatively easily reached, even in situations where the corresponding mathematics becomes analytically intractable. W. M. Sharp[5] has employed these methods on a computer, numerically generating self consistent Vlasov equilibria for an $\underline{\tilde{A}}$ and $\tilde{f}_s$ modeling the Tormac sheath region. Typical results are shown in Fig. 7.

The model taken is one in which the ions have a loss cone form, and supply the parallel current which induces the shear. As discussed at

the end of Sec. I.C.4, this is achievable by taking f_e to vary slowly with $\underline{P}$ as one moves away from its quasiphysical region. Sharp took an f_e with no $\underline{P}$ variation at all, $f_e = (\text{constant}) \times e^{-\gamma_e H}$. This yields $U_e = (\text{constant}) \times e^{\gamma_e e\Phi}$, independent of $\underline{A}$. The electron distribution is thus a local Maxwellian, with zero parallel current, and density variation given by the Boltzmann factor $e^{\gamma_e e\Phi}$.

In Fig. 7a are shown the reference trajectory $\tilde{\underline{A}}$ (dotted line) and self-consistent trajectory $\underline{A}^{SC}$. Sharp allowed for a parallel velocity $w_{\parallel}(X_o)$ in the physical form of the ion distribution, as discussed at the end of Sec. I.C.5, setting up an iterative procedure which adjusted $w_{\parallel}$ to get progressively better fits of $\underline{A}^{SC}$ to $\tilde{\underline{A}}$.

Figure 7b graphs the magnetic field versus x, and Fig. 7c shows $\Phi(x)$. The nonphysical behavior at large x is a deficiency of taking f_e to be independent of $\underline{P}$. Quasi-neutrality requires that $n_e = n_i$. As x becomes large, f_i is modeled to go to zero, as occurs in Tormac. Since $n_e \sim \exp \gamma_e e\Phi$, $-\Phi$ must become large as x does to maintain quasineutrality.

In Figs. 7d-7f are plotted contours of the ion distribution, in the Tormac model interior ($x/\rho_i = -4.0$), sheath center ($x/\rho_i = 0$), and sheath exterior ($x/\rho_i = 4.0$). The left-hand column shows f_i in terms of the reference variables $\tilde{v}_{\parallel}$ $\tilde{v}_{\perp}$, while the right-hand column plots f_i in terms of the true physical variables.

The equilibrium described by these figures is fully self-consistent, yet from Fig. 7b we see that η in the sheath region is quite substantial ($\eta \simeq 1/3$), so that individual particle trajectories are appreciably distorted from their form in the limit of a uniform magnetic field.

The principal objective of this work was to establish a prescription for writing down a canonical form of the distribution function which yields self-consistent equilibria in this appreciable - η regime, consistent with given physical specifications on the forms of the magnetic field and of the physical distribution function. From the proximity of the reference and self-consistent trajectories in Fig. 7a and the close resemblance of the reference distributions to the physical ones in Figs. 7d-7f, we see that this goal has been achieved.

APPENDIX A: SEMIQUANTITATIVE ESTIMATE OF U

In this Appendix we shall proceed further with the explicit evaluation of U_s from f_s, to check that the qualitative expectations of the previous Sec. I.C.4 were accurate.

We use f_{LC} from Eq. (I-29) in (I-30), first obtaining $F_s = F_{LC}$. For simplicity, we neglect the Φ dependences. Then $\tilde{H}' = H' = H = h + \frac{1}{2}(\underline{P} - \underline{A})^2$. For a typical $\underline{A}$, e.g., $\underline{A}_2$ of Fig. 4a, the region of integration in h lies above the boundary 2 of Fig. 4b. In part of this region, namely below the quasiphysical boundary (labeled 1 in Fig. 4b), $f_{LC} = 0$, and so the resultant region of integration is the intersection of these two regions.

Neglecting Φ turns the guiding center condition (I-17) into

$$0 = \hat{b}_2 \cdot [\underline{P} - \underline{A}(X_o)] \equiv (P_2 - A_2)\ (X_o), \tag{A-1}$$

and so Eq. (I-19) says

$$v_{\parallel}^{\ 2}\ (X_o, \underline{P}) = V(X_o|\underline{P}) = \frac{1}{2}[\underline{P} - \underline{A}(X_o)]^2. \tag{A-2}$$

This says that we are neglecting the ExB drift velocity, $v_E \equiv v_2(X_o) << v_x(X_o)$. If we now generalize the meaning of $v_{\parallel}^{\ 2}$ from (A-2) to arbitrary $\underline{A}$,

$$v_{\parallel}^{\ 2}\ (\underline{P}, \underline{A}) \equiv \frac{1}{2}(\underline{P} - \underline{A})^2, \tag{A-3}$$

we can write (I-29) in the form

$$f_{LC} = \max \begin{cases} |\Omega I^{-1}| n\bar{g}_o\ e^{-\gamma v_{\parallel}^{\ 2}/2}\ (e^{-\gamma h} - e^{-R\gamma h}\ e^{(R-1)\gamma h_1}) \\ 0 \end{cases} \tag{A-4}$$

where $\quad h_1 \equiv (\tilde{v}_{\parallel}^2 - v_{\parallel}^{\ 2})/2.$

The region of integration of h runs from $h = h_2$ to ∞, where

$$h_2 \equiv \max\ (0,\ h_1).$$

The only term in (A-4) which has an h-dependence besides those which explicitly appear is Ω, whose dependence on h is quite weak, as we shall see in Parts II and III. (The $v_\parallel$- dependence of Ω is more important.) Neglecting this weak h-dependence, we perform the h-integration, obtaining

$$\begin{aligned} F_{LC}\ (\underline{P},\underline{A}) = {} & |\Omega I^{-1}| n\ \bar{g}_o\ e^{-\gamma v_\parallel^2/2}\ (8\pi\sqrt{2}\ \gamma^{-3/2}) \\ & \times [\Gamma(\tfrac{3}{2},\ \gamma h_2) - R^{-3/2}\ e^{(R-1)\gamma h_1}\Gamma(\tfrac{3}{2}, R\gamma h_2)], \end{aligned} \tag{A-5}$$

where $\Gamma(n, y) \equiv \int_y^\infty dx\ x^{n-1}\ e^{-x}$ is the incomplete Γ - function of index n.[4] Since we have designed f_{LC} in (A-4) to be nonnegative, we know that the quantity in square brackets in (A-5) is also nonnegative, equaling zero only when $R = 1$.

In Fig. 8a we sketch the behavior of $\Gamma(\frac{3}{2}, y)$. Using this form, we show in Fig. 8b the appearance of the term $-R^{-3/2} \exp[(R-1)\gamma h_1]$ $\Gamma(\frac{3}{2}, R\gamma h_2)$ in Eq. (A-5). It is a lopsided downward spike, peaking at $\tilde{v}_\parallel = v_\parallel$, having magnitude $R^{-3/2}\ \pi^{1/2}/2$. Because of the factor $R^{-3/2}$, this term will be negligible for large R. Even for smaller R, since the basic dependence of F_{LC} comes from the overall factor $\exp(-\gamma v_\parallel^{\ 2}/2)$, we drop this factor in our present effort to get a rough estimate of $U(\underline{A})$. We are thus evaluating U for a Maxwellian distribution, i.e., we are

letting $R \to \infty$. Equation (I-27) then gives

$$\bar{g}_o = (\gamma/2\pi)^{3/2}. \qquad \text{(A-6)}$$

Now we perform the $\underline{P}$ integration to get U from F. From Eqs. (I-30), (A-5) and (A-6), we have

$$U(\underline{A}) = 4 \int d\underline{P}\; n|\Omega I^{-1}|\, \pi^{-1/2}\, e^{-\gamma v_{\parallel}^2/2}\; \Gamma(3/2, R\gamma h_2). \qquad \text{(A-7)}$$

As discussed in Sec. I.C.4, the factor $\exp(-\gamma v_{\parallel}^2/2) \equiv \exp[-\gamma(\underline{P}-\underline{A})^2/2]$ is a diffusive propagator-like term, smearing out the form of the rest of the integrand over a region of radius $v_{th} \equiv \gamma^{-1/2}$. The Γ-factor gives the ridge with crest at $\underline{\tilde{A}}$, and the factor n modulates the height of this ridge. To evaluate (A-7), we assume that the falloff radius v_{th} is sufficiently small in comparison with the scales over which n, Ω and I vary, that we may take them outside the integral. In particular, this means that the shear, which gives rise to the elbow in $\underline{\tilde{A}}$, is sufficiently weak that, at least for $\underline{A}$ in the neighborhood of $\underline{\tilde{A}}$ (where Γ, and so U, will be nonnegligible), we may set up a locally rectangular coordinate system (P_2, $\tilde{v}_{\parallel} = P_3$), centered at that $\underline{\tilde{A}}$ ($\tilde{X}_o$) such that $\underline{A}$ lies on the line $\tilde{X}_o$ = constant, and with coordinate axes in the increasing $\tilde{X}_o$ (unit vector $\hat{b}_2$) and $\tilde{v}_{\parallel}$ (unit vector $\hat{b}_3$) directions. This is illustrated in Fig. 9. In this coordinate system, vectors $\underline{P}$ and $\underline{A}$ are described by (P_2, $P_3 = \tilde{v}_{\parallel}$) and (o, A_3), respectively, and so $v_{\parallel}^2 \equiv (\underline{P}-\underline{A})^2 = P_2^2 + (P_3 - A_3)^2$, and $h_1 \equiv \tilde{v}_{\parallel}^2 - v_{\parallel}^2 = 2P_3A_3 - A_3^2 - P_2^2$. The ridge from Γ has a flat top out to the point P_{31}, given by $o = h_1(P_{31})$. This yields $P_{31} = (P_2^2 + A_3^2)/2A_3$.

As P_s increases beyond this, the Γ ridge falls off, over a falloff distance given by $h_2 = h_1\ (P_{32}) = R^{-1}V_{th}^2)/2\ A_3$.

In this coordinate system, and within the approximations made, Eq. (A-7) becomes

$$U(\underline{A}) = 4\pi^{-1/2}\ n|\Omega I^{-1}|\int dP_2\ e^{-\gamma P_2^2/2}\int dP_3\, e^{-\gamma(P_3-A_3)^2}\ \Gamma(\tfrac{3}{2},R\gamma h_2)\,. \quad \text{(A-8)}$$

The factor h_2 has both P_2 and P_3 dependence. The factor $\exp(-\frac{1}{2}\gamma P_2^2/2)$ makes $P_2 \lesssim v_{th}$ in regions of significant contribution to U. Considered as a function of A_3, P_{21} goes to infinity both as A_3 goes to zero and infinity, with a minimum value of P_2 at $A_3 = P_2 \lesssim v_{th}$. Thus for $A_3/v_{th} \to 0$ (i.e., $\underline{A}$ near $\tilde{\underline{A}}$), $P_{31} \to \infty$, and $h_2 = 0$, making $\Gamma = \pi^{1/2}/2$ over the entire integration region, and so

$$U(A_3/v_{th} \ll 1) \simeq 4\,\pi\, n\,|\Omega I^{-1}|\,v_{th}^2. \quad \text{(A-9)}$$

For $A_3/v_{th} \gg 1$, $2P_{31} \simeq A_3$, and so Γ has fallen off to zero in the region where $\exp[-\gamma(P_3 - A_3)^2/2]$ is substantial. Thus

$$U(A_3/v_{th} \gg 1) \simeq 0\,. \quad \text{(A-10)}$$

Finally, in the intermediate region $A_3/v_{th} \sim 1$, we eliminate the P_2 dependence of h_2 in (A-8) by letting P_2 take a typical value $P_{21} \lesssim v_{th}$. Then $P_{31} \simeq (P_{21}^2 + A_3^2)/2A_3 \sim v_{th}$, and so Γ will introduce a cutoff in one of the integration limits in the P_3 integral of the Gaussian:

$$\int dP_3\ e^{-\gamma(P_3-A_3)^2/2}\ \Gamma \simeq \Gamma(\tfrac{3}{2},0)\int_{-\infty}^{(P_{31}-A_3)} du\, e^{-\gamma u^2/2} \equiv \left(\frac{\pi v_{th}}{\sqrt{2}}\right) E[(P_{31}-A_3)/v_{th}]\,. \quad \text{(A-11)}$$

(E is normalized to satisfy $E(z \to \infty) = 1$. One may readily express it in terms of the error function, erf (z). Inserting (A-11) into Eq. (A-8). we obtain

$$U(A_3/v_{th} \sim 1) \simeq 4\pi\ n|\Omega I^{-1}|v_{th}^2\ E\left\{[P_{31}(P_{21}) - A_3]/v_{th}\right\}. \qquad \text{(A-12)}$$

From Eqs. (A-9), (A-10) and (A-12) we see the qualitative expectations for $U(\underline{A})$ developed in Sec. I.C.4 are born out; U has the form of a ridge, with height proportional to $n(\tilde{X}_o)$ and width on the order of v_{th}, and with ridge crest at $\underline{A} = \underline{\tilde{A}}$.

Finally, we can put Eqs. (A-9) and (A-12) into a more recognizable form, by evaluating the factor $|\Omega I^{-1}|$. This factor is to be evaluated at $\underline{P}$ on or near the reference trajectory $\underline{\tilde{A}}$. For such points $\underline{P}$, we may again use the locally rectangular coordinate system (P_2, P_3). Thus $I \equiv \partial(P_y, P_2)/\partial(\tilde{X}_o, \tilde{v}_{\|}) = \partial(P_2, P_3)/\partial(\tilde{X}_o, \tilde{v}_{\|}) = (\partial P_2/\partial\tilde{X}_o) = (\partial\tilde{A}_2/\partial\tilde{X}_o)$, where the last equality follows from the guiding center condition, Eq. (A-1) (with $\underline{A}$ replaced by $\underline{\tilde{A}}$, and X_o by $\tilde{X}_o$). The potential V is given approximately by $V \simeq \frac{1}{2}(\delta x)^2\ (\partial\tilde{A}_2/\partial\tilde{X}_o)^2$, where $\delta x \equiv x - \tilde{X}_o$, and so the associated gyrofrequency Ω satisfies $\Omega^2 = (\partial\tilde{A}_2/\partial\tilde{X}_o)^2$. We thus see that (restoring factors of m) $|\Omega I^{-1}| = m$, and so the coefficient in Eqs. (A-9), (A-12) is given by

$$4\pi\ n v_{th}^2\ |\Omega I^{-1}| = 4\pi\ nT, \qquad \text{(A-13)}$$

4π times the usual scalar pressure.

PART II. PARTICLE MOTION IN REALISTIC GEOMETRIES AND THE FINITE GYRORADIUS GUIDING CENTER HAMILTONIAN

II.A. Introduction to Part II

In the present portion of this thesis, we will study particle motion in realistic geometries, with particular interest in cases in which the perpendicular inhomogeneity parameter η, defined (as in the Introduction to Part I) as the ratio of gyroradius ρ to the magnetic scale length $L_\perp$ perpendicular to the magnetic field, is a nonnegligible parameter. Such strong inhomogeneity occurs, for example, in mirror machines and in the Tormac sheath region, where η may take on values as large as 1/3. In contrast to the slab geometry assumed in Part I, the geometries considered here are assumed to have a small but finite inhomogeneity along the field lines. We characterize this longitudinal inhomogeneity by a second parameter ε, the ratio of ρ to the parallel magnetic scale length $L_\parallel$ (or equivalently, the ratio of the longitudinal bounce frequency ω_b to the gyrofrequency Ω). We shall consider systems which have a single symmetry direction, as does any axisymmetric device. The resultant particle motion problem then has a single exactly conserved canonical momentum (versus two for the slab geometry), and two nontrivial degrees of freedom.

The assumption that $\varepsilon \ll 1$ ("near-slab" geometry) implies a separation between the time scales on which the gyromotion and longitudinal motion occur. This time scale separation enables one to find a good adiabatic invariant, the gyroaction J. In the limit of $\eta \to 0$ (along

with certain additional restrictions to be discussed), J reduces to mc/e times the usual magnetic moment $\frac{1}{2}mv_\perp^2/B$, but it is important to note that, as long as $\varepsilon \ll 1$, a good invariant J can be found, even for η comparable to one.

Use of J in the equations of motion reduces the number of non-trivial degrees of freedom in the problem by one, giving a guiding center theory. Previous work[6,7,8] on finite gyroradius ($\eta \neq 0$) guiding center motion has made the assumption that the scale lengths $L_\parallel$ and $L_\perp$ were of the same order, and an expansion in the single parameter $\varepsilon \sim \eta$ was performed. The assumption $\varepsilon \sim \eta$ is insufficiently flexible for the configurations to which the theory to be developed here is applicable.

Some inroads have been made in studying finite gyroradius guiding center motion. Gardner[6] outlined a prescription by which one could in principle obtain a finite gyroradius guiding center Hamiltonian K, in which one performs a series of canonical transformations on the original Hamiltonian H to remove the dependence of H on the gyrophase Θ to successively higher orders. Stern[7] then carried out this scheme explicitly to first order. Even to this order, the algebra involved becomes cumbersome, and finding the appropriate canonical transformations is an unsystematic process. Yet the first order result is simple; K has the same form as its zero order limit, but with the variables having a slightly different meaning.

More recently, Northrop and Rome[8] studied guiding center motion to second order, using a non-canonical framework. In so doing they dispense with the need for transformations which are canonical, but sacrifice

the advantages of a Hamiltonian theory, where all of the equations of motion come from the single function K, and symmetries and conservation laws are most readily apparent. As a result, these authors find a breakdown at second order in conservation of canonical angular momentum p_ϕ, which a canonical formulation automatically avoids.

The findings of Ref. 8 parallel those of Ref. 7; to first order, the equations of motion have the same form as the zero order equations, but with a generalized form of gyroaction. The second order corrections to the perpendicular drifts are also obtained in Ref. 8, but require, in the words of those authors, an amount of algebra which is "truly extensive and presents an almost infinite opportunity for mistakes."

Here we describe a treatment of the guiding center motion problem which attempts to minimize the difficulties and shortcomings encountered in the work just described. We adopt a canonical framework, to take advantage of the elegance and power of canonical methods, but employ a rather different method from that of Refs. 6 and 7 for canonically transforming H. This method (described in Sec. II.B) is systematic, and enables one to obtain explicit expressions for the guiding center Hamiltonian K to arbitrarily high order in both ε and η. The transformation process relies heavily on Lie perturbative methods,[9] which greatly simplify the task of making canonical transformations.

The organization of Part II is as follows. In Sec. II.B, we describe in general terms the procedure we shall use to transform from H to K. Sections II.C through II.I supply the specifics of the procedure outlined in Sec. II.B, obtaining the guiding center Hamiltonian K,

valid to $O(\varepsilon^2, \varepsilon\eta, \eta^2)$, in Eq. (II-89) of Sec. II.I. This is the most important result of Part II.

In Sec. II.J we use the form of K and transformation equations obtained in previous sections to derive another important result, a generalized expression for the gyroaction J, valid for finite ε and η, and in the presence of magnetic inhomogeneity, including shear. Section II.K applies the formalism developed to that point. Specializing to the negligible ε regime, we obtain more explicit expressions for J, as well as developing results describing particle motion in an inhomogeneous magnetic field, which will be of use in Part III. Finally, in Sec. II.L we discuss possible directions for extension and application of the formalism.

II.B. Overview of the Procedure for Obtaining K

We now describe the general nature of the transformation process which takes us from H to K. The more specific mechanics of each stage of the process will be presented in the following sections.

The theory is first phrased in a form which is perturbative in ε, but non-perturbative in η, reflecting the fact that it is only $\varepsilon \ll 1$ which is necessary for a valid guiding center theory. To obtain explicit expressions for K, one must then also do an expansion in η. Two variants of Lie methods are employed to facilitate the two expansions.

The procedure is carried out to second order in both ε and η. The resultant guiding center Hamiltonian K will then be expressed as a double expansion,

$$K = \sum_{\ell,m=0}^{\infty} \varepsilon^{\ell} \eta^{m} K_{\ell m}, \tag{II-1}$$

where we explicitly calculate K_{00}, K_{01}, K_{10}, K_{11}, K_{02}, and K_{20}. K_{00} turns out to have the same form as the zero-gyroradius guiding center Hamiltonian usually calculated. In addition, we find that $K_{01} = K_{10} = K_{11} = 0$, so that K equals K_{00} plus terms second order in ε and η. This is in accordance with the findings of Refs. 7 and 8, whose first order guiding center equations were of the same form as the zero order ones, but with modified definitions of the variables.

The transformation process from H to K proceeds in two stages, designated S and T. The original coordinates and momenta $\underline{z}$, in terms of which H is expressed, include α, which measures the poloidal radius, and its conjugate momentum p_α. These variables correspond to x and p_x of the slab model Hamiltonian ($\varepsilon = 0$), Eq. (I-4) of Part I; they are the variables of the gyromotion. Stage S of the transformation process transforms from (α, p_α) to action angle variables (Θ^o, J^o). The resultant Hamiltonian H^o has an $O(\varepsilon)$ dependence on Θ^o, so that J^o is not quite the adiabatic invariant J in terms of which K is expressed. The remaining portion T of the transformation process is perturbative in character. The angle dependence of the Hamiltonian is removed to successively higher orders in ε, by the sequence of transformations $\{T_1, T_2 \cdots\}$. These transformations are induced using Lie perturbative methods, which are computationally much more efficient than the mixed-variable generating function method traditionally employed.[10]

The transformation S from (α, p_α) to (Θ^o, J^o) is a finite transformation, i.e., nonperturbative in character, and so is induced using a generating function $S(q, \underline{P})$ of the mixed variable type. Our "near-slab" Hamiltonian H is formally the same as the slab Hamiltonian of Part I [compare Eq. (II-5) to Eq. (I-4)]. The slab problem is one dimensional, and so can be solved exactly, for arbitrary η, in terms of quadratures involving the potential V, which is in general anharmonic. Explicit evaluation of these quadratures is difficult, and will involve an expansion in η.

Analogously, exploiting the formal similarity of the near-slab to the real-slab H, we choose the generating function S for the near slab problem in the same manner which has been used to solve the slab problem,[11] i.e., $S = \int^{\alpha} d\alpha' \, p_\alpha(\alpha')$. As for the slab problem, this S induces a transformation to action angle variables which is valid for arbitrary η. And as for the slab problem, obtaining explicit expressions from the formal ones involves an expansion in η, difficult to achieve using standard methods.

The standard approach to solving the one dimensional problem by action angle variables possesses two principal difficulties. One of these, alluded to just above, is the evaluation of integrals of the form $\int dx\,[1-ax^2-bx^3-cx^4 \cdots]^{1/2}$, needed to calculate S, and the action J as a function of the energy E = H. The second major difficulty is solving the mixed-variable coordinate transformations and performing the functional inversion of J(H) to obtain H in terms of the new variables.

Here again Lie methods are of great use. The <u>one</u> dimensional problem, soluble as just described using generating function S, may be more easily solved using Lie techniques. One first makes a trivial finite canonical transformation R_o to action angle variables (θ^o, j^o), in terms of which the Hamiltonian h^o would be θ^o independent only for a harmonic potential V. The remainder of the problem is perturbative in character, and so may be readily solved using Lie methods, inducing a sequence $R = \{R_1, R_2, \cdots\}$ of perturbative transformations, which remove the angle dependence of the Hamiltonian to successively higher orders in η, fully analogous to the sequence of transformations $T = \{T_1, T_2, \cdots\}$ used in stage two of the transformation process. The result is an angle independent Hamiltonian $h(j)$, expressed as an expansion in η, and the coordinate transformation, with the original and final variables already in unmixed form. The difficulties of the standard solution procedure just described are thereby completely circumvented.

In trying to generalize this approach in a straightforward way to the <u>near</u>-slab problem, one finds that the initial transformation R_o to action-angle variable used for the slab problem is no longer canonical. The way around this difficulty is the following. We know that the mixed variable generating function $S = \int^{\alpha} d\alpha' p_\alpha$ properly induces a canonical transformation from (α, p_α) to (Θ^o, J^o); the problem was explicitly calculating S. We also note that S has precisely the form as that for a one-dimensional problem, where the <u>old</u> coordinate β and <u>new</u> momentum $P_b{}^o$ associated with the second degree of freedom are treated simply as

constant parameters. We can solve this one-dimensional problem using Lie methods, as previously described, circumventing the usual difficulties encountered in solving by calculating S, using the coordinate transformation R': $(\alpha, p_\alpha) \to (\theta, j)$ (so $R' = R \cdot R_0$) in $S = \int^{\theta(\alpha)} d\theta' (d\alpha/d\theta') \times p_\alpha [\alpha(\theta')]$, is a simple matter. For a truly one-dimensional problem, calculation of S after transformation R' is obtained is superfluous. In the near-slab problem, however, we need S to be able to go on to stage two; it insures that the new variables $\Theta^0 = \theta$, $J^0 = j$, and the others of the set $\underline{Z}^0$, are canonically conjugate.

To summarize the procedure then: from the original Hamiltonian H, developed in Sec. II.C, we set up a one degree of freedom problem, involving a simply related Hamiltonian h defined on the restricted, one-degree-of-freedom phase space. This 1-D problem is treated in Sec. II.F. The Lie methods needed for the treatment are developed in Sec. II.E.

From the transformation R' obtained in solving the 1-D problem, in Sec. II.G we explicitly calculate the generating function S, which was given formal definition in Sec. II.D. S enables us to transform from $H(\underline{z})$ to $H^0(\underline{Z}^0)$; this is achieved in Sec. II.H. $H^0(\underline{Z}^0)$ emerges as an expansion in both η (from the R procedure) and ε (from the fact that H^0 is not completely independent of Θ^0), analogous to Eq. (II-1). Finally, given H^0, in Sec. II.I we apply Lie-induced transformations $\{T_n\}$ $(n = 1, 2, \cdots)$, obtaining the guiding center Hamiltonian K.

II.C. Coordinate System, Initial Hamiltonian

We consider an axisymmetric magnetic geometry, independent of toroidal angle ϕ. We choose orthogonal particle coordinates $q^\mu = (\alpha, \beta, \phi)$, with conjugate momenta $p_\mu = (p_\alpha, p_\beta, p_\phi)$. Coordinates α and β describe the particle variable, constant along poloidal field lines (specifically, we shall take α to be the poloidal flux enclosed by a poloidal field line), and β measuring the position along the poloidal field lines. The coordinate system is illustrated in Fig. 10.

The α and β used here are <u>not</u> the Euler potentials used in Refs. 6 and 7. Use of the Euler potentials, allows one to pick an A_β which depends only on α. However this choice in general makes the q^μ nonorthogonal ($\nabla\alpha \cdot \nabla\beta \neq 0$). We sacrifice the nice properties of the Euler potentials in favor of the orthogonality property. The metric tensor $g^{\mu\nu}(\alpha, \varepsilon\beta) \equiv \underline{\nabla} q^\mu \cdot \nabla q^\nu$ is then diagonal, with diagonal elements $g^\alpha \equiv |\nabla\alpha|^2$, $g^\beta \equiv |\nabla\beta|^2$, and $g^\phi \equiv |\nabla\phi|^2 = R^{-2}(\alpha, \varepsilon\beta)$. The "near-slab" nature of the problem is expressed by having all quantities in the theory (e.g., $g^{\mu\nu}$ and Hamiltonian H) depend on $\varepsilon\beta$, instead of simply β.

Defining $\underline{e}^\mu \equiv \underline{\nabla} q^\mu$, $\underline{e}_\mu \equiv d\underline{x} / \partial q^\mu$, we have

$$\underline{B} = (\underline{\nabla} \times \underline{A}) = (g^1 g^2 g^3)^{1/2} \left\{ \underline{e}_1 \left(\frac{\partial A_3}{\partial q^2} - \frac{\partial A_2}{\partial q^3}\right) + \text{cyclic permutations} \right\}, \quad \text{(II-2)}$$

where $A_\mu \equiv \underline{e}_\mu \cdot \underline{A}$.

Equation (II-2) is valid for a general orthogonal coordinate system. Specializing to our present choice of coordinates, we know that $B^\alpha \equiv B^1 \equiv \underline{e}^1 \cdot \underline{B} = 0$. Our hypothesis that the A_μ are independent of ϕ,

and in particular our choice that α be the poloidal flux enclosed by a poloidal flux line, allows the choice of gauge

$$A_\mu = [o,\ A_\beta(\alpha,\ \varepsilon\beta),\ \alpha], \tag{II-3}$$

with magnetic field

$$B^\mu \equiv \underline{e}^\mu \cdot \underline{B} = (g^1 g^2 g^3)^{1/2} \quad (o,\ -\partial A_\phi/\partial\alpha = -1,\ \partial A_\beta/\partial\alpha). \tag{II-4}$$

An important feature of Eq. (II-3) is that $A_\alpha = 0$. This causes the Hamiltonian to be of the same form as the slab Hamiltonian used in part I [Eq. (I-4)]. Setting $m = c = 1$, we have

$$\begin{aligned} H(\alpha,\ p_\alpha;\ \varepsilon\beta,\ p_\beta;\ p_\phi) &= \frac{1}{2}\,(p - eA)_\mu g^{\mu\nu}(p - eA)_\nu + e\Phi \\ &= \frac{1}{2}\, g^\alpha(\alpha,\ \varepsilon\beta)p_\alpha^2 + V(\alpha|\varepsilon\beta, p_\beta,\ p_\phi), \end{aligned} \tag{II-5}$$

where

$$\begin{aligned} V(\alpha|\varepsilon\beta, p_\beta, p_\phi) \equiv \frac{1}{2}\left\{g^\beta\ (\alpha,\varepsilon\beta)[\,p_\beta - eA_\beta(\alpha,\varepsilon\beta)]^2 + g^\phi(\alpha,\varepsilon\beta)[\,p_\phi - e\alpha]^2\right\} \\ + e\ \Phi\,(\alpha,\varepsilon\beta) \end{aligned} \tag{II-6}$$

is the effective potential for the oscillation of the particle in α. The first form given for H in Eq. (II-5) is valid for a coordinate system q^μ of arbitrary metric; the second form is restricted to our specific choice.

The independence of H from ϕ implies p_ϕ = constant. If we let $\varepsilon = 0$, then p_β would also be a constant, and our Hamiltonian would be precisely of the slab type used in Part I. In this case, the potential well $V(\alpha)$ has a constant position, with well bottom α_o parametrically dependent

on the constants p_ϕ and p_β. The particle gyromotion then corresponds to the high frequency oscillation in this constant well. For $\varepsilon \equiv 0$, p_β is no longer exactly conserved. Driven by the $\mathcal{O}(\varepsilon)$ dependence of $p_\beta = -\partial H/\partial\beta$ on β, p_β has an $\mathcal{O}(\varepsilon)$ jitter at the gyrofrequency. In addition, the slow variation of H with β as a particle moves along a field line produces a corresponding variation of p_β on the bounce period time scale. If the particle is trapped, oscillating in β and ϕ, p_β will also experience a periodic variation on this time scale. This in turn produces an oscillation of the well $V(\alpha)$ and, in particular, of the well bottom α_o, occurring once every bounce period. This is the explanation of the banana orbits of trapped particles in axisymmetric geometries, expressed in the canonical terminology we are using. Taking the α surface at which $\dot{\phi} = 0$ as the "banana center" α_b, we find from Hamilton's equation for $\dot{\phi}$ that $\alpha_b = p_\phi/e$. The guiding center position $\alpha_o = \alpha_o(\varepsilon\beta, p_\beta, p_\phi)$ oscillates about α_b, with the slow variation of p_β and $\varepsilon\beta$.

The jitter of p_β and $\varepsilon\beta$ at the gyrofrequency induce a jitter of V and the metric coefficient g^α. This modifies the particle gyromotion, as well as giving rise to particle drifts.

In the following sections, we shall see the specific mathematics of these processes.

II.D. Generating Function S; Formal Description

In this section we describe stage one of the transformation process outlined in Sec. II.B, in the formal framework mentioned there, which is nonperturbative in η. That is, we shall write down the mixed variable

generating function S, which transforms to action angle variables (θ^o, J^o), in terms of a certain integral. This integral is difficult to perform directly, if V is anharmonic or g^α depends on α.

The angular momentum $p_\phi = P_\phi$ is exactly conserved, and retains its value through all the transformations to be performed. We therefore drop it from explicit notation. We denote by $\lambda \equiv (\lambda_1, \lambda_2)$ the pair of variables describing motion along the field line. Before transformation S is applied, these are the "old" variables $\lambda^o \equiv (\varepsilon\beta, p_\beta)$ and after S they are the corresponding "new" pair $\lambda^N \equiv (\varepsilon b, P_b)$ (where both b and P_b may have a superscript j, denoting which of the sequence of tranformations T_j was last applied). We shall also have need for the "mixed" pair $\lambda^M \equiv (\varepsilon\beta, P_b)$, in writing down S.

S is then obtained in the following manner. The energy E of the particle is given by

$$E = H(\alpha, p_\alpha; \lambda^o) = \frac{1}{2} g^\alpha p_\alpha^{\,2} + V(\alpha|\lambda^o). \tag{II-7}$$

We solve this for p_α:

$$p_\alpha\ (\alpha, E, \lambda) \equiv \left\{2\,[E - V(\alpha|\lambda)] / g^\alpha\ (\alpha, \lambda)\right\}^{1/2}, \tag{II-8}$$

and define the functional form of the action J^o by

$$J^o\ (E, \lambda) \equiv (2\pi)^{-1} \oint d\alpha\ p_\alpha\ (\alpha, E, \lambda). \tag{II-9}$$

We then invert this, $J^o(E, \lambda) \to E(J^o, \lambda)$, and finally define S by

$$S\ (\alpha, J^o; \lambda^M) \equiv \beta\ P_b^{\,o} + S_1, \tag{II-10}$$

$$S\ (\alpha, J^o\ ;\ \lambda^M) \equiv \beta \int^\alpha d\alpha'\ p_\alpha\ [\alpha', E(J^o, \lambda^M), \lambda^M]. \tag{II-11}$$

Note that it is $E(J^o, \lambda^o)$ and not $E(J^o, \lambda^M)$ which is the particle's true energy. These two are different by a term of $O(\varepsilon)$, and so in the slab limit $\varepsilon \to 0$ this choice for S becomes the usual choice used[11] to solve the one dimensional particle motion problem.

The transformation equations follow from $S(\underline{q}, \underline{P})$ via the usual description,[10]

$$\underline{p} = \partial S/\partial \underline{q}, \qquad \underline{Q} = \partial S/\partial \underline{P}. \tag{II-12}$$

Of particular interest are

$$p_\alpha = \partial S/\partial \alpha = p_\alpha[\alpha, E(J^o, \lambda^M), \lambda^M] = \left\{2[E(J^o, \lambda^M) - V(\alpha|\lambda^M)]/g^\alpha(\alpha, \varepsilon\beta)\right\}^{1/2}, \tag{II-13}$$

$$p_\beta = \partial S/\partial \beta = P_b{}^o + \varepsilon \partial S_1/\partial(\varepsilon\beta) \equiv P_b{}^o + \varepsilon\delta p_\beta(\alpha, J^o, \lambda^M), \tag{II-14}$$

and

$$\beta = b^o - \partial S_1/\partial P_b{}^o \equiv b^o + \delta\beta(\alpha, J^o, \lambda^M). \tag{II-15}$$

Using Eq. (II-13) in (II-5), we obtain a non-canonical form H' for the Hamiltonian which we shall need in getting an explicit expression for H^o (cf. Sec. II-H):

$$H(\alpha, p_\alpha; \lambda^o) = \frac{1}{2} g^\alpha(\alpha, \varepsilon\beta) p_\alpha{}^2 + V(\alpha|\lambda^o) = [E(J^o, \lambda^M) - V(\alpha|\lambda^M)] + V(\alpha|\lambda^o),$$

and thus

$$H(\alpha, p_\alpha; \lambda^o) = H'(\alpha, J^o, \lambda^o, \lambda^M) \equiv E(J^o, \lambda^M) + V(\alpha|\lambda^o) - V(\alpha|\lambda^M). \tag{II-16}$$

One then uses Eqs. (II-14) and (II-15) in H' to remove the variables λ^o in favor of the new ones λ^N, getting in addition terms in $\delta\beta$, δp_β which are higher order in ε. Since $\delta\beta$ and δp_β are themselves functions of the mixed pair λ^M, this process must be applied iteratively, to

as high order in ε as desired. The result is $H^o[Z^o \equiv (\Theta^o, J^o; \varepsilon b^o, P_b{}^o)]$, expressed as an expansion in ε. In Sec. II.4 we carry out this procedure in detail.

II.E. Lie Methods; Preparation for Explicit Calculation

Now that stage one of the transformation process has been described in terms of the formal definition of S [Eqs. (II-10) and (II-11)], we are ready to begin obtaining more explicit results. In order to do this, we will make use of two variants of Lie methods, the type used being appropriate to the particular form of the Hamiltonian being treated. In stage one, the appropriate variant is essentially the normal form type used by Dragt and Finn.[9] Stage two requires a second variant, which we call the "near-slab" type. The philosophy underlying the two is, however, just the same, and many of the important equations dictating their use are formally identical.

In this section we present our formulation of both types of Lie methods. We first describe the Dragt-Finn variety, and then more briefly describe the near-slab type, pointing out the most important differences between the two.

II.E.1. Dragt-Finn Lie Method

We assume a Hamiltonian h', expressed as a power series expansion in its canonical variables $\underline{z}'$. The phase point $\underline{z}' = 0$ is a fixed point of the system. The "distance" of phase point $\underline{z}'$ away from this fixed point, expressed in terms of some suitable metric, is proportional

to the parameter η, and so the mth order, term in this multivariable expansion for h' is proportional to η^m.

Because $\underline{z}' = 0$ is a fixed point, $h_1' \equiv \partial h'/\partial \underline{z}'\big|_{\underline{z}'=0} = 0$. The zero-order term in h', $h_o' \equiv h'(\underline{z}'=0)$, will turn out to be of the order of the parallel kinetic energy at the particle guiding center, and so for a typical particle is of the same size as the perpendicular kinetic energy at the guiding center, which is on the order of the second order term in the expansion. To indicate that these two terms are of comparable size, we express, this second order term without an accompanying η, $h_2' \equiv \underline{z}'\,\underline{z}' : (\frac{1}{2}\,\partial^2 h'/\partial \underline{z}'\,\partial \underline{z}')_{\underline{z}'=0}$, even though evidently $h_2' \sim (\underline{z}')^2$, while $h_o' \sim (\underline{z}')^o$. We thus write the expansion of h' as

$$h'(\underline{z}') = h_o' + \sum_{m=2}^{\infty} \eta^{m-2}\, h_m'\,(\underline{z}'), \qquad \text{(II-17)}$$

where it is understood that η is only present in the actual expression for h' <u>implicity</u>, i.e., it is a formal device which we use to concisely describe our ordering.

The term h_o' is independent of the dynamical variables in the restricted space, and so could normally be thrown away. In the present application, however, h_o' is dependent upon the variables λ, which are frozen in the one-dimensional problem of stage one, but which are dynamical variables for the full problem, and so we retain h_o', though it has no effect on the 1-D problem.

The second order term h_2' describes a set of N simple harmonic oscillators. (N is the number of degrees of freedom in the phase space.

In Sec. II.F, we shall be taking N = 1.) Discarding all h_m' for $m > 2$ yields the unperturbed problem, exactly soluble, and expressible, if desired, in terms of action variables $\underline{j}^o$. The conjugate angle variables $\underline{\theta}^o$ enter only in the $m > 2$ terms of the full problem. The Hamiltonian becomes

$$h'(\underline{z}') = h^o[\underline{z}^o \equiv (\underline{\theta}^o, \underline{j}^o)] \equiv h_o + h_2^o(\underline{j}^o) + \sum_{m=3}^{\infty} \eta^{m-2} h_m^o(\underline{z}), \qquad \text{(II-18)}$$

where $h_o \equiv h_o'$, $h_2^o = \underline{\omega}^o \cdot \underline{j}^o$, and $\underline{\omega}^o$ is the set of frequencies of the N unperturbed oscillators.

The objective of the perturbative technique is to apply canonical transformations $\{R_1, R_2, R_3, \cdots\}$ to transform away the dependence of the Hamiltonian on angles $\underline{\theta}$ to successively higher orders. Transformation R_n transforms from $h^{n-1}(\underline{z}^{n-1})$ to $h^n(\underline{z}^n)$, removing the dependence of Hamiltonian on $\underline{\theta}$ up to $O(\eta^n)$:

$$h^n(\underline{z}) = k^n(\underline{j}) + \sum_{m=n+1}^{\infty} \eta^m h_{m+2}^n(\underline{z}) \qquad \text{(II-19)}$$

$$k^n(\underline{j}) \equiv h_o + \sum_{m=0}^{n} \eta^m h_{m+2}^n(\underline{j}).$$

For any phase function $G(\underline{z}')$, we define the corresponding Lie operator $\check{G}$ by

$$\check{G} \equiv \{G, \ \}, \qquad \text{(II-20)}$$

where $\{\ ,\ \}$ denotes Poisson brackets. The key to Lie perturbative methods lies in the fact that the operator

$$R_G \equiv \exp \check{G} \equiv 1 + \check{G} + \frac{1}{2}\check{G}^2 + \cdots \tag{II-21}$$

induces a canonical transformation.

We therefore take the R_n to be of the form

$$R_n \equiv \exp\,(\eta^n\,\check{G}_{n+2}). \tag{II-22}$$

The generator $\eta^n\,G_{n+2}$ is a polynomial in $\underline{z}'$ of order (n+2), $\eta^n\,G_{n+2} \sim (z')^{n+2}$ (just as for $\eta^n\,h_{n+2}$), and so the operator $\eta^n\,\check{G}_{n+2} \sim (z')^n$. The specific form of the generators will be determined shortly.

One can show[9], for any two phase functions f_1 amd f_2, that the operator R_G satisfies $R_G(f_1 f_2) = (R_G f_1)(R_G f_2)$, and $R_G(f_1+f_2) = (R_G f_1) + (R_G f_2)$. Therefore, writing any phase function g as a power series expansion in the phase variables z, R_G has the additional property $R_G g(\underline{z}) = g[R_G \underline{z}]$. We shall use this in Eq. (II-24).

The coordinate transformations are given by

$$\underline{z}^n = R_n^{-1}\underline{z}^{n-1} \equiv \exp(-\eta^n\ \check{G}_{n+2})\ \underline{z}^{n-1}. \tag{II-23}$$

If we have any phase function $f^o(\underline{z}^o)$ which we wish to express in terms of $\underline{z}^n$, i.e., $f^n\ (\underline{z}^n) \equiv f^o(\underline{z}^o)$, then

$$f^n(\underline{z}^n) = f^0(R_n\ R_{n-1}\cdots R_1\underline{z}) = (R_n\ R_{n-1}\cdots R_1)\ f^o(\underline{z}). \tag{II-24}$$

In particular, we want to take $f^o = h^o$, and choose the G_n's to eliminate the $\underline{\theta}$ - dependence of h^n.

We carry out this procedure explicitly to second order, i.e. we obtain R_1, R_2, or G_3, G_4, and thereby k^1, k^2.

Expanding the exponential in (II-22), for n = 1 we have

$$h^1(\underline{z}) \equiv R_1 h^o(\underline{z}) = (1+\eta\check{G}_3 + \frac{1}{2}\eta^2 \check{G}_3^{\,2} + \cdots)(h_o^{\,o} + h_2^{\,o} + \eta h_3^{\,o} + \eta^2 h_4^{\,o} + \cdots)$$

$$= k^o + \eta(h_3^{\,o} + \check{G}_3 h_2^{\,o}) + \eta^2(h_4^{\,o} + G_3^{\,o} + \frac{1}{2}\check{G}_3^2 h_2^{\,o}) + \cdots. \qquad \text{(II-25)}$$

Henceforth we drop the superscript "0" on the $h_m^{\,o}$. G_3 is determined by requiring that the term $h_3^{\,1} \equiv h_3 + \check{G}_3 h_2$ in Eq. (II-25) be independent of $\underline{\theta}$, and that the generator G_3 have no secular terms. From the definition (II-20 of a Lie operator, $\check{G}_3 h_2 = -\check{h}_2 G_3$, and from the same definition, $-\check{h}_2$ is just the time derivative $(d/dt)_o$ along the unperturbed trajectory in phase space. The above definition of $h_3^{\,1}$ thus becomes a differential equation for G_3:

$$(d/dt)_o\, G_3 \equiv -\check{h}_2 G_3 = h_3^{\,1} - h_3, \qquad \text{(II-26)}$$

with solution

$$G_3 = -\check{h}_2^{-1}(h_3^{\,1} - h_3) \equiv \int^t dt\,(h_3^{\,1} - h_3). \qquad \text{(II-27)}$$

The nonsecularity condition on G_3 requires that $h_3^{\,1}$ be chosen to be the average $\bar{h}_3$ along the unperturbed trajectory of h_3:

$$h_3^{\,1} \equiv \bar{h}_3. \qquad \text{(II-28)}$$

This choice of $h_3^{\,1}$ is independent of $\underline{\theta}$, as required.

One proceeds analogously to second order. We have $h^1 = k^1 + \eta^2 h_4^{\,1} + \cdots$, where, from Eqs. (II-21), (II-25), (II-26) and (II-28)

$$k^1 \equiv k^o + \eta h_3^{\,1} = (h_o + h_2) + \eta\bar{h}_3, \qquad \text{and} \qquad \text{(II-29)}$$

$$h_4^{\,1} \equiv h_4 + \check{G}_3 h_3 + \frac{1}{2}\check{G}_3^2 h_2 = \frac{1}{2}\check{G}_3(h_3 + \bar{h}_3). \qquad \text{(II 30)}$$

Analogous to Eq. (II-25), then, we have

$$\begin{aligned} h^2 \equiv R_2 h^1 &= (1 + \eta^2 \check{G}_4 + \cdots)(k^1 + \eta^2 h_4{}^1 + \cdots) \\ &= k^1 + \eta^2 (h_4{}^1 + \check{G}_4\, h_2) + \cdots \\ &\equiv k^1 + \eta^2\, h_4{}^2 + \cdots \equiv k^2 + \cdots . \end{aligned} \tag{II-31}$$

We determine G_4 in the same manner as G_3 was determined before, requiring $h_4{}^2 \equiv h_4{}^1 - \check{h}_2\, G_4$ to be $\underline{\theta}$-independent, and G_4 to be nonsecular. Then we have

$$h_4{}^2 \equiv \overline{h_4{}^1} = \bar{h}_4 + \overline{\tfrac{1}{2}\, \check{G}_3 (h_3 + \bar{h}_3)}, \text{ and} \tag{II-32}$$

$$-\check{h}_2 G_4 = h_4{}^2 - h_4{}^1 = \overline{h_4{}^1} - h_4{}^1 \tag{II-33}$$

with solution

$$G_4 = -\check{h}_2^{-1}\, (\bar{h}_4{}^1 - h_4{}^1) \equiv \int^t dt' (\bar{h}_4{}^1 - h_4{}^1). \tag{II-34}$$

The generalization of these equations to arbitrary order is obvious. We note that at each stage R_n leaves k^{n-1} unchanged, and gives an additional term $h''_{n+2} \equiv \overline{h^{n-1}_{n+2}}$ which, added to k^{n-1}, yields k^n. This and most other important features are present in the near-slab type of Lie method, which we now describe.

II.E.2. Near-Slab Lie Method

At the beginning of stage two of the transformation process, we shall be faced with a Hamiltonian H^o the near-slab form

$$H^o(\underline{Z}^o) = H_o(J^o;\ \varepsilon b^o,\ P_b{}^o) + \sum_{\ell=1}^{\infty} \varepsilon^\ell H_\ell(\Theta^o,\ J^o;\ \varepsilon b^o,\ P_b{}^o). \tag{II-35}$$

This type of Hamiltonian was treated by McNamara and Whiteman,[12] using Lie methods to find a good adiabatic invariant J. Here we want to do more; we shall develop a Lie perturbative technique to transform the Hamiltonian H^o and all its variables $\underline{z}^o$, to eliminate its Θ-dependence and thereby obtain the guiding center Hamiltonian K, in the process finding J.

There are two important features which distinguish this near-slab form from that of $h^o(\underline{z}^o)$ of the preceding section. The first is that, whereas η there was only present in h^o implicitly, here ε is an <u>explicit</u> small parameter, i.e., a particular small constant. In this respect the appropriate Lie method should resemble the sort employed by Deprit and Cary[13] more than that of Dragt and Finn. The second important feature, and the principal reason a different variant of Lie methods is needed to treat this problem, is the appearance of ε in <u>two</u> places in Eq. (II-35); accompanying the term H_ℓ, as did η in Eq. (II-17), <u>and</u> also multiplying the variable b in the arguments of each H_ℓ. This second ε dependence, which gives H^o its near-slab nature, changes the ordering somewhat from that of Sec. II.E.1. Without this dependence, we would take H_o to be our unperturbed Hamiltonian, giving the unperturbed trajectories one must integrate along in calculating the transformation generators. The trajectories of H_o are not of the simple harmonic nature which those of $k^o \equiv h_o + h_2$ were, and so these orbit integrals would be difficult to perform explicitly. We shall see shortly that the shuffling in the ordering which this near-slab ε dependence introduces is just such as to make a simple integral over gyrophase the appropriate one.

The procedure parallels that of the Dragt-Finn method of the last section. We use a sequence $\{T_1, T_2, T_3 \cdots\}$ of transformations to remove the dependence of the Hamiltonian on Θ, the transformation T_n transforming $H^{n-1}(\underline{Z}^{n-1})$ into $H^n(\underline{Z}^n)$. Analogous to Eq. (II-19), H^n is of the form

$$H^n(\underline{Z}^n) = K^n(J, \varepsilon b, P_b) + \sum_{\ell=n+1}^{\infty} \varepsilon^{\ell} H_{\ell}^n (\Theta, J, \varepsilon b, P_b), \qquad \text{(II-36)}$$

$$K^n(J, \varepsilon b, P_b) + \sum_{\ell=0}^{n} \varepsilon^{\ell} (J, \varepsilon b, P_b),$$

so that $H^{\infty} = K^{\infty} = K$, the guiding center Hamiltonian. The T_n are given by

$$T_n \equiv \exp \varepsilon^n \check{W}_n. \qquad \text{(II-37)}$$

As in the Dragt-Finn technique, the generators W_n are taken to be of the same functional form as the terms in the Hamiltonian they are designed to eliminate, $W_n = W_n(\Theta, J, \varepsilon b, P_b)$. Paralleling Eq. (II-25), we write

$$H^1 \equiv T_1 H^0 = (1 + \varepsilon \check{W}_1 + \frac{1}{2} \varepsilon^2 \check{W}_2^{\,2} + \cdots)(H_o + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots)$$

$$= H_o + \varepsilon (H_1 + \check{W}_1 H_o) + \varepsilon^2 (H_2 + \check{W}_1 H_1 + \frac{1}{2} \check{W}_1^{\,2} H_o) + \cdots. \qquad \text{(II-38)}$$

It is here that the second, near-slab dependence of H on ε enters, changing the ordering. We write

$$-\check{W}_1 H_o = H_o \check{W}_1 = H_{o\theta} \check{W}_1 + \varepsilon H_{ob} \check{W}_1, \qquad \text{(II-39)}$$

where the operators $\check{H}_{o\theta}$, $\check{H}_{ob}$, both of $\mathcal{O}(1)$, are defined by

$$\check{H}_{o\theta} \equiv \frac{\partial H_o}{\partial \theta}\frac{\partial}{\partial J} - \frac{\partial H_o}{\partial J}\frac{\partial}{\partial \theta} = -\frac{\partial H_o}{\partial J}\frac{\partial}{\partial \theta}, \quad \check{H}_{ob} \equiv \frac{\partial H_o}{\partial \varepsilon b}\frac{\partial}{\partial P_b} - \frac{\partial H_o}{\partial P_b}\frac{\partial}{\partial \varepsilon b}. \tag{II-40}$$

Defining the zero-order gyrofrequency by $\Omega_o \equiv \partial H_o/\partial J$, we see that $-\check{H}_{o\theta} = \Omega_o\, \partial/\partial\Theta \equiv d/dt)_o$, the time derivative along the simple trajectory in which Θ increases linearly in time, and the other variables $(J, \varepsilon b, P_b)$ are frozen. Since $\check{H}_{ob}$ enters at higher order in ε than $\check{H}_{o\theta}$, it is these trajectories which are appropriate to integrate along, to calculate the W_n.

Using Eq. (II-39) in (II-38), we find that the first order part of H^1 is only part of the factor $(H_1 + \check{W}_1 H_o)$ there, $H_1{}^1 = (H_1 - \check{H}_{o\theta} W_1)$, the remainder $-\varepsilon \check{H}_{ob}\, W_1$ going into $H_2{}^1$. Imposing the same conditions as in Sec. II.E.1 gives the defining equations for W_1; $H_1{}^1 = \bar{H}_1$, and

$$d/dt|_o\, W_1 \equiv -\check{H}_{o\theta}\, W_1 = \bar{H}_1 - H_1, \tag{II-41}$$

with solution

$$W_1 = -\check{H}_{o\theta}(\bar{H}_1 - H_1) \equiv \int^t dt'(\bar{H}_1 - H_1), \tag{II-42}$$

analogous to Eqs. (II-26) and (II-27). This gives a W_1 with the same near-slab dependences as the H_ℓ, and so we write $\check{W}_1 = \check{W}_{1\theta} + \varepsilon\, \check{W}_{1b}$, analogous to (II-39) and (II-40), to maintain the proper ordering in the terms of order two and higher. Using this and Eqs. (II-39), (II-42) in Eq. (II-38), we find

$$H_2{}^1 = H_2 + \frac{1}{2}\check{W}_{1\theta}(H_1 + \bar{H}_1) - \check{H}_{ob}\, W_1, \tag{II-43}$$

slightly different from Eq. (II-30) for h_4^1.

The procedure to second order is just as in Sec. II.E.1. One obtains

$$H_2^2 = \bar{H}_2^1, \qquad \text{and} \tag{II-44}$$

$$-\check{H}_{o\theta} W_2 = \bar{H}_2^1 - H_2^1, \tag{II-45}$$

with solution

$$W_2 = -\check{H}_{o\theta}^{-1} (\bar{H}_2^1 - H_2^1) \equiv \int^t dt' (\bar{H}_2^1 - H_2^1). \tag{II-46}$$

II.F. The "Frozen λ" Problem

With the methods of Sec. II.E.1 in hand, we are prepared to set up and solve the 1-D, "frozen λ" problem sketched in Sec. II.B. We begin with the initial Hamiltonian $H(\alpha,p_\alpha;\lambda)$, and treat the longitudinal variables λ as constant parameters. This leaves us with a one degree of freedom phase space, with Hamiltonian H.

We define the particle's α-center α_o as in Eq. (I-17):

$$0 = \partial/\partial\alpha\ V(\alpha|\lambda)|_{\alpha_o}, \tag{II-47}$$

which may be solved to give $\alpha_o = \alpha_o(\lambda)$. We then make the simple transformation $(\alpha,p_\alpha) \to \underline{z}' \equiv (\delta\alpha \equiv \alpha - \alpha_o, p_\alpha)$, which is canonical in the reduced phase space. The new Hamiltonian $h'(z')$ is given by

$$h'(\underline{z}';\lambda) = H(\alpha,p_\alpha;\lambda) = \frac{1}{2} g^\alpha p_\alpha^2 + V[\alpha_o(\lambda)+\delta\alpha|\lambda]$$

$$= V_o + \sum_{m=2}^{\infty} [\tfrac{1}{2} g_{m-2} p_\alpha^2 (\delta\alpha)^{m-2} + V_m (\delta\alpha)^m], \tag{II-48a}$$

where

$$g^{\alpha}(\alpha_o + \delta\alpha, \lambda) \equiv \sum_{m=o}^{\infty} g_m(\alpha_o, \lambda)(\delta\alpha)^m, \qquad \text{and} \tag{II-48b}$$

$$V(\alpha_o + \delta\alpha, \lambda) \equiv \sum_{m=o}^{\infty} V_m(\alpha_o, \lambda)(\delta\alpha)^m. \tag{II-48c}$$

Comparing Eq. (II-48a) with (II-17), we make the identifications

$$h_o' = V_o \ , \quad h_1' = V_1 = 0, \text{ and}$$

$$\eta^{m-2} h_m' = \frac{1}{2} g_{m-2} p_\alpha^2 (\delta\alpha)^{m-2} + V_m(\delta\alpha)^m, \quad m > 1.$$

Following the discussion in Sec. II.E.1, we trivially make the finite transformation R_o: $\underline{z}' \to \underline{z}^o \equiv (\theta^o, j^o)$ to action-angle variables, canonical in the reduced space, given by

$$\delta\alpha = a(\theta^o, j^o|\lambda), \; p_\alpha = p(\theta^o, j^o|\lambda), \tag{II-49}$$

where the phase functions a and p are defined as

$$a(\theta, j|\lambda) \equiv (\Omega_o/V_2)^{1/2} j^{1/2} \sin\theta \equiv \bar{a}(\lambda) j^{1/2} \sin\theta,$$

$$p(\theta, j|\lambda) \equiv (2\Omega_o/g_o)^{1/2} j^{1/2} \cos\theta \equiv \bar{p}(\lambda) j^{1/2} \cos\theta.$$

Here $\Omega_o^2 \equiv 2g_o V_2$. The frequency Ω_o is the $\eta \to o$ gyrofrequency, including effects due to finite $(v_\parallel/\Omega_c L_s)$, which exist even in the zero gyroradius limit. (Here $\Omega_c \equiv eB/mc$, the unsheared gyrofrequency, and we recall that L_s is the shear scale length.) Using (II-49) in (II-48), we find the Hamiltonian $h^o(\underline{z}^o)$:

$$h^o(\underline{z}) = (V_o + \Omega_o j) + \sum_{m=3}^{\infty} [\frac{1}{2}\bar{g}_{m-2}\bar{p}^2\bar{a}^{m-2}\cos^2\theta \sin^{m-2}\theta + V_m \bar{a}^m \sin^m\theta] j^{m/2}. \tag{II-50}$$

Comparing this with Eq. (II-18), we find (dropping the superscript O on

the $h_m^{\,o}$, as before)

$$h_o\ (\lambda)\ \equiv V_o,\quad h_2(\underline{z};\lambda)\ \equiv \Omega_o j \tag{II-51}$$

$$\begin{aligned}\eta h_3(z;\lambda) &\equiv \Omega_o \bar{a}\ j^{3/2}\ [(g_1/g_o)\ \cos^2\theta \sin\theta + (V_3/V_2)\ \sin^3\theta] \\ &= \Omega_o \bar{a}\ j^{3/2}\ [\,c_3\ \sin 3\theta + c_1\ \sin\theta]/4\end{aligned} \tag{II-52}$$

$$\begin{aligned}\eta^2 h_4(\underline{z};\lambda) &\equiv \Omega_o \bar{a}^2 j^2\ [(g_2/g_o)\ \cos^2\theta \sin^2\theta + (V_4/V_2)\sin^4\theta] \\ &= -\ \Omega_o \bar{a}^2 j^2\ [\,d_4\ \cos 4\theta + d_2\ \cos 2\theta - d_o]/8.\end{aligned} \tag{II-53}$$

Here c_m (in h_3) and d_m (in h_4) are the coefficients accompanying $\sin m\theta$ or $\cos m\theta$ and in particular

$$c_3 \equiv (g_1/g_o - V_3/V_2),\ c_1 \equiv (g_1/g_o + 3V_3/V_2),$$

$$d_4 \equiv (g_2/g_o - V_4/V_2),\ d_2 \equiv 4V_4/V_2,\ d_o \equiv (g_2/g_o + 3\ V_4/V_2).$$

The parameters c_m, d_m, $\bar{a}$, Ω_o, g_m, and V_m are all functions of λ, and the dependence of the h_m on $\underline{z} \equiv (\theta,j)$ is explicitly displayed.

We now apply the general formulae obtained in Sec. II.E.1. From Eqs. (II-28) and (II-52), we see that

$$h_3^{\,1} = \bar{h}_3 = 0. \tag{II-54}$$

Using this in (II-27) yields

$$\begin{aligned}\eta\ G_3 &= -\ \Omega_o^{-1} \int^{\theta} d\theta'\ \eta\ h_3 \\ &= \bar{a}\ j^{3/2}\ [\,c_3(\cos 3\theta)/12 + c_1(\cos\theta)/4].\end{aligned} \tag{II-55}$$

From Eqs. (II-55), (II-53) and (II-30), we then find

$$\eta^2 h_4{}^1 = \eta^2 (h_4 + \tfrac{1}{2}\check{G}_3 h_3)$$

$$= -\Omega_o \bar{a}^2 j^2 \left\{ (d_4 + c_3 c_1/4)\cos 4\theta + (d_2 + c_3 c_1)\cos 2\theta - [d_o - (3/8)(c_3{}^2 + c_1{}^2)] \right\} /8. \tag{II-56}$$

From this and (II-32), we read off $\eta^2 h_4{}^2$:

$$\eta^2 h_4{}^2 = \eta^2 \overline{h_4{}^1} = \Omega_o \bar{a}^2 j^2 \, [d_o - (3/8)(c_3{}^2 + c_1{}^2)]/8. \tag{II-57}$$

Equations (II-56), (II-57) and (II-34) give us G_4:

$$\eta^2 G_4 = \Omega_o^{-1} \int^{\theta} d\theta' \, \eta^2 (\overline{h_4{}^1} - h_4{}^1)$$

$$= \bar{a}^2 j^2 \, [(d_4 + c_3 c_1/4)(\sin 4\theta)/32 + (d_2 + c_3 c_1)(\sin 2\theta)/16] . \tag{II-58}$$

This is to as high order as we shall explicitly carry the calculation, though clearly the extension to arbitrarily high order is straightforward. To this order $h^\infty(j;\lambda) \simeq k^2(j;\lambda) = h_o + h_2 + \eta h_3{}^1 + \eta^2 h_4{}^2$, which we can read off from Eqs. (II-51), (II-54) and (II-57). In addition, because h^∞ is just the energy for the 1-D particle motion problem specified by parameter λ, and $j \equiv j^\infty$ is the action, these are to be identified with $E(J^o, \lambda)$ and J^o respectively, which resulted from solving this I-D problem using the traditional mixed-variable generating function prescription, as discussed in Sec. II.D. Therefore

$$E(j,\lambda) = h^\infty(j,\lambda) \simeq k^2(j,\lambda) = (h_o + h_2) + \eta^2 h_4{}^2$$
$$= (V_o + \Omega_o j) + \Omega_o \bar{a}^2 j^2 \, [d_o - (3/8)(c_3{}^2 + c_1{}^2)]/8, \tag{II-59}$$

and

$$j \equiv j^{\infty} = J^{o}. \tag{II-60}$$

We observe that the Lie method of solution has completely circumvented the difficulties of cumbersome integrals and functional inversions encountered in the traditional solution procedure, as discussed in Sec. II.B.

Note that knowledge of G_4 was not necessary for writing out the form of k^2; the stipulation that $h_4{}^2 = \overline{h_4{}^1}$ in Eq. (II-32) completed our knowledge of k^2, and only then is G_4 determined. However if we desire the <u>interpretation</u> of the variables $\underline{z}^2$ appearing in k^2, we need the transformation R_2, and so G_4.

In the next section we shall calculate the generating function S, which requires use of the coordinate transformations just obtained. To the order we are working, we will only need the transformation equations for $\delta\alpha$ and p_α to first order, i.e., we will need only G_3. From Eqs. (II-23) and (II-49),

$$\delta\alpha \equiv \alpha - \alpha_o(\lambda) = a(\underline{z}^o|\lambda) = R_1\, a(\underline{z}^1|\lambda) \simeq R_1\, a(\underline{z}|\lambda) \simeq (1 + \eta\, \check{G}_3)\, a(\underline{z})|\lambda), \tag{II-61}$$

and similarly

$$p_\alpha = p(\underline{z}^o|\lambda) \simeq (1 + \eta\, \check{G}_3)\, p(\underline{z}|\lambda). \tag{II-62}$$

Denoting by y any of the variables $\delta\alpha$, p_α, $\delta\beta$, δp_β, we may write y as an expansion in η,

$$y = \sum_{m=o}^{\infty} \eta^m y_m. \tag{II-63}$$

Using Eq. (II-55) in (II-61) and (II-62), we explicitly write out the first two terms in this η-expansion of $\delta\alpha$, p_α:

$$\delta\alpha_o = a(\underline{z}|\lambda), \qquad p_{\alpha o} = p(\underline{z})|\lambda), \tag{II-64}$$

$$\eta\delta\alpha_1 \equiv \eta\check{G}_3 a(\underline{z}|\lambda) = -\bar{a}^2 j(c_1 + c_\alpha \cos 2\theta)/4 \tag{II-65}$$

$$\eta p_{\alpha 1} \equiv \eta\check{G}_3 p(\underline{z}|\lambda) = \bar{p}\,\bar{a}\, j\, c_p \sin 2\theta/2 = j\, c_p \sin 2\theta \tag{II-66}$$

where

$$c_\alpha \equiv (c_1 + c_3)/2 = (g_1/g_o + V_3/V_2), \text{ and } c_p \equiv (c_1 - c_3)/4 = V_3/V_2.$$

II.G. Generating Function; Explicit Form

In the present section, we calculate the generating function S, and the associated functions $\delta\beta$, δp_β, defined in Eqs. (II-14) and (II-15). These will be necessary to obtain Hamiltonian H^o from H', as described in Sec. II.D. We want to obtain H^o explicitly to $\mathcal{O}(\epsilon^2, \epsilon\eta, \eta^2)$ which will necessitate finding $\delta\beta$, δp_β up to one order further than the $\eta \to o$ simple harmonic limit, i.e., we will need up to ηy_1, in the terminology of Eq. (II.63). For this, we need S to one order higher than its simple harmonic value, and this in turn requires $\delta\alpha$ and p_α up to first order as well.

We calculate S_1 using the coordinate transformations obtained in the last section:

$$S_1 \equiv \int^\alpha d\alpha'\, p_\alpha(\alpha') = \int^\theta d\theta'(d\delta\alpha/d\theta')p_\alpha(\theta', j|\lambda^M). \tag{II-67}$$

This integral over θ has integrand

$$(d\delta\alpha/d\theta)p_\alpha(\theta,j|\lambda^M) = (\bar{a}j^{1/2}\cos\theta + \tfrac{1}{2}\bar{a}^2 j\, c_\alpha \sin 2\theta)(\bar{p}j^{1/2}\cos\theta + \tfrac{1}{2}\bar{p}\,\bar{a}\, j c_p \sin 2\theta) \tag{II-68}$$

$$+ \text{ higher order},$$

and so we compute S_1 by a trivial integration:

$$S_1(\theta,j|\lambda^M) = j(\theta + \tfrac{1}{2}\sin\theta) - \tfrac{1}{2}\bar{a}\, j^{3/2} c_s(\cos\theta + \cos 3\theta/3) + \text{higher order}, \tag{II-69}$$

where $\quad c_s \equiv c_\alpha + c_p = (g_1/g_o + 2V_3/V_2)$.

From Eqs. (II-14), (II-15), we have $\delta p_\beta \equiv \partial S_1/\partial\lambda_1^M$, $\delta\beta \equiv \partial S_1/\partial\lambda_2^M$, where $\lambda^M \equiv (\lambda_1^M, \lambda_2^M) \equiv (\varepsilon\beta, P_b^o)$, and so we are interested in the derivative $\partial S_1/\partial\lambda_{1,2}$, where α, j and the other λ_i are to be held constant. We want to express these derivatives in terms of the form $S_1(\theta,j|\lambda)$ we now have for S_1 [cf. Eq. (II-70)]. We have

$$\partial S_1/\partial\lambda_1)_{\alpha,j,\lambda_2} = (\partial S_1/\partial\theta)_{j,\lambda}(\partial\theta/\partial\lambda_1)_{\alpha,j,\lambda_2} + (\partial S_1/\partial\lambda_1)_{\theta,j,\lambda_2}.$$

From (II-67)

$$(\partial S_1/\partial\theta)_{j,\lambda} = (d\,\delta\alpha/d\theta)p_\alpha(\theta,j|\lambda).$$

We have $\delta\alpha\,(\theta,j|\lambda)$, which we can in principal invert to obtain $\theta\,(\delta\alpha \equiv \alpha - \alpha_o,\ j|\lambda)$, whence

$$(\partial\theta/\partial\lambda_1)_{\alpha,j,\lambda_2} = -(\partial\alpha_o/\partial\lambda_1)_{\lambda_2}\,(\partial\theta/\partial\delta\alpha)_{j,\lambda} + (\partial\theta/\partial\lambda_1)_{\delta\alpha,j}$$

$$= -[(\partial\alpha_o/\partial\lambda_1)_{\lambda_2} + (\partial\delta\alpha/\partial\lambda_1)_{\theta,j,\lambda_2}](\partial\delta\alpha/\partial\theta)^{-1}_{j,\lambda}.$$

Putting these pieces together, we find, finally,

$$(\partial S_1/\partial\lambda_1)_{\alpha,j,\lambda_2} = -p_\alpha(\theta,j|\lambda)[(\partial\alpha_o/\partial\lambda_1)_{\lambda_2} + (\partial\delta\alpha/\partial\lambda_1)_{\theta,j,\lambda_2}] \qquad \text{(II-70)}$$

$$+ (\partial S_1/\partial\lambda_1)_{\theta,j,\lambda_2}.$$

There are three terms in (II-70), the two in the square brackets, and a third term $(\partial S_1/\partial\lambda_1)_{\theta,j,\lambda_2}$. From Eq. (II-69), this last term is given by

$$(\partial S_1/\partial\lambda_1)_{\theta,j,\lambda_2} = -\frac{1}{2}\partial(\bar{a}\ c_s)/\partial\lambda_1 j^{3/2}(\cos\theta + \cos 3\theta/3).$$

From the ordering scheme outlined in Appendix B, this term is down from the first by η^2, and so we neglect it completely, to the order we are working. Similarly the second term is down by a factor η from the first, and so contributes to $\delta p_{\beta 1}$. Analogous equations to those above hold for $\partial S_1/\partial\lambda_2$. Using the η expansion of p_α in (II-70) gives

$$\delta\beta_o = \bar{p}\ j^{1/2}\cos\theta\ (\partial\alpha_o/\partial\lambda_2),\ \delta p_{\beta o} = -\bar{p}\ j^{1/2}\cos\theta(\partial\alpha_o/\partial\lambda_1), \qquad \text{(II-71)}$$

$$\eta\delta\beta_1 = d_p j\ \sin 2\theta,\quad \eta\delta p_{\beta 1} = -\ d_b\ j\ \sin 2\theta, \qquad \text{(II-72)}$$

where $\quad d_p \equiv c_p(\partial\alpha_o/\partial\lambda_2) + \partial(\ln\bar{a})/\partial\lambda_2,$ and

$$d_b \equiv c_p(\partial\alpha_o/\partial\lambda_1) + \partial(\ln\bar{a})/\partial\lambda_2.$$

Here λ is understood to be evaluated at λ^M. The dependence of the $\delta\beta_m$, $\delta p_{\beta m}$ on θ and j appears explicitly, as will be necessary in finding the explicit dependence of H^o on these variables. In the next section, we proceed to do this.

II.H. Calculation of H^o

We have now completed the work of stage one of the transformation procedure, and are ready to proceed to stage two. Once we have obtained an explicit expression for H^o (the task of this section), stage two is a simple matter, using the Lie method of Sec. II.E.2.

As described following Eq. (II-16), H^o is obtained by using $\delta\beta$ and δp_β in the mixed-variable form H' of the Hamiltonian, to eliminate the old variables in favor of the new. This iterative procedure is in principle executable to arbitrary order in both ε and η, yielding H^o as a double expansion analogous to (II-1). For higher order, however, the expansion rapidly becomes complicated. This is the price to be paid for using a mixed-variable generating function; the transformation equations must subsequently be unmixed. Because the transformation is finite, Lie methods are not useful, and we have been forced to pay this price.

We carry out the expansion to $\mathcal{O}(\varepsilon^2, \varepsilon\eta, \eta^2)$. The first iteration of the expansion of Eq. (II-16) yields

$$\begin{aligned} H' &= E(\lambda^M) + V(\alpha^M|\lambda^o) - V(\alpha^M|\lambda^M) \\ &= E(\lambda^N) \\ &+ \varepsilon[\delta\beta/\partial\varepsilon b)\ E(\lambda^N) + \delta p_\beta(\partial/\partial P_b)\ V(\alpha^M|\lambda^M)] \\ &+ \frac{1}{2}\varepsilon^2\ [(\delta\beta)^2(\partial/\partial\varepsilon b)^2 E(\lambda^N) + (\delta p_\beta)^2(\partial/\partial P_b)^2 V(\alpha^M|\lambda^M)] + \text{higher order.} \end{aligned} \tag{II-73}$$

We have suppressed some of the arguments here for notational simplicity, and the notation α^M denotes that the transformation S gives us α in terms of the mixed variables, $\alpha = \alpha_o(\lambda^M) + \delta\alpha(\theta, j|\lambda^M) \equiv \alpha^M$, so that we must expand this function about λ^N in the next iteration.

The $O(\varepsilon^o)$ term in (II-73), $E(\lambda^N) = k^o + \eta^2 h_4{}^2 + O(\eta^3)$, is already in terms of the new variables and so we identify

$$H_{oo} = k^o, \quad \eta H_{o1} = 0, \quad \eta^2 H_{o2} = \eta^2 h_4{}^2. \tag{II-74}$$

From Eq. (II-6), we have

$$\begin{aligned} V^1(\alpha|\lambda^M) &\equiv (\partial/\partial P_b)\ V(\alpha|\lambda^M) = g^\beta(\alpha,\lambda^M)[P_b - eA_\beta(\alpha|\lambda^M)]\,, \\ (\partial^2/\partial P_b{}^2)\ V(\alpha|\lambda^M) &= g^\beta(\alpha,\lambda^M). \end{aligned} \tag{II-75}$$

In the $O(\varepsilon^2)$ term, we may replace λ^M by λ^N, $\delta\beta$ by $\delta\beta_o$, δp_β by $\delta p_{\beta o}$, and α by $\alpha_o{}^N \equiv \alpha_o(\lambda^N)$ in g^β. The $O(\varepsilon)$ term in (II-73) gives rise to a number of terms, contributing to H_{10}, H_{11}, and H_{20}. One must expand $\delta\beta$, δp_β, and V^1 in both ε and η:

$$\delta\beta(\lambda^M) = \delta\beta_o(\lambda^N) + \varepsilon\delta\beta_o(\partial/\partial\varepsilon b)\ \delta\beta_o(\lambda^N) + \eta\ \delta\beta_1(\lambda^N) + \text{higher order (h.o.)}$$

$$\delta p_{\beta o}(\lambda^M) = \delta p_{\beta o}(\lambda^N) + \varepsilon\delta\beta_o(\partial/\partial\varepsilon b)\delta p_{\beta o}(\lambda^N) + \eta\delta p_{\beta 1}(\lambda^N) + \text{h.o.},$$

$$V'(\alpha^M|\lambda^M) = V' + \varepsilon\delta\beta_o[\partial/\partial\varepsilon b + (\partial\alpha_o/\partial\varepsilon b)\partial/\partial\alpha_o]\ V' + \delta\alpha_o\partial/\partial\alpha_o V' + \text{h.o.},$$

where V' on the right hand side is evaluated at $\alpha_o{}^N, \lambda^N$. Combining the various contributions yields

$$H_{10} = \delta\beta\ (\partial k^o/\partial\varepsilon b) + \delta P_{\beta o}\ V'(\alpha_o{}^N|\lambda^N) \tag{II-76}$$

$$\eta H_{11} = \eta\delta\beta_1(\partial k^o/\partial\varepsilon b) + \eta\delta p_{\beta 1}V'(\alpha_o{}^N|\lambda^N) + \delta p_{\beta o}\delta\alpha_o(\partial/\partial\alpha_o)V' \tag{II-77}$$

$$\begin{aligned} H_{20} &= \tfrac{1}{2}[(\delta\beta_o)^2(\partial/\partial\varepsilon b)^2 k^o + (\delta p_{\beta o})^2\ g^\beta(\alpha_o,\lambda^N)] \\ &\quad + \delta\beta_o\ [(\partial k^o/\partial\varepsilon b)(\partial/\partial\varepsilon b)\delta\beta_o + V'(\partial/\partial\varepsilon b)\partial p_{\beta o}] \\ &\quad + \delta p_{\beta o}\ \delta\beta_o\ [(\partial/\partial\varepsilon b) + (\partial\alpha_o/\partial\varepsilon b)(\partial/\partial\alpha_o)]V'. \end{aligned} \tag{II-78}$$

All terms here are evaluated at the new variables $\underline{Z}^o$. The explicit dependence of all quantities here on $(\theta, j) = (\theta^o, J^o)$ are known from results already obtained. The dependences on $\lambda^N = (\varepsilon b^o, P_b{}^o)$ are determined by the magnetic geometry in the particle's vicinity.

II.I. Calculation of K from H^o

In this section we complete the transformation process, performing the transformations T_n of stage two, using the formalism developed in Sec. II.E.2, to obtain the finite ε, finite η, guiding center Hamiltonian K.

We proceed explicitly to $\mathcal{O}(\varepsilon^2)$, i.e., we shall calculate K_{20}. As noted in Sec. II.F in the context of the frozen -λ problem, to obtain the second-order term, $\varepsilon^2 K_2$, of K, we only need to calculate the generator εW_1, and not $\varepsilon^2 W_2$. In addition, just as each K_ℓ is an expansion in η, $K_\ell = \sum_{m=0}^{\infty} \eta^m K_{\ell m}$, the generators W_ℓ are also expressed as η expansions, $W_\ell = \sum_{m=0}^{\infty} \eta^m W_{\ell m}$. Because we are seeking K_{20} but neglecting ηK_{21}, it will only be necessary to calculate W_{10}.

To begin with, from Eqs. (II-35), (II-36), (II-74), and (II-60), we have the $\mathcal{O}(\varepsilon^o)$ term of H^o,

$$\begin{aligned} K^o(J;\lambda^N) \equiv H_o(J;\lambda^N) = E(J;\lambda^N) &\cong k^o + \eta^2 h_4{}^2 \\ &= [V_o(\lambda^N) + \Omega_o J] + \Omega_o\, \bar{a}^2 J^2 [d_o - (3/8)(c_3{}^2 + c_1{}^2)]/8. \end{aligned} \tag{II-79}$$

For $\varepsilon \to 0$, we have $K = H^o = K^o = E$, and (II-79) is the full guiding center Hamiltonian, to $\mathcal{O}(\eta^2)$.

At $\mathcal{O}(\varepsilon)$, we begin by using the expressions obtained for $\delta\alpha_o$, $\delta\alpha_1$, $\delta\beta_o$, $\delta\beta_1$, $\delta p_{\beta o}$, $\delta p_{\beta 1}$ [Eqs. (II-64), (II-65), (II-71), (II-72)] in expres-

sions (II-76) and (II-77), and rewrite H_{10} and H_{11}, explicitly displaying their dependences on Θ and J:

$$H_{10} = (C_{13}J^{3/2} + C_{11}J^{1/2}) \cos\Theta \tag{II-80}$$

$$\eta H_{11} = (C_{24}J^2 + C_{22}J) \sin 2\Theta. \tag{II-81}$$

Here C_{mn} designates the coefficient of the term in H_1 in $\cos m\Theta$ or $\sin m\Theta$ and $J^{n/2}$. In particular

$$C_{13} \equiv \bar{p}\ (\partial\alpha_o/\partial P_b)(\partial\Omega_o/\partial\varepsilon b),$$

$$C_{11} \equiv \bar{p}\ [(\partial\alpha_o/P_b)(\partial V_o/\partial\varepsilon b) - (\partial\alpha_o/\partial\varepsilon b)V'],$$

$$C_{24} \equiv d_p(\partial\Omega_o/\partial\varepsilon b), \qquad \text{and}$$

$$C_{22} \equiv d_p(\partial V_o/\partial\varepsilon b) - d_b V' - (\partial\alpha_o/\partial P_b)(\partial V'/\partial\alpha_o).$$

Following Sec. II.E.2, we can now read off K_{10}, K_{11}:

$$K_{10} = \bar{H}_{10} = 0, \quad \eta K_{11} = \eta H_{11} = 0. \tag{II-82}$$

In addition, from Eq. (II-74) or (II-79), we have

$$K_{00} = k^o, \ \eta K_{01} = \eta H_{01} = 0, \ \eta^2 K_{02} = \eta^2 H_{02} = \eta^2 h_4{}^2. \tag{II-83}$$

The only term remaining to be found, to the order we are working, is K_{20}. To $\mathcal{O}(\varepsilon, \varepsilon\eta, \eta)$, we see that $K = K_{oo} = V_o + \Omega_c J$, formally the same as the $\varepsilon, \eta \to 0$ guiding center usually used, but with slightly different definitions of the variables, in agreement with the findings of Refs. 7 and 8. This difference in meaning is dealt with in Sec. II.J.

We now proceed to calculate K_{20}. Using Eq. (II-80) in (II-42), we find the $\mathcal{O}(\eta^0)$ part, W_{10}, of W_1 to be

$$W_{10} = -\Omega_o^{-1} \int d\Theta' \, H_{10} = -\Omega_o^{-1}(C_{13}J^{3/2} + C_{11}J^{1/2})\sin\Theta, \qquad \text{(II-84)}$$

where we have used that $\partial H_o/\partial J = \partial H_{oo}/\partial J + O(\eta^2) = \Omega_o + O(\eta^2)$. From (II-43) we see that we will need H_{20} and $1/2\, \check{W}_{10\theta}\, H_{10}$ for H_{20}^1. From Eqs. (II-84) and (II-80) we calculate the latter of these, finding, after a bit of algebra,

$$\begin{aligned} \frac{1}{2}\check{W}_{10\theta}\, H_{10} &\equiv \frac{1}{2}\,[(\partial W_{10}/\partial\Theta)(\partial H_{10}/\partial J) - (\partial W_{10}/\partial J)(\partial H_{10}/\partial\Theta)] \\ &= -\,(4\Omega_o)^{-1}\,(C_{11} + C_{13}J)(C_{11} + 3\,C_{13}J). \end{aligned} \qquad \text{(II-85)}$$

From Eq. (II-78) we rewrite H_{20}, explicitly displaying its Θ, J dependences:

$$H_{20} = 2(D_2J + D_4J^2)\,\cos^2\Theta, \qquad \text{(II-86)}$$

$$\begin{aligned} D_2 \;&\equiv \frac{1}{2}\bar{p}^{-2}\Big\{\frac{1}{2}[\partial\alpha_o/\partial P_b)^2(\partial/\partial\varepsilon b)^2 V_0 + (\partial\alpha_o/\partial\varepsilon b)^2\, g^{\beta}(\alpha_o,\varepsilon b)] \\ &\quad + (\partial\alpha_o/\partial P_b)^2(\partial V_o/\partial\varepsilon b)(\partial/\partial\varepsilon b)\,\ell n[\bar{p}(\partial\alpha_o/\partial P_b)] \\ &\quad - (\partial\alpha_o/\partial P_b)(\partial\alpha_o/\partial\varepsilon b)[V'(\partial/\partial\varepsilon b)\,\ell n[\bar{p}(\partial\alpha_o/\partial\varepsilon b)] + (d/d\varepsilon b)V']\Big\}, \\ D_4 \;&\equiv \frac{1}{2}\bar{p}^{\,2}(\partial\alpha_o/\partial P_b)^2\Big\{\frac{1}{2}(\partial/\partial\varepsilon b)^2\Omega_o + (\partial\Omega_o/\partial\varepsilon b)(\partial/\partial\varepsilon b)\,\ell n[\bar{p}(\partial\alpha_o/\partial P_b)]\Big\}, \end{aligned}$$

where $$(d/d\varepsilon b) \equiv (\partial/\partial\varepsilon b) + (\partial\alpha_o/\partial\varepsilon b)(\partial\alpha_o).$$

From Eq. (II-86) we see that

$$\bar{H}_{20} = (D_2\, J + D_4\, J^2). \qquad \text{(II-87)}$$

The final term in Eq. (II-43), $-\check{H}_{ob}W_1 = \check{W}_{1b}\, H_0$, gives a contribution to H_{20}^1 of

$$\check{W}_{10b}\, H_{00} \equiv (\partial W_{10}/\partial \varepsilon b)(\partial H_{00}/\partial P_b) - (\partial W_{10}/\partial P_b)(\partial H_{00}/\partial \varepsilon b).$$

From Eqs. (II-74) and (II-84), one sees that this term is proportional to $\sin\Theta$, and so its contribution $\overline{\check{W}_{10b}H_{00}}$ to $K_{20}=H_{20}{}^2$, is zero. From Eqs. (II-43), (II-44), (II-85) and (II-87), therefore, we obtain K_{20}:

$$\begin{aligned} K_{20} &= H_{20}{}^2 \equiv \overline{H_{20}{}^1} \\ &= (D_2\, J + D_4 J^2) - \Omega_o{}^{-1}(C_{11} + C_{13}J)(C_{11} + 3C_{13}J)/4. \end{aligned} \qquad \text{(II-88)}$$

Assembling the various contributions from Eqs. (II-82), (II-83), and (II-88), then, we write out the guiding center Hamiltonian K, valid to $\mathcal{O}(\varepsilon^2, \varepsilon\eta, \eta^2)$:

$$\begin{aligned} K &= K_{oo} + \eta^2 K_{o2} + K_{20} + \text{h.o.} \\ &= (V_o + \Omega_o J) + \Omega_o \bar{a}^2 J^2 [d_o - (3/8)(c_3{}^2 + c_1{}^2)]/8 \\ &\quad + \varepsilon^2[-\Omega_o^{-1} C_{11}{}^2/4 + (D_2 - \Omega_o^{-1} C_{11}C_{13})J + (D_4 - 3\Omega_o^{-1} C_{13}{}^2/4J^2]. \end{aligned} \qquad \text{(II-89)}$$

The term K_{00} is the guiding center Hamiltonian, valid to $\mathcal{O}(\varepsilon, \varepsilon\eta, \eta)$. It is formally the same as the lowest order Hamiltonian usually used. As already mentioned in Sec. II.F, however, the gyrofrequency Ω_o is more general than its $L_s{}^{-1} \to 0$ limit. We shall evaluate this more general form of Ω_o more explicitly in Sec. II.K.

This lowest order term in K yields the expressions commonly used for the particle drifts, including the drift in α_o which produces orbits with finite banana width, as described in Sec. II.C. The higher order terms $\eta^2 K_{o2}$ and $\varepsilon^2 K_{20}$ give corrections to these drifts. (We note that the part of this slow shift of $\alpha_o(\lambda)$ which is due to a change (with β) of the

contribution $e\Phi$ to V corresponds physically to a particle drift in the direction of $\underline{E}$, due to the particle seeing a time-varying $\underline{E}$-field, i.e., to a polarization drift.)

The term $\eta^2 K_{02}$ gives the lowest order nonzero corrections to K from the anharmonicity of the potential well V, or from the α dependence of g^{α}. It is thus a simple finite gyroradius (finite η) correction, present even in the slab limit $\varepsilon \to 0$, in which the parameter λ is truly frozen.

The term $\varepsilon^2 K_{20}$ comes from the fact that λ is <u>not</u> frozen, but has a jitter at the gyrofrequency. The contribution $\varepsilon^2 \partial K_{20}/\partial J$ of this term to the gyrofrequency Ω is the mathematical expression of the modification of Ω due to the jitter in the potential well $V(\alpha)$, described qualitatively at the end of Sec. II.C. The contribution of the term $\varepsilon^2 K_{20}$ to the particle drifts in b, P_b [and so in $\alpha_o(\lambda)$], is analogous to the pondermotive force F_p for a particle jittering in an electric field $\varepsilon E(x,t)$ $(E \sim 1)$. F_p arises from a nonzero time-averaged acceleration at second order in ε. Solving the force equation $\ddot{x} = \varepsilon E\ (x,t)$ iteratively, one finds a first order jitter in x(t) by freezing x at $x^{(o)}$ on the right hand side: $\ddot{x}^{(1)} = \varepsilon E(x^{(o)},t)$. Integrating this to obtain $x^{(1)}(t) = x^{(o)} + \varepsilon\delta x^{(1)}$, the time average of the next iteration gives F_p:

$$\overline{\ddot{x}^{(2)}} = \varepsilon \overline{E\ (x^{(o)} + \varepsilon\delta x^{(1)},t)} \simeq \varepsilon \overline{E(x^{(o)},t)} + \varepsilon^2 \overline{\delta x^{(1)}\ \partial E(x^{(o)},t)/\partial x}.$$

Analogously, one may calculate the frozen $-\lambda$ gyroorbit $\alpha^{(o)}(t)$ (for which η is arbitrary, but ε is effectively set equal to zero), and from this calculate the first order jitter $\varepsilon\delta\beta^{(1)}$, $\varepsilon\delta p_\beta{}^{(1)}$ in β and p_β from

Hamilton's equations for $\dot{\beta}$ and $\dot{p}_\beta$, e.g., $\dot{\beta}^{(1)} = (\partial/\partial p_\beta)H(\alpha^{(o)}, p_\alpha{}^{(o)}; \varepsilon\beta^{(o)}, p_\beta{}^{(o)})$. Solving for $\varepsilon\delta\beta^{(1)}(t)$ and taking the time average of the next iteration again gives a nonzero contribution of order ε^2.

The present canonical form for K could be readily employed numerically to find guiding center orbits. Specifying the drift surface α_b near which the guiding center trajectory is to move determines P_ϕ, as explained at the end of Sec. II.C. Further specifying the initial guiding center position $[\alpha_o(t_o), b(t_o)]$ then fixes the value of $P_b(t_o)$, through solving $\alpha_o(t_o) = \alpha_o[\varepsilon b(t_o), P_b(t_o), P_\phi]$. Finally, specifying the value of the action, Hamilton's equations may be integrated in a straightforward manner. The trajectory in terms of the physical variables (α_o, b) may then be easily computed from the canonical variables (b, P_b, P_ϕ) using $\alpha_o(\varepsilon b, P_b, P_\phi)$. This is the only additional step required by the canonical formulation beyond those needed in numerically applying the non-canonical formulation employed by Northrop and Rome;[8] while their heuristically determined expression for P_ϕ in terms of the physical guiding center variables is found not to be conserved, the time invariance of P_ϕ in this canonical formulation is automatic.

II.J. Generalized Adiabatic Invariant

In the previous section we obtained the guiding center Hamiltonian K, to $O(\varepsilon^2, \varepsilon\eta, \eta^2)$, expressed in terms of the gyroaction J. Since it is the constancy of J which makes K useful, reducing the number of nontrivial degrees of freedom by one, it is important to be able to express J in terms of more directly physical variables. This expression is the

generalization of the expression $J = (mv_\perp^2/2\Omega_c) \equiv (mc/e)\mu$ usually used, to the finite ε, finite η regime, and allowing for shear in the magnetic field.

We shall find J valid to $O(\varepsilon,\eta^2)$, ignoring terms of order $\varepsilon\eta$, ε^2, and higher. As for the calculation of K, the procedure to higher order is the same, but grows rapidly more complicated.

We assume knowledge of the coordinate system $g^\mu(\underline{x}) \equiv |\partial q^\mu/\partial \underline{x}|^2$, of the potentials $\underline{A}$ and Φ which give the magnetic and electric fields, and of the particle position $\underline{x}$ and velocity $\underline{v} \equiv \dot{\underline{x}}$ at any instant t that we want to compute J. Given the physical specification $\underline{x},\underline{v}$ of the particle, the values of the variables $\underline{z} \equiv (q^\mu, p_\nu)$ in terms of which our initial Hamiltonian H are directly determined, through $q^\mu(\underline{x})$, and $p_\nu = (g^\nu)^{-1}\dot{q}_\nu + eA_\nu$. We therefore want to express J in terms of $\underline{z}$.

We begin by writing out the particle energy $E = H(\underline{z})$ in terms of J. From Eq. (II-89), to the order we are working, E is given by

$$\begin{aligned} E = K &= K_{oo} + \eta^2 K_{o2} + \text{h.o.} \\ &= V_o + \Omega_o J + \Omega_2 J^2 + \text{h.o.}, \end{aligned} \tag{II-90}$$

where V_o, Ω_o, and $\Omega_2 \equiv \Omega_o\, \bar{a}^2\, [d_o - (3/8)(c_3^2 + c_1^2)]/8$

are functions of $\lambda^N \equiv (\varepsilon b, P_b)$. To the order we are working, $\underline{Z} = \underline{Z}^1$, and so in particular we take $\varepsilon b = \varepsilon b^1, P_b = P_b^1$, and we want to express J in terms of $\lambda^o \equiv (\varepsilon\beta, p_\beta)$. We thus need to compute the differences between these parameters:

$$\begin{aligned} P_b - p_\beta &= (P_b - P_b^o) + (P_b^o - p_\beta) = -\,\varepsilon[\check{W}_{10} P_b^o + \delta p_{\beta o}] + \text{h.o.} \\ &= -\,\varepsilon[W_{10}/\partial\varepsilon\beta - (\partial\alpha_o/\partial\varepsilon\beta)p_\alpha] + O(\varepsilon^2, \varepsilon\eta). \end{aligned}$$

Using Eq. (II-84), and the relation $J^{1/2}\sin\Theta = \delta\alpha/\bar{a} + \mathcal{O}(\varepsilon,\eta)$ [where $\delta\alpha \equiv \alpha - \alpha_o(\lambda^o)$], this gives

$$P_b - p_\beta = \varepsilon\Big\{[J_o(\partial/\partial\varepsilon\beta)(C_{13}/\Omega_o) + (\partial/\partial\varepsilon\beta)(C_{11}/\Omega_o)](\delta\alpha/\bar{a}) + (\partial\alpha_o/\partial\varepsilon\beta)\,p_\alpha\Big\} + \mathcal{O}(\varepsilon^2,\varepsilon\eta). \tag{II-91}$$

Here $J_o(E,\lambda) \equiv E_\perp/\Omega_o \equiv [E - V_c(\lambda)]/\Omega_o(\lambda)$ is the $\mathcal{O}(\varepsilon^o,\eta^o)$ value of the adiabatic invariant, somewhat generalized over the expression $mv_\perp^2/2\Omega_c$ to allow for finite $(v_\parallel/\Omega_c L_s)$ or $(v_E/\Omega_c L_B)$ effects which enter, even in the zero η limit (c.f. Sec. II.K). $E_\perp$ is the kinetic energy in the α direction which the particle would have at its guiding center position $\alpha_o(\lambda)$, if λ were frozen at its instantaneous value.

Similarly, for the β coordinate we get

$$\varepsilon(b-\beta) = -\varepsilon\delta\beta_o + \mathcal{O}(\varepsilon^2,\varepsilon\eta) = -\varepsilon(\partial\alpha_o/\partial p\quad p_\alpha + \mathcal{O}(\varepsilon^2,\varepsilon\eta). \tag{II-92}$$

Now we invert Eq. (II-90):

$$\begin{aligned} J = J(E,\lambda^N) &\equiv E_\perp/\Omega_o - \Omega_o E_\perp^2/\Omega_o^3 + \text{h.o.} \\ &= J_o(E,\lambda^N) - (\Omega_2/\Omega_o)\,J_o^2 + \text{h.o.} \end{aligned} \tag{II-93}$$

The term in Ω_2 is already $\mathcal{O}(\eta^2)$, and so can be evaluated at $\lambda = \lambda^o$. The term $J_o(E,\lambda^N)$ we expand to $\mathcal{O}(\varepsilon)$ in $\lambda^N - \lambda^o$, using Eqs. (II-91) and (II-92). This gives, finally,

$$\begin{aligned} J &= J(E,\lambda^o) + \varepsilon(b-\beta)(\partial J^o/\partial\varepsilon\beta) + (P_b - p_\beta)(\partial J_o/\partial p_\beta) + \text{h.o.} \\ &= J_o(E,\lambda^o) - (\Omega_2/\Omega_o)J_o^2 \\ &+ \varepsilon\Big\{p_\alpha[(\partial\alpha_o/\partial\varepsilon\beta)(\partial J_o/\partial p_\beta) - (\partial\alpha_o/\partial p_\beta)(\partial J_o/\partial\varepsilon\beta)] \\ &+ (\delta\alpha/\bar{a})[J_o(\partial/\partial\varepsilon\beta)(C_{13}/\Omega_o) + (\partial/\partial\varepsilon\beta)(C_{11}/\Omega_o)]\Big\} \\ &+ \mathcal{O}(\varepsilon^2,\ \varepsilon\eta,\ \eta^3). \end{aligned} \tag{II-94}$$

All parameters here are evaluated at $\lambda = \lambda^o$ and $\alpha_o(\lambda^o)$, i.e., at the true particle position. The term in Ω_2 is the lowest order finite-η correction which is nonzero. It is of order η^2. The term in ε enters to compensate for the jitter in $J_o(E,\lambda^o)$ coming from its dependence on λ^o, which has a corresponding $O(\varepsilon)$ jitter.

The λ dependence of J here enters both explicitly, and through the function $\alpha_o(\lambda)$. Until now we have been assuming α_o is a known function. We can more explicitly express α_o, defining

$$\delta\alpha \equiv \alpha - \alpha_o(\lambda) \equiv \delta\alpha_o + \eta\delta\alpha_1 + \eta^2\delta\alpha_2 + \cdots,$$

and solving for the successive $\delta\alpha_m$ by expanding the α-center condition (II-47) about α:

$$\begin{aligned} 0 = V_1(\alpha_o,\lambda) &= V_1(\alpha,\lambda) - \delta\alpha(\partial/\partial\alpha)V_1(\alpha,\lambda) + \frac{1}{2}(\delta\alpha)^2(\partial/\partial\alpha)^2 V_1 + \cdots \\ &= V_1(\alpha,\lambda) - 2\delta\alpha V_2(\alpha,\lambda) + 3(\delta\alpha)^2 V_3(\alpha,\lambda) - \cdots. \end{aligned} \tag{II-95}$$

Substituting the η expansion of $\delta\alpha$ into (II-95) and collecting like powers in η yields

$$\begin{aligned} \delta\alpha_o &= (V_1/2V_2),\ \eta\delta\alpha_1 = (3V_3/2V_2)(\delta\alpha_o)^2, \\ \eta^2\delta\alpha_2 &= [2(3V_3/2V_2)^2 - (4V_4/2V_2)](\delta\alpha_o)^3, \end{aligned} \tag{II-96}$$

and so forth. The V_m here are evaulated at α,λ. Inserting (II-96) into $\alpha_o = \alpha - \delta\alpha(\alpha,\lambda^o)$, expression (II-94) for J is expressed purely in terms of quantities defined at the particle position. To more explicitly evaluate J, one must assume specific forms for the metric coefficients g^μ and the potentials $\underline{A}$ and Φ. In the following section, we assume specific simple forms for these, namely forms appropriate to the slab geometry of

Parts I and III, and apply the results of Part II, to prepare for Part III.

II.K. Application: Slab Geometry with Shear

The stability work of Part III begins with the conductivity kernel of a slab plasma, with arbitrary inhomogeneity in the inhomogeneity direction x. This kernel requires a knowledge of the particle orbits and drift frequencies in such a magnetic geometry. In preparation for Part III, in this sectiion we apply the results thus far obtained in the present portion of this work, to a slab geometry, allowing for shear and variation in the magnetic field strength ($\rho/L_{\underline{B}} \neq 0$). As part of this application, we shall be able to specialize the results of Sec. II.J, obtaining a more explicit expression for the gyroaction in the presence of strong perpendicular magnetic inhomogeneity, and, in particular, in the presence of magnetic shear.

The appropriate Hamiltonian is that of Part I [Eq. (I-4)], which is in terms of Cartesian coordinates, for which $g^{\mu\nu}$ is the identity matrix $g^{\mu\nu} = \delta^{\mu\nu}$. This is not the same as the $\varepsilon \to 0$ limit of the Hamiltonian of Part II, for which the results of this part were developed. (In the limit $\varepsilon \to 0$, H in Eq. (II-5) is the Hamiltonian of a particle in a system with cylindrical symmetry.) However it is one of the advantages of the canonical formulation we are using that formulae in different geometries tend to look the same. Everything developed in the present part of this work is also valid for the Cartesian slab geometries of Parts I and III.

The adoption of a slab geometry, i.e., taking only the ε^o terms in the expressions obtained thus far, is an extreme limit of the theory, but a limit with relevance to the Tormac sheath, where one may well have $\eta^2 >> \varepsilon$.

II.K.1 Specialization to Slab Geometry

Going back to the notation of Part I, we have the sole inhomogeneity direction x, corresponding to the coordinate α of Part II, the y direction corresponding to β, and the z direction corresponding to ϕ. The expansion coefficients g_m are now all zero, except for $g_o = 1$. We consider a given particle, with guiding center $X_o(\underline{P})$, and go into the Cartesian coordinate system natural to the magnetic field at X_o, evaluating the x-dependent system $[\hat{b}_1, \hat{b}_2(x), \hat{b}_3(x)]$ of Part I at X_o. We recall the definitions of this right handed triad: $\hat{b}_1 \equiv \hat{x}$, $\hat{b}_2 \equiv \hat{b}_2(X_o) \equiv \hat{b}_3 \times \hat{b}_1$, $\hat{b}_3 \equiv \hat{b}_3(X_o) \equiv \hat{B}(X_o)$, the magnetic field direction at X_o. The corresponding coordiates we denote by (x_1, x_2, x_3).

We expand the potentials $\underline{A} = (o, A_y, A_z)$ and Φ about $x \approx X_o$:

$$\delta \underline{A}(\delta x, X_o) \equiv \underline{A}(x) - \underline{A}(X_o) = \sum_{n=1}^{\infty} (\delta x)^n (\partial/\partial X_o)^n A_o/n!$$
$$= [\sum_{n=0}^{\infty} (\delta x)^{n+1} (\partial/\partial X_o)^n \underline{B}_o/(n+1)!] \times \hat{x} \qquad \text{(II-99)}$$

$$\delta \Phi(\delta x, X_o) \equiv \Phi(x) - \Phi(X_o)$$
$$= - \sum_{n=0}^{\infty} (\delta x)^{n+1} (\partial/\partial X_o)^n E_o/(n+1)! , \qquad \text{(II-100)}$$

where $\underline{A}_o \equiv \underline{A}(X_o)$, $\underline{B}_o \equiv \underline{B}(X_o)$, $E_o \equiv E(X_o)$. Defining $\underline{v}_o \equiv \underline{P} - e\underline{A}_0 \equiv v_{02}\hat{b}_2 + v_{03}\hat{b}_3$,

we express the effective potential V of Eq. (I-5) as

$$V(x|\underline{P}) = V(X_o|\underline{P}) + \frac{1}{2} e^2|\delta \underline{A}|^2 - e\,\delta\underline{A}\cdot\underline{v}_o + e\,\delta\phi$$
$$\equiv \sum_{m=0}^{\infty} V_m(X_o, \underline{P})(\delta x)^m. \tag{II-101}$$

Using (II-99) and (II-100), we can read off the V_m s. Proceeding up to $m = 4$, we find

$$V_o \equiv V(X_o|\underline{P}) = \frac{1}{2}\underline{V}_o^{\,2} + e\,\phi_o,$$
$$V_1 \equiv -e(\underline{v}_o \times \underline{B}_o)\cdot\hat{x} - eE_o = -e(v_{o2}B_o + E_o) = 0$$
$$V_2 \equiv \frac{1}{2}[e^2B_o^{\,2} - e\,\underline{v}_o \times (\partial\underline{B}_o/\partial X_o)\cdot\hat{x} - e(\partial E_o/\partial X_o)],$$
$$V_3 \equiv \frac{1}{2} e^2\,\underline{B}_o\cdot(\partial\underline{B}_o/\partial X_o) - (e/3!)[\underline{v}_o\times(\partial^2\underline{B}_o/\partial X_o^{\,2})\cdot\hat{x} + \partial^2E_o/\partial X_o^{\,2}], \tag{II-102}$$
$$V_4 \equiv e[\underline{B}_o\cdot(\partial^2\underline{B}_o/\partial X_o^{\,2})/3! + (\partial\underline{B}_o/\partial X_o)^2/8]$$
$$-(e^2/4!)[\underline{v}_o \times (\partial^3\,\underline{B}_o/\partial X_o^{\,3})\cdot\hat{x} + \partial^3E_o/\partial X_o^{\,3}].$$

Solving the second of Eqs. (II-102) for v_{o2} gives

$$v_{o2} = -\,cE_o/B_o \equiv v_E \tag{II-103}$$

where we have reinstated the c. To $\mathcal{O}(\eta)$ this velocity, $v_{o2} \equiv \dot{x}_2(X_o)$, is the same as the guiding center drift $\overline{\dot{x}_2}$ in the $\hat{b}_2$ direction, which is therefore the usual $E\times B$ drift v_E, in this small η limit.

For convenience we define

$$\hat{b}_\pm(x) \equiv \frac{1}{2}[\pm i\hat{b}_2(x) + \hat{b}_3(x)] \equiv \frac{1}{2}[\pm i\hat{y} \quad \text{[illegible]}] \exp - i\Theta_S(x), \tag{II-104}$$

and $\quad B(x) \equiv |B(x)| \equiv \overline{B}\exp\Theta_B(x).$

Here $\overline{B}$ is a constant, and $\kappa_{B,S}(x) \equiv L_{B,S}^{-1}(x) \equiv d\Theta_{B,S}/dx$. Using these definitions and $\kappa \equiv \kappa_B - i\,\kappa_S$, we have

$$\underline{B}(x) = B(x)(\hat{b}_+ + \hat{b}_-) = B(x)(\hat{b}_+ + c.c.), \qquad \text{(II-105)}$$

$$\partial B/\partial x = B(x)(\kappa\, \hat{b}_+ + c.c.) = B(x)(\hat{b}_3\kappa_B + \hat{b}_2\kappa_S). \qquad \text{(II-106)}$$

If we in addition neglect all derivatives of $\kappa_{B,S}(x)$, we have in general

$$(\partial/\partial x)^n\ \underline{B} = B(x)[\kappa^n\ \hat{b}_+ + c.c.]. \qquad \text{(II-107)}$$

Using Eqs. (II-106), (II-103) in the third of expressions (II-102) gives

$$V_2 = \tfrac{1}{2}[e^2 B_o{}^2 - e(v_{o2}\kappa_B - v_{o3}\kappa_S)\ B_o - e(\partial E_o/\partial X_o)] \qquad \text{(II-108)}$$

$$= \tfrac{1}{2}[e^2 B_o{}^2 + e\kappa_B E_o + e\kappa_S\, v_{o3} B_o - e(\partial E_o/\partial X_o)],$$

and therefore (restoring the m and c),

$$\Omega_o{}^2 \equiv 2g_o\ V_2 = \Omega_c{}^2 + \kappa_S v_{o3}\Omega_c + (\kappa_E - \kappa_B)\ v_E\Omega_c, \qquad \text{(II-109)}$$

where $\kappa_E \equiv d\ \ell n\ E_o/dx$.

We see that the gyrofrequency Ω is different from its homogeneous-$\underline{B}$ value Ω_c, even in the zero gyroradius ($\eta \to 0$) limit, for finite $(\kappa_S v_{o3}/\Omega_c)$ or $(\kappa_B - \kappa_E)v_E/\Omega_c$, as mentioned previously. For substantial $\kappa_S v_{o3}$, the particle experiences a restoring force $\sim -\ v_{o3}B_2(x)$ in the x direction, in addition to the $v_2 B_3$ force which gives the usual Ω_c part of Ω_o. Since $B_2 \sim B_o(\delta x/L_S) \sim \delta x$, this additional force senses finite shear, even in the $\eta \to 0$ limit. Similarly, for substantial $\kappa_B v_E$, the particle senses a restoring force $\sim v_2 \delta B_3(x) \simeq v_E B_o(\delta x/L_B)$ in addition to the force giving Ω_c. Again, this force is linear in δx, and so renormalizes Ω_o.

The shifts in Ω_o due to finite $(\kappa_S\ v_{o3}/\Omega_c)$[14] and $(\kappa_E v_E/\Omega_c)$[15] have been noted by other workers. We can rewrite the last term in Eq. (II-109) as $(\kappa_E-\kappa_B)v_E\Omega_c = \Omega_c(\partial v_E/\partial x)$. This form arose in Ref. 15 for the special case $\kappa_B = 0$.

II.K.2. Particle Motion

In preparation for Part III, we solve the particle motion problem to $\mathcal{O}(\eta)$. Since $\varepsilon = 0$, the pseudomotion problem of Sec. II.F is the full problem, i.e., $K(J,\lambda) = E(J,\lambda) = h^{\infty}(J,\lambda)$, in the terminology of Eq. (II-59). To $\mathcal{O}(\varepsilon^o,\eta^1)$, then, we have

$$K(J,\lambda) = K_{oo}(J,\lambda) = V_o + \Omega_o J. \qquad \text{(II-110)}$$

The x motion has already been solved in Sec. II.F. Replacing $\delta\alpha$ by δx, Eqs. (II-64), (II-65) give us

$$\delta x_o = \bar{x}_o \sin\Theta \qquad \text{(II-111)}$$

$$\eta\delta x_1 = -\ \bar{x}_o^{\,2}\ (V_3/4V_2)(3+\cos 2\Theta), \qquad \text{(II-112)}$$

where $\bar{x}_o \equiv \bar{a}\ J^{1/2}$ is the $\mathcal{O}(\eta^o)$ approximation to the particle gyroradius, and $\bar{a} \equiv (\Omega_o/V_2)^{1/2}$. Explicit expressions for V_2 and Ω_o are given in Eqs. (II-108), (II-109). Using (II-107) in Eqs. (II-102), we obtain a simpler expression for V_3:

$$\begin{aligned} V_3 &= \tfrac{1}{2}\,\kappa_B(eB_o)^2 - (eB_o 3!)[(\kappa_B^{\,2}-\kappa_S^{\,2})v_{o2} - 2\kappa_B\kappa_S\ v_{o3}] \\ &= m\left\{\kappa_B\ \Omega_c^{\,2}/2 + [(\kappa_S^{\,2}-\kappa_B^{\,2})v_E + 2\kappa_B\kappa_S\ v_{o3}]\Omega_c/3!\right\}. \end{aligned} \qquad \text{(II-113)}$$

Here, and henceforth, we neglect all derivatives of E(x). In the second line of (II-113), we have restored the m and c. Similarly, we may write

V_4 as

$$V_4 = (7\,\kappa_B^2-\kappa_S^2)(eB_o)^2/24 - (eB_o/4!)[(\kappa_B^3-3\kappa_B\kappa_S^2)\,v_{o2}-(3\kappa_B^2\kappa_S-\kappa_S^3)\,v_{o3}] \tag{II-114}$$

$$= m\left\{7\kappa_B^2-\kappa_S^2)\Omega_c^2 - [(3\kappa_B\kappa_S^2-\kappa_B^3)v_E + (3\kappa_B^2\kappa_S-\kappa_S^3)v_{o3}]\Omega_c\right\}/4!.$$

Higher order V_m's can be readily obtained, if desired.

Equations (II-111) and (II-112), along with the fact that $\Omega = \partial K/\partial J = \Omega_o + O(\eta^2)$, give a complete description of the x motion. The x_2 and x_3 motion then comes from a simple time integration of Hamilton's equations,

$$\dot{x}_2 = \left\{P - e\underline{A}[x(t)]\right\}\cdot\hat{b}_2, \quad \dot{x}_3 = \left\{P - eA[x(t)]\right\}\cdot\hat{b}_3. \tag{II-115}$$

Using $x = X_o + \delta x_o + \eta\delta x_1$, and the expansion (II-99) for $\delta\underline{A}$, we find

$$x_2(\Theta) = X_{20} + \delta x_{20} + \eta\delta x_{21},\; x_3(\Theta) = X_{30} + \eta\delta x_{31}, \tag{II-116}$$

where the guiding center coordinates X_{20}, X_{30} evolve linearly in time (or Θ), at rates

$$\dot{X}_{20} = v_E + (\bar{x}_o^2\Omega_c/4)(3\,V_3/V_2 - \kappa_B) \equiv v_E + v_B, \tag{II-117}$$

$$\dot{X}_{30} = v_{o3} + \kappa_S\,(\bar{x}_o^2\,\Omega_c/4), \tag{II-118}$$

and the gyromotion $\delta x_{1,2,3}\,(\Theta,J)$ about this guiding center, to $O(\eta)$, is given by Eqs. (II-111) (II-112), and:

$$\delta x_{20} = \bar{x}_{20}\cos\Theta, \qquad \bar{x}_{20} \equiv (\Omega_c/\Omega_o)\bar{x}_o \tag{II-119}$$

$$\eta\delta x_{21} = (\bar{x}_o^2\,\Omega_c/8\,\Omega_o)(3\,V_3/V_2 + \kappa_B)\,\sin 2\Theta \tag{II-120}$$

$$\eta\delta x_{31} = -\,\kappa_S(\bar{x}_o^2\,\Omega_c/8\,\Omega_o)\,\sin 2\Theta. \tag{II-121}$$

This completes the mathematical description of the particle motion, to $O(\eta)$. Several aspects of the motion should be noted. First, already mentioned, is the dependence of the gyrofrequency on $v_{03} \equiv v_{\parallel}(X_0)$, for finite shear. As we shall see in Part III, this dispersion in Ω_0 has a significant effect on the ion Bernstein modes, and the instabilities associated with them. The other two "frequencies" (i.e., time derivatives $\dot{\underline{\theta}} \equiv \dot{\underline{\theta}}$ of the canonical coordinates $\underline{\theta}$) in the problem, $\dot{X}_{20}$ and $\dot{X}_{30}$, are also modified by shear. In the limits of $\kappa \to 0$ and negligible $\kappa_B v_E/\Omega_c$, one has $V_3/V_2 = \kappa_B$, and $\bar{x}_0^{\,2} = (v_\perp^{\,2}/2\Omega_c)$, and so Eq. (II-117) reduces to $v_B \equiv \dot{X}_{20} - v_E = (\kappa_B v_\perp^{\,2}/2\Omega_c)$, the usual $\underline{B} \times \underline{\nabla} B$ drift. From Eq. (II-118) we see that shear also modifies the particle parallel velocity, a fact noted previously in the literature.[16]

We also note modifications in the gyromotion about the guiding center, induced by shear. Comparing Eqs. (II-111) and (II-119), one sees that, even at $O(\eta^0)$, finite shear distorts the gyroorbit from a circular to an elliptical shape. At $O(\eta)$, the corrections $\eta\delta x_{i1}$ $(i = 1,2,3)$ further distort the orbit. In particular, $\eta\delta x_{31}$ causes the orbit to twist in the direction of magnetic shear, turning the elliptical orbit into a "potato-chip" form.

The canonical framework and graphical techniques of Part I enable one to understand qualitatively many of these shear-induced modifications. For example, the correction to $\dot{X}_{30}$ in Eq. (II-118) may be understood from referring to Fig. 2. We recall that $v_2\hat{b}_2 + v_3\hat{b}_3 = \underline{P} - e\underline{A}(x)$. As x oscillates about X_0, $\underline{A}(x)$ oscillates on the $\underline{A}$ trajectory about $\underline{A}(X_0)$. For a particle with positive parallel velocity, such as at point c in Fig. 2, the shear causes the trajectory $\underline{A}(x)$ to bend away from point

P as x moves in either direction from X_o, thereby causing the parallel velocity v_3 to have a positive addition to its value v_{o3} at X_o. Conversely, the curvature of A toward point b causes $|\dot{X}_{30}|$ to be smaller, for particles with negative parallel velocity. In a similar way, one sees that the curvature V_2 of the potential well $V = \frac{1}{2}(P - eA)^2$ is enhanced by shear for a particle with momenta P at point c, and lessened for P at point b. This explains the finite κ_S correction to Ω_o in Eq. (II-109).

II.K.3 Adiabatic Invariant

We now specialize the results of Sec. II.J to the present geometry, calculating J to $\mathcal{O}(\eta)$. Defining $v_{o1} \equiv p_x{}^2(X_o)$ $(m = 1)$, we have $E = \frac{1}{2}(v_{o1}{}^2 + v_{o2}{}^2 + v_{o3}{}^2) + e\Phi(X_o)$, and so $E_\perp \equiv E - V_o = \frac{1}{2} v_{o1}{}^2$, the kinetic energy in the x direction at the guiding center. Equation (II-94) the tells us that $(x \equiv \alpha,\ X_o \equiv \alpha_o)$

$$\begin{aligned} J &= J_o(E, X_o, \lambda) + \mathcal{O}(\eta^2) \\ &= m v_{o1}{}^2/2\,\Omega_o(X_o) + \mathcal{O}(\eta^2). \end{aligned} \tag{II-122}$$

To $\mathcal{O}(\eta^o)$ we may replace $\Omega_o(X_o)$ by $\Omega_o(x)$ and (II-122) evidently reduces to the usual expression $mv_\perp{}^2/2\Omega_c$, valid in the limits of negligible $(\kappa_S v_{o3}/\Omega_c)$ and $(\kappa_B v_{o2}/\Omega_c)$. Even to $\mathcal{O}(\eta^o)$, however, we see that these parameters modify J.

We note that, expressed in terms of $J_o(E, X_o, \lambda)$, the form of J at $\mathcal{O}(\eta)$ is the same as that for $\mathcal{O}(\eta^o)$. If we express X_o as $x - \delta x(x,\lambda)$, however, the first order form is different from the one valid at zero order. Consistent with the ordering described in Appendix B, we have

$$V_o(X_o) = V_o(x) - \delta x V_1(x) + (\delta x)^2 V_2(x) - (\delta x)^3 V_3(x) + O(\eta^2)$$
$$= [V_o - (\delta x_o)^2 V_2] - (\delta x_o)^3\, 3V_3 + O(\eta^2), \qquad \text{(II-123)}$$

where the V_m on the second line are evaluated at x, and $\delta x_o \equiv V_1/2\, V_2$, from the first of Eqs. (II-96).

In addition, for Ω_o we have

$$\Omega_o(X_o) = \Omega_o(x) - \delta x(\partial/\partial x)\, \Omega_o(x) + O(\eta^2)$$
$$= \Omega_o(x)\, [1 - (\delta x_o)(3V_3/2V_2] + O(\eta^2). \qquad \text{(II-124)}$$

Putting (II-123) and (II-124) into (II-122), we obtain an expression for J entirely in terms of the quantities E, V_o, V_1, V_2, V_3, all evaluated at x:

$$J = \Omega_o^{-1}(X_o)[E - V_o(X_o)]$$
$$= \Omega_o^{-1}(x)[E - V_o(x) + (\delta x_o)^2 V_2 + 3(\delta x_o)^3 V_3][1 - (\delta x_o)(3V_3/2V_2)]^{-1} + O(\eta^2). \qquad \text{(II-125)}$$

At $O(\eta^o)$, this reduces to

$$J = \Omega_o^{-1}(x)\, [E - V_o(x) + (\delta x_o)^2\, V_2] + O(\eta)$$
$$= \frac{1}{2}\, [p_x^{\,2} + (\dot{p}_x/\Omega_o)^2]/\Omega_o(x) + O(\eta), \qquad \text{(II-126)}$$

where we have used $V_o(x) \equiv V(x|\lambda) = E - \frac{1}{2}\, p_x^{\,2}$, $V_1(x) \equiv (\partial/\partial x)\, V(x|\lambda) = -\, \dot{p}_x$, and $V_2(x) = \frac{1}{2}\, \Omega_o^{\,2}(x)$. In the limit of negligible v_E/v_i , $\dot{p}_x = v_2\Omega_c(x)$ (where $v_i \equiv b_i(x)\cdot\underline{v}$ is the particle velocity in the i direction of the coordinate system natural to position x), and, using $p_x \equiv v_1$, Eq. (II-126) may be written

$$J = \frac{1}{2}\, [v_1^{\,2} + v_2^{\,2}(\Omega_c/\Omega_o)^2]/\,\Omega_o + O(\eta), \qquad \text{(II-127)}$$

where again, all quantities are evaluated at x. We note the "elliptical distortion" of this form from the unsheared circular limit in which $v_\perp^2 = v_1^2 + v_2^2$ appears, parallel to the distortion in Eqs. (II-111), (II-119) of the $O(\eta^0)$ gyroorbit.

The higher order expression for J, Eq. (II-125), involves $V_3(x)$, which is given in Eqs. (II-102) or (II-113). The calculation of J to still higher orders may be analogously carried out, bringing in higher-m V_m, and correspondingly higher order derivatives of $\underline{B}(x)$.

II.L. Directions for Extension of the Formalism

In the development of the formalism of the present part of this thesis, we have made a number of simplifying assumptions. These were useful in developing the formal procedures, but now that the procedures have been established, the assumptions may be weakened, and the range of applicability of the theory may thereby be enlarged.

One assumption implicity made throughout is that resonance effects between the gyromotion and bounce motion are negligible. Thus, one removes the dependence of H on the gyrophase Θ to all orders in ε ending finally with a Θ-independent Hamiltonian K, for which no resonance effects are possible.

To study possible resonance effects in the framework developed here, we begin with the Hamiltonian $H^0(\underline{Z}^0)$ of Sec. II.H, the starting point for stage two of the transformation process. Instead of neglecting resonance effects as done in stage two, one may instead first solve for the unperturbed bounce motion, taking $H_0^0(J^0, \varepsilon b^0, P_b^0)$ as the unperturbed Hamiltonian. H_0^0 has one nontrivial degree of freedom, and so the bounce

motion may be solved for exactly. Analogous to the way in which the one-dimensional, frozen-λ problem for the gyromotion was solved, we transform from (θ_b^o, P_b^o) to (θ_b^o, J_b^o), the action-angle variables of the bounce motion. The Hamiltonian $\mathcal{H}^o$ in terms of these variables then has a lowest order term $\mathcal{H}_o^o = H_o^o$ which is independent of both angle variables, $\mathcal{H}_o^o = \mathcal{H}_o^o(J^o, J_b^o)$. The higher order terms $\epsilon^\ell \mathcal{H}_\ell^o$ are in general dependent on both angle variables, and this coupling allows for the study of resonance effects. From this, one obtains the resonance condition $o = n_1\bar{\Omega} + n_2\omega_b$ (where $\bar{\Omega}$ is the bounce averaged gyrofrequency), the condition for violation of the perturbative nature of the transformations made in stage two.

Another assumption we have made is that of perfect axisymmetry. One would like to remove this assumption to study geometries for which the toroidal symmetry is weakly broken, either due to unintentional field imperfections, as in Tokamaks, or due to an intentional design for confinement, as in Elmo Bumpy Torus.

The central difficulty overcome by the formalism we have developed here is inducing the finite canonical transformation S to action angle variables (Θ^o, J^o), which have the property of lying sufficiently close to the desired variables (Θ, J) (in terms of which K is expressed) that the remaining transformation T: $(\Theta^o, J^o) \rightarrow (\Theta, J)$ may be achieved by purely perturbative means. Inducing the perturbative transformations comprising T is a relatively simple and straightforward matter, especially when aided by Lie techniques. In view of this, the procedure developed here may be expected to be applicable to particle motion in <u>any</u> geometry in which the procedure we have used for inducing S again gives action-angle variables which are only perturbatively far away from the true gyroaction J (the adiabatic invariant) and its conjugate angle Θ.

In the present context, one may consider breaking the toroidal symmetry in either of two ways. One may allow any or all of the metric coefficients g^{μ} (as well as ϕ and the components A_{μ} of the vector potential), formerly independent of ϕ, to have either a small additional ϕ-dependent part; $g^{\mu} = g_0^{\mu}(\alpha, \epsilon\beta) + \delta g_1^{\mu}(\alpha, \epsilon\beta, \phi)$, or a slow dependence on ϕ, as well as on β; $g^{\mu} = g^{\mu}(\alpha, \epsilon\beta, \delta\phi)$. (The quantity δ here is the parameter of smallness.)

In the former case, a nonzero perturbation $A_{\alpha 1}$ of A_{α} is permissible; in the latter case it is not, since there would then be no acceptable unperturbed Hamiltonian having the same formal similarity to a one-dimensional particle motion problem as does H in Eq. (II-5). In both cases, one may follow a procedure fully analogous to that described in Sec. II.D to induce the transformation S. The resultant Hamiltonian H^{o} will then have an $O(\epsilon^{o},\delta^{o})$ part independent of Θ^{o}, and Θ^{o}-dependent terms higher order in both ϵ <u>and</u> δ. The remainder of the transformation process is thus again perturbative in character, and so should be executable without great difficulty. The generators of these transformations will now be ϕ dependent. Resonance effects may also be studied as outlined above for the ϕ-dependent case, with a more flexible resonance condition, $0 = n_1 \bar{\Omega} + n_2 \omega_b + n_3\omega_{\phi}$, arising from the new characteristic frequency ω_{ϕ}, describing the time required for the particle to traverse one period in the toroidal perturbation.

Finally, one may relax the assumption of time-translational invariance, i.e., one may allow the g^{μ} and potentials Φ and $\underline{A}$ to have a weak time dependence, in either of the two senses just discussed in connection with breaking of rotational symmetry. In this way one may study, for example, the effects of an electrostatic perturbation $\delta\Phi_1 \sim e^{-i\omega t}$ on

the particle motion. The transformations would then be time dependent, and the resonance condition would take the form

$$\sigma = n_1 \bar{\Omega} + n_2 \omega_b + n_3 \omega_\phi + n_4 \omega.$$

APPENDIX B. ORDERING SCHEME OF PART II

In this Appendix we clarify certain aspects of the ordering scheme we use in comparing terms we find in the expansions performed in Part II.

We define the scale length $\bar{\alpha}_o$ of perpendicular inhomogeneity by

$$\bar{\alpha}_o \equiv \min\,(V_2/V_3,\ g_o/g_1) = \min\,[\partial\ \ell n\ V_2/\partial\alpha_o)^{-1}, (\partial\ \ell n\ g_o/\partial\alpha_o)^{-1}]. \quad \text{(B-1)}$$

The perpendicular expansion parameter η is then given by $\eta \equiv \delta\alpha/\bar{\alpha}_o$, and we assume that

$$V_m\ \delta\alpha/V_{m-1} \lesssim \eta < 1\ \ (m \geq 3),\quad g_m\,\delta\alpha/g_{m-1} \lesssim \eta < 1\ \ (m \geq 1). \quad \text{(B-2)}$$

In contrast to the ordering in (B-2), since V_o is on the order of the particle's parallel energy at the guiding center, and $V_2(\delta\alpha)^2$ is on the order of its perpendicular energy, one has $V_o \sim V_2(\delta\alpha)^2$, for a typical particle. From this ordering, and from the definitions [cf. Secs. II.F, II.G] of the c_m (including c_α, c_p, and c_s), all of these parameters are on the order of $\bar{\alpha}_o^{-1}$, and all the d_m (m = 0,2,4) are on the order of $\bar{\alpha}_o^{-2}$.

While the expansion coefficients $V_m(\alpha_o,\lambda)$ have this slow dependence on α_o, $\partial V_m/\partial\alpha_o \sim V_m/\bar{\alpha}_o$, we note that the potential $V(\alpha|\lambda)$ itself has a stronger α dependence: $\partial V/\partial\alpha \sim V_2\delta\alpha \sim V/\delta\alpha = \eta^{-1}\ V/\bar{\alpha}$.

In addition to an α_o dependence, many of the parameters of the theory (including α_o itself) have a dependence on $\lambda_1 \equiv \varepsilon\beta$(or εb), $\lambda_2 \equiv p_\beta$ (or P_b). We define characteristic scale lengths $\bar{\lambda}_i$(i = 1,2) in the variables λ_i by

$$\bar{\lambda}_i \equiv \min\,[\partial \ln V_2/\partial\lambda_i)^{-1},\ (\partial \ln g_o/\partial\lambda_i)^{-1}], \tag{B-3}$$

in analogy with (B1). We assume that the various parameters f of the theory (e.g., $f = V_m$, g_m, $\delta\alpha$, h_m) all have comparable dependences on the λ_i. In ordering the various terms arising in calculations (as in, e.g., Sec. II.G), we shall thus assume

$$\partial f/\partial\alpha_o \sim f/\bar{\alpha}_o\ , \qquad df/\partial\lambda_i \sim f/\bar{\lambda}_i. \tag{B-4}$$

We now apply this ordering scheme to a particular example, ordering the three terms in Eq. (II-70). We have

$$\begin{aligned}(\text{term 2/term 1}) &= (\partial\delta\alpha/\partial\lambda_i)/(\partial\alpha_o/\partial\lambda_i) \sim (\delta\alpha/\bar{\lambda}_1)/(\bar{\alpha}_o/\bar{\lambda}_1) \\ &= \delta\alpha/\bar{\alpha}_o = \eta,\end{aligned} \tag{B-5}$$

and

$$\begin{aligned}(\text{term 3/term 1}) &\sim (\bar{a}\, j^{1/2}\ c_s/\bar{\lambda}_1)\ j/(p_{\alpha o}\ \bar{\alpha}_o/\bar{\lambda}_1) \\ &\sim \eta\ j/(p_{\alpha o}\ \bar{\alpha}_o) \sim \eta^2,\end{aligned} \tag{B-6}$$

where we have used $j \sim p_{\alpha o}\ \delta\alpha$ and $\delta\alpha \sim \bar{a}\ j^{1/2}$. These are the orderings used in Sec. II.G in calculating $\delta\beta$ and δp_β.

PART III. STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS

III.A. Introduction to Part III

The final part of this work will be concerned with the development of a systematic framework for obtaining disperson equations for the study of normal modes in a strongly inhomogeneous plasma. The formalism to be developed begins with Poisson's equation, and is thus restricted in validity to electrostatic pertubations. In addition, we assume a slab geometry, as in Part I. Generalization of the formalism to the study of arbitrary electromagnetic perturbations should not be difficult, and generalization to allow for more complex geometries may also be possible. These matters are discussed further in Sec. III.E.

The formalism to be developed is aimed at characterizing the problem of normal modes in strongly inhomogeneous plasmas in a manner which resembles as closely as possible the more familiar formulations of the simpler problem of stability in weakly inhomogeneous plasmas, while still maintaining an exact formulation, so that when information is finally discarded in order to achieve a mathematically tractable problem, it can be done in a systematic and consistently ordered fashion.

We are aided in this pursuit by two mathematical tools in particular. The first is a canonical framework, as in Parts I and II. The second is a variational formulation of the problem, which is useful both in insuring that approximatiion schemes are applied in a consistent manner,[17] and also in lessening the work involved in obtaining dispersion equations.[18,19,20]

Use of the variational formulation is introduced in Sec. III.B, in which the formalism is developed in a very general but abstract fashion, using only a single requisite property [Eq. (III-4)] of the conductivity kernel $\overline{K}_s$. From the variational formulation, one may follow two routes. One can develop differential equations describing the spatial variation of the eigenmodes $\phi(x)$, which are approximations to the infinite order differential equation (d.e.) which describes the exact problem. Alternatively, if one can make a guess at the form of the solutions of these d.e. s, one can derive dispersion equations directly from the variational principle, even in parameter regimes where truncation of the d.e. at finite order is invalid. We shall follow both of these routes. The two approaches, solving d.e.'s or use of a variational principle, complement each other. The d.e. approach has more direct connection to the physics, and to the original equation we want to solve, Eq. (III.3). The truncated d.e. we will be solving will be formally very similar to the Schrödinger equation, and this formal resemblance will supply additional insight into what sorts of solutions $\phi(x)$ to expect. Given a good guess at how $\phi(x)$ looks, determined from insight derived from the d.e. approach, the variational approach supplies a computationally more efficient means for obtaining dispersion equations.

In Sec. III.C the abstract formalism of Sec. III.B is given physical content, by introducing the explicit form of the conductivity kernel $\overline{K}_s$. It is here that use of a canonical framework becomes particularly helpful. The resultant formalism is applicable to slab geometries with arbitrary inhomogeneity in the density and magnetic field (including

shear), and yet emerges in a form which transparently reduces in the appropriate limits to the weakly inhomogeneous expressions commonly used. Two particularly useful aspects of this should be pointed out. The first of these is the conceptually unifying aspect of such a formalism. Dispersion equations applicable to different particular geometries now arise as different limits of the same dispersion equation, eliminating the need for going through an analogous but new derivation of the dispersion equation for each particular geometry. The second useful aspect is that such a formalism permits a more careful derivation of the various effects present in a given situation. Much previous work has involved ad hoc derivations or statements of the appropriate equation to be studied, obtained by intuitive generalizations of the equations applicable to simpler situations. Most notable among these are the equations given for the study of stability of sheared plasmas. In certain cases the present formalism serves as a more rigorous justification of previous work, or finds limits on the validity of that work. In still other cases, we have found effects (to our knowledge previously undiscovered), which can be significant in certain situations (in particular in sheared plasmas). These emerge from the canonical formulation in an automatic and natural fashion, and the physical origin is discernable. One example of these effects is the fact that, in a sheared magnetic field, the diamagnetic drift frequency ω_*, which plays a central role in dispersion equations for drift modes, is modified, having a dependence on the same parameter $(v_{\parallel}/L_s\Omega_c)$ which, as we saw in Part II, also modifies the gyrofrequency.

In Sec, III.D we apply the formalism to a study of the ion drift cyclotron instability, in both unsheared and sheared magnetic fields. Study of this instability allows some of the useful and novel aspects of the formalism to be illustrated. Discussion of possible useful directions for extension of the formalism is given in Sec. III.E.

For the present part of the thesis, we need to distinguish between the various inhomogeneity parameters designated by η in Parts I and II. Thus we denote by η_n, η_s, η_B and $\eta_{\underline{B}}$ the ratio of ion gyroradius ρ_i to the scale lengths of density (L_n), shear (L_S), magnetic field modulus (L_B), and magnetic field [$L_{\underline{B}} \equiv \min(L_S, L_B)$], respectively, and define $\eta \equiv \max(\eta_n, \eta_{\underline{B}})$.

III.B. Poisson's Equation; General Formalism

III.B.1. Exact formulation

Restricting ourselves to electrostatic perturbations, we take for our wave equation Poisson's equation,

$$\nabla^2\phi\ (\underline{x}) = -\ 4\pi\ \rho(\underline{x}). \qquad \text{(III-1a)}$$

Here $\phi(\underline{x})$ is the perturbed electrostatic potential, and $\rho(\underline{x})$ is the perturbed charge density at point $\underline{x}$.

We assume that $\rho(\underline{x})$ is given by the sum over contributions $\rho_s(\underline{x})$ of the linear response of each species s to ϕ. Laplace transforming in time, we write this response in the frequency domain as

$$-4\pi\ \rho_s(\underline{x};\ \omega) = \int d\underline{x}'\ \overline{K}_s(\underline{x},\underline{x}';\ \omega)\phi(\underline{x}';\ \omega) \qquad \text{(III-1b)}$$

where ω is the component of frequency. Using Eq. (III-1b) in (III-1a),

we obtain the wave equation for ϕ, exact, within the confines of linear theory, regardless of geometry:

$$\nabla^2\phi(\underline{x};\ \omega) = \sum_s \int d\underline{x}'\ \overline{K}_s(\underline{x},\underline{x}';\ \omega)\ \phi(\underline{x}';\ \omega). \qquad \text{(III-1c)}$$

Specializing to a slab geometry makes the dependence of ϕ on the homogeneity directions y and z trivial,

$$\phi(\underline{x}) = \phi(x)\ \exp\ i(k_y y + k_z z). \qquad \text{(III-2)}$$

This leaves us with a one dimensional wave equation to solve, parametrically dependent on the constants k_y, k_z, just as the assumption of slab geometry in Parts I and II in the study of particle motion reduced that problem to a one dimensional one, parametrically dependent on the momenta $\underline{P} \equiv (P_y,\ P_z)$. Inserting (III-2) into (III-1) yields

$$0 = (-\partial_x{}^2 + k_y{}^2 + k_z{}^2)\phi(x) + \sum_s \int dx'\ \overline{K}_s(x,x';\lambda)\phi(x'), \qquad \text{(III-3)}$$

where $\lambda \equiv (\omega,\ k_y,\ k_z)$, and ∂_x denotes derivative with respect to x. In Sec. III.C we will consider the explicit form of the conductivity kernel K_s, derived from first principles in Appendix C. Here we need only use a certain property of $\overline{K}_s$ to be shown there, namely that $\overline{K}_s$ can be written as an integral over contributions K_s to $\overline{K}_s$ from particles having guiding center X_o:

$$\overline{K}_s\ (x,\ x';\ \lambda) = \int dX_o\ K_s(\sigma,\ X_o,\ \sigma';\ \lambda), \qquad \text{(III-4)}$$

where $\sigma \equiv x - X_o$, $\sigma' \equiv x' - X_o$. K_s bears a more direct relation to the expressions for susceptibilities usually used in weakly inhomogeneous stability calculations, as we shall see.

As noted in the Introduction to Part III, it will be computationally useful to phrase the stability problem in terms of a variational principle. For any two functions $\phi(x)$, $\psi(x)$, normalized so that $\int dx|\phi(x)|^2 = 1 = \int dx|\psi(x)|^2$, we define the functional $S(\psi,\phi)$ by

$$S(\psi,\phi) \equiv S_V(\psi,\phi) + \sum_s S_s(\psi,\phi)$$
$$\equiv \int dx\, \psi^*(x)(\overleftarrow{\partial}_x\overrightarrow{\partial}_x + k_y^2 + k_z^2)\phi(x) + \sum \int dx \int dx' \psi^*(x)\overline{K}_s(x,x')\phi(x'), \tag{III-5}$$

where $\overrightarrow{\partial}_x \equiv \partial_x$, and $\overleftarrow{\partial}_x$ denotes an x derivative acting to the left. (The subscript "V" on S_V denotes "vacuum.") $S(\phi,\phi)$ is closely related to the additional energy present in the plasma in the presence of perturbation ϕ, as noted in the variational formulation of Berk and Dominguez.[17] Setting the variation of S with respect to ψ^* equal to zero gives back the basic wave equation (III-3):

$$o = [\delta/\delta\psi^*(x)]\, S(\psi,\phi). \tag{III-6}$$

Here $\delta/\delta f(x)$ denotes functional differentiation with respect to any function $f(x)$. Though we will not be making use of it subsequently, we note that a related equation for $\psi^*(x)$ is obtained by taking $\delta S/\delta\phi(x)$:

$$o = -(\partial_x^2 + k_y^2 + k_z^2)\psi^*(x) + \sum_s \int dx' \psi^*(x')\overline{K}_s(x',x;\lambda). \tag{III-7}$$

From the explicit form [Eq. (III-48)] for $\overline{K}_s(x,x;\omega)$, introduced in Sec. III.C.1, we see that

$$[\overline{K}_s(x,\underline{x}';\omega)]^* = \overline{K}_s(\underline{x}', x;\omega^*), \tag{III-8}$$

where "*" denotes complex conjugation. Actually, this form for $\overline{K}_s$ is valid only for ω in the upper-half plane; for ω in the lower-half plane, an analytic continuation of this form must be made.

However, for the moment we denote by $\overline{K}_s$ only the form given in (III-48), so that Eq. (III-8) holds, as does the analogous relation for the 1-D kernel, $[\overline{K}_s(x,x';\omega,k_y,k_z)]^* = K_s(x',x;\omega^*,k_y,k_z)$. Using this in (III-7) and complex conjugating, one obtains Eq. (III-3), with $\psi(x)$ replacing $\phi(x)$. Thus, $o = \delta S/\delta\phi(x)$ yields Poisson's equation for $\psi(x)$, taking this interpretation of $\overline{K}_s$.

Now we make use of the form (III-4). Putting this form into the expression in Eq. (III-5) for the contribution S_s to S and replacing X_o by x, by the following manipulations we obtain a compact expression for S_s:

$$\begin{aligned} S_s(\psi,\phi) &= \int dx \int d\sigma \int d\sigma' \, \psi^*(x+\sigma) K_s(\sigma,x\sigma') \, \phi(x+\sigma') \\ &= \int dx \int d\sigma \int d\sigma' \, \psi^*(x) \exp(\sigma\overleftarrow{\partial}_x) \, K_s(\sigma,x,\sigma') \exp(\sigma'\overrightarrow{\partial}_x) \, \phi(x) \\ &= \int dx \, \psi^*(x) \, K_s(i\overleftarrow{\partial}_x, x, i\overrightarrow{\partial}_x) \, \phi(x). \end{aligned} \tag{III-9}$$

In the third line, the infinite order differential operator $K_s(i\overleftarrow{\partial}_x,x,i\overrightarrow{\partial}_x)$ is understood to mean the kernel K_s, Fourier transformed in arguments σ and σ';

$$K_s(k, x, k') \equiv \int d\sigma \int d\sigma' \, e^{-ik\sigma} K_s(\sigma,x,\sigma') \, e^{-ik'\sigma'}, \tag{III-10}$$

and with the replacements $k \to i\overleftarrow{\partial}_x$, $k' \to i\overrightarrow{\partial}_x$.

(Using the same symbol K_s for both $K_s(\sigma,x,\sigma')$ and its Fourier transform presents no ambiguity, if one simply notes whether the units in the arguments are $\sigma \sim x$. or $k \sim x^{-1}$. The formal manipulations using the displacement operator $\exp(\sigma\partial_x)$ in (III-9) may be verified, using more standard, but far lengthier procedures, involving Fourier transforming with only numbers (instead of operators) as arguments. Since both the operators $\overleftarrow{\partial}_x$ and $\overrightarrow{\partial}_x$ in (III-9) point outward, neither of them acts on the

x-dependence of K_s itself.

Given the form (III-9) for S_s, expression (III-5) may be more neatly expressed as

$$S(\psi,\phi) = \int dx\ \psi^*(x)\ K(i\overleftarrow{\partial}_x,\ x,\ i\overrightarrow{\partial}_x)\ \phi(x), \qquad \text{(III-11)}$$

where $K(k,x,k') \equiv K_v(k,k') + \sum_s K_s(k,x,k')$, and $K_v(k,k') \equiv kk' + k_y^2 + k_z^2$.

We now use this form of S in the variational equation, Eq. (III-6). To take the variation $\delta/\delta\psi^*(x)$ of S, we must integrate all derivatives $\overleftarrow{\partial}_x$ (which act on ψ^*) by parts, turning them into derivatives $-\overrightarrow{\partial}_x$ acting on ϕ and on K's own x dependence, yielding $S(\psi,\phi) = \int dx\ \psi^*(x)\ K(-i\overrightarrow{\partial}_x,\ x,\ i\overrightarrow{\partial}_x)\ \phi(x)$. (We note the sign change in the left-hand derivative in K.) Poisson's equation, Eq. (III-6), then reads

$$o = K(-i\overrightarrow{\partial}_x,\ x,\ i\overrightarrow{\partial}_x)\ \phi(x). \qquad \text{(III-12)}$$

Without loss of generality, we could at first have allowed ϕ and ψ to have some dominant x behavior, $\psi(x) = \psi_1(x) \exp ikx$, $\phi(x) = \phi_1(x) \exp ikx$. Equation (III-11) then gives

$$\begin{aligned} S(\psi,\phi) &= \int dx\ \psi_1^*(x)\ K(k + i\overleftarrow{\partial}_x, x,\ -k + i\overrightarrow{\partial}_x)\ \phi_1(x) \\ &= \int dx\ \psi_1^*(x)\ K(k - i\overleftarrow{\partial}_x, x,\ -k + i\overrightarrow{\partial}_x)\ \phi_1(x), \end{aligned} \qquad \text{(III-13)}$$

and Poisson's equation appears as

$$o = K(k-i\overrightarrow{\partial}_x,\ x,\ -k + i\overrightarrow{\partial}_x)\ \phi_1(x). \qquad \text{(III-14)}$$

Equations (III-12) and (III-14) are exact formulations of the full nonlocal Poisson's equation, Eq. (III-13). The integral operator in (III-3) has been converted into an infinite order differential operator,

which may be truncated at some finite order to enable analytic progress to be made. These differential formulations could have been obtained directly from the integral equation (III-3), without first going through the variational formulation. We have taken the variational route for two reasons. First, in performing the truncation at finite order of (II-12) or (II-14), one must be careful to perform the truncation in a consistent manner, so as not to introduce terms leading to spurious growth rates and frequency shifts, as pointed out by Berk and Dominguez.[17] This consistency is achieved through the variational principle, performing the truncation on S in a manner which preserves the symmetry property (III-8) of S (i.e., we truncate at the same order in $\overleftarrow{\partial}_x$ and $\overrightarrow{\partial}_x$), then integrating by parts and taking the variation of S, as done above, to obtain a differential equation.

The second reason for the variational formulation is that it is in its own right a powerful means for obtaining dispersion relations, without actually having to solve a differential equation first. Denoting by L_ϕ the mode localization width, if $\rho_s/L_\phi > 1$, the Fourier representation of $\phi(x)$ will have a width $\Delta k \sim L_\phi^{-1} > \rho_s^{-1}$, and so one expects that the operator $K_s(i\overleftarrow{\partial}_x, x, i\overrightarrow{\partial}_x)$ in (III-9) cannot validly be truncated at some finite order in ∂_x about some average value of k. Proper application of K_s thus falls naturally into two regimes. When $\rho_s/L_\phi<1$ is satisfied, the variational principle can be used to obtain results that could also be gotten from solving a differential equation (d.e.), with less labor than is required for the d.e. approach. In the regime $\rho_s/L_\phi>1$, truncation at finite order (in ∂_x) of K_s becomes invalid, and the contribution if K_s to Poisson's equation is truly integral in nature.

Here, the variational formulation is especially useful in obtaining valid approximate dispersion equations. We discuss these matters further in Secs. III.B.2 and II.B.3.

Formulations of Poisson's equation similar to Eq. (III-12) or (III-14) have been derived by other authors. Gerver, Birdsall, Langdon, and Fuss[21] derive a form

$$o = D_G(x,\ k - i\overleftrightarrow{\partial}_x;\ \lambda)\phi_1(x). \qquad \text{(III-15)}$$

Whereas Eq. (III-12) is exact, however, certain approximations were made in obtaining (III-15), which break down in the regime of appreciable η. These are manifested in the fact that the operator in (III-15) has no x derivatives acting on the x dependence of D_G itself. Essentially, D_G is what the operator K would be if the left hand ∂_x there acted only on ϕ_1: $D_G(x,k-i\overleftrightarrow{\partial}_x)\ \phi_1(x) = K(k-i\overleftrightarrow{\partial}_x,\ \xi,\ -k+i\overleftrightarrow{\partial}_x)\phi_1(x)|_{x=\xi}$.

Berk and Book[22] derive an exact formulation, equivalent to (III-14), but without using property (III-4). Rewriting $\overline{K}(x,x')$ as $D_B[s \equiv x'-x,\ \overline{x} \equiv \frac{1}{2}(x+x')]$, they derive a formulation which is exact and which, like Eq. (III-15), has only a two argument form. Operator methods like those used above greatly simplify the derivation and expression of their result. In our terminology, their formulation may be expressed as

$$o = D_B\ (i\partial_x + i\partial_\xi/2,\ \xi)\phi(x)|_{x=\xi}, \qquad \text{(III-16)}$$

where D_B here is the kernel $D_B(s,\overline{x})$ just above, Fourier transformed in the first argument, and with the replacements $k \to i\partial_x + i\partial_\xi/2$, $\overline{x} \to \xi$. Equation (III-16) is equivalent to (III-12), with ∂_x acting on that of D_B. However, the use of variables s, $\overline{x}$ in D_B masks the symmetry in x,x'

which $\overline{K}$ possesses. As a result, when the explicit form for the kernel is put into the formalism, use of the three-argument kernel $K(\sigma,X_0,\sigma')$ yields expressions for the weakly inhomogeneous limit and its higher order corrections in a more direct and transparent fashion (cf. Sec. III.C).

The formalism developed thus far is quite close to that of Berk and Dominguez.[17] At the present stage of development, the advantage of the present formalism lies in the use of operator techniques, which allows for a much quicker derivation and more compact expression of results than previous work. In the development in the following sections, we shall find two principal additional advantages.

The first of these comes from using the variational formulation to obtain dispersion equations directly, similar in spirit to the methods of Refs. 18-20. As already mentioned, the variational method is a labor-saving device in the regime $\rho_i/L_\phi < 1$ where a differential equation approach like that of Ref. 17 is valid, and more importantly, it enables one to obtain dispersion equations valid in the fully nonlocal regime $\rho_i/L_\phi > 1$, where the validity of the differential equation approach fails.

A second chief advantage lies in our use of a canonical formulation. As we shall see in Sec. III.C, the conciseness which the canonical formulation and the operator techniques already employed lend to the formalism, permits various effects of potential significance, some to our knowledge previously unknown, to arise automatically from the mathematics.

III.B.2. Truncation of the exact equations

To obtain an analytically tractable problem, we now begin to apply approximations to the exact equation [Eq. (III-12) or (III-14)] to be solved. We shall expand to second order in ∂_x that portion $K_<$ of the operator $K(i\overleftarrow{\partial}_x + k, x, -k + i\overrightarrow{\partial}_x)$ which is "nearly local," so that the finite-order truncation is valid, obtaining the contribution $S_<$ from $K_<$ to S. (The subscript "<" refers to the "nearly local" criterion $\rho_s/L_\phi < 1$.) These nearly local contributions to S are all but the nonlocal portion of the ion contribution in the regime $\rho_i/L_\phi > 1$. This term, discussed in more detail in Secs. III.B.3 and III.C., is evaluated in Appendix D.

Dropping the subscript on $S_<, K_<$ (or assuming $\rho_i/L_\phi < 1$, so that $S_< = S$, $K_< = K$), we expand K to second order, analogous to the procedure of Ref. 17:

$$\begin{aligned} S(\psi, \phi) = & \int dx\, \psi_1^* K(k,x,-k')\, \phi_1 \\ & + \int dx\, \psi_1^* [i\overleftarrow{\partial}_x \partial_k K(k,x,-k') - \partial_{k'} K(k,x,-k') i\overrightarrow{\partial}_x]|_{k=k'} \phi_1 \qquad \text{(III-17)} \\ & + \int dx\, \psi_1^* [-\tfrac{1}{2}\overleftarrow{\partial}_x^{\,2} \partial_k^{\,2} K(k,x,-k') + \overleftarrow{\partial}_x \partial_k \partial_{k'} K(k,x,k') \overrightarrow{\partial}_x \\ & \qquad - \tfrac{1}{2}\partial_{k'}^{\,2} K(k,x,k') \overrightarrow{\partial}_x^{\,2}]|_{k=k'} \phi_1 \\ & \qquad + O(\partial_x^{\,3} \partial_k^{\,3}). \end{aligned}$$

(We discuss the explicit meaning of the formal expansion parameter $\partial_x \partial_k$, in Sec. III.C.5.)

Integrating certain terms in (III-15) by parts and rearranging terms, we cast it in a more convenient form:

$$
\begin{aligned}
S &= \int dx\, \psi_1^{*}\, K(k,x,-k)\, \phi_1 \\
&\quad + \int dx\, \psi_1^{*}\, (\overleftarrow{\partial}_x\, \Gamma + \Gamma^{*}\, \overrightarrow{\partial}_x)\, \phi_1 \\
&\quad + \int dx\, \psi_1^{*}\, (\overleftarrow{\partial}_x\, \tfrac{1}{2}\, d_k^{\,2}\, K\, \overrightarrow{\partial}_x)\, \phi_1 \qquad + O(\partial_x^{\,3}\partial_k^{\,3}) \\
&\equiv S_o + S_1 + S_2 \qquad + O(\partial_x^{\,3}\partial_k^{\,3}).
\end{aligned}
\tag{III-18}
$$

Here, $\Gamma \equiv \Gamma_1 + \Gamma_2$, $\Gamma^{*} \equiv \Gamma_1^{\,*} + \Gamma_2^{\,*}$, where

$$
\begin{aligned}
&\Gamma_1 \equiv \Gamma_1(k,x) \equiv i\partial_k\, K(k,x,-k')\big|_{k=k'},\ \Gamma_1^{\,*} \equiv i\partial_{k'} K(k,x,-k')\big|_{k=k'}, \\
&\Gamma_2 \equiv \Gamma_2(k,x) \equiv \partial_x\partial_k^{\,2}\, K(k,x,-k')\big|_{k=k'},\ \Gamma_2^{\,*} \equiv \partial_x\partial_{k'}^{\,2}\, K(k,x,-k')\big|_{k=k'},
\end{aligned}
\tag{III-19}
$$

and the symbol d_k denotes total derivative with respect to k, so $d_k^{\,2}\, K \equiv (d/dk)^2\, K(k,x,-k)$. Henceforth, unless explicitly denoted otherwise, K will mean $K(k,x,-k)$ (i.e., evaluated at k=k'). The terms $\Gamma_2, \Gamma_2^{\,*}$ are down from $\Gamma_1, \Gamma_1^{\,*}$ by $O(\partial_x\partial_k)$, and will eventually be neglected, but we may carry them along for the moment at no extra expense. The terms Γ and Γ^{*} are "formal complex conjugates" of each other, in the sense to be described in Sec. III.C.1, and so we write Γ as the sum of its formally real and imaginary parts:

$$\Gamma \equiv \Gamma_R + i\Gamma_I,\quad \Gamma^{*} \equiv \Gamma_R - i\Gamma_I,$$

and similarly for $\Gamma_1 \equiv \Gamma_{1R} + i\Gamma_{1I}$ and $\Gamma_2 \equiv \Gamma_{2R} + i\Gamma_{2I}$.

We now obtain the consistently-truncated form of Poisson's equation (assuming $\rho_i/L_\phi < 1$, so that $S_< = S$), taking $\delta S/\delta\psi_1^{*}(x)$ as described previously. Using (III-18) for S, one finds

$$o = (A - B\, \overrightarrow{\partial}_x - C\, \overrightarrow{\partial}_x^{2})\, \phi_1, \tag{III-20}$$

where A, B, and C are functions (not operators) of x and k, defined by $A \equiv K - \partial_x \Gamma$, $B \equiv \frac{1}{2}\partial_x d_k^2 K + 2i\Gamma_1$, and $C \equiv \frac{1}{2} d_k^2 K$. The term B will be negligible, to the order we will work, and so (III-20) takes the form

$$o = (Q - \partial_x^2)\, \phi_1, \qquad \text{(III-21)}$$

where $Q(k,x,\lambda) \equiv A/C$. This has the form of the Helmholtz equation for heat flow in one dimension, or of the time-independent Schrödinger equation, $o = \{[V(x) - E] + \frac{1}{2} p^2\}\psi$, with (V-E) replaced by Q. We note in addition that, even if B were not negligible, (III-20) could still be put into the form of Eq. (III-21), inserting $\phi_1 \equiv f\, \phi_2$ into (III-20), and choosing the function f(x) so that the coefficient B' of $\partial_x \phi_2$ vanishes. This yields

$$f(x) = f_o \exp(-\tfrac{1}{2}\int^x dx'\, B/C), \qquad \text{and}$$

$$o = (Q' - \partial_x^2)\, \phi_2, \qquad \text{(III-22)}$$

where $Q' \equiv A'/C'$, $A' \equiv A - B(\partial_x f/f) - C(\partial_x^2 f/f)$, and $C' \equiv C$.

We have thus succeeded in putting the truncated problem into the familiar form of the Schrödinger equation, with no assumptions yet made on the shape of the "well" Q(x), or on the localization of ϕ. One must remember that Q(x) is in general complex, more general than the usual Schrödinger equation studied in quantum mechanics.

III.B.3. Variational method

We appeal to the fact that S is variational with respect to both ψ and ϕ, i.e., that it is stationary when ψ^* and ϕ are solutions to their respective equations. When they are solutions to their equations, we also have $S = 0$. The variational method is thus the following. One makes a guess at the form of the solutions ϕ and ψ, parametrically dependent upon the set of variables $\underline{\mu} \equiv \{\mu_1, \mu_2, \cdots \mu_N\}$. Evaluating $S(\psi,\phi;\lambda)$ using these forms gives $S(\underline{\mu};\lambda)$. The stationarity condition on S then corresponds to the N conditions

$$o = \partial S/\partial \mu_i, \quad (i = 1,2 \cdots N). \tag{III-23}$$

As is the case when ϕ,ψ are exact solutions, one then sets

$$o = S(\underline{\mu};\lambda), \tag{III-24}$$

which yields a dispersion equation for $\omega(k_y,k_z)$. If the guess made for ϕ and ψ is a good one, with difference between the exact solutions and optimized trial solutions on the order of some small parameter ε, the error in the resultant dispersion relation (III-24) is $O(\varepsilon^2)$.

We have already adopted a first parameter, $\mu_1 \equiv k$. We now take $\psi_1^* = \phi_1$. This gives a variational formulation paralleling that used by Ross and Mahajan,[18,19] and Hazeltine and Ross,[20] in their variational approach to studying drift waves. Defining $\kappa_\phi(x) \equiv \partial_x \ln\phi_1$, and $\langle f \rangle \equiv \int dx\, \phi_1^2(x)\, f(x)$, for any function f, Eq. (III-18) may be written (restoring the subscript <)

$$S_< = \langle K_< \rangle - \langle \partial_x \Gamma_R \rangle + \langle \tfrac{1}{2}\kappa_\phi^2\, d_k^2 K_< \rangle + O(\partial_x \partial_k^3). \tag{III-25}$$

We recall that for $\rho_i/L_\phi < 1$, one has $S_< = S$. In Sec. III.C, we shall see that the contribution $\bar{K}_s(\underline{x},\underline{x}')$ to $\bar{K}(\underline{x},\underline{x}')$ consists of two parts, a purely local part, the "adiabatic term" $\bar{K}_{sA} = \delta(\underline{x},\underline{x}')\lambda_i^{-2}(x)$, and a non-local part $\bar{K}_{sB}$. In the regime $\rho_i/L\phi > 1$, $\bar{K}_{iA}$ may still be combined with the "nearly local" contributions to S, but $\bar{K}_{iB}$ must be treated differently. In Appendix D, we evaluate the contribution S_{iB} of $\bar{K}_{iB}$ to S, in the regime $\rho_i/L_\phi > 1$, using a straight line orbit approximation for the ion orbits in the region of mode localization. In this regime, we refer to S_{iB} as $S_>$, and so we have

$$S = S_< + S_{iB} \equiv S_< + S_> , \qquad (\rho_i/L_\phi > 1),$$

and from before, (III-26)

$$S = S_< , \quad S_> = 0 \quad (\rho_i/L_\phi < 1).$$

In both regimes, $S_<$ has the form of (III-25).

Before further specifying the parameters μ_i, we can apply Eq. (III-23) for $\mu_i = \mu_1 \equiv k$. In Sec. III.C.2 we will see that, to the order we are working both $K_<$ and Γ_R in $S_<$ are even functions of k, and in Appendix D, we will find that $S_>$ is independent of k, assuming that $k < L_\phi^{-1}$. Therefore S is an even function of k, in both regimes, and so the variational condition for k is

$$o = \partial S/\partial k = 2k\ \partial S(k^2)/\partial k^2, \qquad \text{(III-27)}$$

satisfied by k = 0. We shall choose this solution henceforth.

Each of the variational equations (III-23) we will use has a counterpart in the differential equation approach as used, for example, by Gerver et al.[21] These authors expand Eq. (III-15) to second order

in both $\delta x \equiv x-x_o$ and $\delta k \equiv i\partial_x$, and to first order in $\delta\omega \equiv \omega-\omega_o$, where the values of x_o, k are determined from the conditions

$$o = \partial_k D_G(x_o,k;\lambda), \quad o = \partial_{x_o} D_G(x_o,k;\lambda), \qquad \text{(III-28)}$$

and ω_o, is determined from the local dispersion relation

$$o = D_G[x_o, k; \lambda_o \equiv (\omega_o, k_y, k_z)]. \qquad \text{(III-29)}$$

The first of Eqs. (III-28) corresponds to our Eq. (III-27), and gives the same result, k = 0. Equation (III-29) corresponds to taking a lowest-order approximation to our Eq. (III-24). The higher order contributions to the dispersion relation S = 0 held in $S_<$ in Eq. (III-25) correspond to the shift $\delta\omega$ in frequency found in solving the Weber equation which arises in the differential equation approach of Ref. 21. We refer to these contributions to the dispersion relation as $K_{d.e.}$ (the subscript "d.e." denoting "differential equation").

Now we further specify the form of the trial solutions. We expect solutions localized at some (real) point $x_o \equiv \mu_2$, with half width $L_\phi \equiv |\alpha|^{-1/2}$, given by the parameter $\alpha \equiv \mu_3$, in general complex. Defining $\delta x \equiv x-x_o$, $\xi \equiv \alpha^{-1/2}\,\delta x$, we choose ϕ_1 to be given by the eigenstates of the Weber equation

$$(-\partial_\xi^{\,2} + \xi^2)\phi_1 = f\phi_1. \qquad \text{(III-30)}$$

This has solutions ϕ_{1n} (n = 0, 1,2,···) given by

$$\phi_{1n}(\xi) \equiv N_{xn}(\alpha)\, H_n(\xi)\, e^{-\xi^2/2}, \qquad \text{(III-31)}$$

where the H_n are the Hermite polynomials, and N_{xn} is chosen so that $1 = \int_{-\infty}^{\infty} dx|\phi_{1n}(\xi)|^2$. The ϕ_{1n} are solutions of (III-30) with eigen-

value $f = f_n \equiv (2n+1)$. The solutions may be localized either by a potential well in the real part Q_R of $Q(x)$ in (III-21), or by the effect of the imaginary part Q_I, physically corresponding to a Landau resonance effect, in the manner first elucidated by Pearlstein and Berk[23] in studying the so-called universal instability.

We define the moments M_m of the Gaussian by

$$M_m(\alpha) \equiv \int_{-\infty}^{\infty} dy\, y^m\, e^{-\alpha y^2}, \qquad \text{(III-32)}$$

so that

$$M_o(\alpha) \equiv (\pi/\alpha)^{1/2}, \quad M_1(\alpha) \equiv 0, \quad M_m(\alpha) = (m-1)\, M_{m-2}(\alpha)/2\alpha. \qquad \text{(III-33)}$$

The Hermite polynomials may be defined by

$$H_n(\xi) = e^{\xi^2} (-\partial_\xi)^n\, e^{-\xi^2}. \qquad \text{(III-34)}$$

In particular, $H_o(\xi) = 1$, so that normalizing $\phi_{1o} = N_{xo}\, e^{-\xi^2/2}$ yields

$$N_{xo}(\alpha) = M_o^{-1/2}(\alpha) = (\alpha/\pi)^{1/4}. \qquad \text{(III-35a)}$$

Using (III-34), one can show that[24] in general

$$N_{xn}(\alpha)\ (2^n n!)^{-1/2}. \qquad \text{(III-35b)}$$

Expanding $K_<(x)$ about $x = x_o$

$$K_<(x) = \sum_{m=o}^{\infty} (\delta x)^m\, \partial_{x_o} K_<(x_o)/m!, \qquad \text{(III-36)}$$

we evaluate $\langle K_< \rangle$ in (III-25), for $\phi_1 = \phi_{10}$:

$$\begin{aligned} \langle K_< \rangle &= M_o^{-1} \left\{ M_o\, K_<(x_o) + M_2\, \partial_{x_o}{}^2\, K_</2! + \cdots \right\} \\ &= K_<(x_o) + (4\alpha)^{-1}\, \partial_{x_o}{}^2 K_< + O(L_\phi{}^4/L_Q{}^4). \end{aligned} \qquad \text{(III-37)}$$

Here $L_Q \equiv |\partial_x \ln Q|^{-1}$ is the scale length of the potential Q localizing

the mode. An expansion analogous to (III-37) may be done for the other terms in (III-25) but, to the order we are working (namely, keeping only up to second derivatives on $K_<$), we may keep only the zero order parts of these. In addition, it will be convenient (and consistent with our ordering) to approximate $K_<$ in these higher order terms by K_o, the difference $K_< - K_o$ being of higher order. We shall choose K_o to be all contributions to K except that from K_{iB} which, though important for determining the growth rate of the instability to be studied, is nevertheless small:

$$K_o = K - K_{iB} = K_v + K_e + K_{iA}. \qquad \text{(III-38)}$$

A more explicit form for K_o will be given in Sec. III.D.1. In the regime $\rho_i/L_\phi < 1$, where $K_< = K = K_o + K_{iB}$, one therefore has

$$\langle K_< \rangle \simeq \langle K_o \rangle + K_{iB} \cong K_o + K_{iB} + (4\alpha)^{-1}\, \partial_{x_o}{}^2 K_o \qquad \text{(III-39a)}$$

and

$$\langle -\partial_x \Gamma_R \rangle \simeq -\partial_{x_o} \Gamma_R \simeq \sum_s (-i\partial_{x_o} \partial_k K_{sB} + \text{c.c.}), \qquad \text{(III-39b)}$$

where we have used (III-17), and the facts that K_v is independent of x and that $K_{sA} \sim \lambda_s{}^{-2}(x)$ is independent of k (as already mentioned), so that only the terms K_{sB} in K contribute to $-\partial_x \Gamma_R$. For $\rho_i/L_\phi > 1$, $K_< = K - K_{iB} = K_o$, and therefore

$$\langle K_< \rangle = \langle K_o \rangle \simeq K_o + (4\alpha)^{-1}\, \partial_{x_o}{}^2 K_o, \qquad \text{(III-40a)}$$

and

$$\langle -\partial_x \Gamma_R \rangle \simeq -\partial_{x_o} \Gamma_R \simeq (-i\partial_{x_o} \partial_k K_{eB} + \text{c.c.}). \qquad \text{(III-40b)}$$

In Sec. III.C.4 we will see that the electron contribution to Γ_R is down from that of the ions by a factor $\sim (\Omega_i/\ell\Omega_e)$. Therefore, we may

retain only the ion term in (III-39b), and totally neglect the term in (III-40b).

With these approximations, using (III-26) in (III-24), the dispersion relation assumes the following forms in the small and large $-\rho_i/L_\phi$ regimes:

$$0 = S \simeq K_o + S_{iB} + K_{n.\ell.} + K_{d.e.}, \quad \rho_i/L_\phi < 1, \qquad \text{(III-41a)}$$

and

$$0 = S \simeq K_o + S_{iB} + K_{d.e.}, \quad \rho_i/L_\phi > 1. \qquad \text{(III-41b)}$$

For $\rho_i/L_\phi < 1$, $S_{iB} = \langle K_{iB} \rangle \simeq K_{iB}$. We evaluate S_{iB} for $\rho_i/L_\phi > 1$ in Appendix D, and find that it has a form quite similar to its form in the regime $\rho_i/L_\phi < 1$. This is discussed further in Sec. III.C. The term $K_{n.\ell.} \equiv \langle -\partial_x \Gamma_R \rangle \simeq -i\partial_{x_o} \partial_k K_{iB} + \text{c.c.}$ is the lowest order correction due to the nonlocal (n.ℓ.) nature of the conductivity kernel. We see that $S_{iB}(\rho_i/L_\phi < 1) + K_{n.\ell.}$ in Eq. (III-41a) is replaced by $S_{iB}(\rho_i/L_\phi > 1)$ in (III-41b). Finally, $K_{d.e.}$, whose explicit form will be given shortly, is the contribution arising from solving a differential equation in x, instead of just a local dispersion equation [viz. $K(x) = 0$]. All quantities in Eqs. (III-41) are evaluated at x_o. For the Gaussian form of $\phi_{io}(x)$, and the quadratic approximation of $K_o(x)$ we have made expanding about $x = x_o$, $K_{d.e.}$ corresponds to the ground state energy coming from solving the Schrödinger equation for a simple harmonic oscillator (SHO); $K_{d.e.}$ is then given by $K_{d.e.} = \frac{1}{4}(\alpha^{-1}\partial_{x_o}^{\ 2} K_o + \alpha d_k^{\ 2} K_o)$. Since the differential equation to be solved here is of the same form as the Schrödinger equation, we may use our knowledge of the nature of the solutions of that equation to generalize Eq. (III-41) to be valid both for the higher n

"excited state" solutions ϕ_{1n}, and for anharmonic corrections to the "potential" $K_o(x)$. For the quadratic potential, we generalize the expression for $K_{d.e.}$ to

$$K_{d.e.} = K_{d.e.}^{(n)} \equiv (f_n/4)(\alpha^{-1}\partial_{x_o}{}^2 K_o + \alpha d_k{}^2 K_o), \qquad \text{(III-42)}$$

where we recall $f_n \equiv 2n + 1$. Letting anharmonic corrections enter will shift the eigenvalues from their harmonic well values, which we may account for by allowing the coefficients f_n to shift from their harmonic values $2n + 1$, as the anharmonicity is turned on.

We now write out the variational equations for $\mu_2 \equiv x_o$ and $\mu_3 \equiv \alpha$. The term S_{iB} in both ρ_i/L_ϕ regimes will turn out to be equal to a sum over harmonic number ℓ of terms proportional to $(\omega - \omega_{*i})(\omega - \ell\Omega)^{-1}$. This term is essential for obtaining the growth rates for the modes we shall be studying, but it is nevertheless small in comparison to the other terms in S. We can therefore neglect it in the x_o and α variational equations. For the x_o equation, $K_{d.e.}$ and $K_{n.\ell.}$ are also higher order contributions, leaving only K_o in expressions (III-41) for S:

$$o = \partial S/\partial x_o = \partial_{x_o} K_o + \text{h.o.} \qquad \text{(III-43)}$$

Equation (III-43) is the variational counterpart of the second of conditions (III-28). It localizes the mode at an extremum of K_o, just as solving the differential equation (III-21) localizes the mode at an extremum of Q(x). The difference between these localization criteria of the variatiional and differential equation approaches is negligible, to the order we are working.

Our final variational equation gives us α:

$$\sigma = \partial S/\partial\alpha \simeq (f_n/4)(-\alpha^{-2}\partial_{x_o}{}^2 K_o + d_k{}^2 K_o) \qquad \text{(III-44)}$$

and

$$\alpha = (\tfrac{1}{2}\,\partial_{x_o}{}^2 K_o/\tfrac{1}{2} d_k{}^2 K_o)^{1/2}. \qquad \text{(III-45)}$$

This gives

$$K_{d.e.} = f_n[\tfrac{1}{2}\, d_k{}^2 K_o)(\tfrac{1}{2}\,\partial_{x_o}{}^2 K_o)]^{1/2}. \qquad \text{(III-46)}$$

In Sec. III.d.1 we will find that $d_k{}^2 K_o > 0$. Whether α, and $K_{d.e.}$, are real or imaginary thus depends on whether the mode is localized at a minimum or maximum of K_o (or, approximately, of Q).

In order for the localized mode assumption to be reasonable, the potential Q must be of sufficient width and magnitude that $(L_\phi/L_Q) < 1$. Using $L_\phi{}^2 \simeq f_n\,\alpha^{-1}$, and for the purposes of estimation assuming a well in Q_R which is quadratic from a maximum depth $Q_d \equiv |Q(x_o)|$ up to $Q = 0$, so that $L_Q{}^2 \simeq Q_d/(\tfrac{1}{2}\,\partial_{x_o}{}^2 Q)$, we obtain the criterion for localization

$$1 > (L_\phi/L_Q)^2 \simeq |K_{d.e.}/K_d|, \qquad \text{(III-47)}$$

where $K_d \sim K_o$ is the corresponding depth of the well in K_o.

To apply this abstract formalism, one must look at the specific form of the function K. We proceed to do this in the following section.

III.C. Conductivity K; Explicit Form

In this section we insert the physics into the abstract formalism of Sec. III.B, by using the explicit form for the conductivity kernel in a plasma, expressed in a canonical framework. The work of Parts I and II will prove useful to us here, in providing a description of the particle orbits, in choosing an appropriate distribution function, and

in understanding the physical origin of certain effects, which automatically arise from the mathematics of the canonical formulation. Most notable among these are a generalized expression for the diamagnetic drift frequency in the presence of shear, and ion resonance effects which may occur for appreciable shear, which previous ad hoc derivations have missed.

III.C.1. Explicit expression for K_s

In Appendix C we derive general expressions for the conductivity kernel $\overline{K}_s$ in a three dimensional plasma, in a canonical formulation. As there, we denote by $\underline{I}$ the three canonical momenta whose invariance makes the time rate of change $\underline{\Omega}(\underline{I}) \equiv \dot{\underline{\theta}}(\underline{I}) \equiv \partial H/\partial \underline{I}$ of the corresponding coordinates $\underline{\theta}$ constant in time. For the slab geometry, we take $\underline{I} \equiv (J, P_y, P_z) \equiv (J, \underline{P})$, where J is the generalized gyroaction discussed in Part II. The conjugate coordinates are $\theta \equiv (\Theta, Y, Z)$, with time derivatives $\Omega \equiv (\Omega, \dot{Y}, \dot{Z})$. In Appendix C, we find that

$$\overline{K}_s(\underline{x},\underline{x}';\omega) = \overline{K}_{sA}(\underline{x},\underline{x}') + \overline{K}_{sB}(\underline{x},\underline{x}',\omega), \qquad \text{(III-48a)}$$

where

$$\overline{K}_{sA}(\underline{x},\underline{x}') \equiv \lambda_s^{-2}(x)\ \delta(\underline{x}-\underline{x}'), \qquad \text{(III-48b)}$$

and

$$\overline{K}_{sB}(\underline{x},\underline{x}';\omega) \equiv (4\pi)(2\pi)^3 \sum_{\underline{\ell}} \int d\underline{I}\, \rho_{\underline{\ell}}(\underline{x}|\underline{I})\, \frac{\omega\partial f_s/\partial H - \underline{\ell}\cdot\partial f_s/\partial \underline{I}}{\omega + \underline{\ell}\cdot\underline{\Omega}}\, \rho_{\underline{\ell}}^*(x'|\underline{I}). \qquad \text{(III-48c)}$$

Here, f_s is the equilibrium distribution function (denoted f_{os} in Appendix C), $\ell \equiv (\ell, \ell_y, \ell_z)$, and the other symbols are defined in Appendix C.

First we deal with the nonlocal part, $\bar{K}_{sB}$. Specializing to a slab geometry, we Fourier transform out the y,z,y', and z' dependences in $\bar{K}_{sB}$. This involves the corresponding transforms of $\rho_{\underline{\ell}}(\underline{x})$ and $\rho^*_{\underline{\ell}}(\underline{x}')$. Writing $\underline{r}(\underline{Z}) \equiv \underline{R} + \delta\underline{r}$, where $\underline{R} \equiv (X_o, Y, Z)$ is the guiding center position and $\delta\underline{r} \equiv (\delta x, \delta y, \delta z)$ describes the gyromotion about $\underline{R}$, one finds

$$\rho_{\underline{\ell}}(x,k_y,k_z) = e\delta(\ell_y+k_y)\delta(\ell_z+k_z)\tilde{J}_\ell(\sigma,k_y,k_z|\underline{I}), \quad \text{(III-49)}$$

where $\sigma \equiv x - X_o$, and

$$\tilde{J}_\ell(\sigma,k_y,k_z|\underline{I}) \equiv (2\pi)^{-1}\int_0^{2\pi} d\Theta\,\delta(x-X_o-\delta x)e^{-\ell\Theta}\exp[-i(k_y\delta y(\Theta,\underline{I})+k_z\delta z(\Theta,\underline{I})].$$

Using Eqs. (III-48c), (III-49), and (III-1b), we find the one dimensional analog of (III-1b), for the contribution $\rho_{sB}(\underline{x})$ to $\rho_s(\underline{x})$: $-4\pi\rho_{sB}(x;\lambda) = \int dx'\,\bar{K}_{sB}(x,x';\lambda)$, where

$$\bar{K}_{sB}(x,x';\lambda) \equiv (4\pi)(2\pi)\sum_{\ell=-\infty}^{\infty}\int d\underline{I}\,\frac{\omega\partial f_s/\partial H-\ell\cdot\partial f_s/\partial\underline{I}}{\omega+\ell\cdot\underline{\Omega}}\,\tilde{J}_\ell(\sigma,k_y,k_z|\underline{I})\tilde{J}^*_\ell(\sigma',k_y,k_z|\underline{I}). \quad \text{(III-50)}$$

We assume that we have chosen our distribution function f_s in terms of the canonical variables well, so that the self consistent equilibrium potentials, determined by the formalism of Part I, are precisely the desired reference potentials, and so $X_o = \tilde{X}_o(\underline{P})$, $v_3 \equiv v_{o3} \equiv v_\parallel(X_o) = \tilde{v}_\parallel(\underline{P})$. As in Part I, we shall take $f_s = f_s(H,\underline{P})$, dependent on J only through H. Therefore $-\underline{\ell}\cdot\partial f_s/\partial\underline{I} = -\underline{\ell}\cdot\partial f_s/\partial\underline{P} = \hat{\underline{k}}\cdot\partial f_s/\partial\underline{P}$, where $\hat{\underline{k}} \equiv \hat{y}\,k_y + \hat{z}\,k_z$. Changing from integration variables $\underline{I}$ to $(H,\underline{P})$, then, we have

$$d^3\underline{I} \equiv d^2\underline{P}\,dJ = dX_o\,dv_3 dH\,\Omega^{-1}I, \quad \text{(III-51)}$$

where $I \equiv \partial(P_y, P_z)/\partial(X_o,v_3)$, as introduced in Sec. I.C.2. Putting (III-51) into (III-50), comparison with Eq. (III-4) allows us to read

off K_{sB}:

$$K_{sB}(\sigma,X_o,\sigma',\lambda) =$$

$$8\pi^2 e^2 \sum_{\ell=-\infty}^{\infty} \iint dv_3 d\Omega^{-1} I \frac{\omega\partial f_s/\partial H + \hat{\underline{k}}\cdot\partial f_s/\partial \underline{P}}{\omega + \underline{\ell}\cdot\underline{\Omega}} \tilde{J}_\ell(\sigma,k_y k_z)\tilde{J}_\ell^*(\sigma',k_y,k_z). \tag{III-52}$$

A central quantity of the abstract formalism of Sec. III.B is $K_<(k,X_o,-k')$. We therefore Fourier transform (III-52) with respect to both σ and σ'. We use

$$\begin{aligned}\tilde{J}_\ell(k,k_y,k_z|\underline{I}) &\equiv \int d\sigma\, e^{-k\sigma}\, \tilde{J}_\ell(\sigma,k_y,k_z|\underline{I}) \\ &= (2\pi)^{-1}\int_0^{2\pi} d\Theta\, e^{-i\ell\Theta}\int d\sigma\, \delta(\sigma-\delta x)\exp[-i(k\sigma+k_y\delta y+k_z\delta z)] \\ &= (2\pi)^{-1}\int_0^{2\pi} d\Theta\, e^{-i\ell\Theta}\, e^{-\underline{k}\cdot\delta\underline{r}(\Theta)}, \quad (\underline{k}\equiv k\hat{x}+k_y\hat{y}+k_z\hat{z})\end{aligned} \tag{III-53}$$

in Eq. (III-52), obtaining the important formula

$$K_{sB}(k,\ X_o,-k';\ \lambda) = \tag{III-54}$$

$$8\pi^2 e^2 \sum_{\ell=-}^{\infty} \iint dv_3 d\Omega^{-1} I \frac{\omega\partial f_s/\partial H + \hat{\underline{k}}\cdot\partial f_s/\partial \underline{P}}{\omega + \underline{\ell}\cdot\underline{\Omega}} \tilde{J}_\ell(\underline{k}|I)\tilde{J}^*(\underline{k}'|\underline{I}).$$

Because it is purely local, the contribution S_{sA} of the "adiabatic term" $\overline{K}_{sA}$ to S is easier to calculate. Using (III-48b) in the first line of Eq. (III-9), one finds

$$S_{sA}\ (\psi,\phi) = \int dx\ \psi^*(x)\ \lambda_s^{-2}(x)\ \phi(x). \tag{III-55}$$

Comparing, this with the third line of (III-9), we read off K_{sA}:

$$K_{sA}(k,\ x,\ -k) = K_{sA}(x) = \lambda_s^{-2}\ (x). \tag{III-56}$$

As already noted, these forms for K_{sA} and S_{sA} are valid and useful for any value of ρ_s/L_ϕ, while K_{sB} in (III-45) is only of use in the "nearly local" regime $\rho_s/L_\phi < 1$. In Appendix D we calculate a form for S_{sB} which is valid for the regime $\rho_s/L_\phi > 1$, starting from $K_{sB}(\sigma, X_o, \sigma')$ as given in Eq. (III-52). We thus obtain expressions for S and dispersion equations valid in both regimes, and can then smoothly connect the behavior of S in these two regions to obtain a picture of mode stability valid for all ρ_s/L_ϕ.

III.C.2. On the Structure of K_s; Discussion

In the present section we discuss certain important features of the Kernel K_s [given by Eqs. (III-54), (III-56)] which is a central object in the abstract formalism of Sec. III.B. We note that $K_s(k, X_o, -k')$ has the same structure as the usual expression for the susceptibility χ_s of a weakly inhomogeneous plasma in a uniform magnetic field; indeed, in that limint K_s reduces to just $\underline{k}^2\chi_s$. It is much more general, however, in two basic respects. The first is that all the quantities (e.g. $\underline{I}$, $\underline{\Omega}(\underline{I})$, $\tilde{J}_\ell$) appearing in K_s have a meaning generalized from their uniform $\underline{B}$ limits to arbitrary magnetic inhomogeneity. The second is that $k \neq k'$, so that K_s contains more information than χ_s. This is reasonable, since K_s describes the full nonlocal conductivity properties of the plasma, while χ_s is a quantity derived under a local ($\eta \to o$) approximation. It arises from the present formalism [cf. Eq. (III-14)] by letting the operators $\vec{\partial}_x$ in K act only on the x dependence of ϕ_1 and not of K itself. This corresponds to setting $k = k'$.

The factor $J_\ell^2(k_\perp \rho_s)$ (J_ℓ is the Bessel function of the first kind), appearing in the uniform $\underline{B}$ expressions for the conductivity, has been replaced in Eq. (III-54) by the factor $\tilde{J}_\ell(\underline{k}|\underline{I})\tilde{J}_\ell^*(\underline{k}'|\underline{I})$. This is more general than J_ℓ^2 in both of the ways just mentioned. We recover J_ℓ by letting $\delta\underline{r}(\Theta)$ go to its simple harmonic, small gyroradius ($\eta_{\underline{B}} \to o$) limit. We use the Bessel identity

$$J_\ell(\mu) = (2\pi)^{-1}\int_o^{2\pi} d\Theta e^{-i\ell\Theta} e^{i\mu\sin\Theta} = i^\ell (2\pi)^{-1}\int_o^{2\pi} d\Theta e^{-i\ell\Theta} e^{-i\mu\cos\Theta}, \quad \text{(III-57)}$$

and Eqs. (II-111), (II-119) of Part II for $\delta\underline{r}$, to obtain the $\eta_{\underline{B}} \to o$ limit of $\tilde{J}_\ell$. These equations tell us that, in the coordinate system $[\hat{b}_1 \equiv \hat{x},\ \hat{b}_2 \equiv \hat{b}_3 \times \hat{b}_1,\ \hat{b}_3 \equiv \hat{B}(X_o)]$ natural to the magnetic geometry at $x = X_o$, the $\eta_{\underline{B}} \to o$ limit of the gyromotion is given by

$$\delta\underline{r}(\Theta) = [\delta x_o(\Theta),\ \delta x_{20}(\Theta),\ 0],$$

where $\delta x_o(\Theta) = \bar{x}_o \sin\Theta$ is the excursion in the $x \equiv \underline{x}\cdot\hat{b}_1$ direction, $\delta x_{20}(\Theta) = \bar{x}_{20}\cos\Theta$ is the excursion in the $x_2 \equiv \underline{x}\cdot\hat{b}_2$ direction, and the excursion in the $x_3 \equiv \underline{x}\cdot\hat{b}_3$ direction is zero. The oscillation amplitudes $\bar{x}_o$ and $\bar{x}_{20}$ are related by $\bar{x}_{20} = \bar{x}_o\ (\Omega_c/\Omega_o)$, where $\Omega_c \equiv eB_o/mc$ is the usual cyclotron frequency at X_o, and Ω_o is the $\eta_{\underline{B}} \to o$ limit of the gyrofrequency which, we recall from Sec. II.K.1, is not in general equal to Ω_c.

From these results of Part II, we write

$$\underline{k}\cdot\delta\underline{r}(\Theta) \equiv k\delta x_o(\Theta) + k_2\delta x_{20}(\Theta) \equiv \mu_o\cos(\Theta-\Theta_o), \quad \text{(III-58a)}$$

where $k \equiv \underline{k}\cdot\hat{b}_1$ and $k_2 \equiv k\cdot\hat{b}_2$. (Similarly, we define $k_3 \equiv \underline{k}\cdot\hat{b}_3$). From this follow the definitions of μ_o, Θ_o:

$$\mu_o^2 \equiv k^2\bar{x}_o^2 + k_2^2\bar{x}_{20}^2,\ k\bar{x}_o = \mu_o\sin\Theta_o,\ k_2\bar{x}_{20} = \mu_o\cos\Theta_o. \quad \text{(III-58b)}$$

Using (III-58a) and (III-57) in (III-53), we obtain, finally,

$$\lim_{\eta_{\underline{B}} \to o} \tilde{J}_\ell(\underline{k}|\underline{I}) = (-i)^\ell e^{-i\ell\Theta_o} J_\ell(\mu_o). \qquad \text{(III-59)}$$

For $\delta\underline{r}(\Theta)$ describing anharmonic gyromotion, we can still obtain explicit expressions for the $\tilde{J}_\ell$. Proceeding only up to $\mathcal{O}(\eta_{\underline{B}})$ as in Sec. II.K.2, we can write

$$\underline{k} \cdot \delta\underline{r}(\Theta) = \mu_o \cos(\Theta-\Theta_o) + \mu_1 \cos 2(\Theta-\Theta_1), \text{ where } \mu_1/\mu_o \sim \mathcal{O}(\eta_{\underline{B}}).$$

Using the Bessel identity

$$\exp(i\mu\sin\Theta) = \sum_\ell e^{i\ell\Theta} J_\ell(\mu), \text{ or } \exp(-i\mu\cos\Theta) = \sum_\ell (-i)^\ell e^{i\ell\Theta} J_\ell(\mu), \qquad \text{(III-60)}$$

one finds $\tilde{J}_\ell$ is given by

$$\tilde{J}_\ell(\underline{k}|\underline{I}) = (-i)^\ell e^{-i\ell\Theta_o} \sum_m e^{2im(\Theta_o-\Theta_1)} J_{\ell-2m}(\mu_o) J_m(\mu_1). \qquad \text{(III-61)}$$

One could, if desired, proceed similarly to arbitrary order in the $\eta_{\underline{B}}$ expansion of $\delta\underline{r}\ (\Theta)$.

In general, one may define J_ℓ by

$$\tilde{J}_\ell(\underline{k}|\underline{I}) \equiv J_\ell(\underline{k}|\underline{I}) e^{-i\ell\Theta_k}, \qquad \text{(III-62)}$$

where Θ_k is chosen to make $J_\ell(\underline{k}\to o)$ real and positive. Then $\Theta_k \to \Theta_o$, and $J_\ell \to J_\ell$ in the $\eta_{\underline{B}} \to o$ limit. The J_ℓ retain certain of the useful properties of their simple harmonic limits, the J_ℓ. For example, using Parseval's theorem and the definition (III-53) of $\tilde{J}_\ell$, one finds the relation

$$1 = e^{-i\underline{k}\cdot\delta\underline{r}} e^{i\underline{k}\cdot\delta\underline{r}} = \sum_\ell \tilde{J}_\ell(\underline{k}) \tilde{J}^*_\ell(\underline{k}) = \sum_\ell J^2_\ell(\underline{k}). \qquad \text{(III-63)}$$

For $k \neq k'$, there is a phase factor $\exp i\ell(\Theta_{k'}-\Theta_k)$ present in expression (III-54), absent in the weakly inhomogeneous limit ($k = k'$)

usually used. For substantial ℓ, this factor can give a significant correction to the dispersion relation, contributing to the term $-\partial_x \Gamma_R$ in Eq. (III-25) or (III-41) (cf. Sec. III.D.2). This correction has been found previously by other workers.[17, 27]

For $k = k'$, one sees from Eq. (III-54) that K_{sB} is a "formally real" quantity, i.e. if we replaced the resonance denominator $(\omega + \ell\cdot\Omega)$ (which gives K_{sB} a real and imaginary part) by a real function of the momenta $\underline{I}$, K_{sB} itself would be real. Similarly, one may speak of functions which are "formally imaginary" or "formally complex conjugates," as in the discussion of Γ and Γ^* in Sec. III.B.2. For example, from (III-54) one sees that $K_{sB}(k,X_o,-k') = K^*_{sB}(k',X_o,-k)$, where the "*" denotes the formal complex conjugate.

The entire k dependence of $K_{sB}(k,X_o,-k)$ is contained in the factor $J^2_\ell(\underline{k})$. For anharmonic orbits, this factor, and so K_s, are not in general even in k, as is the $\eta_{\underline{B}} \to o$ limit $J^2_\ell(\mu_o)$. However, these anharmonic corrections, of order $\eta_{\underline{B}}$, are negligible in our ordering scheme, as will be discussed in Sec. III.C.5.

We shall break K into a lowest order part K_h, plus a term $K_{o.m.}$ which is of higher order, and may be neglected, consistent with our ordering (cf. Sec. III.C.5). The subscript "h" in K_h indicates that in evaluating K_h we use the small gyroradius, harmonic limits of the particle motion, and therefore the J_ℓ limit of the J_ℓ. The subscript "o.m." in $K_{o.m.}$ refers to all the various anharmonic orbit modification effects which enter expression (III-54) for K_{sB}, when $s = i \equiv$ ion species. Thus $K_{o.m.}/K_h \sim \eta_{\underline{B}}$. The term kept, $K_h(k,X_o,-k')$, is an integral over

the factor $e^{-i\ell(\Theta_k-\Theta_{k'})}J_\ell(\mu_o)J_\ell(\mu_o')$, where $\mu_o' \equiv \mu_o(k')$. Therefore, $K_h(k,X_o,-k)$ is even in k, as is $d_k^2K_h$. From Eqs. (III-19), we see that the remaining term, $-\partial_x\Gamma_R \approx -\partial_x\Gamma_{1R}$ in Eq. (III-25), has a k dependence given by

$$\begin{aligned}\Gamma_{1R} &\equiv \frac{1}{2}(\Gamma_1+\Gamma_1^*) \equiv \frac{1}{2}(i\partial_k - i\partial_{k'})K(k,x,-k')\big|_{k=k'} \\ &\sim (i/2)(\partial_k-\partial_{k'})e^{-i\ell(\Theta_k-\Theta_{k'})}J_\ell(\mu_o)J_\ell(\mu_o')\big|_{k=k'} \\ &= \ell(\partial_k\Theta_k)J_\ell^2(\mu_o). \end{aligned} \tag{III-64}$$

From Eqs. (III-59), the factor

$$\begin{aligned}\partial_k\Theta_k &\approx \partial_k\Theta_o = \partial_k \arctan(k\bar{x}_o/\mu_o) \\ &= k_2^2\bar{x}_{20}(\bar{x}_o/\mu_o)[\mu_o^2 + (k\bar{x}_o)^2]^{-1}\end{aligned} \tag{III-65}$$

is even in k, as is $J_\ell^2(\mu_o)$. Thus each of the terms in expression (III-25) for $S_<$ is even in k, to the order we are working, as assumed in Eq. (III-27) in obtaining the condition k = o. For k = o, Eq. (III-65) reduces to

$$\partial_k\Theta_k = \bar{x}_o/\mu_o = (\Omega_o/\Omega_c)k_2^{-1}. \tag{III-66}$$

III.C.3. Specializing K_{sB}: Choice of Distribution f_s

We proceed further with expression (III-54) for K_{sB}, by choosing a particular form for the distribution fucntion f_s, so that the integrals over momenta appearing in K_{sB} can be performed. We are guided in this choice by the work of Part I. As in Sec. III.C.1, we assume we have adjusted the variable parameters in f_s [such as $w_{\|s}(X_o)$, the average

parallel velocity at X_o of particles of species s having guiding center X_o], so that the self-consistent and reference variables are the same. Then the physical and quasiphysical regions of the $(H,\underline{P})$ space are also the same, and so the value of f_s outside this region is irrelevant.

We begin by specifying f_s quite generally, and then progressively specialize it to get more spcific results. We modify the form of f_s from that of Part I to account for the parallel drift velocity $w_{\parallel s}(X_o)$ (just defined in more precise terms) of the distribution, and for the perpendicular drift $v_{02} = v_E$. As in Part I, we want to express f_s in terms of functions of the invariants which have a readily discernable physical meaning. One of these is X_o, already incorporated into the formalism. The second is the parallel velocity u of a particle at X_o, relative to the average parallel velocity $w_{\parallel s}$ of the particles of that species with guiding center X_o, $u(\underline{P}) \equiv v_3 - w_{\parallel s}$. Finally, we need a function of the invariants describing the perpendicular kinetic energy (again, at the guiding center). From Sec. II.K.3, we recall that the perpendicular kinetic energy at X_o is $\frac{1}{2}(v_{01}^2 + v_{02}^2) = \frac{1}{2}(v_{01}^2 + v_E^2)$. Going into the frame drifting with perpendicular velocity $v_{02} = v_E$ leaves only the kinetic energy in the $\hat{b}_1 \equiv \hat{x}$ direction, $h' \equiv \frac{1}{2} v_{01}^2$ (which is what one means by perpendicular kinetic energy when writing, for example, $\mu = mv_\perp^2/2B$, as we saw in Secs. II.J and II.K.3). We shall thus express f_s in terms of X_o, u, and h' or $H' \equiv h' + \frac{1}{2}u^2 = H - e_s\Phi(X_o) - \frac{1}{2} v_E^2 - w_\parallel v_3 + \frac{1}{2}w_\parallel^2$.

We now perform manipulations on the factor $\omega\partial f_s/\partial H + \hat{k}\cdot\partial\underline{P}$ appearing in (III-45), and will then work on the denominator $\omega + \underline{\ell}\cdot\underline{\Omega}$. Expressing f_s in terms of H' instead of H, we have

$$\partial f_s/\partial \underline{P})_H = \partial f_s/\partial \underline{P})_{H'}$$

$$+ \partial f_s/\partial H' \left\{(\partial X_o/\partial \underline{P})[e_s E_o - u(\partial w_\parallel/\partial X_o) - v_E(\partial v_E/\partial X_o)] - w_\parallel(\partial v_3/\partial \underline{P})\right\}, \quad \text{(III-67)}$$

where $E_o \equiv E(X_o) \equiv -\partial\Phi/\partial X_o$, as in Part II.

We shall take f_s as a sum over terms f_{sj}, as in Sec. I.C.3. Eq. (III-67) then also holds for each of the f_{sj}. Using (III-67), we find that

$$\omega\partial f_{sj}/\partial H)_{\underline{P}} + \hat{\underline{k}}\cdot\partial f_{sj}/\partial \underline{P})_H = -\bar{\gamma}_{sj}(\omega' - \omega_{*sj})f_{sj} + [k_3 - k_2(\partial w_\parallel/\partial X_o)]\partial f_{sj}/\partial u, \quad \text{(III-68)}$$

where $k_j \equiv \hat{b}_j \cdot \underline{k}$ $(j = 2, 3)$, and where we have the generalized forms of familiar symbols:

$$\omega' \equiv \omega - k_2 u_E - k_3 w_\parallel, \quad \text{(III-69)}$$

$$u_E \equiv (\partial X_o/\partial P_2)[-e_s E_o + u(\partial w_\parallel/\partial X_o) + v_E(\partial v_E/\partial X_o)] \quad \text{(III-70)}$$

$$\omega_{*sj} \equiv k_2 \kappa_{fj}(\partial X_o/\partial P_2)\bar{\gamma}_{sj}^{-1}, \quad (\partial X_o/\partial P_2) \equiv \hat{b}_2 \cdot (\partial X_o/\partial \underline{P}), \quad \text{(III-71)}$$

$$\kappa_{fj} \equiv (\partial/\partial X_o)\ln f_{sj}, \qquad \bar{\gamma}_{sj} \equiv -(\partial/\partial H')\ln f_{sj}, \quad \text{(III-72)}$$

and used $f_s = f_s(H', u, X_o)$ in computing $\partial f_s/\partial \underline{P})_{H'}$. We note that ω_{*sj}, κ_{fj}, and $\bar{\gamma}_{sj}$ are not functions of X_o, u, and h. Taking the H' dependence of f_{sj} to be $\exp(-\gamma_j H')$ (i.e., assuming a local Maxwellian form, with uniform temperature), we have $\bar{\gamma}_{sj} = \gamma_j$ = constant. The variation of $f_s(H, \underline{P})$ with respect to P_2, i.e., $\hat{b}_2 \cdot (\partial f_s/\partial \underline{P})$, is dominated by the variation of the density $n(X_o)$ [cf. Eq. (III-83)], and thus $\kappa_{fj} \simeq \kappa_n \equiv (\partial/\partial X_o)\ln n(X_o)$. In the limits of no shear and $\partial v_E/\partial X_o \to 0$, $\partial X_o/\partial P_2 \to (m\,\Omega_c)^{-1}$, and so

$$u_E \to v_E = -cE_o/B_o, \qquad \text{and} \quad \omega_{*sj} \to k_2\kappa_n v_{sj}^2/\Omega_c$$

($v_{sj}^2 \equiv \gamma_{sj}^{-1}$), the usual diamagnetic drift frequency. With shear, the factor $(\partial X_o/\partial P_2)$ becomes strongly v_3 - dependent [cf Eq. (III-85)], in fact becoming singular as $(1 + \kappa_s v_3/\Omega_c)$ goes to zero. Accordingly, the expressions for u_E and ω_{*sj} are modified by finite $\kappa_s v_3/\Omega_c$, just as found in Sec. II.K for the gyrofrequency Ω. In particular, for values of $\underline{P}$ where the function $X_0(\underline{P})$ approaches the point of double-valuedness (as for point a in Fig. 2), $\partial X_o/\partial P_2$ becomes infinite, as do u_E, ω_{*sj}, and Ω_o^{-1}. The approach of Ω to zero is a real one, arising as the septum creating the two wells in Fig. 3 appears, as $|v_3|$ is increased from zero, so that even a particle with $\eta_{\underline{B}} \equiv \rho_i/L_{\underline{B}} \simeq 0$ can make very large excursions in x, in this very flat-bottomed well. In this unstable situation, even a very slight perturbation $\delta E_2 \sim k_2\phi$ can cuase large changes in X_o; hence the large contributions to the response $k_2 u_E$, $\omega_{*sj} \sim k_2$ in (III-68) and (III-69). The response cannot really become infinite, of course; as these terms go to infinity, the linearization approximation upon which the conductivity kernel was derived breaks down.

We turn now to expressing the resonant denominator $\omega + \underline{\ell} \cdot \underline{\Omega}$ in terms of the new variables (u,h'). We rewrite Eqs. (II-117) and (II-118) as

$$\dot{X}_{20} = v_E + v_B\,(u,h'), \qquad \dot{X}_{30} = w_\parallel + u, \tag{III-73}$$

where v_B, the generalization of the $B \times \nabla B$ drift, may be read off from Eq. (II-117). We also expand the gyro-frequency Ω (which, we recall, is v_3-dependent) about its value at the drift frame velocity $\hat{b}_2 v_E + \hat{b}_3 w_\parallel$.

From Eq. (II-109), we have:

$$\Omega = \Omega_c[1 + (\partial v_E/\partial X_o)/\Omega_c + \kappa_s v_3/\Omega_c]^{1/2} + O(\eta_{\underline{B}}^{\ 2})$$
$$\simeq \Omega_o + \frac{1}{2}(\Omega_c/\Omega_o)(\kappa_s u) \equiv \Omega_o + \delta\Omega \qquad \text{(III-74)}$$

where we redefine Ω_o slightly from its meaning in Part II, by $\Omega_o \equiv \Omega(u = 0) \equiv \Omega_c[1 + (\partial v_E/\partial X_o)/\Omega_c + \kappa_s w_{\parallel}/\Omega_c]^{1/2}$. Using (III-73), (III-74), we then have

$$\omega + \underline{\ell} \cdot \underline{\Omega} = \omega'' - \ell\Omega_o - (k_2 v_B + k_3 u + \ell\delta\Omega),$$
$$= \omega'' - \ell\Omega_o - (k_2 v_B + k_3' u), \qquad \text{(III-75)}$$

where $\omega'' \equiv \omega - k_2 v_E - k_3 w_{\parallel}$ is the wave frequency as seen from the drift frame, and $k_3' \equiv k_3 + \ell(\Omega_c/\Omega_o)\kappa_s/2$. We see that the dispersion in Ω i.e., its dependence upon u) has introduced a term which makes $\delta k_3 \equiv k_3' - k_3 \neq 0$. Since this term is proportional to ℓ, even for weak shear, δk_3 may be substantial, for sizeable ℓ. This can have significant effects on growth rates of instabilities such as the ion drift cyclotron instability (to be studied in Sec. III.D), where the assumption $k_3 \simeq 0$ is usually made.

Using the results obtained thus far in this section, we may write Eq. (III-54), valid for arbitrary functions $f_{sj}(H', u, X_o)$, as

$$K_{sB} = \sum_j K_{sjB}, \quad K_{sjB} = K_{sjB1} + K_{sjB2}$$
$$K_{sj(B1,B2)}(k, X_o, -k') \equiv 8\pi^2 e^2 \sum_\ell \int du \int dh' |\Omega^{-1}| \tilde{J}_\ell(\underline{k}) \tilde{J}_\ell^{\ *}(\underline{k}') F_{B1.B2}, \qquad \text{(III-76)}$$

where the factors $F_{B1,B2}$ are given by

$$F_{B1} \equiv -\bar{\gamma}_{sj}\ (\omega' - \omega_{*sj}) f_{sj}\ (\omega + \underline{\ell} \cdot \underline{\Omega})^{-1},$$
$$F_{B2} \equiv [k_3 - k_2\ (\partial w_{\parallel}/\partial X_o)](\partial f_{sj}/\partial u)(\omega + \underline{\ell} \cdot \underline{\Omega})^{-1}, \tag{III-77}$$

and $(\omega + \underline{\ell} \cdot \underline{\Omega})$ is given by (III-75).

Now we further specialize the form of f_s. Following (I-21) and (I-26), we take

$$f_s(H' = h' + \frac{1}{2} u^2,\ u, X_o) = n_s |\Omega I^{-1}| e^{-\gamma_s u^2/2} (\bar{g}_o e^{-\gamma_s h'} - \bar{g}_1 e^{-R\gamma_s h'}). \tag{III-80}$$

The normalization, analogous to (I-23), is taken as

$$n_s(X_o) = 2\pi \int_0^\infty dh' \int_{-\infty}^\infty du |\Omega^{-1} I| f_s. \tag{III-81}$$

Putting (III-70) into (III-71) gives

$$1 = (2\pi/\gamma_s)^{3/2}\ (\bar{g}_o - \bar{g}_1/R), \tag{III-82}$$

as in (I-27).

By taking $0 < \bar{g}_1 < \bar{g}_o$, one may consider a loss cone type distribution, with loss cone partially filled $(\bar{g}_1 < \bar{g}_o)$, if desired. In the application to be made in Sec. III.D, we shall take $\bar{g}_1 = 0$, a drifting Maxwellian distribution. We assume $\bar{g}_1 = 0$ henceforth, and so may drop the j subscript on $f_{sj=o} = f_s$. Equation (III-80) then reduces to

$$f_s(H', X_o) = n_s |\Omega I^{-1}| \bar{g}_o\ e^{-\gamma_s H'}, \tag{III-83}$$

where $\bar{g}_o = (\gamma_s/2\pi)^{3/2}$. Since $f_s(H', X_o)$ is not explicitly u-dependent, F_{B2} in (III-77) and K_{sB2} in (III-76) vanish, so that $K_{sB} = K_{sB1}$. In the following section, we use our choice (III-83) for f_s in expressions (III-76) and (III-77) to explicitly evaluate the integrals prescribed there.

III.C.4. K_{sB} for the Drifting Maxwellian

In this section we use the specialized form (III-83) for f_s to obtain a more explicit expression for the contribution $K_{sB}(k,x,-k')$ to the conductivity. From (III-76) and (III-77),

$$K_{sB}(k,X_o,-k') = -\lambda_s^{-2} \sum_{\ell} (2\pi)\int du\int dh' \; \bar{g}_o \; e^{-\gamma_s u^2/2} \; e^{-\gamma_s h'} \times e^{-i\ell(\Theta_k-\Theta_{k'})} \; J_\ell(\underline{k}) J_\ell(\underline{k}') (\omega'-\omega_{*s})(\omega''-\ell\Omega_o-k_2v_B-k_3'u)^{-1}. \tag{III-84}$$

Orbit modification effects enter here in several places, all of which we lump into the term $K_{o.m}$. There is the $O(\eta_{\underline{B}})$ correction to $\tilde{J}_\ell \equiv J_\ell \; e^{-i\ell\Theta_k}$ from its harmonic limit, as already discussed. There is also the u dependence of u_E and ω_{*s}, in addition to that for Ω, already expressed in Eq. (III-74). From Eqs. (III-70), (III-71), we see that both these are proportional to $(\partial X_o/\partial P_2) \equiv (\partial X_o/\partial \underline{P})\cdot\hat{b}_2$. From Part I, we know that along the pseudotrajectory $\underline{A}^{sc}(\bar{x})$, $(\partial X_o/\partial P_2) = (m\Omega_c)^{-1}$. Off this trajectory $(v_3 \neq 0)$, $(\partial X_o/\partial P_2)$ is shifted from this value by the local radius of curvature $P = eB/\kappa_s$ of $\underline{A}^{sc}$ in the $\underline{P}$ plane:

$$(\partial X_o/\partial P_2) = (m\Omega_c)^{-1} \; (1 + v_3/P)^{-1} = (m\Omega_c)^{-1} \, (1+\kappa_s v_3/\Omega_c)^{-1}. \tag{III-85}$$

As in (III-74), we may thus expand u_E, ω_{*s} about $v_3 = w_{\|}$, the first order corrections being $O(\kappa_s u/\Omega_c) \sim O(\eta_{\underline{B}})$ down from the zero order terms. In the case of Ω, this $O(\eta_{\underline{B}})$ correction appears in K_{sB} multiplied by harmonic number ℓ, and so we retain it in K_h. The corrections to K_{sB} due to this dispersion in u_E, ω_{*s} are only $O(\eta_{\underline{B}})$, and so are put into $K_{o.m.}$. Each of these $O(\eta_{\underline{B}})$ terms comprising $K_{o.m.}$ can be explicitly evaluated, if desired. These terms are of higher order than we require, as already

mentioned, however, and so shall not be explicitly displayed here. Also lumped into the neglected $K_{o.m.}$ contribution is the term from $k_2 v_B$ in (III-84). Finally, we neglect the terms in $\partial w_{\|}/\partial X_o$ and $\partial v_E/\partial X_o$ in (III-70), so that u_E may be written

$$u_E(v_3) = v_E(1 + \kappa_s v_3/\Omega_c)^{-1}. \qquad \text{(III-86)}$$

We denote by u_E, ω_{*s} the zero-order ($v_3 = w_{\|}$) values of these quantities. From Eqs. (III-58b) we note that $\Theta_o \equiv \arcsin(k\bar{x}_o/\mu_o)$ is independent of the action J, and hence of h', and for $k/k_2 << 1$, of u. We may thus approximate (III-84) by its contribution to K_h:

$$K_{sB}(k,X_c,-k') = -\lambda_s^{-2}\sum_\ell e^{-i\ell(\Theta_k-\Theta_{k'})} 2\pi \int du \int dh' \, \bar{g}_o \, e^{-\gamma_s u^2/2} \, e^{-\gamma_s h'}$$
$$\times J_\ell(\mu_o) J_\ell(\mu_o')(\omega' - \omega_{*s})(\omega'' - \ell\Omega_o - k_3' u)^{-1} \qquad \text{(III-87)}$$
$$= -\lambda_s^{-2}\sum_\ell e^{-i\ell(\Theta_k-\Theta_{k'})}[1-W(z_{\ell s})](\Omega_1/\Omega_2) I_\ell(\mu_s\mu_s')\exp[-\tfrac{1}{2}(\mu_s^2+\mu_s'^2)].$$

Here $\Omega_1 \equiv \omega'-\omega_{*s}$, $\Omega_2 \equiv \omega''-\ell\Omega_o$, $z_{\ell s} \equiv \Omega_2/|k_3' v_s|$, $v_s \equiv \gamma_s^{-1/2}$, $\mu_s^2 \equiv \mu_s^2(k) \equiv (v_s/\Omega_o)^2\,[k^2 + k_2^2(\Omega_c/\Omega_o)^2]$, $\mu_s' \equiv \mu_s(k')$, I_ℓ is the modified Bessel function of index ℓ, and

$$W(z) \equiv (2\pi)^{-1/2}\int_{-\infty}^{\infty} dx\, x\, e^{-x^2/2}(x-z)^{-1} = 1 + zZ(z), \qquad \text{(III-88)}$$

for $z \equiv z_R + i\, z_I$ in the upper half plane, where

$$Z(z) \equiv (2\pi)^{-1/2}\int_{-\infty}^{\infty} dx\, e^{-x^2/2}(x-z)^{-1} \qquad \text{(III-89)}$$

is the plasma dispersion function. In obtaining (III-87) we have made use of the Bessel identity[28]

$$\int_0^\infty dx\, x\, e^{-x^2/2}\, J_\ell(qx)\, J_\ell(q'x) = I_\ell(qq')\, \exp[-\tfrac{1}{2}(q^2+q'^2)], \qquad \text{(III-90)}$$

and of the fact that $\bar{x}_o^{\,2} = v_{o1}^2/2V_2 = (v_{o1}/\Omega_o)^2$, so that

$$\mu_o^{\,2} = \bar{x}_o^{\,2}[k^2+k_z^{\,2}(\Omega_c/\Omega_o)^2] = (v_{o1}/\Omega_o)^2[k^2+k_z^{\,2}(\Omega_c/\Omega_o)^2]. \qquad \text{(III-91)}$$

From (III-87), the contribution of K_{sB} to $K \equiv K(k,X_o-k)$ is

$$K_{sB}(k,X_o,-k) = -\lambda_s^{-2}\sum_{\ell}(1-W_{\ell s})(\Omega_1/\Omega_2)\Lambda_{\ell s}, \qquad \text{(III-92)}$$

where $\Lambda_{\ell s} \equiv \Lambda_\ell(b_s)$, $b_s \equiv \mu_s^{\,2}$, $\Lambda_\ell(b) \equiv I_\ell(b)\,e^{-b}$, and $W_{\ell s} \equiv W(z_{\ell s})$. For $K_{n.\ell.}$, we also need the contribution of species s to $\Gamma_1 \equiv i\partial_k K(k,X_o,-k')|_{k=k'}$. Since K_{sA} in Eq. (III-56) is k- independent, this contribution is given by

$$i\partial_k K_{sB}(k,X_o,-k')|_{k=k'} = -\lambda_s^{-2}\sum_{\ell}(1-W_{\ell s})(\Omega_1/\Omega_2)[\ell\partial_k\Theta_k)\Lambda_{\ell s} + (i/2)(\partial_k b_s)\Lambda'_{\ell s}], \qquad \text{(III-93)}$$

where $\Lambda'_{\ell s} \equiv \partial\Lambda_\ell(b_s)/\partial b_s$. The term in $(\ell\partial_k\Theta_k)$ yields Γ_{1R}, and the one in $(\partial_k b_s)$ gives Γ_{1I}. In the frequency regime we shall be considering, we have $\omega < \ell\Omega_i << \Omega_e$, and so $\Omega_2 \lesssim \ell\Omega_i$ for the ions, and $\Omega_2 \sim \ell\Omega_e$ for the electrons. Therefore the electron contribution to Γ_1 is down from the ion contribution by a factor $\lesssim (\Omega_i/\ell\Omega_e)$. The electron contribution is thus completely negligible, reflecting the fact that the small electron gyroradius causes the electron contribution to the conductivity to be almost truly local. Since in addition K_{sA} does not contribute to $K_{n.\ell.}$, nor does the vacuum term K_v (being x independent), one has the form

$$K_{n.\ell.} = -\partial_x[i\partial_k K_{iB}(k,x,-k')|_{k=k'} + \text{c.c.}],$$

already stated in Sec. III.B.3. Using

$$\ell\Omega_1/\Omega_2 = -\Omega_1/\Omega_o + (\omega''/\Omega_o)(\Omega_1/\Omega_2) \qquad \text{(III-94)}$$

and the Bessel identity

$$1 = \sum_{\ell=-\infty}^{\infty} \Lambda_\ell(b), \tag{III-95}$$

we find

$$K_{n.\ell.} = \partial_x[\lambda_i^{-2} (\partial_k\Theta_k) \sum_\ell (1-W_{\ell i})(\ell\Omega_1/\Omega_2)\Lambda_{\ell i}]$$

$$= \partial_x[-\lambda_i^{-2}(\partial_k\Theta_k) I_1(\Omega_1/\Omega_o) + \lambda_i^{-2}(\partial_k\Theta_k)(\omega''/\Omega_o)\sum_\ell(1-W_{\ell i})(\Omega_1/\Omega_2)\Lambda_{\ell i}], \tag{III-96}$$

where the species label s=i on Ω_1, Ω_2, ω'', and Ω_o is implicit, and $I_1 \equiv 1- \sum_\ell W_{\ell i} \Lambda_{\ell i}$. In the limit of no shear ($\eta_s \to 0$), we will consider modes in which $k_3' = k_3 = 0$, $W_{\ell i} = W(z=\infty) = 0$, for all ℓ, and thus $I_1 = 1$. In Fig. 11 we sketch the behavior of the real and imaginary parts, W_R and W_I, of $W(z)$. These functions are substantial in the region $|z| \lesssim 1$, going rapidly to zero for larger $|z|$. We write $\omega'' = (\ell+\Delta\ell)\,\Omega_{oi}$, with $\Delta\ell$ some substantial fraction of one. Thus $z_{\ell' i} \equiv (\omega'' - \ell'\Omega_{oi})/[k_3+\frac{1}{2}\ell'\kappa_s)v_i]$ $\simeq (\ell + \Delta\ell - \ell')/[(k_3+ \frac{1}{2}\ell'\kappa_s)\rho_i]$. Again assuming $k_3\rho_i \ll 1$, we sketch the ℓ' dependence of this in Fig. 12. We see that it is only for ℓ' in the vicinity of ℓ that $|z_{\ell' i}| \sim 1$, and so it is only a few terms of the sum in I_1 that contribute. Thus, even in the presence of shear, we expect $|I_1|$ to be some number $\lesssim 1$. We also note that $K_{n.\ell.}$, and hence Γ_{1R}, have both real <u>and</u> imaginary parts, reflecting the distinction between truly real and formally real.

We have now evaluated the terms called for by the dispersion relation, Eq. (III-41). Before going on to an application, we can now use the explicit expressions just obtained to make more precise the various assumptions on ordering we have made up to this point. We proceed to do this is in the following section.

III.C.5. Some Estimates and Ordering

In the development thus far, we have made a number of assumptions on the relative sizes of the various terms under consideration. In the application of the formalism in Sec. III.D, we shall make a number of additional ordering assumptions. Now that we have derived explicit expressions for the quantities called for by the abstract formalism of Sec. III.B, we are in a position to consider more precisely what these approximations imply.

There are a number of small parameters which enter into the theory. To begin with, there are the parameters η_s, $\eta_{\underline{B}}$, η_n, etc., defined in the Introduction to Part III, and used frequently throughout each part of this thesis. As has been noted, in Sec. III.D we shall be considering mode frequencies $\omega \sim \ell\Omega_i$, with ℓ a moderately large number, perhaps in the range $5 \lesssim \ell \lesssim 20$, for which an analysis using separate Bernstein harmonics is preferable to one involving the straight line orbit approximation. Thus ℓ^{-1} is another small parameter.

In the truncation procedure of Sec. III.B.2, we neglected terms with derivatives of $K_<$ of higher than second order, first in k, and later in x. From the structure of K_{sB} in Eqs. (III-54), (III-84), or (III-87), we see that ∂_k acts on two places in K_{sB}: on the factor $e^{-i\ell\Theta_k}$, and on the factor $J_\ell(\mu_o)J_\ell(\mu_o')$ or $\Lambda_\ell(\mu_s\mu_s')$. (Here, ℓ satisfies $\ell\Omega_s \simeq \omega$, hence $5 \lesssim \ell \lesssim 20$ for the ions, as noted above, and $\ell \simeq 0$ for the electrons.) Using Eqs. (III-65) or (III-66), one has that

$$\partial_k^n e^{-i\ell\Theta_k} \sim (\ell\partial_k\Theta_k)^n e^{-i\ell\Theta_k} \sim (\ell/k_2)^n e^{-i\ell\Theta_k}, \qquad \text{(III-97)}$$

and recalling that $b_s \equiv \mu_s^2 \simeq (k_\perp \rho_s)^2$, one finds

$$\partial_k^{2n} \Lambda_{\ell s} \lesssim k_2^{-2n} \Lambda_{\ell s}, \quad \partial_k^{2n+1} \Lambda_{\ell s} \lesssim (k/k_2^{2n}) \Lambda_{\ell s}. \qquad \text{(III-98)}$$

In view of these scalings, the action of ∂_k on the former factor dominates the latter, for the ions, by the harmonic number ℓ. Also, by Eq. (III-27), (k/k_2) is equal to (or very nearly equal to) zero. Thus, we have

$$\Gamma_{1I}/\Gamma_{1R} \sim (\partial_k \ln b_s)/(\ell \partial_k \Theta_k) \sim (k/k_2 \ell) << 1. \qquad \text{(III-99)}$$

For $K(k,x,-k)$, the factor in Θ_k is not present, and so ∂_k acts purely on the b_s dependence. Thus

$$\frac{1}{2} d_k^2 K \sim k_2^{-2} K, \qquad \text{(III-100)}$$

Now we can discuss the condition for convergence of the infinite order operator K of Eq. (III-13) or (III-14), and the validity of truncation at finite order. In Sec. III.B, these conditions were abstractly symbolized by the condition $\partial_x \partial_k < 1$. Making the straightforward replacements $\partial_k \to \ell/k_2$, $\partial_\chi \to L_\phi$, we arrive at the convergence condition

$$1 > \partial_x \partial_k \sim (\ell/k_2 L_\phi) \equiv \varepsilon_{\phi\ell}. \qquad \text{(III-101)}$$

For the electrons, where $\ell \approx 0$, this condition is easily satisfied, even for very small L_ϕ. The truncation at second order for the electrons is thus a very good approximation, reflecting the highly local nature (small $k\rho_e$) of the electron response. We therefore henceforth consider (III-101) only in connection with the ions. For the ions, this condition can be rather stringent. The ratio of the parameter $\varepsilon_{\phi\ell}$ to ρ_i/L_ϕ

(which was the parameter for determining whether the nearly local or fully integral form of S_{iB} should be employed) is given by $\varepsilon_{\phi\ell}/(\rho_i/L_\phi) = (\ell/k_2\rho_i)$. In Eq. (III-104), we shall find that $(\ell/k_2\rho_i) \sim \eta_n < 1$. Thus Eq. (III-101) is a somewhat more stringent condition than $\rho_i/L_\phi < 1$ for loss of validity of the nearly-local form of S_{iB}. At the end of this section [see Eq. III-116]], we discuss why condition (III-101) is probably more stringent than necessary, and may be replaced by $\varepsilon_{\phi 1} < 1$. This criterion is of stringency comparable to $\rho_i/L_\phi < 1$. Thus, when the validity of the nearly-local approximation for S_{iB} breaks down, the regime where $\rho_i/L_\phi > 1$ is entered. In this regime, dispersion equation (III-41b) becomes valid.

Applying the same sort of scaling argument as used to obtain (III-101), and using (III-45) for $L_\phi^{-2} = |\alpha|$, we reexpress condition (III-47) for mode localization:

$$1 > (L_\phi/L_Q)^2 \simeq (k_2 L_Q)^{-1}. \qquad \text{(III-102)}$$

This condition is met whenever (III-101) is.

We can estimate the size of k_2, for the application of Sec. III-D, using the fact that we shall be considering modes in a frequency regime where

$$\omega \sim \ell\Omega_i \simeq \omega_{*i} \simeq k_2\, \kappa_n\, \rho_i\, v_i. \qquad \text{(III-103)}$$

This gives

$$k_2\rho_i \simeq \ell L_n/\rho_i \equiv \ell\,\eta_n^{-1} > \ell \qquad \text{(III-104)}$$

and so (III-93) gives a condition on L_Q:

$$\eta_Q \equiv \rho_i/L_Q < \ell\,\eta_n^{-1}. \qquad \text{(III-105)}$$

Before we can compare the relative sizes of the various terms in dispersion equations (III-41), we must estimate the size of L_Q. Without shear, L_Q is given by the scale length $L_p \sim L_n$ of the plasma parameters. As $\kappa_s \equiv L_s^{-1}$ is increased from zero, however, a new scale length Loe, defined by

$$L_{oe} \equiv [\partial_x |\ln z_{oe}|]^{-1} \simeq L_s(\omega/k_2 v_e) << L_s \qquad \text{(III-106)}$$

moves rapidly in from infinity, becoming the dominant (shorter) scale length of the potential Q. Thus,

$$L_Q \equiv \min\ (L_n, L_{oe}). \qquad \text{(III-107}$$

The changeover from $L_Q = L_n$ to $L_Q = L_{oe}$ occurs when

$$\kappa_s/\kappa_n \equiv L_n/L_s \simeq \omega/k_2 v_e \simeq \eta_n\ (v_i/v_e) << 1, \qquad \text{(III-108)}$$

where we have used Eqs. (III-103) and (III-104) in the second approximation. For $T_e \simeq T_i$, and $\eta_n \simeq \frac{1}{5}$, this gives $\kappa_s/\kappa_n \simeq (1/200)$ at changeover. Thus, essentially because $v_i/v_e << 1$, shear effects become strong long before the naive guess $\kappa_s/\kappa_n \sim 1$ one might make.

We now compare the relative sizes of the terms appearing in the dispersion relation, using the scalings discussed in this section. We find

$$\begin{aligned} &K_{d.e.}/K \sim f_n(k_2 L_Q)^{-1}, \quad K_{n.\ell.}/K \sim (\ell/k_2 L_p), \\ &K_{o.m.}/K \sim \eta_{\underline{B}}\ , \ K_o \sim K_n \sim K. \end{aligned} \qquad \text{(III-109)}$$

We note that $K_{n.\ell.}$ scales as the reciprocal of $L_p \equiv \min(L_n, L_{\underline{B}})$ instead of as L_Q^{-1}, as does $K_{d.e.}$. This reflects the fact, noted in Sec. III.C.4, that the electron contribution to Γ_{1R} is down from the ion contribution by $(m_e/m_i \ell)$. The action of ∂_x in $K_{n.\ell.} = -\partial_x \Gamma_{1R}$ brings in (through L_{oe})

a factor $(m_i/m_e)^{1/2}$ which partially compensates, but still leaves the ion contribution dominant. The dependence of $K_{n.\ell.}$ on L_s^{-1} is thus much weaker than that of $K_{d.e.}$.

From Eqs. (III-109) and (III-104), we have that $K_{o.m.}/K_{d.e.} \lesssim \eta_{\underline{B}}\, k_2\, L_Q \sim (L_Q/L_{\underline{B}})\ell\eta_n^{-1}$. For both the regime $L_Q = L_n$ and $L_Q = L_{oe}$, using this condition and expression (III-108) gives

$$K_{o.m.}/K_{d.e.} \lesssim (v_i/v_e)_- << 1, \qquad \text{(III-110)}$$

justifying our assumption that $K_{o.m.}$ may be neglected in (III-41), replacing K there by K_h.

In Fig. 13 we display the scaling of these various ratios (K_x/K_h) (x = h, d.e., n.ℓ., o.m.) with κ_s/κ_n. We employ the parameters $\varepsilon_{p\ell} \equiv \ell/k_2L_p$ $\varepsilon_{p1} \equiv 1/k_2L_p$, and η_n as calibrations on the ordinate axis (not drawn to scale). We note the elbow in $(K_{d.e.}/K_h)$ due to the transition from $L_Q = L_n$ to $L_Q = L_{oe}$, occurring at $\kappa_s/\kappa_n \sim (\omega/k_2v_e)$, and see that $K_{o.m.}/K_{d.e.} << 1$, as shown just above algebraically. Using $L_\phi \simeq \alpha^{-1/2} \sim L_Q(k_2L_Q)^{-1/2}$ and (III-104), we express (III-101) as

$$1 > \ell(k_2L_Q)^{-1/2} \simeq (\ell\eta_n\eta_Q)^{1/2}. \qquad \text{(III-111)}$$

This convergence condition is violated at $\kappa_s/\kappa_n \simeq (v_i/v_e)(\ell\eta_n)^{-1} \simeq (L_{oe}/L_s)(\ell\eta_n^2)^{-1}$, indicated by line a in Fig. 13. To the right of line b, the less stringent condition (III-116) or (III-117) breaks down, and the criterion $\rho_i/L_\phi < 1$ fails at some point in the vicinity of this line. To the right of this line, use of the form S_{iB} valid for $\rho_i/L_\phi > 1$ becomes appropriate.

From the point of view of the ions, the appearance of the localized mode in the nearly local ($\rho_i/L_\phi < 1$ and fully integral ($\rho_i/L_\phi > 1$)

regimes is quite different. However, we now show that the dispersion equations for these two different regimes are, nevertheless, quite similar. Using the large argument limit of Λ_ℓ,

$$\Lambda_\ell(b) \xrightarrow[b>\ell^2]{} (2\pi b)^{-1/2}, \qquad \text{(III-112)}$$

and putting this into Eq. (D29), one has

$$\Lambda_{\ell i}\ (\rho_s/L_\phi < 1) \simeq (2\pi b_i)^{-1/2} \simeq (2\pi k_\perp^2 \rho_i^2)^{-1/2}. \qquad \text{(III-113)}$$

On the other hand, taking the n=0 value of (D30),

$$\Lambda_{\ell i}(\rho_s/L_\phi > 1) \simeq (2\pi^2)^{-1/2}\ [\alpha\rho_i^2 + k_2^2\rho_i^2]^{-1/2}. \qquad \text{(III-114)}$$

If we write $\alpha = \langle k_x^2 \rangle$, some average value of the wavevector in the $\hat{x} = \hat{b}_1$ direction, we see that (III-113) is the same as (III-114), up to a factor of $\pi^{1/2}$. Thus, up to this factor, the contribution of S_{iB} to dispersion equations (III-41a) and (III-41b) is the same. Moreover, from Fig. 13 we see that $K_{n.\ell.}$ has become small compared to $K_{d.e.}$ in the regime where $\rho_s/L_\phi > 1$ becomes valid, and so may be neglected in (III-41a). Equations (III-41a) and (III-41b) are then almost identical, though applying to quite different regimes. This similarity in form is corroborated by the numerical findings of Ref. 21. These authors perform a stability analysis for a plasma with strong density inhomogeneity using two methods, one (which they call the "local method") employing truncation of the exact kernel at second order, just as we do here, and the other (termed the "nonlocal method") solving Poisson's equation in k space, taking account of the coupling of wavenumbers caused by ∇n. Applying these methods numerically, these authors find the results of their local method to be in good agreement with the nonlocal method into the regime ($k_2 L_\phi > 1$) where

the latter method is strictly valid, but the former is not.

The authors of Ref. 21 postulate a much more easily satisfied convergence condition than (III-101), namely $1 > \varepsilon_{Q1}$, where $\varepsilon_{Q\ell} \equiv \ell/k_2 L_Q$. (Harmonic number ℓ does not enter into their considerations because, as noted in Sec. III.B.1, they make the approximation $k = k'$ from the beginning, and so set $e^{-i\ell(\Theta_k - \Theta_{k'})} = 1$.) Estimating the sizes of the n=0,1,2 terms in Eq. (III-18), we find

$$\begin{aligned} &(n=0 \text{ term}) \sim K, \ (n=1 \text{ term}) \sim i\partial_x \partial_k K \sim \varepsilon_{Q\ell} K, \\ &(n=2 \text{ terms}) \sim L_Q^{-2} d_k^{\,2} K \sim \varepsilon_{Q1}^{\,2} K, \text{ and } \partial_x^{\,2} \partial_k^{\,2} K \sim \varepsilon_{Q\ell}^{\,2} K. \end{aligned} \tag{III-115}$$

We thus see that, by manipulating terms as done in obtaining (III-18) from (III-17), we can change the parameter $\varepsilon_{\phi\ell}$ of (III-101) to either $\varepsilon_{\phi 1}$ (by collecting terms in $\partial_k^{\,m} \partial_{k'}^{n-m}$ to yield a term in $d_k^{\,n}$), or to $\varepsilon_{Q\ell}$ (by integrating derivatives ∂_x on ϕ_1 by parts so that they act only on K). If we thus postulate that the nth order terms of S scale as $\varepsilon_{\phi 1}^{\,m} \varepsilon_{Q\ell}^{\,n-m}$ $(m = 0, 1, \cdots n)$, where $\varepsilon_{Q\ell} \equiv \ell/k_2 L_Q$, condition (III-101) should be replaced by

$$1 > \varepsilon_{\phi 1}, \ \varepsilon_{Q\ell}. \tag{III-116}$$

Using this weaker condition, we obtain the convergence boundary

$$1 \simeq \varepsilon_{Q\ell} \simeq \eta_n \, \eta_Q, \text{ or } \kappa_s/\kappa_n \simeq (L_{oe}/L_s) \, \eta_n^{-2} , \tag{III-117}$$

instead of that given by Eq. (III-111).

III.D. Application: Ion Drift Cyclotron Instability

In this Section we apply the formalism thus far developed, studying the ion drift cyclotron (IDC) instability,[28] in both unsheared and sheared magnetic geometries. This instability is closely related to the drift cyclotron loss cone (DCLC) instability,[21,27] but finds its free energy source only in the density gradient, as opposed to the DCLC, where a loss cone ion distribution is the primary free energy source. The application displays some (but not all) of the useful features of the formalism. In Sec. III.D.1, we characterize the different shear regimes, into which the scaling of the various terms entering into dispersion equations (III-41) divides the problem. The subsequent sections consider each of these regimes in turn.

III.D.1. Overview

In the application of this section, we use dispersion equation (III-41a) in the regime where $\rho_i/L_\phi > 1$. However, in light of the discussion in Sec. III.C.5, the form of the dispersion equation in these two regimes is almost the same, as are the expressions for mode growth rates. The analysis divides itself naturally into three regimes, as a function of shear strength:

a) "Weak Shear" ($\kappa_s/\kappa_n < L_{oe}/L_s$, i.e. $L_Q \simeq L_n$): In this regime, the shear scale length L_{oe} is too long to significantly affect the form of the potential $Q(x)$. Modes are then localized in a potential well, of width on the order of L_n. Then $K_{d.e.}$ and α are real quantities. From Fig. 13 or Eq. (III-109), we see that in this range $K_{d.e.}^{(n=0)} << K_{n.\ell.}$ and

so $K_{d.e.}$ becomes appreciable only for the higher-n modes ($2n \sim \ell$). Equation (III-41a) is the appropriate dispersion equation.

b) "Intermediate Shear" [$L_{oe}/L_s < \kappa_s/\kappa_n < (L_{oe}/L_s)(\ell/\eta_n^2)$]: In this range, the density and shear scale lengths L_n and L_{oe} determining L_Q are comparable. Shear causes the real part Q_R of the potential Q to have a hump at the point where $k_z = 0$, surrounded by two wells at $|z_{oe}| \sim 1$(cf. Fig. 17b). Away from this hump, the mode is strongly Landau damped by electrons, and so the mode is localized in the Berk-Pearlstein fashion.[23] (This will be described in more detail at the end of this section.) Correspondingly, $K_{d.e.}$ and α are imaginary. $K_{d.e.}^{(n=0)}$ is now comparable to $K_{n.\ell}$, and so both should be retained in dispersion equation (III-41a), for $\rho_i/L_\phi < 1$. As the shear is increased still further in this regime, $L_Q \simeq L_{oe}$ becomes small enough that the range $\rho_i/L_\phi > 1$ is entered. Then Eq. (III-41b) is the appropriate dispersion equation, the nonlocal corrections entering through S_{iB}.

c) "Strong Shear" [$\kappa_s/\kappa_n > (L_{oe}/L_s)(\ell/\eta_n^2)$]: Here L_{oe} has become so small that Q(x) cannot localize modes, even in the Berk-Pearlstein manner. (The terms "weak", "intermediate", and "strong" here refer to the effect on the problem of $L_{oe} << L_s$. Thus L_s/L_n is said to be "small" when the effect of shear on Q(x) dominates that of L_n.) In the following sections, we shall discuss each of these regimes in turn.

Before proceeding to consider the IDC instability in these specific parameter regimes, we develop some more general expressions which are of use for more than one regime of κ_s/κ_n. The IDC instability arises from the interaction of a drift mode, the "ion-shielded electron convection" (ISEC) wave, and the ion Bernstein harmonics. We are thus in a

frequency regime $\omega \sim \ell\Omega_i << \Omega_e$, and so keep only the $\ell = 0$ term in the electron contribution K_{eB}. From Eqs. (III-11), (III-56), (III-92), and (III-96), we write

$$\begin{aligned} K + K_{n.\ell.} &\simeq K_h + K_{n.\ell.} \\ &= \underline{k}^2 + \lambda_e^{-2}\left\{1 - \Lambda_{oe}\left(\frac{\omega_e' - \omega_{*e}}{\omega_e''}\right)(1 - W_{oe})\right\} \\ &+ \lambda_i^{-2}\left\{I_2 - I_3 \sum_\ell \Lambda_{\ell i}\left(\frac{\omega_i' - \omega_{*i}}{\omega_i'' - \ell\Omega_{oi}}\right)(1 - W_{\ell i})\right\}. \end{aligned} \tag{III-118}$$

Here, $\underline{k}^2 \equiv k^2 + k_2^2 + k_3^2$, and we have summed K_{iB} and $K_{n.\ell.}$, yielding the factors $I_2 \equiv 1 - (\kappa_p \partial_k \Theta_k)(\Omega_1/\Omega_{oi})I_1$, and $I_3 \equiv 1 - (\kappa_p \partial_k \Theta_k)(\omega_i''/\Omega_{oi})$. We have made the replacement $\partial_x \to \kappa_p$ here, where κ_p is the inverse plasma scale length (for the ions). In the weak shear regime, and well into the intermediate shear regime, $\kappa_p = \kappa_n$. Even into the strong shear regime, where κ_s is only becoming comparable to κ_n, one still has $\kappa_p \sim \kappa_n$. Noting that $\omega_i' \sim \omega_i'' \sim \ell\Omega_i$, $\Omega_i \equiv \omega_i' - \omega_{*i} \sim \Omega_i$, and $\partial_k \Theta_k \sim k_2^{-1}$, we see that $I_2 \sim 1 - \varepsilon_{p1}$, $I_3 \sim 1 - \varepsilon_{p\ell}$.

For ω_i'' not near $\ell\Omega_{oi}$, one may neglect the K_{iB} contribution (the term Σ_ℓ) in (III-118), which is the origin of the ion Bernstein harmonics ($\omega \sim \ell\Omega_{oi}$). The remaining expression is our choice for K_o in the regime $\rho_i/L_\phi < 1$, whose form is dominantly responsible for determining the mode localization:

$$K_o \equiv k^2 + I_2\lambda_i^{-2} + \lambda_e^{-2}\,[1 - \Lambda_{oe}(1 - \omega_{*e}/\omega_e')(1 - W_{oe})]. \tag{III-119}$$

For the regime $\rho_i/L_\phi > 1$ where we have no contribution $K_{n.\ell.}$, K_o is given by (III-119), replacing I_2 there by unity. In (III-119) we have used (III-86), noting that $|\omega_e' - \omega_e''| \simeq |k_2(u_E - v_E)| \sim |(k_2 v_E)(\kappa_s \rho_e)| << |k_2 v_E|$.

Setting the (formally) real part of this to zero gives the dispersion relation for the ISEC mode:

$$\omega'_{ISEC} \equiv \omega_{ISEC} - k_2 v_E - k_3 w_{\|e} = -\omega_{*e}\Lambda_{oe}\, g_{Re}[(\kappa\lambda_e)^2 + \tau I_2 + (1-\Lambda_{oe}\, g_{Re})]^{-1}, \tag{III-120}$$

where $\tau \equiv \lambda_e^2/\lambda_i^2 \equiv T_e/T_i$, and $g_{Re} \equiv R_e g_e \equiv R_e(1-W_{oe})$. In the usual limit in which this dispersion relation is derived, $\varepsilon_{pl} \to 0$, $k_3 \to 0$, and $\omega'_{ISEC} \to \omega_{ISEC}$, so that $g_e \to 1$, $I_2 \to 1$, and Eq. (III-120) reads

$$\begin{aligned} \omega_{ISEC} &= -\,\omega_{*e}\Lambda_{oe}[(\underline{k}\lambda_e)^2 + \tau + b_e]^{-1} \\ &\simeq -\,\omega_{*e}\Lambda_{oe}[k_2^2(\lambda_e^2 + \rho_e^2) + \tau]^{-1}, \end{aligned} \tag{III-121}$$

where we have used $(1-\Lambda_{oe}) \simeq b_e \simeq k_2^2\rho_e^2$.

We consider ω_{ISEC} as a function of $k_2 \simeq |\underline{k}|$. From Eqs. (III-71) and (III-121), we have that $\omega_{ISEC} \propto k_2\,[k_2^2(\lambda_e^2 + \rho_e^2) + \tau]^{-1}$, and thus ω_{ISEC} has a maximum value ω_{ISEC}^{max}, occurring at $k_2^2(\lambda_e^2 + \rho_e^2) \simeq \tau$. This gives a maximum harmonic number, ℓ_{max}, for which the IDC instability can occur:

$$\ell_{max} \simeq \omega_{ISEC}^{max}/\Omega_i \simeq \tfrac{1}{2}(\kappa_n\rho_i)(\rho_i/\lambda_e)\tau^{1/2}(1+\omega_e^2/\Omega_e^2)^{-1/2}, \tag{III-122}$$

where $\omega_e(x) \equiv [4\pi n_e(x)e^2/m_e]^{1/2}$ is the electron plasma frequency for a plasma with density given by the guiding center density $n_e(x)$. As the density gradient κ_n weakens, ℓ_{max} decreases, until finally ℓ_{max} drops below unity, and the IDC instability disappears altogether.

For calculating $K_{d.e.}$, we need to compute $\frac{1}{2}\, d_k^2 K_o$. The derivative d_k acts only on $\underline{k}^2$ and $\Lambda_{oe} \simeq 1-b_e$ in (III-119), giving

$$\tfrac{1}{2} d_k^2 K_o = 1 + (\omega_e^2/\Omega_{oe}^2)(1+\tau\,\omega_*'/\omega_e')g_e, \tag{III-123}$$

where $\omega_*' \equiv k_2\kappa_n v_i^2/\Omega_{ci}$, so $\omega_{*e}/\omega_*' = -\tau$. Since from (III-120), $\omega_e' \sim \omega_{ISEC}' \sim \omega_*$, we see from (III-123) that $R_e(\frac{1}{2} d_k^2 K_o)$ is a positive quantity, a fact quoted in Sec. III.B.3.

Now using (III-123) and (III-119) in (III-46), we obtain an explicit expression for $K_{d.e.}$:

$$K_{d.e.} = f_n[\tfrac{1}{2}(d_k^2 K_o)\tfrac{1}{2}\partial_{x_o}^2 K_o]^{1/2}$$

$$= f_n\left\{[1+(\omega_e^2/\Omega_{oe}^2)(1+\tau\omega_*'/\omega')g_e]\tfrac{1}{2}\partial_{x_o}^2[L_2\lambda_i^{-2}+\lambda_e^{-2}(1-\Lambda_{oe}g_e(1+\tau\omega_*'/\omega'))]\right\}^{1/2}, \tag{III-124}$$

where $\omega' \equiv \omega_e''$. (In the present shear regime, $\omega' = \omega_e' = \omega_i' = \omega - k_2 v_e$.)

Taking g_e and ω' real, we have just seen that $\frac{1}{2} d_k^2 K_o$ is real and positive. The term $\frac{1}{2}\partial_{x_o}^2 K_o$ will also be positive if the mode is localized at a minimum of $K_o(x)$, and negative if localized at a maximum. Thus the product of these terms, $K_{d.e.}^2$, and their ratio, α^2, will be positive or negative according as the mode is localized at a well bottom or a hill top of K_o, respectively. Moreover from Eq. (III-41), $K_{d.e.} = (\frac{1}{2} d_k^2 K_o)\,\alpha$, so that $(K_{d.e.}/\alpha)$ will be positive. In the case of weak shear, we will find that the mode is localized at a well bottom. For the assumed form of the eigenmodes, $\phi \sim \exp(-\alpha x^2/2)$, we see that α must be real and positive to behave suitably as $|x| \to \infty$, and accordingly so must $K_{d.e.}$. For stronger shear, the modes will be localized at a hill top. The appropriate boundary conditions in this case are outgoing wave solutions,[23] for which α, and so $K_{d.e.}$, have negative imaginary parts ($\alpha \simeq i\alpha_I$, $\alpha_I < 0$).

III.D.2. IDC Instability with Negligible Shear

In this section we begin our study of the IDC instability in the simply case in which it was originally studied,[30] a uniform magnetic field ($\eta_B \to 0$), with density gradient only. We shall retain the correction $K_{n.\ell.}$, however, making the results valid for larger η_n than in the work of Ref. 30. Study of this simple configuration will serve to clarify the nature of the instability, in preparation for the more complicated problem of a sheared field, in addition to showing the effect of $K_{n.\ell.}$ on the instability.

For $k_3 \neq 0$, electron Landau damping enters (III-118) [through $W_{Ioe} \equiv I_m(W_{oe})$] to stabilize the mode. To maximize the growth rate, one therefore sets $k_3 = 0$ in the dispersion equation. Thus $W_{\ell s} \to W(\infty) = 0$ (cf. Fig. 11), and $g_e \equiv 1 - W_{\ell i} \to 1$. For future use, we retain g_e, and g_ℓ in the dispersion equation, remembering that in this unsheared case these factors equal unity.

Using Eq. (III-118) in this unsheared limit, Eq. (III-41a) becomes

$$0 = K + K_{n.\ell.} + K_{d.e.}$$

$$= \underline{k}^2 + \lambda_e^{-2}[1-\Lambda_{oe}g_e(1+\tau\omega_*'/\omega')] + \lambda_i^{-2}\,[I_2 - I_3 \sum_\ell \Lambda_{\ell i} g_\ell \frac{\omega'-\omega_{*i}}{\omega'-\ell\Omega_{ci}}] + K_{d.e.} \quad \text{(III-125)}$$

where $\omega_e' = \omega_i' = \omega' \equiv \omega - k_2 v_E$. For $I_2, I_3 \to 1$, $K_{d.e.} \to 0$, this is the standard dispersion equation of Ref. 30 for the IDC instability. It is to be evaluated at $x = x_o$, where x_o is found from application of (III-43).

We solve Eq. (III-125) for the IDC growth rate γ. We first write it in a slightly different form:

$$0 = [\underline{k}\,\lambda_e)^2 + \tau I_2 + (1-\Lambda_{oe}g_e) + \lambda_e^2 K_{de}] - \tau\Lambda_{oe}\,\omega_*/\omega' - \tau I_3 \sum_\ell \Lambda_{\ell i} g_\ell \frac{\omega' - \omega_*}{\omega' - \ell\Omega_{ci}}, \qquad \text{(III-126)}$$

where we define $\omega_* \equiv \omega_{*i}$, and note that in this regime, $\omega_* = \omega_*'$. Designating the three terms in (III-126) by $A' \equiv A + \lambda_e^2 K_{d.e.}$, B, and C, in Fig. 14 we graphically visualize the dispersion equation to be solved. In each frequency interval $\ell' \leq \omega'/\Omega_{ci} < \ell' + 1$ except for the one containing ω_* ($\ell \leq \omega_*/\Omega_{ci} < \ell + 1$), term C runs from $-\infty$ to ∞, and so an intersection of B+C with -A' in Fig. 14, hence a real root of Eq. (III-126), is guaranteed in each of these intervals ℓ'. The IDC instability can thus arise only in the ℓth interval, if B+C does not extend far enough down (for ω' real) to reach -A'.

From this graphical representation of the origin of the instability, we see that retaining only the two terms ℓ and $\ell+1$ in factor C accurately represents the dispersion relation in the frequency interval where the instability arises. This two term approximation is graphically displayed in Fig. 15 . Previous work on the IDC instability has solved the IDC dispersion equation both numerically[27] and analytically.[30,31,32] In contrast to the two-term approximation we shall use here, the previous analytic solutions retained only a <u>single</u> term of factor C, illustrated in Fig. 15b. The radically different behavior of this one-term approximation of C around the ℓth frequency interval yields an expression for γ which, as we shall see, scales quite differently with I_2, I_3, g_e, g_ℓ and $K_{d.e.}$ than the γ from the two term approximation. In the unsheared, small $\eta_\parallel$ limit of Ref. 30 the first four of these become unity; $K_{d.e.}$ is

neglected, and this improper scaling is irrelevant. In the present work, however, we are interested in the corrections arising from these factors, and so the two-term approximation becomes essential.

Using the two-term approximation to C, we see that C has two zeroes, one at $\omega' = \omega_*$, and the other at

$$\bar{\omega} \equiv (\Lambda_\ell\, g_\ell\, \omega_{\ell+1} + \Lambda_{\ell+1}\, g_{\ell+1}\omega_\ell)/(\Lambda_\ell g_\ell + \Lambda_{\ell+1} g_{\ell+1}) \simeq \omega_{\ell+1/2}, \quad \text{(III-127)}$$

where $\omega_\ell \equiv \ell\Omega_{ci}$. The final near equality follows from (III-104), and the large argument limit (III-112) on the Λ_ℓ. We define $\bar{\Omega}$ as the value of ω' at which C is a minimum; this occurs roughly midway between ω_* and $\bar{\omega}$:

$$\bar{\Omega} \simeq \frac{1}{2}\,(\omega_* + \bar{\omega}). \quad \text{(III-128)}$$

We need the derivatives

$$\partial_\omega C = -\tau I_3[\Lambda_\ell g_\ell(\omega_*-\omega_\ell)(\omega'-\omega_\ell)^{-2} + \Lambda_{\ell+1} g_{\ell+1}(\omega_*-\omega_{\ell+1})(\omega'-\omega_{\ell+1})^{-2}], \quad \text{(III-129)}$$

$$\partial_\omega^2 C = 2\tau I_3[\Lambda_\ell g_\ell(\omega_*-\omega_\ell)(\omega'-\omega_\ell)^{-3} + \Lambda_{\ell+1} g_{\ell+1}(\omega_{\ell+1}-\omega_*)(\omega_{\ell+1}-\omega')^{-3}], \quad \text{(III-130)}$$

$$\partial_\omega B = \tau\Lambda_{oe} g_e\, \omega_*/\omega'^2, \quad \text{(III-131)}$$

$$\partial_\omega^2 B = -2\tau\Lambda_{oe} g_e\, \omega_*/\omega'^3. \quad \text{(III-132)}$$

Now we solve (III-126), expanding in $\delta\omega \equiv \omega' - \bar{\Omega}$. At $\omega' = \bar{\Omega}$, we have $\partial_\omega C = 0$, and

$$\partial_\omega^2 B/\partial_\omega^2 C \sim (\Lambda_{oe} g_e\, \omega_\ell^2)/(2\Lambda_\ell g_\ell\, \omega_{1/2}^2) = (8\ell^2)^{-1}(\Lambda_{oe} g_e/\Lambda_\ell g_\ell) << 1, \quad \text{(III-133)}$$

so that we may neglect $\partial_\omega^2 B$. The ISEC mode satisfies

$$0 = (A + B)\big|_{\omega_{ISEC}}, \qquad \text{(III-134)}$$

so the full dispersion equation (III-126) may be written

$$0 \simeq [\delta\omega + (\bar{\Omega}-\omega_{ISEC})]\partial_\omega B + \frac{1}{2}(\delta\omega)^2\partial_\omega^2 C + C + \lambda_e^2 K_{d.e.} \qquad \text{(III-135)}$$

where all coefficients are evaluated at $\bar{\Omega}$. This is a quadratic equation in $\delta\omega$ with solution

$$\delta\omega \equiv \omega'-\bar{\Omega} = -(\partial_\omega B/\partial_\omega^2 C)$$
$$\pm \left\{(\partial_\omega B/\partial_\omega^2 C)^2 - 2[(\bar{\Omega}-\omega_{ISEC})\partial_\omega B + C + \lambda_e^2 K_{d.e.}]/\partial_\omega^2 C\right\}^{1/2}. \qquad \text{(III-136)}$$

Instability arises when the expression under the radical is negative. The destabilizing terms are $-2\lambda_e^2 K_{de}/\partial_\omega^2 C$ and $-2(\bar{\Omega}-\omega_{ISEC})\partial_\omega B/\partial_\omega^2 C$. For instability, we must have

$$(\bar{\Omega}-\omega_{ISEC}) + \lambda_e^2 K_{d.e.}/\partial_\omega B > \frac{1}{2}\partial_\omega B/\partial_\omega^2 C - C/\partial_\omega B. \qquad \text{(III-137)}$$

We maximize the growth rate γ by letting $\omega_* = \bar{\omega}$, so that $C(\bar{\Omega}) = 0$. From Eqs. (III-130), (III-31), we estimate the size of the other stabilizing term, $(\partial_\omega B/\partial_\omega^2 C)^2$, from

$$\frac{1}{2}\,\partial_\omega B/\partial_\omega^2 C \simeq (\omega_* \Lambda_{oe} g_e\, \omega_\ell^{-2})/(8\, I_3 \Lambda_\ell g_\ell\, \omega_{1/2}^{-2})$$
$$\simeq (32\ell^2)^{-1}(\Lambda_{oe} g_e/I_3 \Lambda_\ell g_\ell)\,\omega_* << \Omega_i. \qquad \text{(III-138)}$$

Neglecting this term as well gives us a maximized growth rate:

$$\gamma_{mx}^2 \simeq 2\bar{\Omega}\,(1-\omega_{ISEC}/\bar{\Omega})\,\partial_\omega B/\partial_\omega^2 C + 2\lambda_e^2 K_{d.e.}/\partial_\omega^2 C$$
$$\simeq (\bar{\Omega}^2/8 I_3 \Lambda_\ell g_\ell \ell^2)[1-\omega_{ISEC}/\bar{\Omega})\,\Lambda_{oe} g_e + \lambda_i^2 K_{d.e.}]. \qquad \text{(III-139)}$$

In Appendix E we use an analogous procedure for the one term approximation to the dispersiion equation, obtaining Eq. (E5) for γ_{mx}^2, which (neglecting $K_{d.e.}$) has a form quite similar to (III-139), but with a factor $(I_3\Lambda_\ell g_\ell/\Lambda_{oe}g_e)$, instead of the factor $(\Lambda_{oe}g_e/8I_3\Lambda_\ell g_\ell \ell^2)$ of (III-139). Using (III-112), and writing $b_i \simeq \ell^2$, the ratio of these two factors is $(8I_3^2\Lambda_\ell g_\ell^2\ell^2/g_e^2\Lambda_{oe}^2) \simeq 8/2\pi \simeq 1$, where we have taken the limits I_3, g_ℓ, $g_e \to 1$. As mentioned previously, therefore, though the scaling of γ_{mx} with these factors is very different for the two approximations, in the limit that they are usually taken, the two expressions for the growth rate are approximately the same.

For the negligible shear regime, $g_e = g_\ell = 1$. From (III-139), we see that γ^2 is affected by I_3, and also implicity by I_2, which changes ω_{ISEC} as in (III-120). These effects are readily seen graphically. Increasing I_2 increases term A' in Eq. (III-126), thereby depressing the line -A' in Figs. 14 or 15a. This in turn increases the gap between (-A') and the minimum in (B+C), enhancing γ^2, as well as decreasing ω_{ISEC} [and so increasing $(1-\omega_{ISEC}/\bar{\Omega})$]. Increasing I_3 increases the curvature $\partial_\omega^2 C|_{\bar{\Omega}}$, and thereby is stabilizing, allowing the same gap in (B+C)-(-A) to be bridged by $\gamma^2\partial_\omega^2 C$, for smaller γ^2.

We must no consider the dependence of the quantities I_2, I_3 on the physical parameters. Without loss of generality, we may restrict ourselves to positive k_2, relying on the very general symmetry property[25,26] of $\bar{K}_s$ in Eq (III-48), that if $[\phi(\underline{x})\omega]$ satisfies Eq. (III-1), so does $[\phi^*(x),-\omega^*]$, i.e. under the mapping $(\underline{k},\omega)\to(-\underline{k},-\omega^*)$, the physics is unchanged. In addition, we may without loss of generality take $\kappa_n>0$. Taking $\kappa_n<0$ will merely cause the drift mode $(\omega \sim \omega_*)$ to interact with a Bernstein harmonic

$\omega \sim -\ell\Omega_i$, instead of $\omega \sim \ell\Omega_i$. This change leaves $\varepsilon_{p\ell} = \ell\kappa_n/k_2$ unchanged.

From the definitions following (III-118) of I_2, I_3, we see that $1-I_2 \sim \Omega_1 = \omega'-\omega_*$. At $\gamma = \gamma_{mx}$ we have $\omega' \simeq \bar{\Omega} = \omega_*$, and so $\Omega \simeq 0$. The effect of I_2 on γ_{mx} is therefore negligible. Also, we have $I_3 \simeq 1 - \ell\kappa_n/k_2 = 1 - \varepsilon_{p\ell}$. Since ℓ, κ_n, and k_2 are all positive, increasing κ_n decreases I_3 which, from the foregoing discussion, enhances γ_{mx}^2 over the value one would expect if the nonlocal correction $K_{n.\ell.}$ were neglected, especially for larger ℓ. The validity of expression (III-139) and the concomitant scaling, $\gamma_{mx}^2 \sim I_3^{-1} \simeq (1-\varepsilon_{p\ell})^{-1}$, are limited, however, by the convergence condition (III-116), and by the validity of the truncation at second order in $\delta\omega$ used to obtain Eq. (III-135).

Just as for I_2, the effect of increasing $K_{d.e.}$ in Figs. 14 or 15a is to further depress the line (-A') thereby enhancing γ^2, as is algebraically corroborated by Eq. (III-139). We shall see shortly that the eigenmodes are localized at a minimum of K_o. Accordingly, as explained in Sec. III.D.1, $K_{d.e.}$ is real and positive for this shear regime. Thus, for fixed ℓ, the higher -n modes (which have larger $K_{d.e.}^{(n)} = f_n K_{d.e.}^{(o)}$) should be more unstable. This enhancement may be understood in the following manner.[33] The factor $K_{d.e.}$ arises in the dispersion equation as an average of $k_x^2(x)$ over the extent of the mode, and so many be grouped with the vacuum term k_2^2 which also appears there. Neglecting the explicit factor $K_{d.e.}$ in (III-139), we have that $\gamma_{mx}^2 \sim \Omega^{-2} \sim \omega_*^2 \sim k_1^2 \equiv k_2^2 + \langle k_x^2 \rangle$. Thus higher - n modes, having larger $\langle k_x^2 \rangle$, should have larger γ^2. From Fig. 15b, we note that use of the one term approximation would indicate

that these higher n modes are more stable than those with lower n, just the opposite behavior from that just described.

This increasing instability with mode number n has limits, however. The first of these is the mode localization condition, Eq. (III-47), which gives a maximum value n_{max} to the mode index n. As will be shown shortly (cf. Fig. 16b), $K_o(x)$ has a well of depth $K_d = K_d = r\lambda_i^{-2}$, where r is some fraction $0 \leq r < 1$, whose exact value depends upon the particular density profile. As n increases from zero, $K_{d.e.}^{(n)} = f_n K_{d.e.}^{(o)}$ increases, and so does the mode growth rate. Finally, as $n \to n_{max}$, $K_{d.e.}^{(n)}$ takes on the maximum value for which (III-47) is not violated, i.e. $K_{d.e.}^{(n)}\big|_{n=n_{max}} \simeq K_d = r\lambda_i^{-2}$. This determines n_{max}. From Eq. (III-139), the contribution of this to $(\gamma_{mx}/\Omega_i)^2$ is about $(r/8L_3\Lambda_\ell g_\ell) \sim (\ell r/8)$.

The second restriction on this analysis is that, if (γ_{mx}/Ω_i) becomes greater than unity, the approximation of separate harmonics we have been using becomes invalid,[34] and one should perform the sum over ℓ to obtain a valid dispersion equation, i.e., one should employ a straight line orbit approximation for the ions. This approximation is considered in connection with the IDC instability by Zelenyi.[32] Here we shall employ only the separate harmonic approximation, our principal interest being to illustrate new effects brought out by the present formalism, rather than exhaustively studying this instability. Applying this separate-harmonic limitation

$$1 > (\gamma_{mx}/\Omega_i)^2 \qquad \text{(III-140)}$$

to the effect on $(\gamma_{mx}/\Omega_i)^2$ from $K_{d.e.}$ alone, we find the restriction

$$1 > \ell r/8 \qquad \text{(III-141)}$$

for the validity of the present analysis.

If (III-141) is satisfied, we may imagine decreasing κ_n from some appreciable value, and observing the instability of the modes for some fixed value of ℓ. As the well width ($\sim L_n$) is increased, keeping its depth about constant, γ_{mx} for a mode with a given n decreases, but higher n modes arise, with the highest n mode roughly satisfying $K_{d.e.}^{(n)} \sim r\lambda_i^{-2}$. Accordingly, we expect the maximum growth rate from all modes of a given ℓ to decrease only slowly, due solely to the effect on Eq. (III-139) of increasing I_3. This slow stabilization of the ℓ-modes continues until κ_n becomes so small that ℓ_{max} of Eq. (III-122) drops below ℓ. At this point, all modes with harmonic number ℓ become stable.

We now proceed to apply the variational condition (III-43), to determine the position x_o at which the mode is localized, and accordingly where (III-139) is to be evaluated. Taking $g_e = 1$, Eq. (III-119) becomes

$$K_o = \underline{k}^2 + I_2\lambda_i^{-2} + \lambda_e^{-2}\,[1-\Lambda_{oe}(1+\tau\omega_*/\omega')]. \qquad \text{(III-142)}$$

Noting that $\lambda_e^{-2}\,\omega_* \sim n(x)\,\kappa_n(x) \equiv n'(x) \equiv \partial_x n$, we see that the factor $\lambda_e^{-2}\,\tau\,\omega_*/\omega'$ reaches a maximum where $n'(x)$ does. At this point $\omega_*/\omega' \simeq 1$. From this information, in Fig. 16a we sketch the factor $\lambda_e^{-2}\,\omega_*/\omega'$, as well as $\lambda_i^{-2} = \lambda_e^{-2}$, versus x. Using this with Eq. (III-142), in Fig. 16b we sketch $K_o(x)$. We see that K_o has a single minimum $x=x_o$, occurring a bit on the low density side of the point x_1 where n' is maximum, in agreement with the numerical findings of Ref. 21. The maximum in $K_o(x)$ to the low density side of x_o is unsuitable for Berk-Pearlstein type mode localization, since in this shear regime $g_e = g_\ell = 1$, for all x.

III.D.3. IDC Instability with Moderate Shear

Now we are ready to consider the problem of the IDC instability in the regime of intermediate shear, where the scale lengths L_n, L_{oe}, which enter into $K_o(x)$ and $Q(x)$, may be comparable. For $\eta_n < 1$ (which is a valid assumption even for strongly inhomogeneous plasmas such as the Tormac sheath), it is in this regime that the transition from $\rho_i/L_\phi < 1$ to $\rho_i/L_\phi > 1$ occurs. However, as discussed in Sec. III.C.5, the change in the actual form of the dispersion equation in making this transition is quite minor, so that the analysis in either range is essentially the same. For definiteness, we shall consider the regime $\rho_i/L_\phi > 1$, so that Eq. (III-41b) is the appropriate dispersion equation.

We can put the dispersion equation into a form very similar to Eqs. (III-125) or (III-126) (which applied to the negligible shear regime), by some generalized definitions of the symbols used there. We define ω_*' as $k_2\kappa_n v_i^2/\Omega_i$, as before, and generalize the shearless definition of ω' as follows: $\omega' \equiv \omega_e' \equiv \omega - k_2 v_E - k_3 w_{\parallel e}$. We also write $\omega_* \equiv \omega_{*i} + k_2(u_{Ei} - v_E) + k_3\,\Delta w_\parallel$, and $\omega_\ell \equiv \ell\Omega_{oi} + k_3\,\Delta w_\parallel$, where $\Delta w_\parallel \equiv w_{\parallel i} - w_{\parallel e}$, the difference in parallel drifts, which produces the shear. The dispersion equation (III-41b) then appears as

$$\begin{aligned} 0 &= K_o + S_{iB} + K_{d.e.} \\ &= \underline{k}^2 + \lambda_e^{-2}\,[1 - \Lambda_{oe} g_e\,(1 + \tau\omega_*'/\omega')] \\ &\quad + \lambda_i^{-2}\,[1 - \sum_\ell \Lambda_{\ell i} g_\ell \left(\frac{\omega' - \omega_*}{\omega' - \omega_\ell}\right)] + K_{d.e.}, \end{aligned} \tag{III-143}$$

where $\Lambda_{\ell i}$ is given by Eq. (D30) for the regime $\rho_i/L_\phi > 1$.

We consider the size of $\Delta w_{\|}$, and the effects on the dispersion equation caused by it. Using Ampere's law, we estimate the size of κ_s or $\eta_s \equiv \kappa_s \rho_i$:

$$\kappa_s \simeq (4\pi ne/B)(\Delta w_{\|}/c), \quad \text{or} \quad \eta_s \simeq \beta\ (\Delta w_{\|}/v_i), \tag{III-144}$$

where $\beta \equiv 8\pi\ p/B^2$ is the usual plasma β. For the stronger-shear portion of this intermediate-shear range, [e.g., from the point $(\kappa_s/\kappa_n) = (L_{oe}/L_s \ell \eta_n^2)$ where the strong convergence condition (III-111) breaks down, to the point $(\kappa_s/\kappa_n) = (\ell L_{oe}/L_s\ \eta_n^2)$ where localization condition (III-102) is violated], using the values $\eta_n \sim .2$, $\ell \sim 10$, $\tau \sim 1$ as in Sec. III.C.5, (κ_s/κ_n) lies in the range .01 to 1. Therefore $\eta_s = (\kappa_s/\kappa_n)\eta_n$ may take on values from about .002 to .2. Strictly speaking, because it is based on Poisson's equation instead of the full Maxwell equations, the present formalism is valid only for low β plasmas. Taking $\beta \sim .01$, using (III-144) tells us that $(\Delta w_{\|}/v_i)$ falls in the range .2 to 20.

In addition to the term $k_3\ \Delta w_{\|}$ in the definition of ω_*. finite $w_{\|i}$ enters ω_* through the terms ω_{*i} and $k_2(u_{Ei} - v_E)$. Using Eq. (III-86) for u_{Ei} and the analogous equation for ω_{*i}, i.e.,

$$\begin{aligned} \omega_{*i} \equiv \omega_{*i}(v_3 = w_{\|i}) &= \omega_{*i}(v_3 = 0)(1 + \kappa_s w_{\|i}/\Omega_{ci})^{-1} \\ &\equiv \omega_*'\ (1 + \kappa_s w_{\|i}/\Omega_{ci})^{-1}, \end{aligned} \tag{III-145}$$

one finds

$$\omega_* \equiv \omega_{*i} + k_2(u_{Ei} - v_E) + k_3 \Delta w_{\|} \simeq \omega_*' - (\omega_*' + k_2 v_E)(\kappa_s w_{\|i}/\Omega_{ci}) + k_3 \Delta w_{\|}. \tag{III-146}$$

Assuming $w_{\|i} \sim \Delta w_{\|}$, we see that ω_* may experinece a fractional shift from its shearless limit ω_*' of $(\kappa_s w_{\|i}/\Omega_{ci}) \sim (\Delta w_{\|}/v_i)\ \eta_s \simeq \beta(\Delta w_{\|}/v_i)^2$ which, with the parameter values just estimated, falls in the range 4×10^{-4} to 4. The shift in ω_* away from the value ω_*' used in previous analyses of instabilities involving this diamagnetic drift frequency may therefore be quite significant.

As noted in Sec. III.D.2, for finite $k_3(x)$, electron Landay damping will set in, localizing the mode. We therefore expect the mode amplitude (and source o' free energy) to be greatest at that point x_2 where $k_3(x_2) = 0$. In contrast to the unsheared case, we are thus at liberty change the point of mode localization, by varying k_y or k_z.

The localization of the mode is determined by K_o, which, from (III-119), is given by (recalling $I_2 \to 1$ for $\rho_i/L_\phi > 1$)

$$K_o = \underline{k}^2 + \lambda_i^{-2} + \lambda_e^{-2}\,[1 - \Lambda_{oe} g_e (1 + \tau\omega_*'/\omega')], \qquad \text{(III-147)}$$

the same as Eq. (III-142), but with the factor $g_e \equiv 1 - W_{oe}$ now x - dependent. Taking x_2 as our origin ($x_2 = 0$), in the neighborhood of $x = x_2$ we have $z_{oe}^{-} = \omega/|k_3(x)v_e| \simeq L_{oe}/|x|$. Using Fig. 11, therefore, in Fig. 17a we draw $-g_e(x)$. Since $W_I(z) \sim e^{-z^2/2}$, at $x = x_2 = 0$ ($-g_{Ie}$) and all its derivatives are zero, giving it a flat bottomed appearance.

For $x/L_{oe} << 1$, $g_e \to 1$, and K_o goes to its unsheared value, sketched in Fig. 16b. For $x/L_{oe} >> 1$, $g_e \to 0$, and $K_o \to \underline{k}^2 + \lambda_e^{-2}(I_2 + \tau)$. In the region $x/L_{oe} \sim 1$, the variation of g_{Re} gives the real part K_{Ro} of K_o a double-well form. This is sketched in Fig. 17b, for L_{oe} a bit shorter than L_n, and for $x = x_2$ taken to the higher-density side of the point x_1

where $n' = 0$. Also drawn (dashed curve) is the form of K_o for the unsheared case, from Fig. 16b.

For weaker shear, the position of the double wells about $x = x_2$ caused by g_e will move out toward $|x| = \infty$, the resultant shape of K_{Ro} moving toward its shearless value in the region of the well induced by n', where a free energy source is available. Thus, for L_{oe} greater than L_n, but comparable to it, we may still have a potential well ($\partial_x^2 K_{Ro} > 0$ at $x = x_2$ instead of the hump ($\partial_x^2 K_{Ro} < 0$) shown in Fig. 17b. In the following, we shall assume $L_{oe} < L_n$, so that the situation is as in Fig. 17b. The analysis for the weaker shear case is, however, similar, connecting the shearless regime of Sec. III.D.2 to the shear-dominated one.

The variational condition (III-43) for x_o allows x_o to be at any extremum of K_o. However modes centered at any x_o besides $x_o \simeq x_2$ will be strongly damped, and so we consider only this local maximum. In addition, in seeking the most unstable case, we choose k_y, k_z so that $x = x_2$ falls at the bottom of the well we found in the shearless regime.

As discussed at the end of Sec. III.D.1, modes localized at a maximum of K_o will have Berk-Pearlstein type localization, with α and $K_{d.e.}$ being (formally) imaginary and negative ($\alpha \simeq -i|\alpha|$). Using the asymptotic value $W(z) \underset{z\to\infty}{\to} -z^2$, and our assumption that the curvature of K_o at x_o is dominated by the factor g_e, we find

$$\frac{1}{2}\,\partial_{x_o}^2 K_o \simeq -L_{oe}^{-2}\,\lambda_2^{-2}\Lambda_e^{-2}\,(1 + \tau\omega_*'/\omega'), \qquad \text{(III-148)}$$

and (III-124) becomes

$$K_{d.e.} \simeq -iL_{oe}^{-1}\,\lambda_e^{-1}\left\{[1 + (\omega_e^2/\Omega_{oe}^2)(1+\tau\omega_*'/\omega')]\Lambda_{oe}(1+\tau\omega_*'/\omega')\right\}^{1/2}. \qquad \text{(III-149)}$$

Evaluating the dispersion equation (III-143) at $x_o = x_2$, we again have $g_e = 1$. Analogous to (III-126), we rewrite (III-143) as

$$0 = [(\underline{k}\lambda_e)^2 + \tau + (1-\Lambda_{oe}g_e) + \lambda_e^2 K_{d.e.}] - \tau\Lambda_{oe}g_e\,\omega_*'/\omega'$$

$$- \tau \sum_\ell \Lambda_{\ell i}\, g_\ell \left(\frac{\omega'-\omega_*}{\omega'-\omega_\ell}\right) \qquad \text{(III-150)}$$

$$\equiv A' + B + C, \qquad A' \equiv A + \lambda_e^2 K_{d.e.}.$$

This has essentially the same form as Eq. (III-126), and the same method of solution may be employed. There are certain differences worthy of note, however.

To begin with, while $K_{d.e.}$ was real for the shearless regime, here it is imaginary, and the corresponding effect of $K_{d.e.}$ on the growth rate will be quite different.

Secondly, while $g_e = 1$ for both the moderate shear and the unsheared cases, g_ℓ is <u>not</u> equal to one, due to the dependence of Ω_i on v_3 or u. We have $g_\ell \equiv 1 - W(z_{\ell i})$, where $z_{\ell i} \equiv \Omega_2/|k_3' v_i|$, $\Omega_2 = \omega'-\omega_\ell$, and $k_3' \equiv k_3 + \ell\kappa_s/2 = \ell\kappa_s/2$. As described in the discussion following Eq. (III-96) and illustrated in Fig. 12, for ω' near the ℓth Bernstein harmonic ($\omega' - \omega_\ell \simeq \Delta\ell\Omega_{oi}$, $0 < \ell < 1$), the ℓth harmonic amd nearby harmonics ℓ' may have $|z_{\ell'}| \lesssim 1$, and so $g_{\ell'}$ is appreciably different from unity, with both real and imaginary parts. For the ℓ and $\ell+1$ terms kept in the two-term approximation, one has $z_{\ell i} \simeq 2\Delta\ell/\ell\eta_s$ and $z_{\ell+1} \simeq 2(\Delta\ell-1)/(\ell+1)\eta_s$. If we let $\Delta\ell \simeq \frac{1}{2}$, this gives $z_{\ell i} \simeq -z_{\ell+1,i} \simeq (\ell\eta_s)^{-1}$. Therefore we shall take $g_{\ell+1} = g_\ell$ in the following analysis.

In addition, although expressions (D29) and (D30) for $\Lambda_{\ell i}$ are quite similar, they are not entirely identical. Even if one writes $\alpha = \langle k_x^2 \rangle$ and sets $n = 0$, as done in Sec. III.C.5 to achieve a formal resemblance, $\Lambda_{\ell i}(\rho_i/L_\phi > 1)$ is still down from $\Lambda_{\ell i}(\rho_i/L_\phi < 1)$ by a factor of $\pi^{1/2}$. Also, α is now an imaginary number, and so will affect the mode growth rate differently, in the stronger-shear end of the moderate-shear regime, where $\alpha \sim k_2^2$. And lastly, the higher -n values of $\Lambda_{\ell i}$ are equal to $(-1)^n c_n$ times the $n=0$ value, $\Lambda_{\ell i}(n=0)$. For n odd, $c_n=0$, and therefore $S_{iB} = 0$, and the IDC instability vanishes. We shall therefore henceforth consider only even-n perturbations, for which $\Lambda_{\ell i}(n) = c_n \Lambda_{\ell i}(0)$.

Finally, the effects of the finite parallel currents $w_{\|s}$ present with finite shear make themselves known in (III-150), through the generalized meanings of ω_*, ω_ℓ, and Ω_{io}. Each of these effects may be seen either graphically from Fig. 14, or algebraically, in Eq. (III-136) or (III-139), which, with suitable reinterpretation, are valid for the moderate shear regime, as well as for the negligible-shear case for which they were derived.

For fixed ω_*', Eq. (III-146) tells us that as $w_{\|i}$ is increased, ω_* decreases, while $\Omega_{oi} \simeq \Omega_c(1 + \frac{1}{2}\kappa_s w_{\|i}/\Omega_c)$ increases. Thus the point $\omega' = \omega_*$ at which term C in (III-150) or Fig. 14 passes through zero shifts toward lower values, while the spacing Ω_{oi} between consecutive poles of C increases. Terms A and B are unaffected. Thus the values of $\bar{\Omega}$ and k_e at which maximum growth occurs will also be shifted.

The situation is different if instead the shear is induced by increasing $-w_{\|e}$, taking $w_{\|i}$ equal zero. In this case $\omega_* \simeq \omega_*'$, $\Omega_{oi} = \Omega_{ci}$ and, aside from the difference induced by $K_{d.e.}$, the dispersion equation

is the same as in the unsheared case.

Now we solve (III-150) for the growth rate, using the two-term approximation, as in Sec. III.D.2. We again define $\bar{\Omega}$ as the point at which $\partial_\omega C = 0$. Using the facts that $g_\ell \simeq g_{\ell+1}$, and that $\Lambda_\ell \simeq \Lambda_{\ell+1}$ [cf. Eqs. (III-114) or (III-113)], we find from (III-129) the relation

$$(\bar{\Omega} - \omega_{\ell+1})^2/(\bar{\Omega} - \omega_\ell)^2 = (\Lambda_{\ell+1} g_{\ell+1}/\Lambda_\ell g_\ell)(\omega_{\ell+1} - \omega_*)/(\omega_* - \omega_\ell) \qquad \text{(III-151)}$$

$$\simeq (\omega_{\ell+1} - \omega_*)/(\omega_* - \omega_\ell).$$

Thus $\bar{\Omega}$ again lies on the real ω' axis, between ω_ℓ and $\omega_{\ell+1}$. Everything now follows as in Sec. III.D.2. In particular, Eqs. (III-134) through (III-136) are again valid. The growth rate is again maximized by the expression on the right hand side of (III-139), but now this expression is no longer real, due to $K_{d.e.}$, α in Λ_ℓ, and g_ℓ. Instead, one must write

$$\gamma_{mx} \simeq \mathrm{Re}\left\{(\bar{\Omega}^2/8I_3\Lambda_\ell g_\ell \ell^2)[(1-\omega_{ISEC}/\bar{\Omega})\Lambda_{oe} g_e + \lambda_i^2 K_{d.e.}]\right\}^{1/2}. \qquad \text{(III-152)}$$

From Fig. 11, one sees that as $z_{\ell i} \simeq (\ell\eta_s)^{-1}$ moves in from infinity, $g_{R\ell} \equiv \mathrm{Re}\, g_\ell$ first increases from one, and then decreases to values below one in the (strong shear) regime where $z_{\ell i} < 1$, while $g_{I\ell} \equiv \mathrm{Im} g_\ell$ is negative, becoming largest around $z_{\ell i} = 1$. Combining this with the facts that $K_{d.e.} \simeq -i\,|K_{d/e.}|$, $\alpha \simeq -i|\alpha|$, and the dependence of $\Lambda_{\ell i}$ in (D30) on n and α, we schematically represent (III-152) as

$$\gamma_{mx}^{(n)} = c_n^{-1/2}\, \gamma_{mx}^{(n=0)} = (n!)^{1/2}\, \gamma_{mx}^{(n=0)},$$

$$\gamma_{mx}^{(n=0)} = k_e\left\{(k_2^2 - i|\alpha|)^{1/2}\,(g_R - ig_I)(R - iI)\right\}^{1/2}, \qquad \text{(III-153)}$$

where $g_R - ig_I \equiv g_\ell = g_{R\ell} - i|g_{I\ell}|$, R corresponds to the term with $(1-\omega_{ISEC}/\bar{\Omega})$, and I corresponds to the term in $K_{d.e.}$. From this we see that the effect of nonzero α, $K_{d.e.}$, and g_I is principally to cause a shift in the real part ω_R' of ω'. The effect of increase (decrease) in g_R from unity is stabilizing (destabilizing), and the higher n (n even) modes again have larger growth rates than the n=0 mode. The limits on this increase of γ_{mx} with n, due to an upper limit n_{max} of n, and due to the breakdown of the validity of the present two-term analysis at $\gamma_{mx}/\Omega_i \sim 1$, are the same as discussed in Sec. III.D.2.

The effect of shear on the IDC instability in the intermediate shear regime is thus the following. The distance L_ϕ over which a given mode is localized narrows, as L_{oe}^{-1} dominates L_n. However, there is now a spread in the positions x_o at which a given eigenmode may be localized. An eigenmode is localized at the point x_o where $k_3 = 0$, and its growth rate, being proportional to the free energy source $[\sim n'(x) \equiv dn/dx]$, is greatest near the position where n' is greatest. Taking a superposition of the unstable modes localized near the most unstable position, one obtains a perturbation which is broader in x, which twists with the magnetic field, but which has a spread in frequency and growth rate. This is the "quasimode" picture, introduced by Roberts and Taylor[35] in connection with MHD resistive instabilities.

For a given ℓ, the larger n modes (n even) have larger growth rates than the n=0 mode, as found in the negligible shear regime. The growth rate of the n=0 mode is not radically reduced by shear in this regime. It maybe somewhat enhanced, in fact, for $z_{\ell i}$ becoming small

enough that $g_R < 1$ (which, we recall, can occur due to the dependence of Ω_{oi} on $u \equiv v_3 - w_{\|i}$). Finally, the values of $\bar{\Omega}$ and k_2 at which the maximum growth appears is modified by the fact that $\Omega_{oi} \neq \Omega_{ci}$, for appreciable $w_{\|i}$.

To obtain a criterion for shear stabilization, therefore, we must proceed to stronger shear. This is done in the following section.

III.D.4. Instability with Strong Shear

In this section we consider what happens when shear becomes so strong that localization condition (III-47) is violated, even for the n=0 mode. In this regime, we find from Eq. (III-147) the well depth $K_d \sim (\lambda_e^{-2} + \lambda_i^{-2}) = \lambda_e^{-2}(1 + \tau)$, and from (III-149), $|K_{d.e.}| \sim L_{oe}^{-1}\lambda_e^{-1} \times [1 + \omega_e^{e}/\Omega_{oe}^{2}]^{1/2} \simeq L_{oe}^{-1}[\lambda_e^{-2} + \rho_e^{-2}]^{1/2} \sim L_{oe}^{-1}[\min(\lambda_e, \rho_e)]^{-1}$. Condition (III-47) then gives the localization criterion

$$1 > (\lambda_e/L_{oe}) \max (1, \lambda_e/\rho_e). \qquad \text{(III-154)}$$

When (III-154) is violated, the final (n=0) mode can no longer be localized by K_o, so that no modes satisfying appropriate boundary conditions exist. The IDC mode is then stabilized. For typical parameters of confinement devices, $\lambda_e/\rho_e \sim 1$, and so (III-154) gives the stabilization criterion

$$1 \sim \lambda_e/L_{oe}. \qquad \text{(III-155)}$$

Combining the results of this and the preceding section, for a given ℓ, as shear is increased from zero, the maximum IDC growth rate is relatively insensitive to the shear, until the regime satisfying Eq. (III-155) is approached, at which point the instability disappears.

This behavior, as well as stabilization criterion (III-155), is in qualitative agreement with the results of Zelenyi.[32]

III.E. Further Comments on the Formalism

The application of Sec. III.D made use of some, but not all, of the novel features of the stability formalism. One principal restriction on the formalism as developed here is that it is based on Poisson's equation, and therefore electromagnetic corrections which one expects for perturbations in higher β plasmas are not taken into account. Many of the interesting features of the formalism (e.g., the v_3 dependences of Ω_{oi}, u_E, and ω_*) become significant only for substantial shear, and for larger shear one needs larger β, for realistic plasmas.

The formalism should be readily generalized to allow for arbitrary electromagnetic perturbations, replacing Poisson's equation (III-1) with the full Maxwell equations

$$\nabla \times [\nabla \times E(\underline{x})] - (\omega/c)^2 E(\underline{x}) = (4\pi i\omega/c^2)\underline{j}(\underline{x}) = \sum_s \int dx' \underset{\sim}{\overline{K}}_s(\underline{x},\underline{x}',\omega) \cdot \underline{E}(x'). \tag{III-156}$$

The structure of the tensor conductivity $\underset{\sim}{K}_s$ is analogous to that of the scalar conductivity $\overline{K}_s$ we have used, with the $\rho_{\underline{\ell}}$'s which appear in expressions (III-48) for $\overline{K}_s$ replaced by $\underline{j}_{\underline{\ell}}$'s, where $\underline{j}(\underline{x}\,|\underline{Z}) \equiv e\,\underline{v}\,(Z) \times \delta[\underline{x} - \underline{r}\,(\underline{Z})]$, analogous to the definition of phase function $\rho(\underline{x}|\underline{Z})$.[26] This analogous structure insures that $\underset{\sim}{\overline{K}}_s$ can be written in a form parallel to Eq. (III-4), and given this crucial property, the formalism developed here follows.

Thus, though the specific results which the present formalism yields cannot be taken too seriously for higher-β plasmas (such as Tormac), the novel effects of appreciable shear indicated by the formalism will also arise in the electromagnetic formalism, and should be taken into account in any careful treatment of the stability of sheared plasmas.

In the present work, the variational formulation was used to obtain dispersion equations in the regime $\rho_i/L_\phi < 1$, and in the regime $\rho_i/L_\phi > 1$, where the mode equation is intrinsically integral in nature H. Berk has suggested[33] that in fact perhaps $S_{i\beta}$ can be evaluated in closed form for <u>arbitrary</u> ρ_i/L_ϕ, using the Gaussian form $\phi_{10} \sim \exp[-\frac{1}{2}\alpha(\delta x)^2]$ for the trial function. If such an evaluation proved possible, the smooth interpolation done here between the small and large - ρ_i/L_ϕ regimes could be checked, and additional insights might be gained on the treatment of such fully nonlocal problems.

Another possibly useful direction for generalization of the formalism is to relax the assumption of a slab geometry, considering, for example, the near slab geometry of Part II, or perhaps even breaking the toroidal symmetry as well. Provided one can find a complete set of particle invariants $\underline{I}$, expressions (III-48) for $\overline{K}_s$, upon which much of the formalism is founded, remain valid. For example, for the near slab case, the slab invariants $\underline{I} = (J, P_y, P_z)$ used here are replaced by $\underline{I} = (J, J_b, P_\phi)$, where the canonical angular momentum P_ϕ specifies the α surface α_b lying at the banana center, and the bounce action J_b characterizes the area enclosed by the banana. In addition, expressions

(III-13) and (III-14), which are the abstract expression of the formalism for the one degree of freedom (slab) case, are readily generalized formally to the two or three degree of freedom case, merely by replacing the scalars k, x, and ∂_x appearing there by vectors. To give physical content to these formal expressions, however, one must be able to write Eqs. (III-48) for these geometries in a form analogous to Eq. (III-4). This entails, basically, being able to associate some average position $\underline{X}_0$ with any given value of the momenta $\underline{I}$. The attainment of a physically meaningful generalization of this sort, and applications to modes in which inhomogeneity parallel to the magnetic field plays an essential role, are problems for future study.

APPENDIX C: DERIVATION OF THE CONDUCTIVITY KERNEL

In this Appendix we derive an expression for the conductivity Kernel $\overline{K}_s$ defined by Eq. (III-1.b), in a canonical framework. The expression derived is quite similar to those obtained in Refs. 25 and 26, but has a form which is more useful for our purposes. We perform the derivation in a bra-ket notation, similar to that of Lewis and Symon,[36] which allows for a very concise expression of the results.

As in Sec. III.C.1, we consider a three-dimensional plasma, with unperturbed Hamiltonian $H_o(\underline{I})$ [denoted by $H(\underline{I})$ in the main text), where $\underline{I}$ are three canonical momenta, constant in time, with conjugate coordinates $\underline{\theta}$. For any phase functions $F(\underline{Z}), G(\underline{Z})$ [$\underline{Z} = (\theta, \underline{I})$], we denote by $\check{F}$, $\check{G}$ the corresponding Lie operators as in Part II,

$$\check{G}\, F \equiv \left\{G,\, F\right\} = -\left\{F,\, G\right\} \equiv -\,\check{F}\, G. \tag{C1}$$

Then the Vlasov equation reads (species label suppressed)

$$(\partial_t - \check{H})\, f = 0, \tag{C2}$$

and writing $H = H_o + H_1$, $f = f_o + f_1$, the linearized Vlasov equation is

$$(\partial_t - \check{H}_o)\, f_1 = \check{H}_1\, f_o = -\,\check{f}_o\, H_1. \tag{C3}$$

Henceforth we shall for notational convenience employ bra-ket notation, with states defined over the phase space, spanned by the orthonormal basis $|\underline{Z}\rangle = |\underline{\theta},\, \underline{I}\,\rangle$. Thus, for example, $f(\underline{Z}) \equiv \langle \underline{Z}|f\,\rangle$. Then Eq. (C3) becomes

$$(\partial_t - \check{H}_o)|f_1\rangle = -\,\check{f}_o|H_1\rangle\,. \tag{C4}$$

Solving this for $|f_1\rangle$,

$$|f_1\rangle = -(\partial_t - \check{H}_o)^{-1}\ \check{f}_o|H_1\rangle, \tag{C5}$$

and therefore the perturbed charge density $\rho_s(\underline{x};t)$ due to species s is given by

$$\rho_s(\underline{x};t) = -\langle \rho\ (x)|(\partial_t - H_o)^{-1}\ \check{f}_{os}|H_1\rangle\ , \tag{C6}$$

or in the frequency domain

$$\rho_s(\underline{x};\omega) = \langle \rho(\underline{x})|(i\omega + \check{H}_{os})^{-1}\ \check{f}_{os}|H_1\rangle\ . \tag{C7}$$

Here the state $|\rho(\underline{x})\rangle$ is the contribution to charge density of a particle at a given phase point $\underline{Z}$:

$$\langle \underline{Z}|\rho(\underline{x})\rangle \equiv e\delta[\underline{x} - \underline{r}(\underline{Z})] = e\int d\underline{p}'\delta[\underline{p}'-\underline{p}\ (\underline{Z})]\ \delta[\underline{x} - \underline{r}\ (Z)] \tag{C8a}$$

$$= \langle\ \underline{Z}\,|e\int d\underline{p}'|\ \underline{x},\ \underline{p}'\rangle, \qquad \text{i.e.}$$

$$|\rho(\underline{x})\rangle = e\ \int d\underline{p}'|\underline{x},\ \underline{p}'\rangle. \tag{C8b}$$

We consider electrostatic perturbations,

$$\langle \underline{Z}|H_1\rangle = e\phi[\underline{r}(\underline{Z})] = \int d\underline{x}'\ \langle \underline{Z}|\rho(\underline{x})\rangle\ \phi(x'). \tag{C9}$$

Putting this into (C7) and comparing with Eq. (III-1b) yields

$$\overline{K}_s(\underline{x},\underline{x}';\omega) = -\ 4\pi\ \langle \rho(x)|(i\omega + \check{H}_{os})^{-1}\ \check{f}_{os}|\rho(x')\rangle. \tag{C10}$$

To put (C10) into a more explicit form, we introduce another orthonormal basis $|\underline{\ell},\ \underline{I}\rangle$:

$$\langle\ \underline{Z}_o|\underline{\ell},\ \underline{I}\rangle \equiv N\ e^{-i\underline{\ell}\cdot\underline{\theta}_o}\ \delta(\underline{I} - \underline{I}_o). \tag{C11}$$

The normalization constant N is determined by the requirement of orthonormality, $\delta(\underline{Z}_1 - \underline{Z}_2) = \langle \underline{Z}_1 | \underline{Z}_2 \rangle$, to be

$$N = (2\pi)^{-3/2}. \tag{C12}$$

The basis $|\underline{\ell}, \underline{I}\rangle$ is chosen because these states are the eigenstates of the Liouville operator $\check{H}_o$. One has

$$\check{H}_o | \underline{\ell}, \underline{I}\rangle = i\underline{\ell}\cdot\partial H_o/\partial \underline{I} | \underline{\ell}, \underline{I}\rangle \equiv i\underline{\ell}\cdot\underline{\Omega} | \underline{\ell}, \underline{I}\rangle . \tag{C14}$$

We shall assume that f_o is a function of <u>both</u> H_o and $\underline{I}$, in general. Then since $H_o = H_o(\underline{I})$, one has

$$\check{f}_o = (\partial f_o/\partial H_o)\, \check{H}_o + \check{f}_{oI}, \quad \check{f}_{oI} \equiv -(\partial f_o/\partial \underline{I})\cdot \partial/\partial \underline{\theta}. \tag{C15}$$

The $|\underline{\ell}, \underline{I}\rangle$ are also eigenstates of $\check{f}_o$ and $\check{f}_{oI}$:

$$\check{f}_{oI} | \underline{\ell}, I\rangle = i\underline{\ell}\cdot \partial f_o/\partial \underline{I} | \underline{\ell}, I\rangle. \tag{C16}$$

Also, one has that

$$\begin{aligned}(i\omega + \check{H}_o)^{-1}\, \check{f}_o &= (i\omega + \check{H}_0)^{-1}\, [\partial f_o/\partial H_o (i\omega + \check{H}_o) - i\omega \partial f_o/\partial H_o + \check{f}_{oI}] \\ &= \partial f_o/\partial H_o + (i\omega + \check{H}_o)^{-1}\, (-\, i\omega\, \partial/\partial \check{H}_o + \check{f}_{oI}).\end{aligned} \tag{C17}$$

In less formal terms, in (C17) we have written the factor $(\partial f_o/\partial H_o)\, \check{H}_o$ in $\check{f}_o$ as the sum of a total time derivative $(\partial f_o/\partial H_o)(i\omega + \check{H}_o)$ and a remainder, so that the integration along unperturbed trajectories induced by $(i\omega + \check{H}_o)^{-1}$ becomes trivial. This technique has been used previously by other authors,[32,35] in a less formal, non-canonical framework.

Inserting the identity dyadic $\underline{1} = \sum_{\underline{\ell}} |\underline{\ell}, I\rangle\langle \underline{\ell}, \underline{I}|$ between $(i\omega + \check{H}_o)^{-1} \check{f}_o$ and both $\langle \rho(\underline{x})|$ and $|\rho(\underline{x}')\rangle$ in (C10), and using Eqs. (C14), (C16) and (C17), one obtains the decomposition:

$$\overline{K}_s\ (x,\underline{x}';\ \omega) = \overline{K}_{sA}\ (\underline{x},\underline{x}') + \overline{K}_{sB}\ (\underline{x},\underline{x}';\ \omega). \tag{C18a}$$

The "adiabatic term" $\overline{K}_{sA}$ is purely local, and ω-independent, coming from the part $\partial f_o/\partial H_o$ of $(i\omega + \check{H}_o)^{-1} \check{f}_o$:

$$\overline{K}_{sA}(\underline{x},\underline{x}') \equiv 4\pi\ \langle \rho(\underline{x})|\partial f_{o_s}/\partial H_{o_s}|\rho(\underline{x}')\rangle = \lambda_s^{-2}(\underline{x})\ \delta(x-x'), \tag{C18b}$$

where $\lambda_s^2\ (\underline{x}) \equiv T_s(\underline{x})/4\pi n_s(\underline{x})\ e^2$, $n_s(\underline{x}) \equiv \int d\underline{Z}\ \delta[\underline{x}-\underline{r}\ (\underline{Z})]\ f_{o_s}$, and $T_s^{-1}(\underline{x}) \equiv -n_s^{-1}(\underline{x})\int d\underline{Z}\ \delta[\underline{x}-\underline{r}\ (\underline{Z})]\ \partial f_{o_s}/\partial H_{o_s}$. The remainder, $\overline{K}_{sB}$, is given by

$$\begin{aligned}\overline{K}_{sB}(\underline{x},\underline{x}';\ \omega) &\equiv -4\pi\ \langle \rho(\underline{x})|(i\omega+\check{H}_{o_s})^{-1}(-i\omega\partial f_{o_s}/\partial H_{o_s} + \check{f}_{oI})|\rho(\underline{x}')\rangle \\ &= 4\pi \sum_{\underline{\ell}} \int dI\ \langle \rho(\underline{x})|\underline{\ell},\underline{I}\rangle \frac{\omega\partial f_{o_s}/\partial H_{o_s} - \underline{\ell}\cdot\partial f_{o_s}/\partial \underline{I}}{\omega + \underline{\ell}\cdot\underline{\Omega}} - \langle \underline{\ell},\underline{I}|\rho(x')\rangle.\end{aligned} \tag{C18c}$$

Defining $\rho_{\underline{\ell}}\ (\underline{x}|\underline{I})$ by

$$\begin{aligned}\rho_{\underline{\ell}}(\underline{x}|\underline{I}) &\equiv (2\pi)^{-3/2}\ \langle \rho(\underline{x})|\underline{\ell},\underline{I}\rangle = (2\pi)^{-3} \int d\underline{\theta}\ e^{-i\underline{\ell}\cdot\underline{\theta}}\ \rho(x|\underline{Z}) \\ &= (2\pi)^{-3} \int d\underline{\theta}\ e^{-i\underline{\ell}\cdot\underline{\theta}}\ e\,\delta[\underline{x} - \underline{r}\ (\underline{Z}),\end{aligned} \tag{C19}$$

we write (C18c) in the more standard form

$$\overline{K}_{sB}(\underline{x},\underline{x}';\ \omega) = (4\pi)(2\pi)^3 \sum_{\underline{\ell}} \int d\underline{I}\ \rho_{\underline{\ell}}(\underline{x}|\underline{I})\ \frac{\omega\partial f_{o_s}/\partial H_{o_s} - \underline{\ell}\cdot\partial f_{o_s}/\partial \underline{I}}{\omega + \ell\cdot\Omega}\ \rho_{\underline{\ell}}^*(\underline{x}'|\underline{I}). \tag{C20}$$

Dropping the explicit dependence of f_o on H_o, the adiabatic term $\overline{K}_{SA}$ and the term in $\partial f_o/\partial H_o$ of (C20) vanish, and one obtains a form paralleling that of Refs. 25 and 26. In our work, it is more convenient to instead drop the explicit dependence on one of the canonical momenta I_I (the gyroaction J). The term $\partial f_o/\partial \underline{I}$ in (C20) then has only two components.

APPENDIX D: EVALUATION OF S_{sB} IN THE REGIME $\rho_s/L_\phi > 1$

We here obtain an approximate expression for the contribution S_{sB} of $\bar{K}_{sB}$ or K_{sB} to S, valid in the regime where the mode localization length L_ϕ is small compared to the gyroradius ρ_s of a typical particle of species s. We take for our distribution function the drifting Maxwellian form (III-83) used in the applications of Sec. III.D, though this particular form is inessential to the evaluation method we use.

From the discussion in Sec. III.B, S_{sB} is given by

$$S_{sB}(\psi,\phi) \equiv \int dx \int d\sigma \int d\sigma' \, \psi^*(x+\sigma) K_{sB}(\sigma,x,\sigma') \, \phi(x+\sigma'), \tag{D1}$$

analogous to Eq. (III-9). We use expression (III-83) for f_s in Eq. (III-52) for K_{sB}. Performing the same manipulations as done in Sec. III.C.3 and III.C.4 to evaluate K_{sB} (k,x,-k'), and setting $\phi(x) = \phi_1(x)\, e^{ikx}$, $\psi(x) = \psi_1(x)\, e^{ikx}$, with $\psi_1^*(x) = \phi_1(x)$ as in Secs. III.B.1 and III.B.3, we have

$$S_{sB} = \int dx \; K_{sB} \; (\psi, \; x, \; \phi), \tag{D2}$$

where

$$K_{sB}(\psi,x,\phi) \equiv -\lambda_s^{-2}(x) \sum_\ell 2\pi \int_{-\infty}^{\infty} du \int_0^{\infty} dh' \, \bar{g}_o \, e^{-\gamma_s u^2/2} \, e^{-\gamma_s h'}$$
$$\times \; \tilde{J}_\ell(\psi) \, \tilde{J}_\ell^*(\phi) \, (\omega' - \omega_{*s}) \, (\omega'' - \ell\Omega_o - k_2 v_B - k_3' u)^{-1}. \tag{D3}$$

The only difference between the quantity $K_{sB}(\psi,x,\phi)$ here and the expression for $K_{sB}(k,x,-k')$ in Eq. (III-84), is in the replacement of $\tilde{J}_\ell(\underline{k})$ defined by

$$\begin{aligned}\tilde{J}_\ell(\phi|\underline{l}) &\equiv \int d\sigma\, \tilde{J}_\ell(\sigma,k_y,k_z|l)\, \phi^*(x = X_o + \sigma) \\ &= (2\pi)^{-1} \int_0^{2\pi} d\Theta\, e^{-il\Theta}\, e^{-\hat{\underline{k}}\cdot\delta\underline{r}(\Theta)}\, \phi^*\,[r_x(\Theta)],\end{aligned} \tag{D4}$$

where we recall that $\hat{\underline{k}} \equiv \hat{y}\, k_y + \hat{z}\, k_z$. The difference between $\tilde{J}_\ell(\underline{k})$ and $\tilde{J}_\ell(\phi)$, then, comes just in replacing a plane wave form e^{ikx} by the form $\phi(x) = e^{ikx}\phi_1(\xi)\,[\xi \equiv \alpha^{-1/2}(x-x_o)]$, with $\phi_1(\xi) = \phi_{1n}(\xi) = N_{xn}H_n(\xi)\,e^{-\xi^2/2}$, as specified in Eq. (III-31). Putting this form for ϕ into (D4) and using $r_x(\Theta) = X_o + \delta x(\Theta)$, we have

$$\tilde{J}_\ell(\phi) = e^{-ikX_o}(2\pi)^{-1} \int_0^{2\pi} d\Theta\, e^{-il\Theta}\, e^{-\underline{k}\cdot\delta\underline{r}(\Theta)}\, \phi_1^*[\alpha^{-1/2}(\delta x(\Theta)-\sigma_o)], \tag{D5}$$

where $\sigma_o \equiv x_o - X_o$, the distance between the point x_o of mode localization and the guiding center X_o of any given particle. The factor e^{-ikX_o} in (D5) is unimportant, cancelling a corresponding factor e^{ikX_o} in $J_\ell^*(\phi)$ in (D3).

The integration $\int du$ over parallel velocity in (D3) is unchanged from that done in Sec. III.C.4 to evaluate $K_{sB}(k,x,-k')$. The integration $\int dh'$ over perpendicular velocity done there used a factor $\gamma_s/2\pi$ from $\bar{g}_o \equiv (\gamma_s/2\pi)^{3/2}$ to yield the integral

$$2\pi(\gamma_s/2\pi)\int dh'\, e^{-\gamma_s h'}\, J_\ell^{\,2}(\mu_o) \equiv \Lambda_\ell(b_s) \tag{D6}$$

in expression (III-92) for $K_{sB}(k,x,-k)$, in the approximation of harmonic gyromotion. Our result for K_{sB} here wiill thus be the same as that expression, but with the J_ℓ in (D6) replaced by $\tilde{J}_\ell(\phi)$. It is therefore convenient to extend the definition of $\Lambda_{\ell s}$ from its value $I_{\ell s}(b)\, e^{-b_s}$ for the regime $\rho_s/L_\phi > 1$, namely

$$\Lambda_{\ell s}(\sigma_o;\ \rho_s/L_\phi > 1) \equiv \gamma_s \int dh'\ e^{-\gamma_s h'}\ \tilde{J}_\ell(\psi)\ \tilde{J}_\ell^*(\phi). \tag{D7}$$

(We note the dependence of this quantity on $\sigma_o \equiv x_o - X_o$.)

To evaluate $\tilde{J}_\ell(\phi)$, we refer to Fig. 18. Assuming harmonic gyromotion in this regime also, with

$$\begin{aligned} \delta\underline{r}\ (\Theta) &= \hat{b}_1 \delta x_o(\Theta) + \hat{b}_2\ \delta x_{20}\ (\Theta) \\ \delta x_o(\Theta) &= \bar{x}_o \sin\Theta, \quad \delta x_{20}(\Theta) = \bar{x}_{20}\cos\Theta, \quad \bar{x}_{20} = \bar{x}_o(\Omega_c/\Omega_o), \end{aligned} \tag{D8}$$

as in Sec. III.C.2, we see that the particle trajectory passes through the region of wave localization at two places ($\Theta \simeq \Theta_\pm$) for $\sigma_o/\bar{x}_o < 1$, and not at all, for $\sigma_o/\bar{x}_o > 1$. For particles having $\sigma_o/\bar{x}_o < 1$, we approximate the true trajectory in the region of wave localization by straight line orbits. From Fig. 18, one has that

$$\sin\Theta_\pm = \sigma_o/\bar{x}_o \tag{D9}$$

and so, calling the particle velocities at $\Theta_\pm$ in the $\hat{b}_1$ and $\hat{b}_2$ directions $v_{1\pm}$, $v_{2\pm}$, using (D8) gives $v_{1+} = -v_{1-}, v_{2+} = v_{2-} \equiv v_2$, with

$$v_2 = -\Omega_o \bar{x}_{20} \sin\Theta_\pm = -\sigma_o\Omega_c,\ v_{1+} = \Omega_o\bar{x}_o \cos\Theta_\pm = \Omega_o(\bar{x}_o^2 - \sigma_o^2)^{1/2}. \tag{D10}$$

The integration over Θ in (D5) breaks into two parts, which we denote by $\tilde{J}_{\ell\pm}$: $\tilde{J}_\ell = \sum_\pm J_{\ell\pm}$. For each of these, we replace $\Theta - \Theta_\pm$ by $\Omega_o t$, and extend the limits of integration to $\pm\infty$ for both, permissible since the Gaussian factor in ϕ_{1n} introduces a cutoff in the integration. Using (D5), we then obtain

$$\tilde{J}^*_{\ell\pm} = e^{ikX_o} e^{i\ell\Theta_\pm} e^{-ik_2\delta x_{2\pm}} (\Omega_o/2\pi) N_{xn} \int_{-\infty}^{\infty} dt\, e^{-\kappa t} e^{-At^2/2} H_n(A^{1/2}t), \qquad \text{(D11)}$$

where $\kappa \equiv \ell\Omega_o + k_2 v_2 + k v_{1+}$, $A \equiv \alpha v_{1+}$ and $\delta x_{2\pm} \equiv \delta x_{20}\ (\Theta_\pm)$. The exponential factors outside the integral, which for shorthand we refer to as $e_\pm$, will cancel against the factor in J_ℓ. The integral itself is simply the Fourier transform from t-space to κ-space of the function $e^{-At^2/2} H_n(A^{1/2}t)$. For n = 0 we have just the Fourier transform of a Gaussian,

$$\int_{-\infty}^{\infty} dt\, e^{-i\kappa t} e^{-At^2/2} = (2\pi/A)^{1/2} e^{-\kappa^2/2A}. \qquad \text{(D12)}$$

Given the $(n-1)^{th}$ solution $\phi_{1(n-1)}$ of the Weber equation (III-3), it is easily verified that ϕ_{1n} is given by

$$\phi_{1n}(\xi) = (2n)^{-1/2} (-\partial_\xi + \xi)\, \phi_{1(n-1)}(\xi). \qquad \text{(D13)}$$

(In fact, this is where expressions (III-34) and (III-35b) come from.) Letting the "raising operator" $(2n)^{-1/2}(-\partial_\xi + \xi)$ act on th Fourier representation of $\phi_{1n}(\xi) = (2\pi)^{-1} \int dq\, e^{iq\xi} \phi_{1n}(q)$, one f nds that

$$\phi_{1n}(q) = i\, (2n)^{-1/2} (-\partial_q + q)\, \phi_{1(n-1)}(q). \qquad \text{(D14)}$$

Therefore

$$\phi_{1n}(q) = i^n (2^n n!)^{-1/2} H_n(q)\, \phi_{1o}(q), \qquad \text{(D15)}$$

i.e., the Fourier transform of the eigenfunctions of the Weber equation are again eigenfunctions of the Weber equation, a result to be expected, because of the symmetry of that equation with respect to ξ and $-\partial_\xi$. Applying this to Eq. (D11), one has

$$\tilde{J}^*_{\ell\pm} = e_\pm \; (\Omega_o/2\pi) N_{kn} H_n(q) \; e^{-q^2/2}, \tag{D16}$$

where $q \equiv \kappa A^{-1/2}$, and $N_{kn} = i^n (2\pi/A)^{1/2} N_{xn}$.

To evaluate (D7), we need $\tilde{J}_\ell(\psi) \tilde{J}^*_\ell(\phi)$. Each $\tilde{J}_\ell$ here is a sum $\sum_\pm \tilde{J}_{\ell\pm}$ over contributions from the crossings at $\Theta_\pm$. We are here interested in the ion contribution S_{iB} for the ion drift cyclotron modes. For these modes, one has $k_2\rho_i >> 1$. Therefore, in moving from Θ_+ to Θ_-, the particle phase factor $e_\pm$ will be highly sensitive to any sorts of perturbations present, e.g., from other modes, from orbit modification due to magnetic inhomogeneities, etc. We thus assume that, in averaging over all the particles in the distribution, that contributions from cross-terms $\tilde{J}_{\ell\pm}\tilde{J}^*_{\ell\mp}$ are negligible, i.e., we assume that

$$\tilde{J}_\ell(\psi) \; \tilde{J}_\ell^*(\phi) \simeq \sum_\pm \tilde{J}_{\ell\pm} \; (\psi) \; \tilde{J}^*_{\ell\pm} \; (\phi). \tag{D17}$$

Using (D16), we then find

$$\begin{aligned} \tilde{J}_\ell \; (\psi) \; \tilde{J}^*_\ell \; (\phi) &\simeq 2 \tilde{J}_{\ell+}(\psi) \; \tilde{J}_{\ell+}(\phi) = 2 \; [\Omega_o/2\pi) \; N_{kn} \; H_n(q) \; e^{-q^2/2}]^2 \\ &= (-1)^n \; \Omega_o^2 \; (\pi\alpha v_{1+}^2)^{-1} \; N_{xn}^2 \; H_n^2(q) \; e^{-q^2}, \end{aligned} \tag{D18}$$

where $q^2 \equiv \kappa^2/A \equiv (\ell\Omega_o + k_2 v_2 + k v_{1+})^2 \; (\alpha v_{1+}^2)^{-1}$.

We assume the orderings

$$\rho_s^{-1} k << L_\phi^{-1} \equiv \alpha^{1/2} \lesssim k_2, \tag{D19}$$

of which only the assumption $k << \alpha^{1/2}$ has not yet already been discussed. With this assumption, we neglect the term kv_{1+} in q^2, making K_{sB} and S_{sB} independent of k. This assumption was used in the variational equation (III-27), which lead to the result that $k = 0$, consistent with $k << \alpha^{1/2}$.

From (D10), the particle velocity v_{o1} in the $\hat{b}_1$ direction at the guiding center is given by $v_{o1} = \Omega_o \bar{x}_o$, and the perpendicular energy of gyration h' is given by $h' = \frac{1}{2} v_{o1}^2$. Thus from (D10), we have

$$h \equiv \frac{1}{2} v_{1+}^2 = h' - \frac{1}{2} (\Omega_o \sigma_o)^2. \tag{D20}$$

Using this and (D18) in (D7) yields

$$\Lambda_{\ell s} (\sigma_o) = (-1)^n e^{-\gamma_s(\Omega_o\sigma_o)^2/2} \int_o^\infty d\eta (e^{-\eta}/\eta) (N_{xn}^2/2\pi) (\Omega_o^2\gamma_s/\alpha) H_n^2(q) e^{-q^2}, \tag{D21}$$

where $\eta \equiv \gamma_s h$, and using Eqs. (D10) and the above approximations,

$$q^2 \simeq (\Omega_o^2\gamma_s/2\alpha)[\ell - k_2\sigma_o (\Omega_c/\Omega_o)]^2 \eta^{-1} \equiv \eta_o/\eta. \tag{D22}$$

If we let η(and so v_{1+}) go to zero, corresponding to the particle trajectory which has $\sigma_o/\bar{X}_o = 1$ (so that the particle moves straight along the crest of the Gaussian), the factor $e^{-\eta}/\eta$ in (D21) becomes infinite, but this factor is more than compensated for by the term $e^{-q^2} = e^{-\eta_o/\eta}$, which arises from the rapid phase oscillations a particle traveling

directly along the Gaussian crest sees, due to the factor $(\ell\Omega_o+k_2v_2)$. The term e^{-q^2} thus provides a cutoff at $q \simeq 1$ in the η-integration in (D21), and so we replace the factor $e^{-\eta_o/\eta}$ there with a step function, equal to unity for $\eta > \eta_o$, and zero for $\eta < \eta_o$.

If we consider for the moment the $n = 0$ case, the η - integral in (D21) is $\int_{\eta_o}^{\infty} d\eta\, e^{-\eta}/\eta \equiv Ei(-\eta_o)$, where Ei is the exponential integral function.[29] For $\eta_o > 0$, this function falls off monotonically. For $\eta_o > 1$, the falloff is dominated by the exponential factor $e^{-\eta}$ in the integrand, $Ei\,(-\eta_o) \sim e^{-\eta_o}$. As $\eta_o \to 0$, Ei has a logarithmic singularity. This limit corresponds to the situation $(\ell\Omega_o + k_2v_2) \to 0$, for which the straight line orbit approximation we have made yields an artificial resonance effect. In this limit, we expect that the curvature of the particle orbits, as well as the other orbit modification effects, will act to make the η-integration be well behaved as $\eta_o \to 0$. Combining these expectations for the large and small η_o behaviors of the η-integration, we approximate that integral by $e^{-\eta_o}$, for all η_o. For $n = 0$, then, Eq. (D21) becomes

$$\Lambda_{\ell s}(\sigma_o; n=0) \simeq e^{-\gamma_s(\Omega_o\sigma_o)^2/2}\, e^{-\eta_o}\, (N_{x0}^2/2\pi)\,(\Omega_o^2\gamma_s/\alpha). \qquad \text{(D23)}$$

For larger n, the factor $H_n^2(q)$ displaces a larger fraction of the integrand excluding e^{-q^2} in (D21) toward larger q values, or smaller η values, so that a smaller and smaller proportion is left in the region where the cutoff factor e^{-q^2} is substantial. We thus approximate $H_n^2(q)$ in the region of substantial e^{-q^2} (small q) by its $q = 0$ value,

$$H_n^{\,2}\ (q = 0) = \begin{cases} 0, & n \text{ odd} \\ (n!), & n \text{ even.} \end{cases} \tag{D24}$$

We thus see that for larger n, $|\Lambda_{\ell s}(n)|$ falls off rather rapidly from the value $\Lambda_{\ell s}(0)$. Defining $c_n \equiv |\Lambda_{\ell s}(n)/\Lambda_{\ell s}(0)|$, then, we have $c_o = 1$, $c_n \simeq 0$ for n odd, and $c_n \simeq (n!)^{-1}$ for n even, and so from (D21), (D23), and (III-30a)

$$\begin{aligned} \Lambda_{\ell s}(\sigma_o; n) &= (-1)^n c_n \Lambda_{\ell s}(\sigma_o;\ n = 0) \\ &= (-1)^n c_n\ e^{-\gamma_s(\Omega_o\sigma_o)^2/2}\ e^{-\eta_o(\gamma_s\Omega_o^{\,2}/2\pi^{3/2}\alpha^{1/2})}. \end{aligned} \tag{D25}$$

To evaluate S_{sB}, it remains to put expression (D25) for $\Lambda_{\ell s}$ into Eq. (III-92) for K_{sB}, and then perform the x (or σ_o) integration indicated by (D2). Assuming for simplicity that since $\eta_p \equiv \rho_i/L_p < 1$, we may neglect the spatial variation of all terms in K_{iB} but that of $\Lambda_{\ell i}$, we may readily evaluate the integration in S_{sB} for the ion species. Again only for simplicity, we first neglect ℓ in comparison with $k_2\sigma_o$ in (D22), in view of Eq. (III-104). This gives

$$\eta_o \simeq \frac{1}{2}\gamma_s(\Omega_o\sigma_o)^2\ (k_2\ \Omega_c/\Omega_o)^2\ \alpha^{-1}, \tag{D26}$$

and so from (D25), defining $\rho_s^{\,2} \equiv (v_s/\Omega_o)^2 \equiv \gamma_s^{\,-1}\ \Omega_o^{-2}$,

$$\begin{aligned} \Lambda_{\ell s}(n) \equiv \int_{-\infty}^{\infty} d\sigma_o \Lambda_{\ell s}(\sigma_o;n) &= (-1)^n c_n(\gamma_s\Omega_o^{\,2}/2\pi^{3/2}\alpha^{1/2}\int_{-\infty}^{\infty} d\sigma_o\, e^{-\frac{1}{2}\gamma_s\Omega_o^2[1+(k_2\Omega_c/\Omega_o)^2]\sigma_o^2} \\ &= (-1)^n c_n\ 2^{-1/2}\pi^{-1}[\alpha\rho_s^{\,2} + (k_2\rho_s)^2\ (\Omega_c/\Omega_o)^2]^{-1/2}. \end{aligned} \tag{D27}$$

This expression simply replaces the factor $\Lambda_{\ell 1} \equiv I_\ell(b_i)e^{-b_1}$ appearing in the form for S_{iB} valid in the small - ρ_i/L_ϕ regime, i.e.,

$$S_{iB} = -\lambda_i^{-2} \sum_\ell (1 - W_{\ell 1})(\Omega_1/\Omega_2)\Lambda_{\ell i}, \tag{D28}$$

where

$$\Lambda_{\ell i}\ (\rho_i / L_\phi < 1) \equiv I_\ell\ (b_i)\ e^{-b_i}, \qquad \text{and} \tag{D29}$$

$$\Lambda_{\ell i}\ (\rho_i / L_\phi > 1) \equiv (-1)^n c_n 2^{-1/2} \pi^{-1} [\alpha\rho_s{}^2 + (k_2\rho_s)^2 (\Omega_c/\Omega_o)^2]^{-1/2}. \tag{D30}$$

APPENDIX E: ONE TERM APPROXIMATION TO THE IDC GROWTH RATE

Here we derive an expression for the IDC growth rate, using the usual approximation made in analytic analyses of the IDC instability, in which we keep only the ℓth term in factor C in Eq. (III-126). As shown in Figs. 15a, 15b, we here redefine $\bar{\Omega}$ from the definition used in the two term analysis of Sec. III.D.2 by

$$0 = \partial_\omega (B + C)\Big|_{\bar{\Omega}} . \tag{E1}$$

This gives

$$(\bar{\Omega} - \omega_\ell)^2 = \bar{\Omega}^2 \; (I_2 \Lambda_\ell g_\ell / \Lambda_{oe} g_e)(\omega_* - \omega_\ell)\, \omega_*^{-1}. \tag{E2}$$

As in Sec. III.D.2, we expand the dispersion equation (III-126) in $\delta\omega \equiv \omega' - \Omega$, again neglecting $\partial_\omega^2 B$ in view of (III-133). Using (III-134) as in Sec. III.D.2, and neglecting the term in $K_{d.e.}$ for simplicity, we obtain the quadratic equation

$$0 \simeq (\bar{\Omega} - \omega_{ISEC})\, \partial_\omega B + \frac{1}{2} (\delta\omega)^2\, \partial_\omega^2 C + C, \tag{E3}$$

with solution

$$(\delta\omega)^2 = -2[(\bar{\Omega} - \omega_{ISEC})\partial_\omega B + C]/\partial_\omega^2 C. \tag{E4}$$

In contrast to the two term approximation, a glance at Fig. 15b shows that here $\partial_\omega^2 C < 0$, and so here the term $C/\partial_\omega^2 C$ is <u>de</u>stabilizing. Assuming that $\omega_{ISEC} \simeq \bar{\Omega}$, we neglect the term in $(\bar{\Omega} - \omega_{ISEC})$ in (E4) and use (E2) to obtain the desired result:

$$\gamma_{mx}^{2} \simeq 2\, C/\partial_{\omega}^{2}\, C = (\omega'-\omega_{\ell})^{2}(\omega'-\omega_{*})(\omega_{*}-\omega_{\ell})^{-1}\Big|_{\omega'=\bar{\Omega}}$$

$$= \bar{\Omega}^{2}\,(I_3\, \Lambda_{\ell} g_{\ell}/\Lambda_{oe}\, g_{e})(1 - \bar{\Omega}/\omega_{*}). \tag{E5}$$

Using the fact that here $\bar{\Omega} \simeq \omega_{ISEC}$, and that ω_{*} is what we call $\bar{\Omega}$ (at maximum growth) in the two term approximation, we see that the factor $(1 - \bar{\Omega}/\omega_{*})$ here corresponds to the factor $(1 - \omega_{ISEC}/\bar{\Omega})$ of Eq. (III-139).

SYMBOL TABLES FOR PARTS I, II, AND III

The following tables provide an explanation of the symbols used, or indicate where in the text the symbol is defined or introduced. A separate section is given for each of Parts I, II, and III of the text. Symbols whose use is restricted to the immediate vicinity of the place they are defined are not included.

In Table A of each part is given a list of general notational conventions used throughout that part. Table B is a listing, in alphabetical order (Latin alphabet first, then Greek alphabet), of the symbols used in that part, followed by the symbol's meaning or where to find it. The symbol is given, followed by a hyphen, followed by information giving the symbol's meaning. Here, if the defining equation referenced occurs in the same part as that to which the symbols table refers, the part number is not given. For example, in the table for Part III,

$$\Lambda_\ell \text{ - } (83)$$

means symbol Λ_ℓ is defined in Eq. (III-83). The notations "b(83)" and "f(83)" indicate that the definition is made in the text in the immediate vicinity before (b) or following (f) Eq. (III-83).

When a symbol has more than one important equation indicating its meaning, a multiple notation may be given, e.g. for the symbol K_s of part III,

$$K_s \text{ - } [(8),(45)],$$

both Eqs. (III-8) and (III-45) are useful in understanding its meaning. In certain cases, where context should make the meaning clear, the same symbol has been used for different quantities. In these few cases, the symbol itself is repeated in Table B.

Finally, some of the symbols with related meanings fall into "notational groups" of similar symbols, for example the various contributions to the Hamiltonians in Part II. To aid the reader in remembering the significance of and differences among these related quantities, *in Table C of Parts II and III, a description is given of the more* important notational groups of those parts.

PART I

Table I.A) Conventions

Notation	Meaning	Examples
tilde	physical specification of the quantity in question	$\tilde{f}_s$, $\tilde{X}_o$, $\tilde{\underline{A}}$
superscript "sc"	"self consistent"	$\underline{A}^{sc}$, $\underline{B}^{sc}$
components i = 1,2,3 of a vector $\underline{V}$	$\hat{b}_i \cdot \underline{V}$	A_2, P_3, V_1

Table I.B) Symbols for Part I

$\underline{A}$ - [vector potential, b(1)], $\tilde{\underline{A}}$-§I.A., A-§I.C.1, beginning, $\tilde{A}$-§I.C.1, beginning

$\underline{A}^{sc}$-§I.C.1, beginning, A_3-(19)

$B \equiv |\underline{B}|$, $\underline{B}$-[magnetic field, (1)], $\tilde{\underline{B}}$-§I.C.1, beginning, $\underline{B}^{sc}$-§I.C.1, beginning

$\hat{b}_i$. (i = 1,2,3,)-b(19)

e_s - charge of particle of species s

F_s - (30)

f_s - [§I.A, §I.C.2], $\tilde{f}_s$ - [§I.A, §I.C.2]

H - [(4), (species label suppressed)], H'-§I.C.3, beginning,

$\tilde{H}'$-§I.C.3, beginning, H_{min}-f(23)

h - [(7),(species label suppressed)], $h_{1,2}$-f(4)

J - f(21)

J - gyroaction

$L_{B,\underline{P},S,n}$ - §I.A, beginning

$n(X_o)$ - [guiding center density (species label suppressed), (23)]

$\underline{P}$ = (P_y,P_z) - b(4), P_2-(A-1), P_3-(19)

$\underline{p}$ = (p_x,p_y,p_z) - b(4)

R - (26)

T - temperature (subscripts suppressed)

U - [§I.A, (14)]

V - (5)(species label suppressed)

$v_\parallel$ - [§I.A, (19)], $v_\perp$ - [§I.A,(20)], $\tilde{v}_\parallel$ - §I.C.2, beginning

$v_\perp$ - §I.C.2, beginning, $v_{1,2}$-(20)

$w_\parallel$ - [parallel drift velocity, §I.C.5,end]

X_o - [§I.A,(17)], $\tilde{X}_o$- §I.C.2, beginning

x - §I.A, beginning

Y - guiding center y coordinate

y - §I.A

Z - guiding center z coordinate, $\underline{Z}$ - f(22)

z - §I.A

γ - inverse temperature (subscripts suppressed)

η - §I.A, beginning

Θ - gyrophase

$\underset{\sim}{\Pi}$ - (2), Π_{xx} - (3) (species label suppressed)

Φ - electrostatic potential, $\tilde{\Phi}$ - §I.C.3, beginning

Ω - gyrofrequency, f(21)

PART II

Table II.A) Conventions

Sub- and superscripts	Meaning	Examples
ℓ	order in ε expansion	$K = \sum_{\ell} \varepsilon^{\ell} K_{\ell}$
m	order in η expansion	$K_{\ell} = \sum_{m} \eta^{m} K_{\ell m}$
n	number of perturbative transformations applied	H^{n}
overbar	time average over unperturbed trajectory	$\bar{h}_3$, $\bar{H}_1$
inverted carot	Lie operator	$\check{G}$, $\check{K}_1$

Table II.B) Symbols for Part II

$\underline{A}$ - [vector potential, (2),(3)], A_{μ} - [vector potential,(2),(3)],

$\underline{A}_0$ - f(100)

a - f(49), $\bar{a}$ - f(49)

$\underline{B}$ - [magnetic field,(41)], B^{μ} - [magnetic field,(4)], $\underline{B}_0$ - f(100)

$\hat{b}_j$ - $\hat{b}$(99), $b_{\pm}$ - (104)

C_{mn} - f(81)

$c_{1,3}$ - f(53), $c_{\alpha,p}$-f(66), c_s-f(69)

$D_{2,4}$ - (86)

$d_{0,2,4}$ - f(53)

E - [particle energy,(71)], E(x)-[electric field], E_1-f(91), E_o-f(100)

$\underline{e}_\mu$ - b(2), $\underline{e}^\mu$-b(2)

G_n - (22), $\check{G}_n$-(22), $\check{G}$-(20)

$g^{\mu\nu}$ - [metric tensor,§II.C], g_m-(48b), $g^\mu(\mu = \alpha,\beta,\phi)$-b(2)

H - (see also Table II.C for all symbols H,h):

H - [particle Hamiltonian,(5)], H'-(16), H^o-[§II.B,§II.H,(35)], H^n-(36), $\check{H}_{ob}$-(39), $\check{H}_{o\Theta}$-(39)

h - §II.B,h'-[§II.E.1, beginning, (48a)], h^o-[§II.B,(50)], h^n-(19)

J - gyroaction, J^o-f(1), J_o-f(91)

j - §II.B, j^o-§II.B

K - (see also Table II.C for all symbols K,k):

K - [guiding center Hamiltonian,(89)], $K_{\ell m}$-(1), K^n-(36)

k^n - (19)

L_B - magnetic modulus shear length, $\underline{L_B} \equiv \min(L_s,L_B)$, L_s-shear scale length, $L_{1,\parallel}$-§II.A, beginning

P_b - Table II.C, P_ϕ-[§II.D, beginning, Table II.C], P_b^o-[§II.B, Table II.C]

p_x - f(1), $p_\mu(\mu = \alpha,\beta,\phi)$-§II.C

$q^\mu(\mu = \alpha,\beta,\phi)$-§II.C

R - [§II.B,(22)], R_n-[§II.B,(22)], R_o-§II.B, R'-§II.B

S - [§II.B,10]

T - f(1), T_n-§II.B

V - (6), V'-(75), V_m-(48c)

v_B - (117), v_E-(103), v_i-f(126), v_{oi} $(i = 2,3)$-f(100), v_{01}-b(122)

$\underline{v}_o$ - f(100)

$W_{\ell m}$ - b(79), W_n-f(37), $\check{W}_n$-(37), $\check{W}_{1b}$-b(34), $\check{W}_{1\Theta}$-b(43)

X_o - b(99), $\dot{X}_{20}$-(117), $\dot{X}_{30}$-(118)

x - f(1), x_j-b(99), $\overline{x}_o$-f(112), $\overline{x}_{20}$(119)

y_m - (63)

$\underline{Z}$ - (see table II.C for all symbols $\underline{Z},\underline{z}$)

α - [f(1),§II.C], α_b-["banana center," §II.C,end], α_o-[§II.C,end(47], $\overline{\alpha}_o$-(B1), α_o^n-f(75)

β - [§II.B,§II.C]

δx_o - (111), δx_1-(112), δx_{20}-(119), δx_{21}-(120), δx_{31}-(121), δp_β-(14), $\delta p_{\beta 1}$-(72), $\delta\alpha$-[f(47),(49)], $\delta\alpha_o$-(64), $\delta\alpha_1$-(65), $\delta\beta$-(15), $\delta\beta_o$-(71) $\delta\beta_1$-(72)

ε - §II.A, beginning

η - §II.A, beginning

Θ - gyrophase, Θ^o-f(1)

θ - §II.B, θ^o-§II.B

κ - f(104), $\kappa_{B,S}$-f(104), κ_E-(109)

$\lambda \equiv (\lambda_1,\lambda_2)$-§II.D,beginning, $\bar{\lambda}_i$-(B3), $\lambda^{O,M,N}$-§II.D,beginning

μ - magnetic moment

ρ - [particle gyroradius, §II.A]

Φ - [electrostatic potential,(6)]

ϕ - [toroidal angle, §II.C]

Ω - gyrofrequency, Ω_o-[f(40),f(49)], Ω_2-(90)

ω_D- bounce frequency

Table II.C) Notational Groups

1) Hamiltonians: In general, symbols H denote Hamiltonians (or parts of Hamiltonians) for the full (2-D) particle motion problem; symbols h demote those for the frozen -λ (1-D) problem. Superscript n denotes the Hamiltonian after n perturbative transformations. K^n and k^n denote the angle-independent part of H^n and h^n, respectively, and so the guiding center Hamiltonian K, being fully angle-independent, satisfies $K = H^\infty = K^\infty$.

Each h^n is expressible as an expansion in η [cf Eq. (19)], with m denoting the power of η:

$$h^n = h_o^n + \sum_{m=o}^{\infty} \eta^m h_{m+2}^n \ .$$

Each H^n is expressible as an expansion in both η and ϵ, with indices m and ℓ, respectively:

$$H^n = \sum_{\ell=o}^{\infty} \epsilon^\ell H_\ell^n \sum_{\ell,m=o}^{\infty} \epsilon^\ell \eta^m H_{\ell m}^n$$

The original particle Hamiltonian, expressed in terms of the curvilinear coordinates (α,β,ϕ) and their conjugate moment a, is denoted by H. The symbol H' denotes a useful non-canonical expression [Eq. (16)] for the Hamiltonian, and h' denotes the Hamiltonian for the frozen-λ problem, before transformation to action-angle variables [Eq. (17)].

2) Lie Generators: The Lie generators for the frozen-λ transformations R_n are denoted G_n; those for the transformations T_n of stage two are denoted W_n. The corresponding Lie operators are denoted $\check{G}_n$ and $\check{W}_n$.

3) Phase variables: The variables for the restricted (1-D) phase space of the frozen-λ problem are denoted by the general symbol $\underline{z}$, with a superscript matching that of the corresponding Hamiltonian, i.e. $h'(\underline{z}')$, $h^n(\underline{z}^n)$. Analogously, the variables for the full phase space are denoted by $\underline{Z}$. When occurring without superscript, $\underline{Z}$ denotes either the variables of the full problem in general (so that any H^n may be considered as a function of $\underline{Z}$), or the specific phase variables of the guiding center Hamiltonian $K \equiv H^\infty$. These variables are

$$\underline{Z} = (\Theta,J;b,P_b),$$

where we have suppressed the variables of the trivial toroidal degree of freedom, the (constant) canonical angular momentum $P_\phi = p_\phi$, and its conjugate toroidal coordinate, as done through most of the text. When occurring with superscript, i.e. $\underline{Z}^n$, $\underline{Z}$ denotes the phase space variables of H^n.

By the symbol $\underline{z}$, we refer either to the phase variables of the frozen-λ problem, described above, or to the variables $(\alpha,\beta,\phi,p_\alpha,p_\beta,p_\phi)$ in terms of which the original Hamiltonian H is expressed.

PART III

Table III-A) Conventions

Subscripts	Meaning	Examples
s	species s	f_s, ω_s''
sj	contribution from part f_{sj} of distribution f_s	ω_{*sj}, $\bar{\gamma}_{sj}$
ℓ	harmonic number	$W_{\ell s}$, Λ_ℓ
R, I	(formally) real and imaginary parts	$W_{oe}=W_{Roe} + iW_{Ioe}$, $\Gamma_1 = \Gamma_{1R} + i\Gamma_{1I}$

Table III.B) Symbols for Part III

A (20), A-f(126), A'-f(126)

B - (20), B-f(126)

C - (20), C-f(126)

D_G - (15), D_B-b(16)

d_k - [total derivative w.r.t. k,(16)]

E_o - (67), e-f(48)

$F_{B1,P2}$ - (77)

f_n - (42), f_s-(48), f_{sj}-b(68)

$\bar{g}_{0,1}$ - (82), $g_{e,\ell}$-b(125), g_{Re}-(120)

H - b(48), H'-b(68), h'-b(68), $H_n(\xi)$-(34)

$\underline{I}$ - b(48), I-(51), I_1-(96), $I_{2,3}$-(118), I_ℓ-(87)

J - b(48), $_\ell$-§III.C.2, J_ℓ-(62), $\tilde{J}_\ell$-(49)

K - (11), K_d-(47), $K_{d.e.}$-[f(29),(42),(46)], $K^{(n)}_{d.e.}$-(42), K_h-b(64), $K_{n.\ell.}$-(41), $K_{o.m.}$-b(64), K_s-[(10),(54),(56)], $\bar{K}_s$-(48a), $\bar{K}_{sA,B}$-[f(25),(48)], K_{sA}-(56), K_{sB}-[(52),(54)], K_{sj}-(76), K_o-[f(37),(119)]

k - (10), k'-(10), $\underline{k}$-(53), $\hat{k}$-f(50), $k_{y,z}$-(2), $k_{2,3}$-[(58a),(68)], k_3'-(75)

$L_{B,\underline{B},n,S}$ - [end of §III.A], L_p-f(109), L_Q-(37), L_ϕ-b(30), L_{oe}-(106)

$\underline{\ell} = (\ell,\ell_y,\ell_z)$ - (48), ℓ_{max}-(122)

M_m - (32)

m_s - mass of particle of species s

$N(\alpha)$ - (31)

n - [mode index, f(41)], $n(X_o)$-f(72), $n_s(X_o)$-(81), n'-f(142)

$\underline{P} = (P_y,P_z)$-b(2), $P_{2,3}$-f(69)

Q - (21), Q_d-b(47), $Q_{R,I}$-f(31), Q'-(22)

R - (82), $\underline{R}$-b(49)

r - b(140), $\delta\underline{r}$-b(49)

S - (5), S_{iB}-(26), $S_{v,s}$-(5), $S_<$-[b(17),(26)], $S_>$-(26)

s - species label

T_s- temperature of species s

u - b(67), u_E-[(70),(86)]

v_B - [(73),(II-117)], v_E-f(72), v_s-thermal velocity of species s,

v_{sj} - [thermal velocity of component j of species s,f(72)],

v_{01} - [b(67),§II.K.3], v_{02}-[b(67),§II.K3], v_3-[b(51),§II.K.3]

$W(z)$ - (88), $W_{\ell s}$-(92), $W_{R,I}$-f(96)

$w_{\|}$ - [b(67),§I.C.6,(species label implicit)]

X_0 - (4), $\dot{X}_{20}$-(73), $\dot{X}_{30}$-(73)

$\underline{x}$ - (1), $\underline{x}'$-(1), x-(2), x'-(3), x_0-(28), $\bar{x}_0$-[(58),f(II-112)], x_1-f(142), x_2-b(147), $\bar{x}_{20}$-[(58),(II-119)]

Y - b(48)

Z - b(48), $Z(z)$-(89), $\underline{Z}$-f(48)

$z_{\ell s}$-(87)

α - [b(31),(45)], $\alpha_I \equiv \mathrm{Im}\,\alpha$

β - (144)

Γ - (18), $\Gamma_{R,I}$-f(19), $\Gamma_{1,2}$-(19), Γ_{1R}-[f(19),(64)], Γ_{1I}-f(19)

γ - growth rate, γ_j-f(72), γ_{mx}-(139), γ_s-(80), $\bar{\gamma}_{sj}$-(72)

$\Delta\ell$ - f(96), $\Delta w_{\|}$-b(143)

$\partial_{x,k}$ - partial derivative w.r.t x or k, $\vec{\partial}_x \equiv \partial_x$, $\overleftarrow{\partial}_x$-partial derivative w.r.t x acting to the left, $\delta/\delta f(x)$-functional derivative w.r.t. function f(x), δx_o-[f(57),(II-111)], δx_{20}-[f(57),(II-130)], δy-b(49), δz-b(49)

$\varepsilon_{Q\ell}$ - b(116), $\varepsilon_{p\ell}$-f(110), $\varepsilon_{\phi\ell}$-(101)

η - §III.A, $\eta_{B,\underline{B},n,S}$-§III.A, η_Q-(105)

Θ - [gyrophase,b(48)], Θ_k-(62), Θ_o-(58b)

$\underline{\theta}$ - b(48)

$\kappa_{B,\underline{B},n,S}$ - §III.A, κ_{fj}-(72), κ_p-f(118), κ_ϕ-b(25)

Λ_ℓ - (92), $\Lambda_{\ell i}$-[(D29),(D30)], $\Lambda_{\ell s}$[(92),App.D]

$\lambda \equiv (\omega, k_y, k_z)$, λ_s-[Debye length of species s,(C18)]

$\underline{\mu} = \{\mu_1, \ldots \mu_N\}$ - b(23), μ_o-(58b), μ_o'-b(64), μ_s-(87)

$\rho(\underline{x})$ - (1), ρ_i-[ion gyroradius,§III.A], $\rho_{\underline{\ell}}(\underline{x}|\underline{I})$-(48)

σ - (4), σ'-(4)

τ - (120)

$\phi(\underline{x})$ - [(1),§III.A], $\phi(x)$-(2), ϕ_1-(13)

χ_s - susceptibility of species s

$\psi(x)$ - (5), ψ_1-(13)

Ω - [gyrofrequency,b(48)], $\underline{\Omega}$-b(48), Ω_c-(59), Ω_o-(59), $\Omega_{1,2}$-(87), $\overline{\Omega}$-[b(128),(151)]

ω - (1), ω'-(69), $\omega' \equiv \omega_e''$-[124,b(143)], ω''-(75), $\overline{\omega}$-(127), $\omega_* \equiv \omega_{*i}$-(126), ω_*'-(123), ω_{*s}-(84), ω_{*sj}-(69), ω_e-[electron plasma frequency,(122], ω_{ISEC}-(120), ω_{ISEC}'-(120), ω_ℓ-b(143)

[Note: Species label s has been suppressed in the frequencies Ω, Ω_c, Ω_o, $\Omega_{1,2}$, ω', ω''.]

Table III.C) <u>Notational Groups</u>

1) Scale lengths L, reciprocal scale lengths κ, and inhomogeneity parameters of expansion ϵ and η:

$L_y \equiv$ scale length of any quantity $y(x)$, e.g. $L_p \equiv$ plasma scale length, $L_{\underline{B}} \equiv$ scale length for variation of magnetic field.

$\kappa_y \equiv L_y^{-1}$

$\varepsilon_{y\ell} \equiv \ell/k_2 L_y$, $\eta_y \equiv \rho_i/L_y$

2) Contributions K to the dispersion equation:

Subscripts denote origin or function of term:

V - vacuum, s - contribution of species s, sj - contribution from portion f_{sj} of distribution f_s, d - well depth of K(x), for localizing modes, d.e. - contribution due to solving a differential equation, h - contribution from assuming small gyroradius (harmonic) particle orbits, n.ℓ - nonlocal contribution, o.m. - corrections due to anharmonic orbit modification effects, 0 - zero-order approximation to K, for determining mode localization, < - nearly local portion of K.

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FIGURE CAPTIONS

Fig. 1. Reference pseudotrajectory $\underline{\tilde{A}}(x)$ modeling the Tormac sheath region, and the corresponding magnetic field $\underline{\tilde{B}}(x)$, for four values of x.

Fig. 2. Illustration of the graphical method introduced in Sec. I.C.1. For three particular values of x on the self-consistent pseudotrajectory $\underline{A}^{SC}(x)$ (labeled 1, 2, and 3) are drawn lines of constant guiding-center position X_0 in the $\underline{P}$-plane, which lie in the direction of the magnetic field $\underline{B}$ at those points. The parallel velocity $v_{\parallel}$ at X_0 is the distance from the pseudotrajectory to any point $\underline{P}$ considered, as shown for the point labeled "c" on line 2.

Fig. 3. Sketch of the form of the effective potential V(x/P) in which a particle with canonical momentum $\underline{P}$ moves, for points $\underline{P}$ like point "a" of Fig. 2, which lies on two lines of constant guiding-center, correponding to the two minima in V(x). Depending on the value of the energy H, the particle may be confined to one of the two wells, or it may pass through both wells, in the course of an oscillation.

Fig. 4. Pseudotrajectory $\underline{A}(x)$ for the simple example of magnetic field $\underline{B}(x)$ with constant modulus and shear scale length. For this example $\underline{B}$ is parallel to $\underline{A}$, as are the lines of constant guiding-center $X_0(\underline{P})$, here drawn for the particular value $X_0=x_2$.

Fig. 5a. The $\underline{P}$ or $\underline{A}$ plane of the $(H,\underline{P})$ space over which distribution f_s is defined, showing the reference and self-consistent pseudotrajectories $\underline{\tilde{A}}$ and $\underline{A}^{sc}$, a single constant guiding-center line, and a number of sample points $\underline{A}_j$, corresponding to the integration boundaries in Fig. 5b.

Fig. 5b. Slice of the $(H,\underline{P})$ space corresponding to the X_o=constant line in Fig. 5a, showing the H-integration boundaries for points $\underline{A}_j$ of Fig. 5a. The curve labeled "0", corresponding to point $\underline{A}_o=\underline{A}^{sc}(X_o)$ on the self-consistent pseudotrajectory, marks the boundary of the "physical region." The curve labeled "1", corresponding to $\underline{A}_1=\underline{\tilde{A}}(X_o)$, marks the boundary of the "quasi-physical region" (cf. Sec. I.C.2).

Fig. 6. Sketch of contours of constant $f(H,\underline{P})$, along an X_o=constant slice of the $(H,\underline{P})$ plane, for the loss-cone distribution function of Sec. I.C.3. Outside the quasiphysical region, f is taken equal to zero.

Fig. 7. Numerical results of W. M. Sharp, applying the methods of Part I:

(a) Reference and self-consistent pseudotrajectories $\underline{\tilde{A}}$ and $\underline{A}^{sc}$ modeling the Tormac sheath. The proximity of $\underline{A}^{sc}$ to $\underline{\tilde{A}}$ indicates that the goal of resemblance of the self-consistent and the reference magnetic fields has been achieved.

(b) Magnetic field $\underline{B}=\hat{y}B_y + \hat{z}B_z$ and its modulus $|\underline{B}|$ as functions of x (in units of the ion gyroradius ρ_i).

(c) Electrostatic potential $\Phi(x)$. The large values for large

x/ρ_i reflect a deficiency of the particular model used for f_e (cf. Sec. I.C.6).

(d-f) Contours of constant ion distribution f_i, at points in the Tormac interior (d), sheath center (e), and sheath exterior (f), for the model of Sec. I.C.6. The resemblance of the physically prescribed reference distributions (left side) and the self-consistent distributions (right side) reflects the achievement of the goal of Part I.

Figs. 8. Sketches of functions arising in the evaluation of the pseudopotential $U(\underline{A})$ in Appendix A.

Fig. 9. Coordinate system $(\hat{b}_2, \hat{b}_3)$ natural to a particular point $\underline{\tilde{A}}(\tilde{X}_o)$ on the reference pseudotrajectory, used in Appendix A to evaluate pseudopotential $U(\underline{A})$.

Fig. 10. Axisymmetric coordinate system (α,β,ϕ) adopted in Part II (cf. Sec. II.C).

Fig. 11. Real and imaginary parts of the plasma dispersion function $W(z) \equiv (2\pi)^{-1/2} \int dx\, e^{-x^2/2}(x-z)^{-1}$, for z on the real axis.

Fig. 12. Sketch of the argument $z_{\ell' i}$ of the dispersion function $W(z_{\ell' i})$, as a function of harmonic number ℓ', showing that for nonzero shear, a few values of ℓ' may have $z_{\ell' i}$ sufficiently small that $W(z_{\ell' i})$ is appreciable (cf. Secs. III.C.4, III.D.3).

Fig. 13. Sketch of the scaling with shear (κ_s/κ_n) of the various contributions to the dispersion equation (cf. Sec. III.C.5). To the right of line a the stringent convergence condition

(III-101) is violated. To the right of line b the less stringent convergence condition (III-116) is violated. When the convergence condition is satisfied (violated), the former (latter) of Eqs. (III-41) is the appropriate dispersion equation. To the right of line c, mode localization condition (III-102) is violated.

Fig. 14. Graphical depiction of the dispersion equation for the IDC instability, studied in Sec. III.D. Instability arises when the sum of terms B and C does not extend down to the ω'-independent line -A, in the interval containing ω_*.

Figs. 15. Sketches of the two-term (a) and one-term (b) approximations to the full IDC dispersion equation, which is illustrated in Fig. 14. The two-term approximation accurately represents the full equation in the interval containing ω_*, in which instability can occur, while the one-term approximation, used in previous analytic studies of the IDC instability, does not.

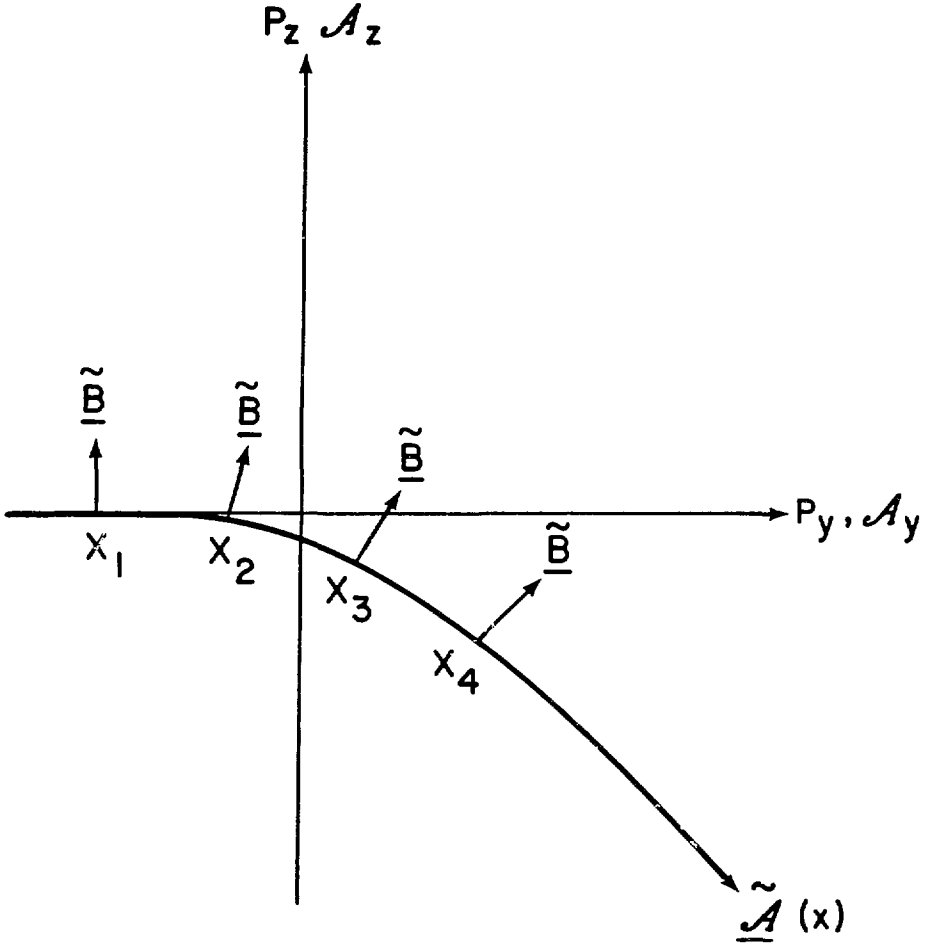
Fig. 16a. Sketch of the factors $\lambda_e^{-2}\omega_*/\omega'$ and λ_i^{-2} versus x, important in determining the form of the function $K_o(x)$ in Fig. 16b.

Fig. 16b. Sketch of the function $K_o(x)$ (for the negligible-shear regime of Sec. III.D.2), which acts as a potential well, at whose bottom unstable eigenmodes are localized.

Fig. 17a. Sketch of the real and imaginary parts of the factor $-g_e \equiv W(z_{oe})-1$, important in determining the position and width of unstable eigenmodes in the appreciable-shear regimes (Secs. III.D.3, III.D.4).

Fig. 17b. Sketch of the real part $K_{RO}(x)$ of the function $K_o(x)$, for shear length L_s large enough that $L_{oe} \approx L_s(\omega/k_2 v_e)$ is somewhat shorter than density scale-length L_n (Sec. III.D.3). Unstable eigenmodes are localized at the local maximum near point x_2 where $k_3(x_2)=0$, in Berk-Pearlstein fashion. Away from this point, the mode is strongly damped through K_{IO} by electron Landau damping.

Fig. 18. Depiction of the gyroorbit of a particle passing through a region of mode localization, in the regime in which the particle gyroradius $\rho_s \approx \bar{x}_o$ is much greater than the mode localization width $L_\phi \approx \alpha^{-1/2}$. In this regime, one may approximate the particle trajectory in the mode localization region by a straight line (cf. Appendix D).



XBL 7811-12796

Figure 1

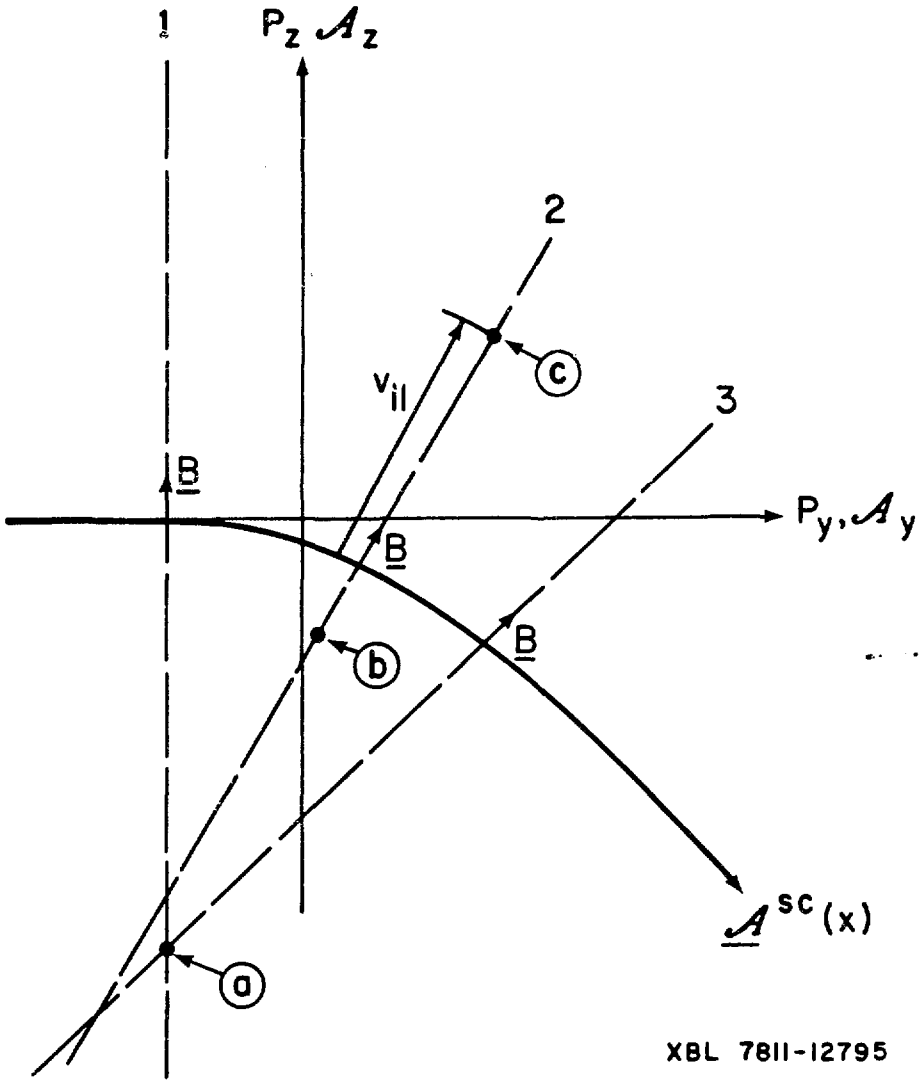

1
$P_z\,\mathcal{A}_z$
2
3
$v_{\parallel}$
c
B
$P_y, \mathcal{A}_y$
B
B
b
$\mathcal{A}^{sc}(x)$
a
XBL 7811-12795

Figure 2

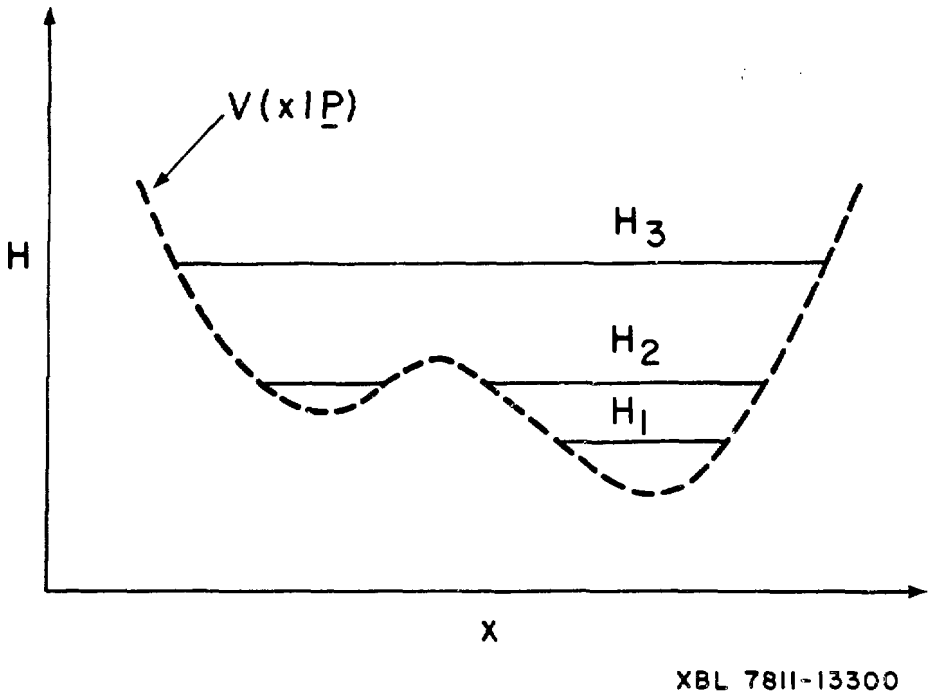


Figure 3

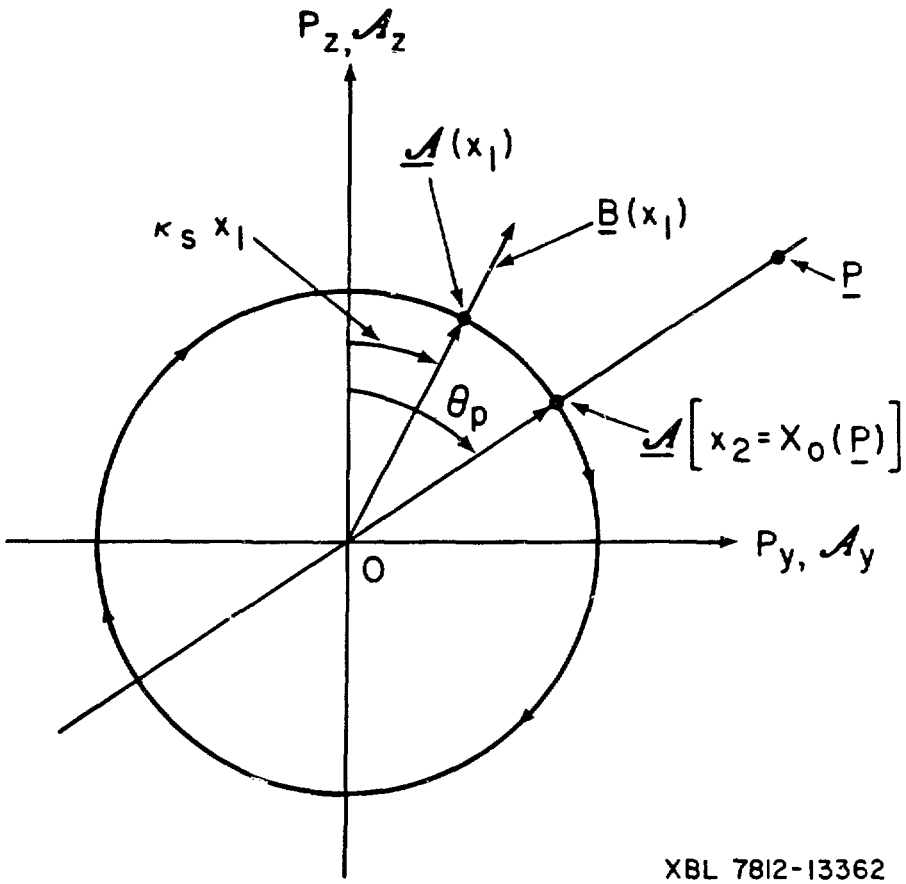
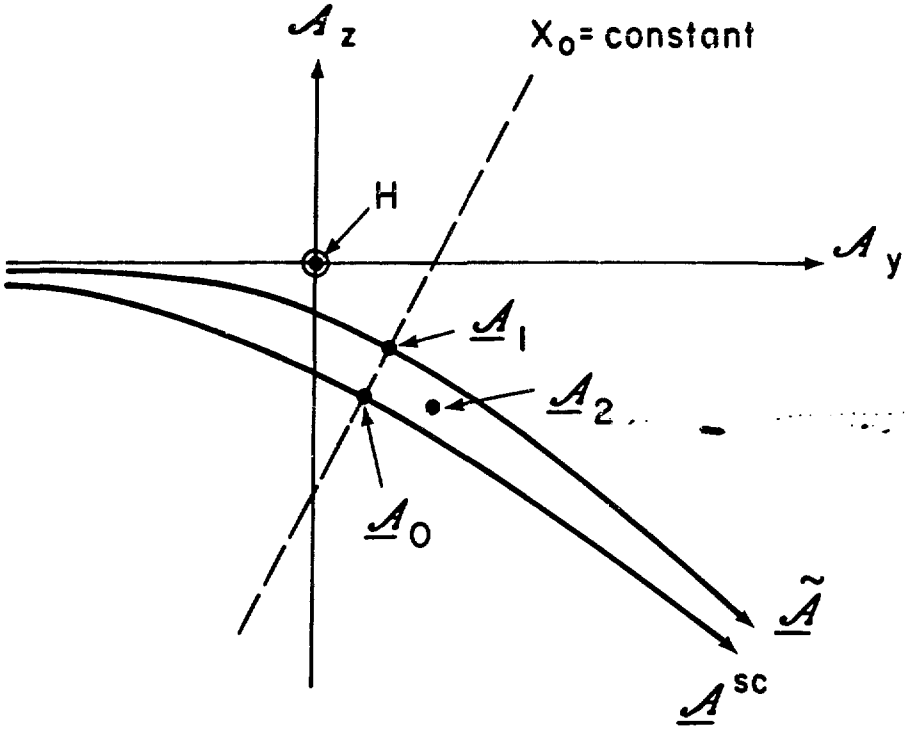

$P_z, \mathcal{A}_z$
$\underline{\mathcal{A}}(x_1)$
$\underline{B}(x_1)$
$\kappa_s x_1$
$\underline{P}$
θ_p
$\underline{\mathcal{A}}\left[x_2 = X_0(\underline{P})\right]$
$P_y, \mathcal{A}_y$
O
XBL 7812-13362

Figure 4



XBL 7811-12793

Figure 5a

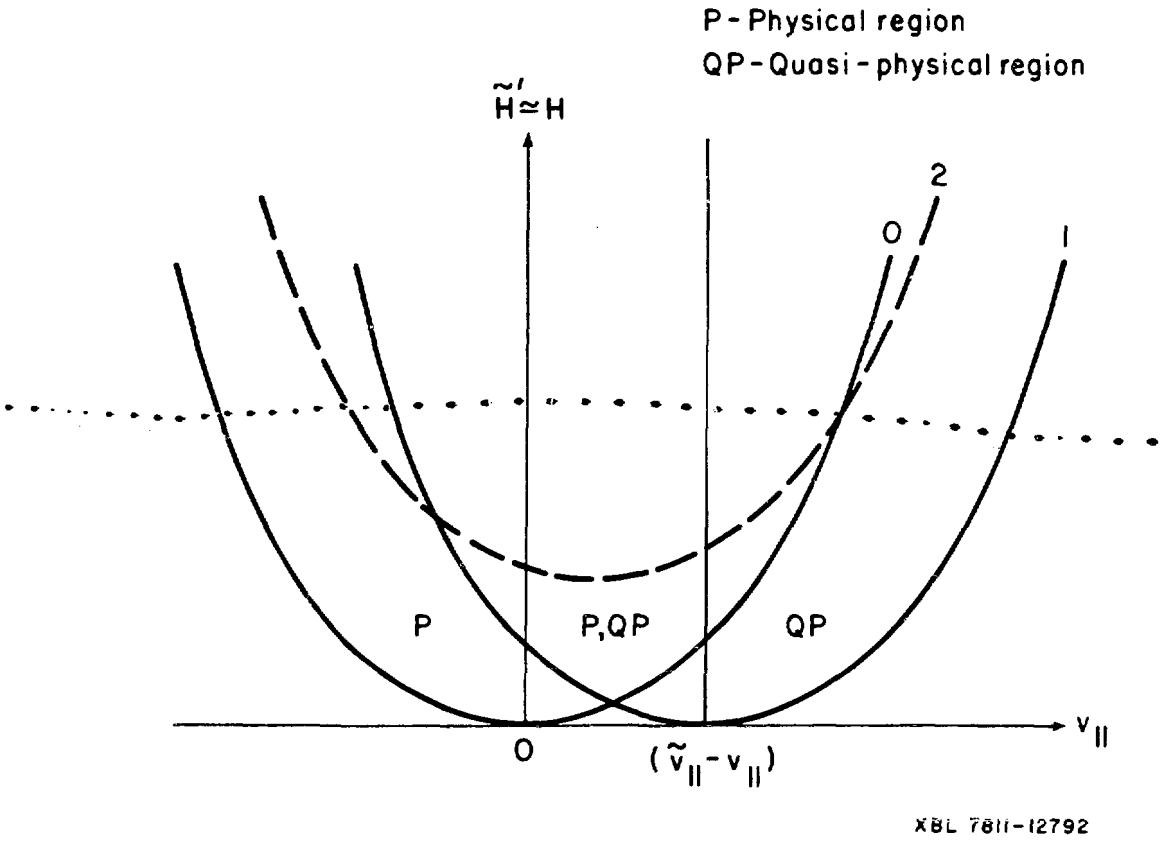

P - Physical region
QP - Quasi - physical region
$\tilde{H}' \simeq H$
2
0
1
P
P,QP
QP
0
$(\tilde{v}_{\parallel} - v_{\parallel})$
$v_{\parallel}$
XBL 7811-12792

Figure 5b

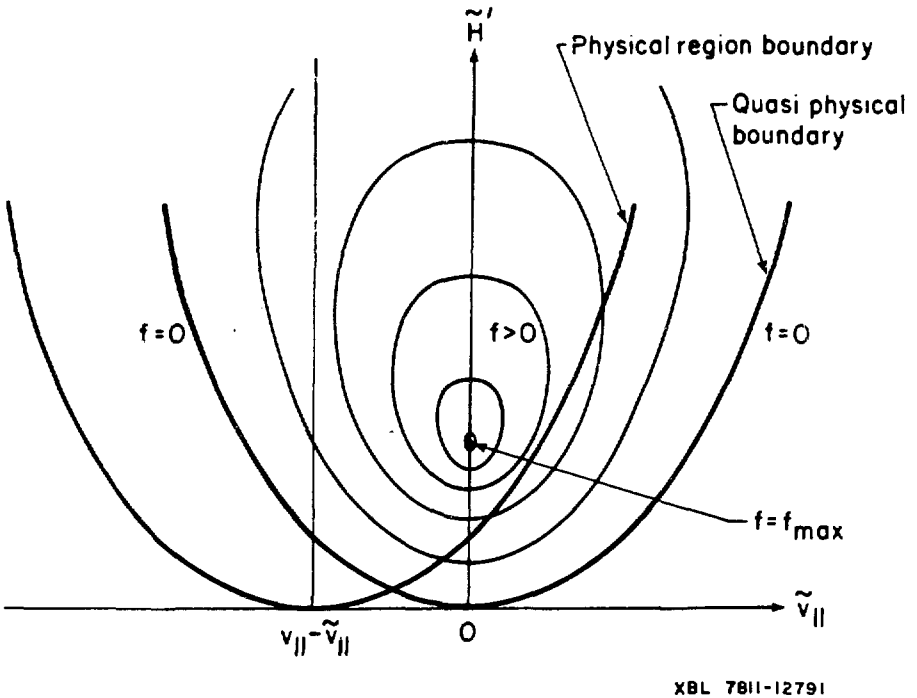
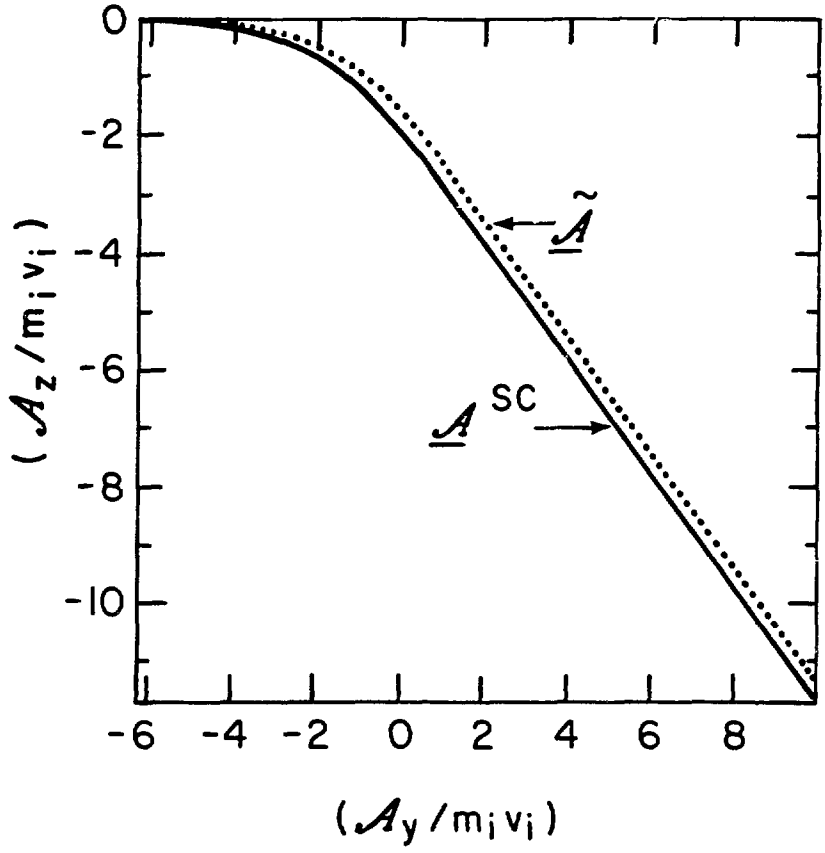
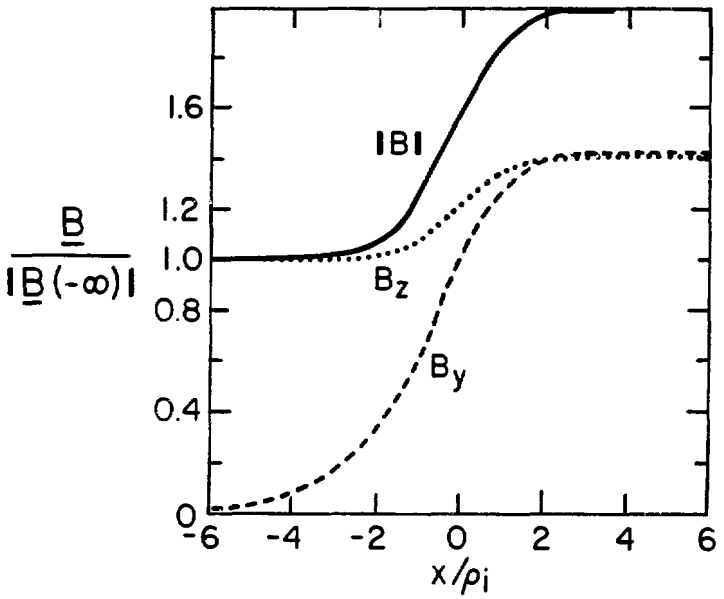

$\tilde{H}'$
Physical region boundary
Quasi physical boundary
f=0
f>0
f=0
f=f$_{max}$
$\tilde{v}_{\parallel}$
$v_{\parallel}-\tilde{v}_{\parallel}$
0
XBL 7811-12791

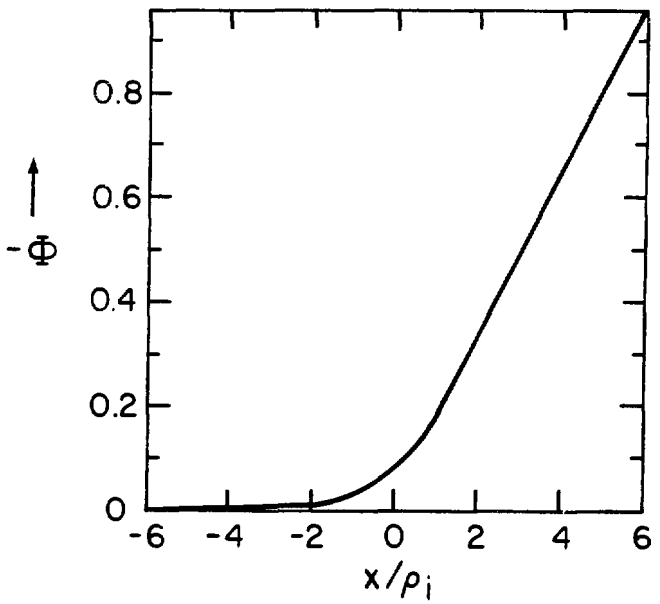
Figure 6



XBL 7811-12798

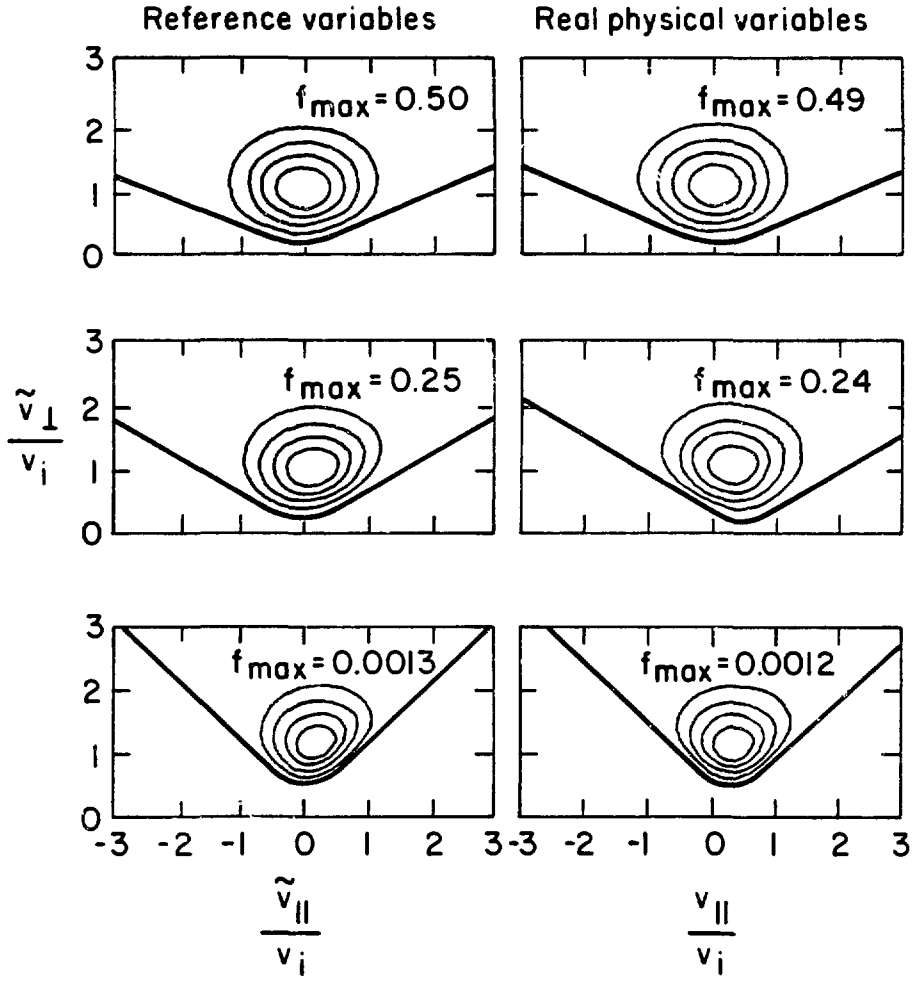
Figure 7a





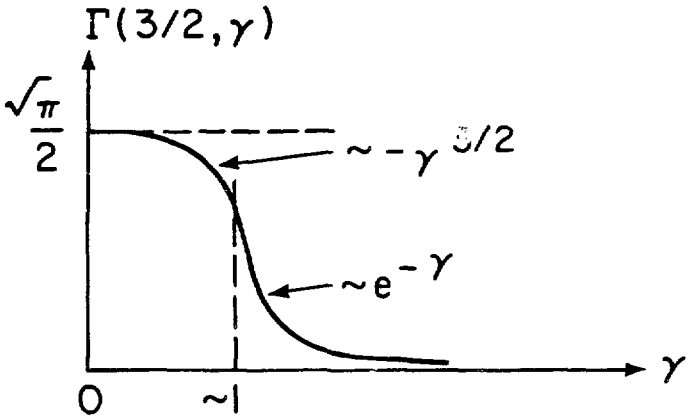
XBL 7811-12799

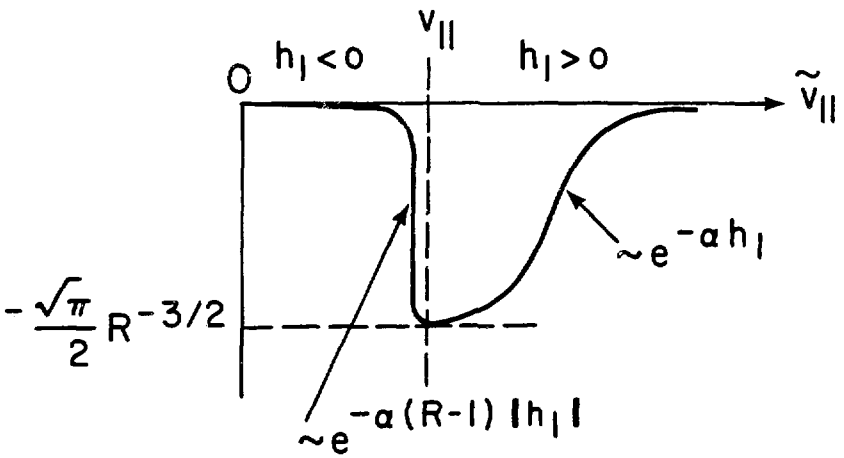
Figure 7b(top), 7c(bottom)



XBL 7811-13302

Figure 7d(top), 7e(middle) and 7f(bottom)





XBL 7811-13301

Figure 8a(top), 8b(bottom)

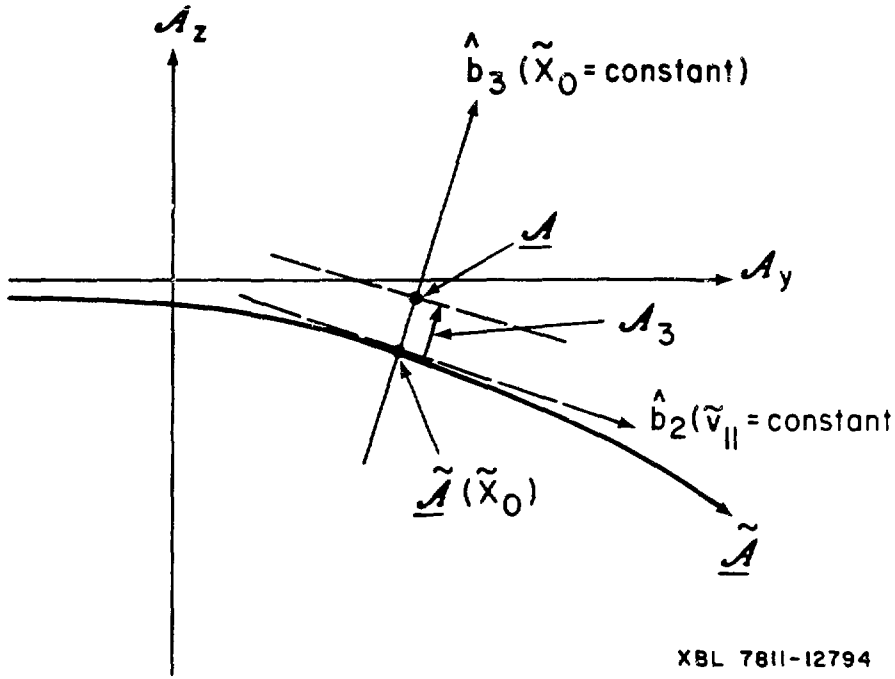
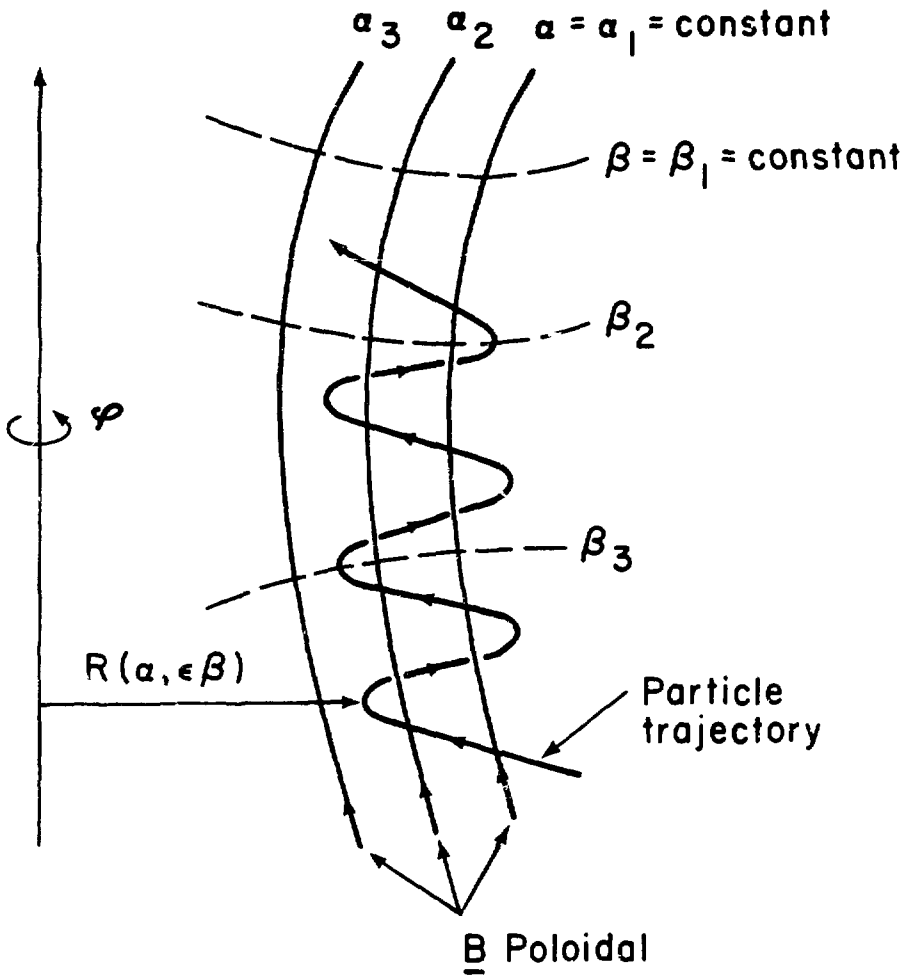

$\mathcal{A}_z$
$\hat{b}_3\,(\tilde{X}_0 = \text{constant})$
$\underline{\mathcal{A}}$
$\mathcal{A}_y$
$\mathcal{A}_3$
$\hat{b}_2(\tilde{v}_{\parallel} = \text{constant})$
$\underline{\tilde{\mathcal{A}}}\,(\tilde{X}_0)$
$\underline{\tilde{\mathcal{A}}}$
XBL 7811-12794

Figure 9



XBL 7811-12797

Figure 10

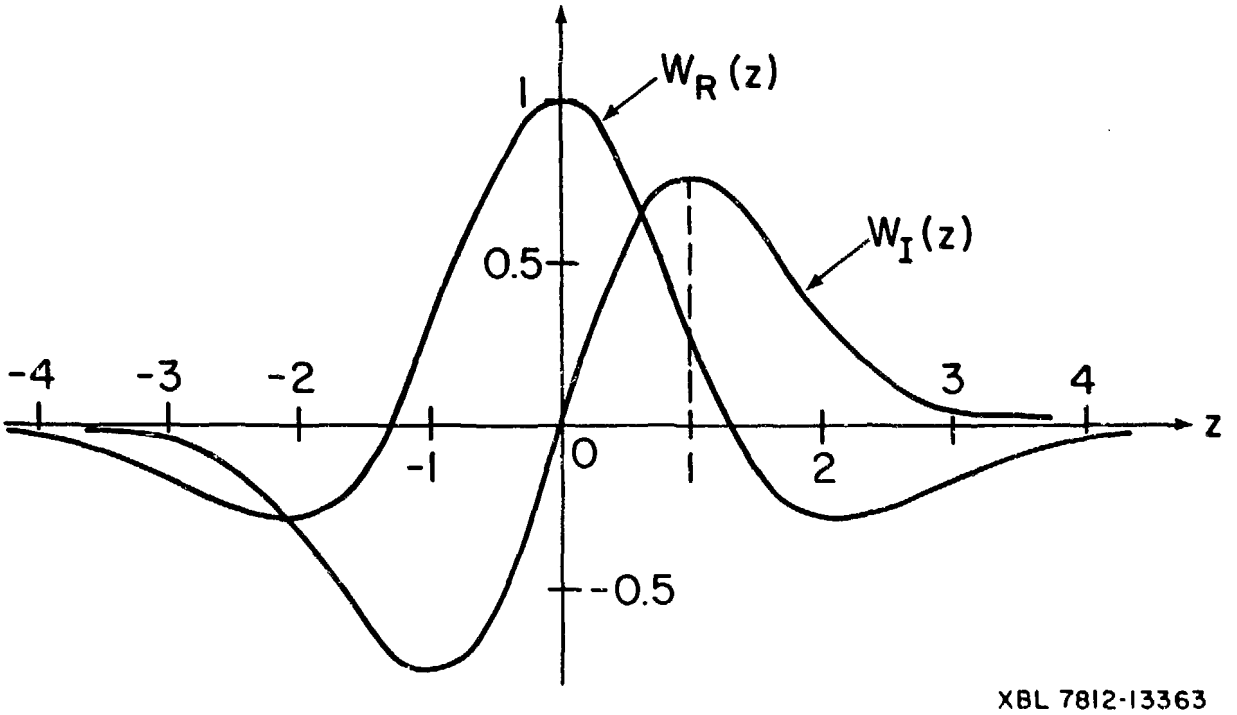


Figure 11

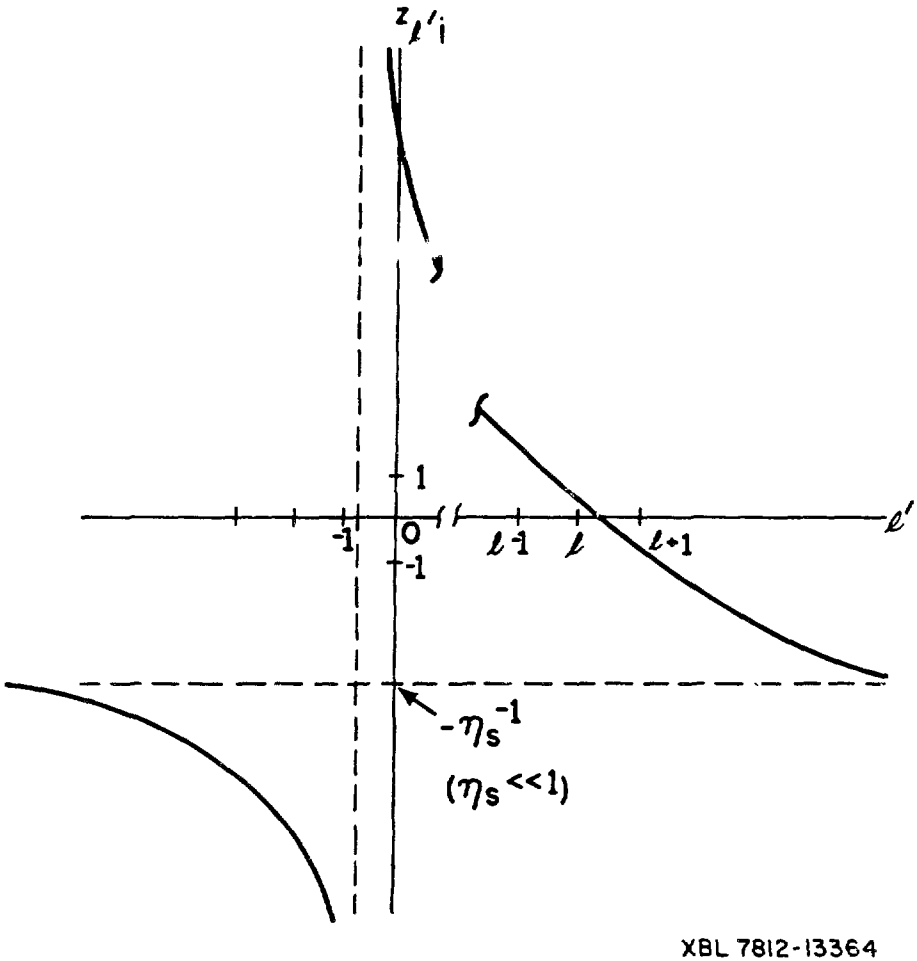
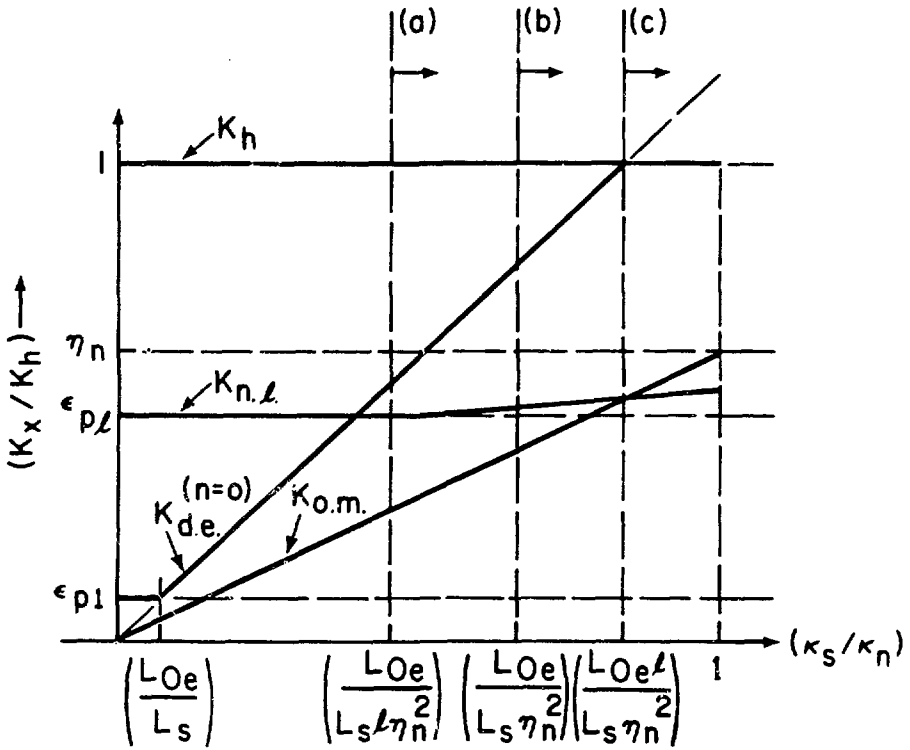
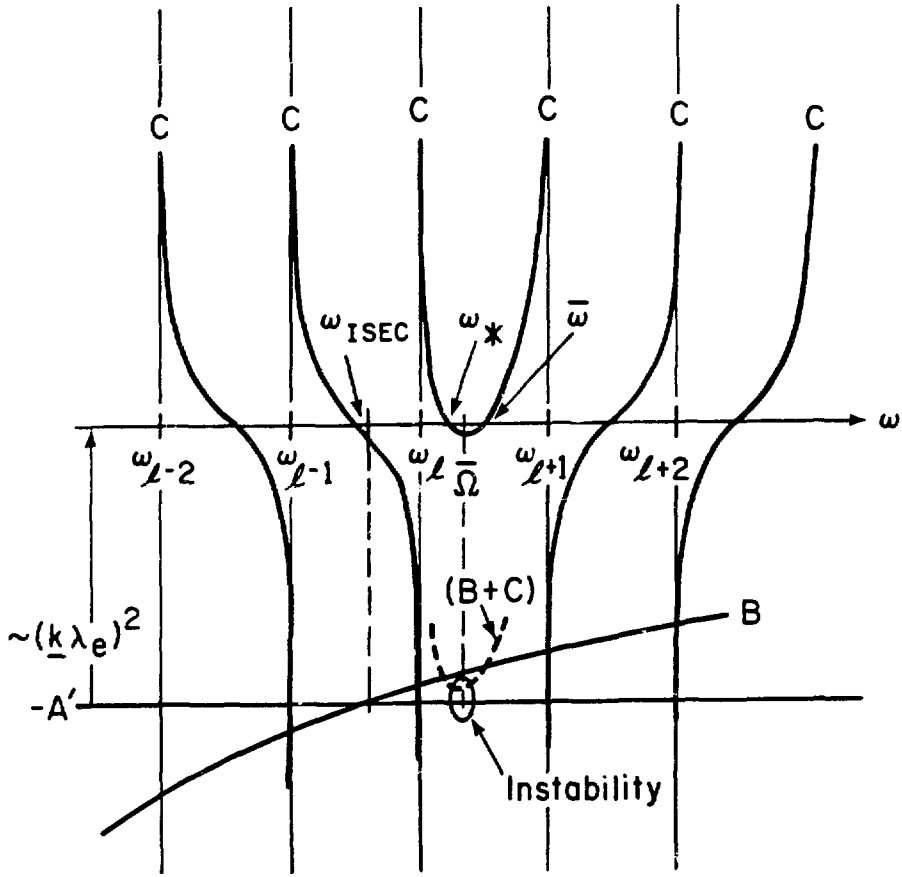


Figure 12



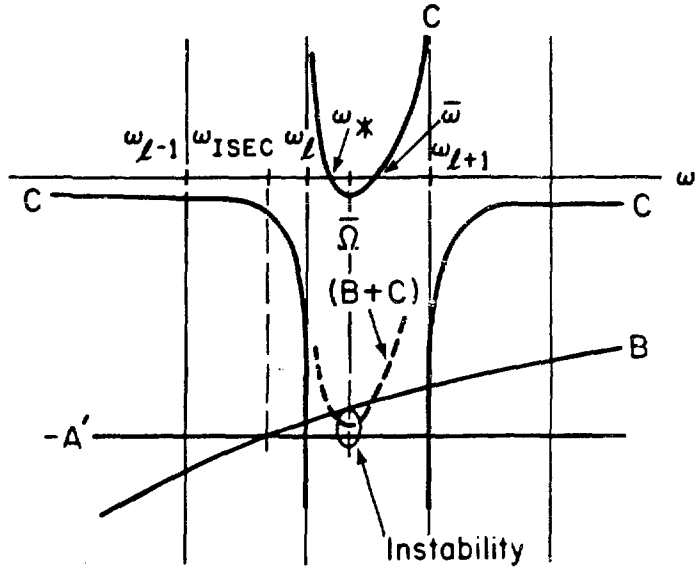
XBL 7812-13365

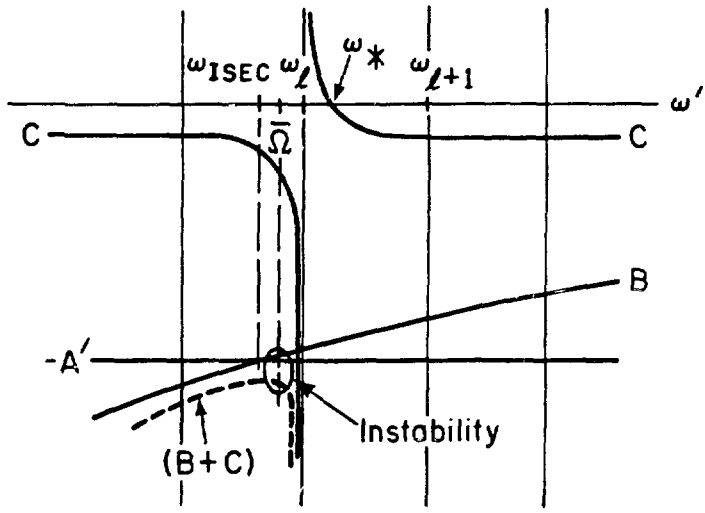
Figure 13



XBL 7812-13367

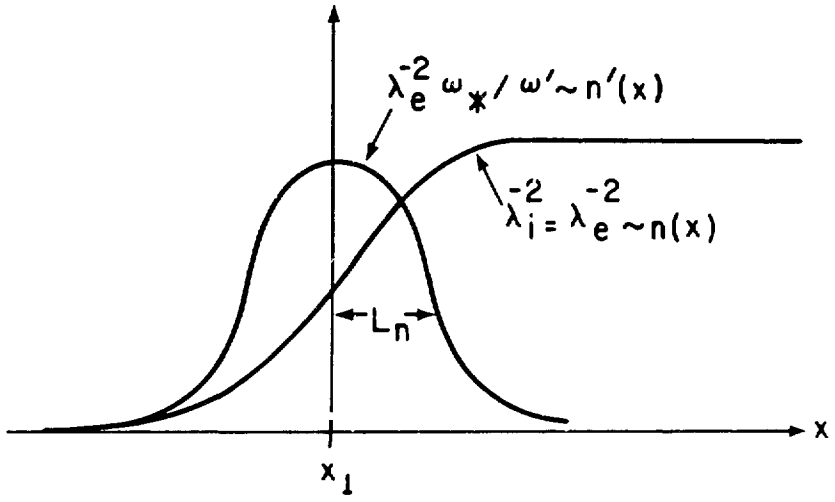
Figure 14

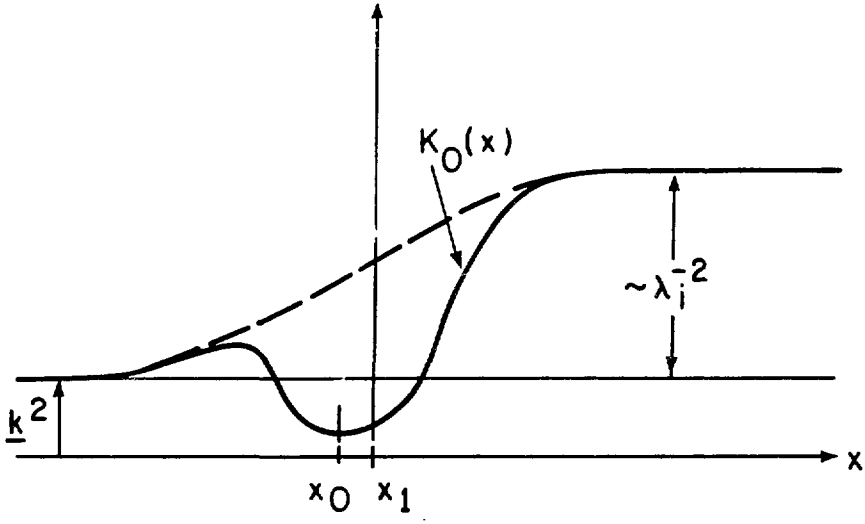




XBL 7812-13368

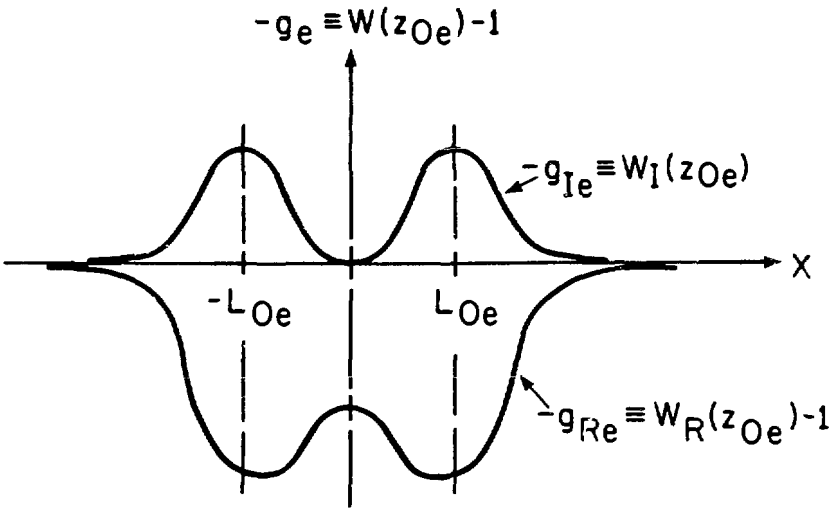
Figure 15a(top), 15b(bottom)

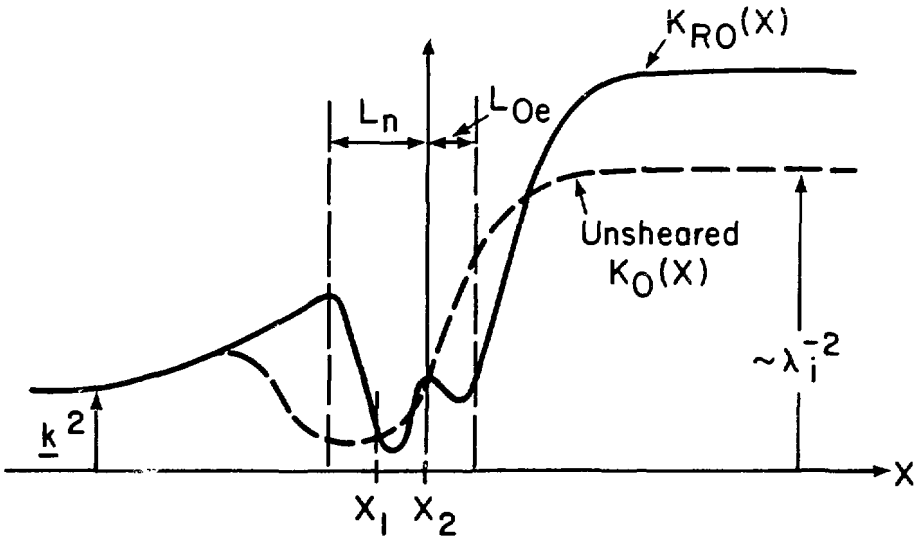




XBL 7812-13370

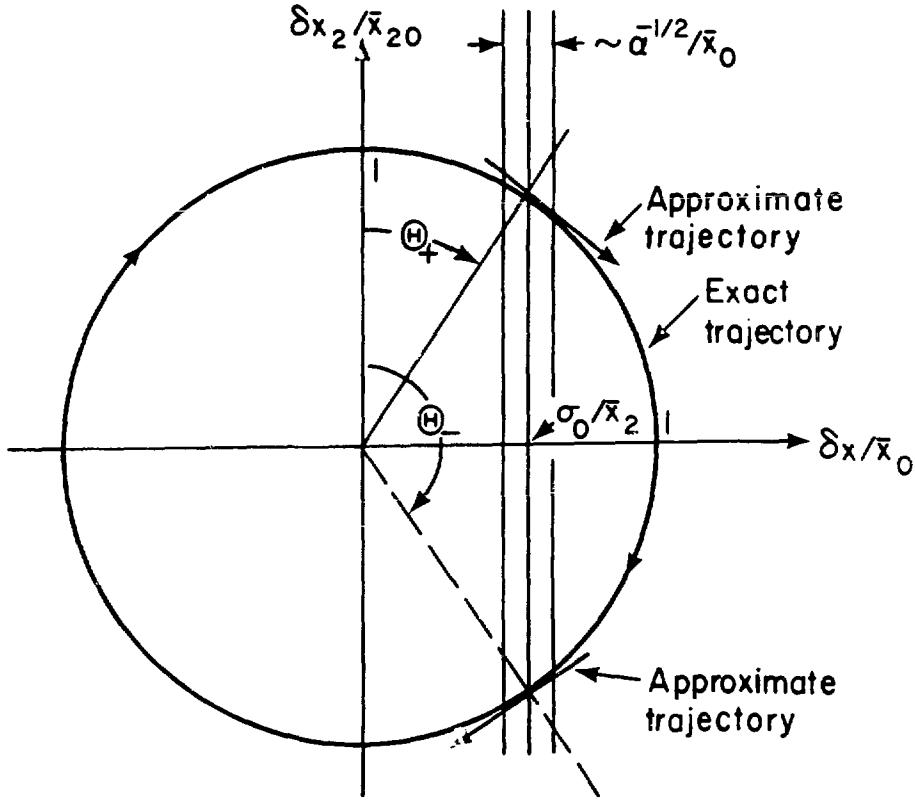
Figure 16a(top), 16b(bottom)





XBL 7812-13369

Figure 17a(top), 17b(bottom)



XBL 7812-13366

Figure 18