UC-32
LBL-9286 c.2

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics, Computer Science & Mathematics Division

Presented at the SIMS 1979 Research Application Conference on Energy Conversion and Fluid Mechanics, Alta, UT, June 25-29, 1979

ON CALCULATING FLOWS WITH SHARP FRONTS IN A POROUS MEDIUM

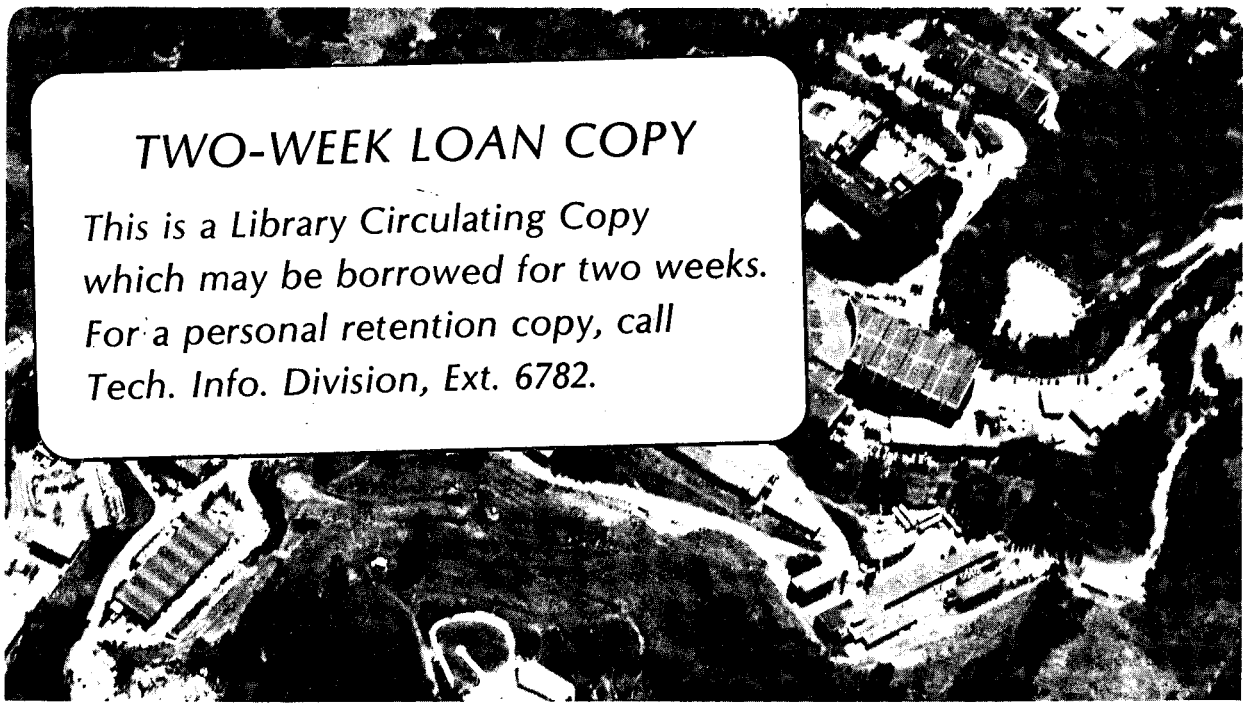
RECEIVED
LAWRENCE
BERKELEY LABORATORY

MAY 9 1980

LIBRARY AND
DOCUMENTS SECTION

N. Albright and P. Concus

June 1979



TWO-WEEK LOAN COPY

This is a Library Circulating Copy which may be borrowed for two weeks. For a personal retention copy, call Tech. Info. Division, Ext. 6782.

LBL-9286 c.2

Prepared for the U.S. Department of Energy under Contract W-7405-ENG-48

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

LBL-9286

ON CALCULATING FLOWS WITH SHARP FRONTS IN A POROUS MEDIUM

by

N. Albright and P. Concus

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Presented at the SIMS 1979 Research Application Conference
on
Energy Conversion and Fluid Mechanics
Alta, Utah, June 25-29, 1979

ON CALCULATING FLOWS WITH SHARP FRONTS IN A POROUS MEDIUM

by

N. Albright and P. Concus*

Abstract. A numerical method capable of tracking sharp fronts is discussed for solving multiphase flow problems in a porous medium. It is used to solve numerically the multidimensional Buckley-Leverett equation and simultaneous saturation-dependent pressure equation for the model problem of a five-spot waterflood of a petroleum reservoir in the absence of capillary pressure. The results indicate that the method performs favorably and that for the model problem the total velocity departs only slightly from that of potential flow.

1. Introduction. In the study of multiphase flow through a porous medium, there arises frequently the difficult task of solving a problem whose solution has sharp fronts or nearly sharp fronts. Examples of these fronts are the ones between different fluids or between regions of differing chemical concentrations. Even though sharp fronts may not be present initially, they can develop with time as a natural consequence of the nonlinearities of a given problem.

The fronts correspond mathematically to discontinuities, or near discontinuities, of solutions of hyperbolic, or nearly hyperbolic, partial differential equations. If one attempts to compute these solutions numerically, difficulty normally is encountered when conventional methods based on discretization of the partial differential equations are employed [2,8,12]. Such methods generally are based on smoothness

*Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720. This work was supported in part by the Engineering, Mathematical, and Geosciences Division of the U.S. Department of Energy under contract W-7405-ENG-48.

assumptions that are violated when discontinuities are present, and they often require devices such as the introduction of artificial dissipation for stability or for ensuring that the correct solution is obtained. We are presently engaged in a study whose purpose is the development of numerical methods that are capable inherently of tracking solution discontinuities in porous flow problems.

2. Buckley-Leverett Problem. The fundamental example of a porous-flow problem with sharp fronts is the Buckley-Leverett problem for immiscible, incompressible, two-phase flow, such as the displacement of petroleum by water in a porous medium. If the effects of gravity and of capillary pressure are neglected, one obtains for one space dimension, with appropriate normalization, the equation [2,3,11]

$$\frac{\partial s}{\partial t} + q \frac{\partial}{\partial x} [f(s)] = 0 . \tag{1}$$

In (1) $s(x,t)$ is the saturation of the displacing fluid (the fraction of available pore volume occupied by that fluid), and x and t are the space and time variables. The quantity q is the total volumetric flow rate, which is independent of x and is usually taken to be a constant, independent of time, so that (1) is, in essence, in conservation-law form. The porosity (not shown explicitly in (1)) also is constant for this problem and for simplicity is assumed to have been absorbed into the other variables.

The quantity $f(s)$ is the fractional volumetric flow rate of the displacing fluid. Typically $f(s)$ is a smooth function whose graph has the S-shape shown in Figure 1. Of particular interest is the presence of the inflection. Because $f(s)$ is not convex, the possible configurations of weak (discontinuous) solutions is more complicated than for the related equations of gasdynamics, such as Burgers's equation ($f(s) = \frac{1}{2}s^2$), for which the function corresponding to $f(s)$ is strictly convex.

These complications are discussed in detail in [7], where the Piecewise Sampling Method (PSM), the numerical method of particular interest in our study, (see § 3) is used to solve (1).

In more than one space dimension, the Buckley-Leverett equation (1) becomes

$$\frac{\partial s}{\partial t} + \underset{\sim}{q} \cdot \mathrm{grad}[f(s)] = 0 \ . \tag{2}$$

Although for one space dimension the total velocity $\underset{\sim}{q}$ is normally constant and could be absorbed into the other variables without appearing explicitly, in two or more dimensions it must be determined simultaneously with s from the incompressibility condition [2,11]

$$-\mathrm{div}[\lambda(s)\,\mathrm{grad}\ p] = Q \ , \tag{3a}$$

$$\underset{\sim}{q} = -\lambda(s)\,\mathrm{grad}\ p \ . \tag{3b}$$

In (3) p is the pressure and Q the sources or sinks. The quantity $\lambda(s)$ is the total mobility (sum of individual phase mobilities) and is positive, bounded strictly away from zero. Boundary conditions on p typically ensure that if s were known, $\underset{\sim}{q}$ would be determined uniquely by solving the elliptic equation (3a).

In [1], PSM is used to solve (2) numerically for two space variables with $\underset{\sim}{q}$ given as the solution of (3) for $\lambda(s)$ constant. That is, $\underset{\sim}{q}$ is taken to be the potential-flow velocity corresponding to the given geometry and distribution of sources. This velocity represents qualitative features of the actual flow and permits the solution of (2) to be obtained using analytic means, for comparison with numerically computed solutions. It is found in [1] that PSM is able to compute efficiently the solution of (2) for a model problem and to obtain a value of the time of breakthrough (of water from injection wells into producing petroleum wells) that is in good accord with the analytically derived one.

In the present paper we continue the study reported in [1] by removing the preliminary restriction of fixed $\underset{\sim}{q}$. We solve the system (2,3) numerically in two space dimensions by coupling PSM with the generalized (preconditioned) conjugate gradient method. We summarize first the features of PSM.

3. Piecewise Sampling Method. The Piecewise Sampling Method is a numerical method that possesses the inherent capability of being able to track solution discontinuities. It is based on a mathematical construction of Glimm [9] and was developed into an efficient computational algorithm by Chorin [4]. It is known also as Glimm's Method and as the Random Choice Method.

Among the features of PSM, which in its basic form is a one space dimensional method, are: (i) The representation of discontinuities is based not on differencing or other discretizations but on local Riemann solutions and a sampling procedure. (ii) Discontinuities are propagated sharply. (iii) Devices such as artificial dissipation are not required. (iv) For a purely hyperbolic problem the correct solution is obtained, corresponding to the limiting solution of parabolic problems as the dissipation approaches zero.

The method advances a solution one step in time by approximating the solution at the initial time by a piecewise-constant function on a spatial grid, solving analytically with the piecewise-constant initial data, and sampling this analytic solution to obtain values for a piecewise-constant approximation at the new time.

The time increments are chosen sufficiently small so that waves propagating from initial discontinuities at the spatial grid points do not interact. The analytic solution may then be obtained by joining together the separate solutions to the Riemann problems for the propagation from each discontinuity. (A Riemann problem is one that has initial data that is constant to the left and to the right of a single step-discontinuity.)

The reader is referred to [4,5,7] and the references therein for

a detailed description of PSM. The extension to two space dimensions using fractional splitting is discussed for the Buckley-Leverett equation in [1].

4. Numerical Solution Procedure. We obtain a numerical solution of (2,3), subject to appropriate boundary and initial conditions, by advancing stepwise in time as follows. Assume the approximate solution for $s^{(n)}$, the saturation at time $t = t_n$, is known. Let $p^{(n)}$ denote the pressure at time t_n. We solve a discrete approximation to

$$-\operatorname{div}[\lambda(s^{(n)}) \operatorname{grad} p^{(n)}] = Q \quad , \tag{4}$$

and its boundary conditions, to obtain an approximation to $p^{(n)}$. Then we advance s one step in time by solving

$$\frac{\partial s^{(n+1)}}{\partial t} + \underset{\sim}{q}^{(n)} \cdot \operatorname{grad}[f(s^{(n+1)})] = 0 \tag{5}$$

using PSM, where $\underset{\sim}{q}^{(n)}$ is obtained using $s^{(n)}$, $p^{(n)}$, and a discrete approximation to (3b).

We illustrate this procedure in detail for the specific example that is studied in [1] of the five-spot configuration of the waterflood of a petroleum reservoir. The function used for $f(s)$ is [1,12]

$$f(s) = \frac{s^2}{s^2 + \alpha(1-s)^2} \quad ,$$

where α is the ratio of water to oil viscosities. Figure 1 corresponds to the value $\alpha = \frac{1}{2}$, which is the value used in [1]. The function used for $\lambda(s)$ is [12]

$$\lambda(s) = s^2 + \alpha(1-s)^2 \quad ,$$

where, for convenience, $\lambda(s)$ is normalized by absorbing the physical constants of absolute permeability and water viscosity into p.

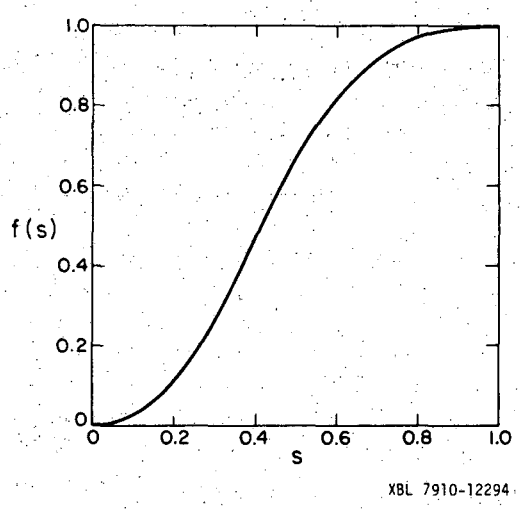


Figure 1: Fractional flow as a function of saturation.

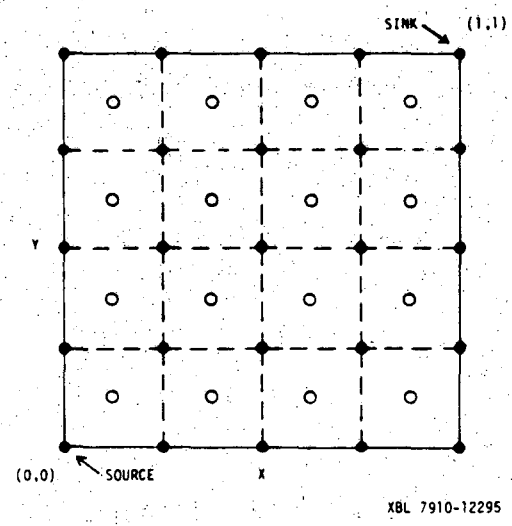


Figure 2: Quarter configuration of the five-spot problem depicting staggered pressure ● and saturation ○ mesh points for a 5×5 grid.

Let unit sources (water injection wells) be placed in the x-y plane at $x = 2\ell$, $y = 2m$ and unit sinks (petroleum production wells) be placed at $x = 2\ell - 1$, $y = 2m - 1$, $\ell, m = \ldots, -2, -1, 0, 1, 2, \ldots$. Then for the quarter configuration of the unit square (see Figure 2) we solve (3) with

$$Q = \delta(x)\delta(y) - \delta(x-1)\delta(y-1) \tag{6}$$

and symmetry boundary condition

$$\partial p/\partial \nu = 0 \tag{7}$$

on the edge of the square, where $\underset{\sim}{\nu}$ is the outer directed normal.

Correspondingly, we take the boundary conditions on s for (2) to be that $\partial s/\partial\nu = 0$ on the edges of the unit square and $s = 1$ at the source $(0,0)$. Initially we take $s \equiv 0$ everywhere except at the source.

A discrete approximation is obtained by approximating p at spatial grid points staggered with respect to the grid points at which s is approximated (see Figure 2). For a uniform square mesh $x_i = ih$, $y_j = jh$, $i = 0,1,\ldots,N$, $j = 0,1,\ldots,N$, one approximates $p(x,y;t)$ at the spatial mesh points (x_i,y_i), whereas $s(x,y;t)$ and $\lambda(s)$ are approximated at the mid-points $(x_i+\frac{1}{2}h, y_j+\frac{1}{2}h)$, $i = 0,1,\ldots,N-1$, $j = 0,1,\ldots,N-1$.

This permits the standard five-point difference approximation to be written for (4). At a general interior point one has, denoting $p_{i,j} = p^{(n)}(x_i,y_j)$, $\lambda_{i+\frac{1}{2},j+\frac{1}{2}} = \lambda(s^{(n)}[x_i+\frac{1}{2}h, y_j+\frac{1}{2}h])$, and $Q_{i,j} = Q(x_i,y_j)$,

$$\begin{aligned}\frac{1}{2h^2}\{ &\lambda_{i+\frac{1}{2},j+\frac{1}{2}}(-p_{i,j+1}-p_{i+1,j}+2p_{i,j}) \\ &+\lambda_{i+\frac{1}{2},j-\frac{1}{2}}(-p_{i+1,j}-p_{i,j-1}+2p_{i,j}) \\ &+\lambda_{i-\frac{1}{2},j+\frac{1}{2}}(-p_{i,j+1}-p_{i-1,j}+2p_{i,j}) \\ &+\lambda_{i-\frac{1}{2},j-\frac{1}{2}}(-p_{i,j-1}-p_{i-1,j}+2p_{i,j})\} = Q_{i,j} .\end{aligned}$$

At boundary points, one obtains similar expressions, with appropriate terms absent from the left-hand side [14].

The resulting system of equations has a symmetric, positive semi-definite coefficient matrix and could be solved by one of several numerical methods that take advantage of its special structure. We have selected for our numerical tests the method of generalized (preconditioned) conjugate gradients with a discrete Laplace operator splitting [6]. This is equivalent to solving iteratively a discrete form of

$$(8) \qquad -\operatorname{div}[\bar{\lambda}\ \operatorname{grad} p_{(\ell+1)}] = -\operatorname{div}[(\bar{\lambda}-\lambda(s))\operatorname{grad} p_{(\ell)}] + Q$$

for the new approximate solution $p_{(\ell+1)}$ from the old $p_{(\ell)}$, where $\bar{\lambda}$ is a constant, with acceleration by the conjugate gradient method. Because the domain is a rectangle, the solution of the discrete form of (8) can be accomplished efficiently using standard, available computer programs. We have used for this purpose the program PWSCRT in the NCAR elliptic partial differential equation software package [13].

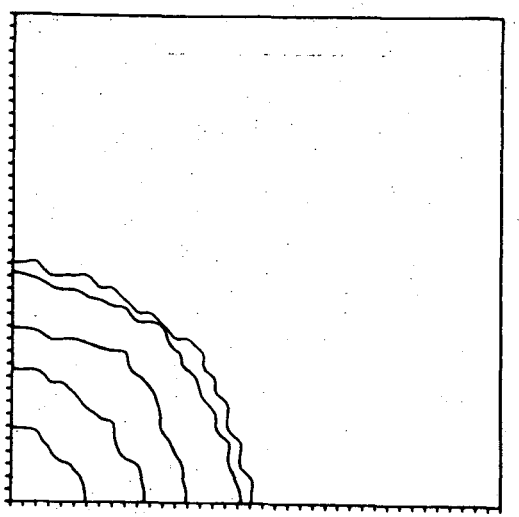
After the approximate solution for p is obtained to desired accuracy, it can be differenced to give the corresponding values for $\underset{\sim}{q}$ on the same grid as for s. At a general interior point $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ we take $\underset{\sim}{q} = (q_x, q_y)$ to be

$$q_x(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) \cong -\lambda_{i+\frac{1}{2},j+\frac{1}{2}}(2h)^{-1}[p_{i+1,j+1} - p_{i,j+1} + p_{i+1,j} - p_{i,j}] ,$$

with a corresponding expression for $q_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$.

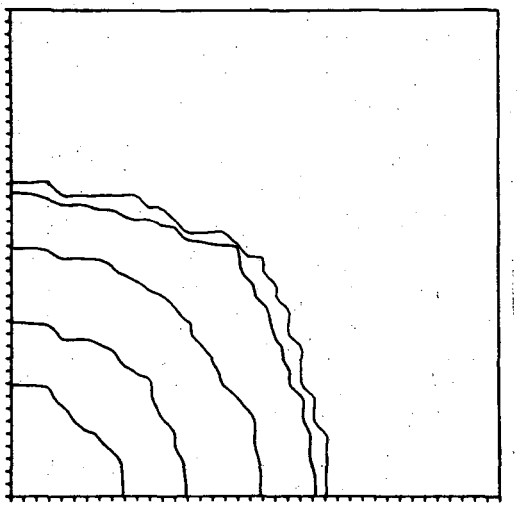
Using the values of $\underset{\sim}{q}^{(n)}$ so obtained, (5) can be stepped ahead an increment in time to obtain $s^{(n+1)}$, the approximation to s at the new time, in the same manner as is described in [1]. This is accomplished by means of fractional steps, which split (5) into a sequence of one-dimensional problems in x and in y. The resulting one-dimensional problems can then be solved by PSM, as described in [1,7]. We have used for this purpose modified forms of the computer programs developed with W. Proskurowski in [1,7].

For the present problem we have found it preferable to order the fractional splittings so that two one-dimensional problems in x are done successively, and correspondingly for y. This permits a favorable placement of the fluctuating PSM grid for s relative to the grid on which $\underset{\sim}{q}$ is given and relative to the boundaries. After a complete time step of PSM -- two one-dimensional x problems and two one-dimensional y problems -- the next stepwise procedure can be initiated starting from the obtained values of $s^{(n+1)}$.



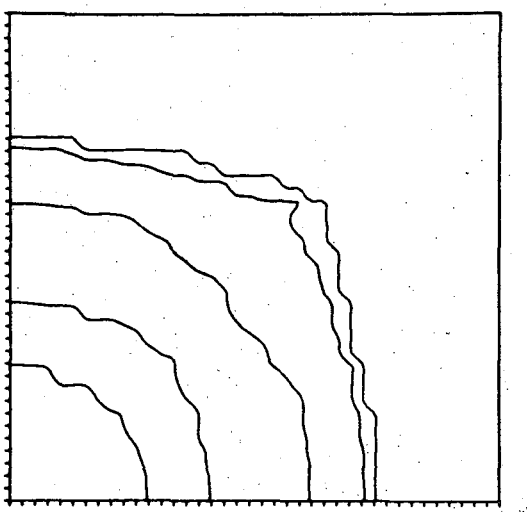
TIME = .50 RECOVERY = .14

Figure 3(a)



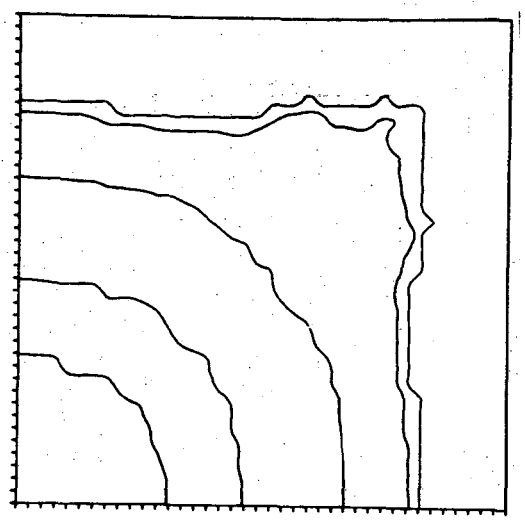
TIME = 1.00 RECOVERY = .26

Figure 3(b)



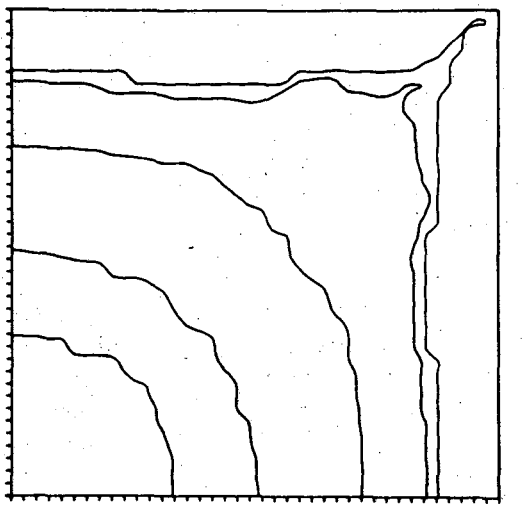
TIME = 1.50 RECOVERY = .37

Figure 3(c)



TIME = 2.00 RECOVERY = .49

Figure 3(d)

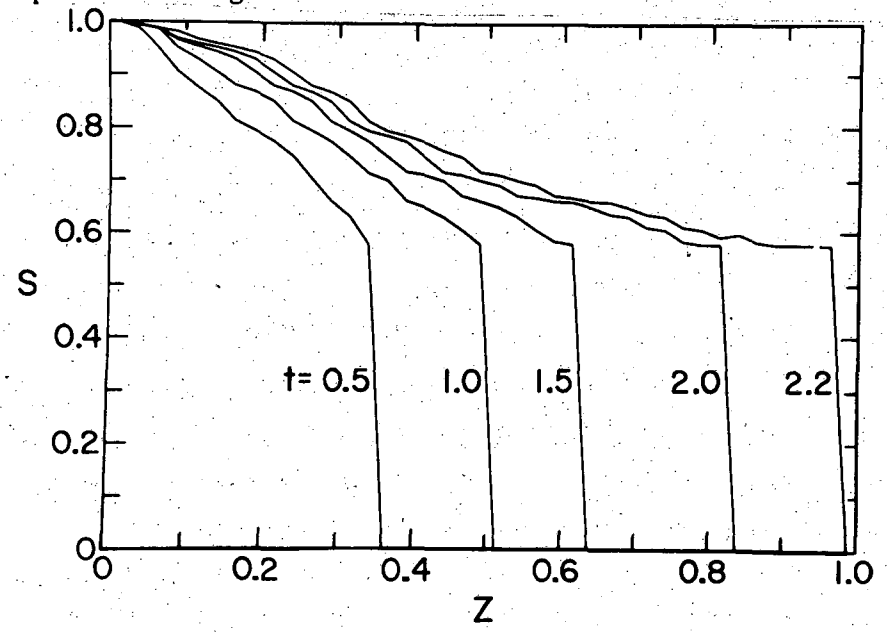


TIME = 2.20 RECOVERY = .56

Figure 3(e)

Figure 3: Saturation contours 0.5(0.1)0.9 of the numerical solution and fraction of initial oil recovered at times t = 0.5, 1.0, 1.5, 2.0, and 2.2.

5. Results: The five-spot problem was solved numerically by the procedure of § 4 on a 41×41 mesh for values of viscosity ratio $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{4}$. The solution obtained is depicted in Figures 3 and 4 for the case $\alpha = \frac{1}{2}$. In Figure 3 contours $s(x,y;t) = 0.5(0.1)0.9$ are plotted at times $t = 0.5$, 1.0, 1.5, 2.0, and 2.2, respectively; the last is the time of breakthrough for this calculation. The fraction of initial oil that has been recovered at the production well at (1,1) is given at each time also. Figure 3 was drawn using subroutine CONREC from the NCAR graphics package. Since that subroutine displaces contours that should lie on top of each other into separate curves, the drawing of the contours for $s = 0.1(0.1)0.4$ was suppressed for clarity. In essentially all cases the tabular output indicated that the front was sharp and that these contours would coincide with the one for $s = 0.5$, which is the first plotted contour ahead of the front. The pinched contours at breakthrough arise in part from the interpolation in the plotting routine. The analytically derived solution obtained in [1] for this problem gives a height at the front of 0.577. In Figure 4 the saturation along the diagonal $y = x$ of the square is plotted for each of the times depicted in Figure 3.



XBL797-2319

Figure 4: Saturation vs. distance $z\sqrt{2}$ along the diagonal $y = x$ at times $t = 0.5$, 1.0, 1.5, 2.0, and 2.2.

One observes from Figure 4 the sharpness with which the front advances. (The non-vertical front results from the linear interpolation between mesh values used in plotting Figure 4.) In Figures 3 and 4 can be seen the fluctuations of the order of one mesh interval that occur with PSM, in part because splitting is used. These fluctuations are stable, however, and do not grow with time. In a problem with dissipation (capillary pressure) the fluctuations would, of course, be smoothed.

Table 1 summarizes some of the information concerning the solution at the time of breakthrough. The columns give, respectively, for each value of α, the value of t at breakthrough, the fraction of the initial oil recovered before breakthrough, the number of full time steps of (5) by PSM required to reach breakthrough, and number of those time steps at which s changed discernably so that a solution of (4) was required.

TABLE 1

Breakthrough Data

α	t	fraction of oil recovered	time steps of (5)	solutions of (4)
¼	1.75	0.47	174	51
½	2.20	0.56	170	51

Fewer solutions of (4) are required, in general, than stepwise advances of (5), because s may remain essentially unchanged after one step of PSM. In our calculations, the time steps were reduced from the maximum allowed by the Courant-Friedrichs-Lewy condition (see [1]) to accommodate a uniformly spaced interval 0.05 of output times; as a consequence for over two-thirds of the time steps the saturation remained essentially unchanged.

Equation (2) was solved for comparison purposes with $\underset{\sim}{q}$ fixed at the potential flow velocity, as obtained by one solution of (3) with λ

constant. It was found that the remaining entries in Table 1 would have differed by only a few percent had this fixed potential-flow velocity been used. In fact, the similarity of the solutions for the fixed and non-fixed velocities for the cases in Table 1 was striking, with at most a few percent difference in velocity and saturation away from the front and a displacement of at most about one mesh interval at the front. (In our test problem with smaller values of α this similarity was not present, because "fingering" of the advancing front occurred for the non-fixed velocity case. See also [10].)

For the cases described in Table 1 one full time step for advancing s in (5) required on the average 0.074 CPU seconds on the CDC 7600. For the same cases one iteration for p in (8), consisting of one solution of Poisson's equation by the NCAR package PWSCRT plus associated conjugate gradient overhead, required on the average 0.117 CPU seconds. Whenever a solution of (4) was required, a sufficiently accurate approximation was obtained after three conjugate gradient iterations of (8). Thus the total CPU time required to reach breakthrough for each value of α was approximately 30 to 31 seconds, of which about 18 seconds was required for solving (4) for the new velocities. Methods other than generalized conjugate gradients with discrete Laplacian splitting for solving (4) might obtain the velocity to required accuracy in less computer time, but none were investigated as part of this study.

In subsequent studies of PSM, we plan to include the effects of capillary pressure and gravity and to investigate alternatives to fractional splitting for multidimensional problems.

Acknowledgment. We wish to thank Alexandre Chorin for several helpful conversations concerning PSM.

REFERENCES

[1] N. ALBRIGHT, P. CONCUS, and W. PROSKUROWSKI, Numerical solution of the multidimensional Buckley-Leverett equation by a sampling method, Paper SPE 7681, Soc. Petrol. Eng. Fifth Symp. on Reservoir Simulation, Denver, CO, Jan. 31 - Feb. 2, 1979.

[2] K. AZIZ and A. SETTARI, Petroleum Reservoir Simulation, Applied Science Publishers Ltd., London, 1979.

[3] S. E. BUCKLEY and M. C. LEVERETT, Mechanism of fluid displacements in sands, Trans. AIME, 146(1942), pp. 107-116.

[4] A. J. CHORIN, Random choice solution of hyperbolic systems, J. Computational Phys., 22(1976), pp. 517-533.

[5] A. J. CHORIN, Random choice methods with applications to reacting gas flow, J. Computational Phys., 25(1977), pp. 253-272.

[6] P. CONCUS, G. H. GOLUB, and D. P. O'LEARY, A generalized conjugate gradient method for the numerical solution of elliptic partial differential equations, Sparse Matrix Computations (J. R. Bunch and D. J. Rose, eds.), 309-332, Academic Press, New York, 1976.

[7] P. CONCUS and W. PROSKUROWSKI, Numerical solution of a nonlinear hyperbolic equation by the random choice method, J. Computational Phys., 30(1979), pp. 153-166.

[8] J. DOUGLAS, D. W. PEACEMAN, and H. H. RACHFORD, JR., A method for calculating multidimensional immiscible displacement, Trans. AIME, 216(1959), pp. 297-306.

[9] J. GLIMM, Solutions in the large for nonlinear hyperbolic systems of equations, Comm. Pure Appl. Math., 18(1965), pp. 697-715.

[10] J. GLIMM, D. MARCHESIN, and O. McBRYAN, The Buckley-Leverett equation: theory, computation and application, Proc. Third Meeting of the International Society for the Interaction of Mechanics and Mathematics, Edinburgh, Sept. 10-13, 1979.

[11] D. W. PEACEMAN, Fundamentals of Numerical Reservoir Simulation, Elsevier, Amsterdam-Oxford-New York, 1977.

[12] A. SPIVAK, H. S. PRICE, and A. SETTARI, Solution of the equation for multidimensional, two-phase, immiscible flow by variational methods, Soc. Petr. Eng. J., 17(1977), pp. 27-41.

[13] P. SWARZTRAUBER and R. SWEET, Efficient FORTRAN subprograms for the solution of elliptic partial differential equations, Tech. Note NCAR-TN/IA-109, National Center for Atmospheric Research, Boulder, CO, 1975.

[14] R. S. VARGA, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720