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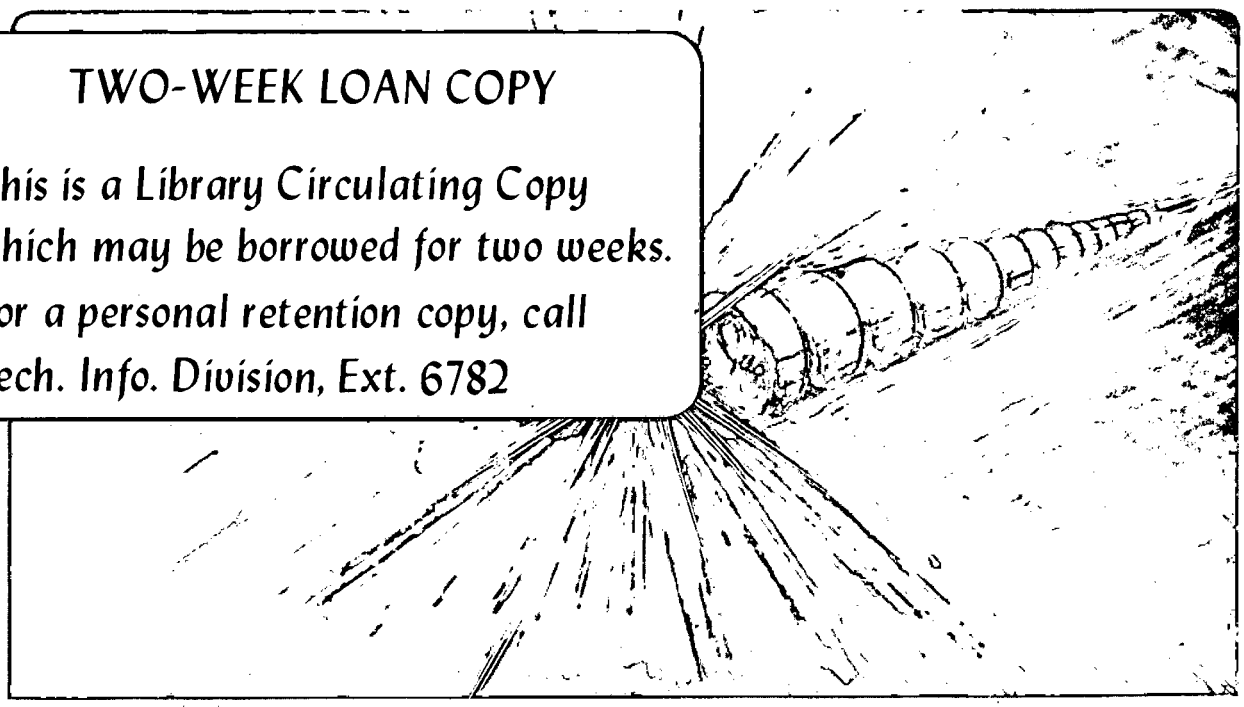

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Unstable continuous spectrum in magnetohydrodynamics

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The continuous spectrum need not be stable for an equilibrium with flow. The case of a noncircular, rotating θ pinch is shown to have an unstable continuum with the instability interpreted as a parametric instability arising from coupling between the rotation frequency and the plasma wave frequencies. On the other hand, equilibria with sub-Alfvénic flow parallel to the magnetic field in incompressible plasmas, as well as static equilibria, have a stable continuum. A formulation to determine the continuous spectrum is given in terms of characteristic surfaces for the equations and does not involve the use of any particular system of coordinates.

I. INTRODUCTION

The study of the temporal behavior of plasma systems under the assumption of a small deviation from an equilibrium state, is equivalent to the study of a spectral problem through the application of the Laplace transform to the linearized equations of motion. The spectral problem can be formally obtained by assuming an exponential behavior $\exp(i\omega t)$ in time. It is known that in the magnetohydrodynamic model, the resulting spectral problem can be cast in a formally self-adjoint form[1] if the equilibrium state is static. As a result ω^2 must be real. When the equilibrium involves plasma flow, this property is lost and the spectral problem is not self-adjoint.[2] We will uncover an additional difference between the spectra of steady and static equilibria. It will be shown that the part of the continuous spectrum known as the Alfvén and Cusp continuum, is always real (and therefore exponentially stable) in the static case, while it may be complex in the general steady equilibrium case.

There seems to be some misunderstanding in the plasma physics literature with regard to the physical and mathematical meaning of the continuous spectrum. Let us now describe the mathematical view of the spectrum.[3] Consider the evolution equation $\partial\xi/\partial t = L\xi$, with L a spatial operator defined in a complete normed space (say a Hilbert space). The corresponding spectral problem is $(L - i\omega)\xi = 0$. For any element f of the space, the integral $\xi(t) = (-1/2\pi)\int_\Gamma e^{i\omega t}(L - i\omega)^{-1} f d\omega$ yields a solution to the evolution equation, where Γ is a closed contour contained in the domain of analyticity of $(L - i\omega)^{-1}$ (viewed as an operator-valued analytic function of ω), and encloses a singularity of $(L - i\omega)^{-1}$. The different types of the spectrum can be defined in terms of the types of the singularities of $(L - i\omega)^{-1}$. Poles are the manifestations of eigenvalues. Continuous singularities give rise to the continuous spectrum. Unlike some views, the continuous spectrum is not a branch cut in the ω plane through which every solution of the Laplace transformed equation can be continued analytically. Rather, only special solutions can be continued. In fact $(L - i\omega)^{-1}$, when it exists as a bounded operator, must be defined uniquely because it is literally the *unique* inverse of $L - i\omega$.

An investigation of the continuous spectrum (the continuum) is of interest for a variety of reasons. It was proposed[4] that radio frequency (rf) heating, by means of resonating with a continuum frequency, could be used as a cheap low-frequency heating mechanism. The continuous spectrum can also be identified analytically in a relatively simple way while the rest of the spectrum can, in general, be treated only numerically. In addition, the continuous spectrum (or actually a somewhat larger set known as the essential spectrum), serves as the only location of possible accumulation points of eigenvalues. In particular, the celebrated ballooning modes[5] are, in fact, limit points of eigenvalues with large toroidal wavenumbers and, significantly, may be unstable.

In the next two sections we describe the origins of the continuous spectrum in magnetohydrodynamics. It arises from the existence of characteristic surfaces in the equations and can be derived without the use of any particular system of coordinates. For a better understanding we treat the special case of a noncircular rotating θ pinch in straight geometry ($\partial/\partial z = 0$). While the circular case is known to have a stable continuum[6,7] for a fixed axial wavenumber, the noncircular case which also describes a toroidal θ pinch will be shown to have an unstable continuous spectrum. This instability will be interpreted as a parameteric instability arising from coupling between the periodic rotation (the driving force) and the plasma wave frequencies based on $k_\parallel$. The discussion will be based on the Hamiltonian nature of the equations of motion and will be presented in Secs. III–V. In the following section we show that the continuum instability is not universal to systems with flow. The case of incompressible flow parallel to the magnetic field will be shown to have a stable continuum. Essentially the same proof is used to show that a general static equilibrium has a stable continuum. This last result was known for the axially symmetric tokamak configuration.[8] The general result was also recently obtained independently by others.[9]

After the results of this investigation were announced,[10] the authors learned of a related work by Hellsten and

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Spies[11] who treated the complementary case of a toroidal plasma rotation with a purely poloidal magnetic field. This case too may have an unstable cusp continuum, but its origin is hydrodynamical and is, in fact, a modification of the unstable continuous spectrum in hydrodynamics. (The cusp continuum has a hydrodynamical counterpart while the Alfvén continuum is genuinely magnetohydrodynamical.) As a result it can be stabilized by a simple requirement such as having isothermal magnetic flux surfaces. In contrast, the parametric instability we describe in this paper disappears in the hydrodynamical limit and, by its nature, will be almost impossible to suppress within the ideal plasma model.

II. THE EQUATIONS

We use the ideal magnetohydrodynamics equations in the following form:

$$\begin{aligned} &\rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})+\nabla(p+\mathbf{B}^2/2)-\mathbf{B}\cdot\nabla\mathbf{B}=0\,,\\ &\mathbf{B}_t+\mathbf{u}\cdot\nabla\mathbf{B}+\mathbf{B}\,\mathrm{div}\,\mathbf{u}-\mathbf{B}\cdot\nabla\mathbf{u}=0\,,\\ &\rho_t+\mathbf{u}\cdot\nabla\rho+\rho\,\mathrm{div}\,\mathbf{u}=0\,,\\ &S_t+\mathbf{u}\cdot\nabla S=0\,,\quad S=p/\rho^\gamma\,, \end{aligned} \tag{1}$$

where B, u, p, and ρ are the magnetic field, the plasma velocity, pressure, and density, respectively. For the determination of the continuous spectrum it is useful to observe that all spatial derivatives appear in the form of two directional derivatives, namely, $\mathbf{u}\cdot\nabla$ and $\mathbf{B}\cdot\nabla$, except for terms involving $\nabla(p+\mathbf{B}^2/2)$ and div u. Hence, only $p+\mathbf{B}^2/2$ and one component of u are differentiated in the direction of $\mathbf{u}\times\mathbf{B}$. It follows that the surface with normal $\mathbf{u}\times\mathbf{B}$ is a characteristic surface of the system which determines the equilibrium states. "Initial conditions" cannot be specified on it arbitrarily since the normal derivative of only two quantities will be determined while the other six equations will impose restrictions on these conditions.

The equilibrium state which is of interest to us, involves an asymmetric rotating θ pinch. Thus, we consider u in the (x,y) plane, $\mathbf{B}=B\hat{z}$. All equilibrium quantities are independent of z. The second and third equations in (1) can be written as

$$\mathrm{div}(B\mathbf{u})=0\,,\quad \mathrm{div}(\rho\mathbf{u})=0\,.$$

We conclude

$$\mathbf{u}\cdot\nabla(B/\rho)=0 \tag{2}$$

so that B/ρ is constant along streamlines. The last equation in (1) implies that S is also constant along streamlines,

$$\mathbf{u}\cdot\nabla S=0\,. \tag{3}$$

Taking the scalar product of the momentum equation with u, we find after using (2) and (3)

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}|\mathbf{u}|^2+\frac{\gamma}{\gamma-1}S\rho^{\gamma-1}+\frac{B^2}{\rho}\right)=0\,. \tag{4}$$

This is the generalized Bernoulli equation for the case of a magnetic field perpendicular to the flow,[12] and it yields another constant along streamlines. It can be shown that, in analogy with fluid dynamics, prescribing the previous three arbitrary functions of the flow flux and the domain of the plasma, is sufficient to determine the equilibrium state.

The linear stability of this state is determined by linearizing system (1) about the equilibrium. We also take the Fourier transform in t and z, or equivalently assume a dependence of $\exp[i(\omega t+kz)]$. For reasons that will presently become apparent, we now determine the characteristic surfaces of the resulting system of equations. These are surfaces on which initial conditions cannot be prescribed arbitrarily. Clearly, the differentiated terms of the linearized equations are exactly those that appear in (1), with coefficients that are equilibrium functions, e.g., the term $\mathbf{u}\cdot\nabla\mathbf{B}$ generates (among others) a term $\mathbf{u}_0\cdot\nabla\mathbf{B}_1$ in the perturbed system, with the subscripts 0, 1 denoting equilibrium and perturbed quantities, respectively. In addition, time derivatives do not appear. Thus, the type of the perturbed system is the same as that of the system which determines the equilibrium state. It is known in the literature[12] that the system (which is of eighth order) is either fully hyperbolic or has two complex characteristic surfaces and six real ones. In our case, the system is fully hyperbolic only for a super-magnetosonic flow, i.e., $|\mathbf{u}_0|^2>\gamma p_0/\rho_0+B_0^2/\rho_0$, which is assumed not to be the case. Using the remark made at the beginning of the section, we make the observation that a surface with normal $\mathbf{u}_0\times\mathbf{B}_0$ is always a six-times characteristic of the linearized, transformed system, and one can find six equations *in* the surface that have to be satisfied. We remark that in the case of a containment equilibrium with magnetic flux surfaces and flow, the flow must of necessity be within the flux surface because of the frozen-in condition, and the flux surface is then the characteristic surface. This is, of course, the case even when the equilibrium is static. It is this property which enables a reduction of the system to a single, second order equation in some sufficiently symmetric cases.[5,13]

The linearized and Fourier transformed form of Eq. (1) is

$$i\omega\rho\mathbf{u}_1+i\omega\rho_1\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}_1+\rho\mathbf{u}_1\cdot\nabla\mathbf{u}+\rho_1\mathbf{u}\cdot\nabla\mathbf{u}+\nabla(p_1+\mathbf{B}\cdot\mathbf{B}_1)=\mathbf{B}\cdot\nabla\mathbf{B}_1+\mathbf{B}_1\cdot\nabla\mathbf{B}\,, \tag{5a}$$

$$i\omega\mathbf{B}_1+\mathbf{u}\cdot\nabla\mathbf{B}_1+\mathbf{u}_1\cdot\nabla\mathbf{B}+\mathbf{B}_1\,\mathrm{div}\,\mathbf{u}+\mathbf{B}\,\mathrm{div}\,\mathbf{u}_1=\mathbf{B}\cdot\nabla\mathbf{u}_1+\mathbf{B}_1\cdot\nabla\mathbf{u}\,, \tag{5b}$$

$$i\omega p_1+\mathbf{u}\cdot\nabla p_1+\mathbf{u}_1\cdot\nabla p+\gamma p\,\mathrm{div}\,\mathbf{u}_1+\gamma p_1\,\mathrm{div}\,\mathbf{u}=0\,, \tag{5c}$$

$$i\omega S_1+\mathbf{u}\cdot\nabla S_1+\mathbf{u}_1\cdot\nabla S=0\,, \tag{5d}$$

where the subscript 0 was suppressed, $\mathbf{B}\cdot\nabla$ can be replaced by $ik|\mathbf{B}|$, and ρ_1 in (5a) can be expressed in terms of p_1, S_1 by using the linearized form of $p=S\rho^\gamma$. Now, replace p_1 by $(p_1+\mathbf{B}\cdot\mathbf{B}_1)-\mathbf{B}\cdot\mathbf{B}_1$, then add $-\mathbf{B}/(\gamma p)$ of Eq. (5c) to (5b) to eliminate the div $\mathbf{u}_1$ term there. The resulting system can be written symbolically as

$$\begin{aligned} &v_\tau+i\omega v=Av+Bw+Cw_\tau\,,\\ &w_\tau+i\omega w+Pw_n=Dv+Ew+Fv_\tau\,, \end{aligned} \tag{6}$$

where the subscripts denote derivatives, $\partial/\partial\tau=\mathbf{u}\cdot\nabla$, w_n is the derivative in the normal direction, A–P are matrix coefficients, w is a 2-vector containing the per-

turbed total pressure $p_1 + \mathbf{B}\cdot\mathbf{B}_1$ and the normal component of $\mathbf{u}_1$. v is a 6-vector which houses the rest of the perturbed quantities and is not differentiated in the normal direction. The second equation in (6) stands for (5c) and the normal component of (5a).

III. THE CONTINUOUS SPECTRUM

The continuous spectrum of the perturbed system with ω the eigenparameter, can be found by taking linear combinations of the equations that involve only derivatives *in* the characteristic surface, and then setting the two dependent variables that are differentiated elsewhere across the surface to zero. In our case it means that we consider

$$dv/d\tau + i\omega v = Av\,. \tag{7}$$

This is a system of six ordinary differential equations along each streamline (characterized, say, by a streamfunction ψ), with periodic boundary conditions. The eigenvalues of this system $\omega_n(\psi)$ determine the continuous spectrum, when ψ is taken as a continuous parameter.

A mathematical proof of this characterization of the continuous spectrum exists, and will be given elsewhere.[14] The formal procedure, however, is equivalent to the ones used by Pao[8] and Grossmann *et al.*[15] for some static equilibria. Reference 15 suggests the useful heuristic argument that the continuous spectrum manifests itself by allowing large derivatives in the solution, which are only possible across characteristic surfaces. However, if one wants to pursue this argument further and try to find the kind of singularity which is generated by using the natural extension of Pao's[8] procedure, it can be shown[14] that the outcome may be $(\psi-\psi_0)^c$ for any real number c (even $c>0$), with $\psi=\psi_0$ the characteristic surface under consideration. Clearly for c sufficiently large, the derivatives across the surface are not large. Nevertheless, this procedure has its merit and will be shown to relate[14] to the existence of the so-called "singular sequences" which characterize points in the essential spectrum.

Going back to system (7), we first write down the simple equation for the perturbed entropy $s = S_1$

$$ds/d\tau + i\omega s = 0, \quad s(0) = s(T)\,, \tag{8}$$

where τ parameterizes the streamline as the time it takes an unperturbed fluid particle to move along its trajectory. T is the period of this motion. The solution to this equation is $s(\tau) = s(0)\exp(-i\omega\tau)$. Either $s\equiv 0$ or $\omega T = 2\pi m$ for some integer m, i.e., $\omega = m\Omega(\psi)$, with $\Omega = 2\pi/T$ the frequency of the unperturbed motion. Since we are interested in the unstable part of the spectrum, we assume $\omega \neq m\Omega$ so that $s\equiv 0$. The same equation (8) holds for the perturbed normal component of B, and it is also set to zero. Equation (7) is now a fourth-order system. A further simplification is obtained by defining $y = v\exp(i\omega\tau)$. This eliminates ω from (7) and it now enters only through the boundary conditions

$$\frac{d}{d\tau}\, y = Ay\,, \quad y(T) = \lambda y(0)\,, \tag{9}$$

where we define

$$\lambda = \exp(i\omega T)\,.$$

The boundary conditions require that a vector solution y return to its initial direction after time T. The numbers λ such that $y(T) = \lambda y(0)$ are known in the literature[16] as the Floquét multipliers of the system (9), and their number is the same as the order of the system (which is 4 in our case). Once the multipliers $\lambda_i (i = 1, \ldots, 4)$ are known, ω is found to be $\omega_{im}(\psi) = \omega_{i0}(\psi) + m\Omega(\psi)$, with m any integer. Clearly, the continuous spectrum is real if, and only if, $|\lambda_i| = 1$ for all i.

The Floquét multipliers can be found in the following way. We arrange four independent solutions of (9) (ignoring boundary conditions) as columns of a matrix Y. $Y' = AY$, with the prime denoting $d/d\tau$. Y is picked such that $Y(0) = I$, the identity matrix. Every solution is a linear combination of the columns of Y and thus can be written as $Y\xi$, with ξ a constant vector. The multipliers are simply the eigenvalues of $Y(T)$, since then there is a ξ such that $Y(T)\xi = \lambda\xi$, and $y(\tau) = Y(\tau)\xi$ satisfies the boundary condition in (9). In the next section we investigate the properties of the multipliers of our system. A good exposition of the theory involved may be found in Ref. 17. We will develop some of the basic properties to make the discussion reasonably self-contained.

IV. HAMILTONIAN PROPERTIES OF THE CONTINUUM

The system (9) can be written as

$$y' = JHy\,, \tag{10}$$

with J an antisymmetric and H a symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H = -\rho'\,\frac{B}{\rho}\left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right] + k\,\frac{B}{\rho}\begin{bmatrix} \beta\rho & & & \\ & \rho q^2 & & \\ & & 1 & \\ & & & 1/q^2 \end{bmatrix}, \tag{11}$$

$$y^T = \left(\frac{1}{\beta\rho}\,B_{1z}\,,\ \frac{i}{\rho q}\,B_{1t}\,,\ iu_{1z}\,,\ -qu_{1t}\right)e^{i\omega\tau}\,,$$

where the subscript t denotes the component along a streamline,

$$q = |\mathbf{u}_0|, \quad \beta = \gamma p/(\gamma p + B^2)\,,$$

and all coefficients are equilibrium quantities. Because of the equilibrium properties (2)–(4), both β and q can be written as functions of ρ for a given streamline, and B/ρ is constant. We note that the coefficients in (10) are real.

A system of the form (10) is recognized as a linear Hamiltonian system. In the more traditional way the Hamiltonian will be defined as $\frac{1}{2}y^T Hy$, and the first and last two components of y will be considered the coordinates and momenta, respectively. Linear Hamiltonian systems give rise to symplectic matrices, i.e., if Y is a solution matrix of (10), $Y(0) = I$, then for all τ

$$Y^T J Y = J\,. \tag{12}$$

This can be shown by noticing that the derivative, with

respect to τ, of the left-hand side of (12) vanishes, that (12) is satisfied at $\tau=0$, and that $J^T=-J$ by its definition. Relation (12) can be written as

$$Y^T=JY^{-1}J^{-1}. \tag{13}$$

This similarity relation implies that the set of eigenvalues of Y is the same as that of Y^{-1}. Also, since the system (10) is real, if λ is an eigenvalue of Y, so is its complex conjugate. Thus, $\lambda, \lambda^*, 1/\lambda, 1/\lambda^*$ are all Floquét multipliers if λ is such. Notice that $1/\lambda^*$ is the reflection of λ in the unit circle. Since system (10) is of fourth order, it is possible that all multipliers are related to one λ [Fig. 1(c)]. The system is stable only if all multipliers are on the unit circle. Because of (9), the relation between the multipliers translates into the statement that $\omega, \omega^*, -\omega, -\omega^*$ are all eigenvalues if one of them is, and there is always an exponentially growing mode if an eigenvalue is complex.

Before investigating the multipliers more closely we can determine them asymptotically for large wavenumbers k (or actually for $kB/\sqrt{\rho} \gg \Omega$). Using a standard method[16] we can obtain solutions of the form

$$\begin{aligned} y&=f\exp\left[k\int_0^\tau \mu(\tau)d\tau\right],\\ f&=f_0+\frac{1}{k}f_1+\frac{1}{k^2}f_2+\cdots, \end{aligned} \tag{14}$$

where f is a vector function and μ is a scalar. In view of (11), system (10) can be written symbolically as

$$y'=(kA+\rho'B)y \tag{15}$$

with A a matrix function of ρ and B a constant matrix. The kth and zeroth-order terms of (15) yield

$$(A-\mu I)f_0=0, \quad (A-\mu I)f_1=f_0'-\rho'Bf_0. \tag{16}$$

The first equation implies that $\mu=\mu(\rho)$ is an eigenvalue of A with f_0 as an eigenvector determined up to a scalar multiple, $f_0=\alpha(\tau)\phi(\rho)$ and ϕ is a known vector. The second equation can be solved for f_1 only if the right-hand side satisfies a solubility condition which requires that it be orthogonal to the eigenvectors of A^T corresponding to the eigenvalue μ. If $g(\rho)$ is such an eigenvector, the orthogonality condition yields an equation for $\alpha(\tau)$ which, after replacing τ by ρ as an independent variable, can be written as

$$g^T\left(\frac{d}{d\rho}f_0-Bf_0\right)=0 \tag{17}$$

Clearly, $\alpha=\alpha(\rho)$ and is periodic. To leading order, the Floquét multipliers are therefore $\exp(k\int_0^T \mu\, d\tau)$, or

$$\begin{aligned} \lambda_{1,2}&=\exp(\pm ikT\langle B/\sqrt{\rho}\rangle)[1+O(1/k)],\\ \lambda_{3,4}&=\exp[\pm ikT\langle B(\beta/\rho)^{1/2}\rangle]\ [1+O(1/k)], \end{aligned} \tag{18}$$

where

$$\langle f\rangle=\frac{1}{T}\int_0^T f d\tau=\frac{1}{T}\oint f\frac{dl}{q}$$

is the time average of f. The logarithm of (18) yields

$$\begin{aligned} \omega_{1,2}&=m\Omega\pm k\langle B/\sqrt{\rho}\rangle+O(1/k),\\ \omega_{3,4}&=m\Omega\pm k\langle B(\beta/\rho)^{1/2}\rangle+O(1/k). \end{aligned} \tag{19}$$

The first two eigenvalues are identified as the Alfvén continuum and the last two as the cusp continuum. In a circular cylinder the average symbols can be dropped, the correction term $O(1/k)$ vanishes and (19) is the familiar expression for this case[6] with m being the separation variable from the $\exp(im\theta)$ dependence. Returning to (18) we see that for large k, and when the exponents do not differ by approximately a multiple of $2\pi i$, the multipliers cannot be the reflections of each other with respect to the unit circle, and thus must be *on* the circle. Hence, the continuous spectrum is real for large k, except perhaps for narrow bands in k of width of order $1/k$ around the points where, to leading order in k, two multipliers coincide.

Before continuing with a complete description of the continuum, we point out that for $k=0$, Eq. (15) can be solved explicitly as $Y=e^{\rho B}$ with Y a matrix solution. Thus, all solutions are periodic and the multipliers are $\lambda_i=1$, $i=1,\ldots,4$, which is a stable state.

V. A PARAMETRIC INSTABILITY

A full description of the Floquét multipliers for all ranges of k will now be given. It is based on general results in the theory of linear Hamiltonian systems. A comprehensive and simple description of the theory can be found in Ref. 17.

A simple multiplier on the unit circle will remain and move along the circle as k is varied continuously. Because the matrix H in (11) (essentially the Hamiltonian) is monotonic in k, the motion of the multiplier will be monotonic in its direction as k increases. A multiplier can deviate from the circle only after it coalesces with another multiplier moving in the opposite direction. Deviation from the circle will then occur only if the two eigenvectors corresponding to the two multipliers also coalesce (which will be the case in general). After deviating from the circle, they will move in opposite directions so as to remain the reflections of each other in the circle. After some interval in k they will coalesce again on the circle and continue their motion along it in their original opposite directions. Since λ and λ^* are both multipliers, there must be two eigenvalues moving clockwise and two moving in the opposite direction. At $k=0$ it was found previously that all multipliers coalesce at $\lambda=1$ but do not then deviate from the circle because they do have a full set of eigenvectors. As k increases, the system remains stable until the first two multipliers coalesce at $\lambda=-1$. From then on the system goes through unstable bands in k which become infinitely narrow as $k\to\infty$. The initial behavior of the system is described in Fig. 1. The general case does involve deviations from the unit circle and thus, an unstable continuous spectrum. This was also confirmed by some numerical checks. For a better understanding of this phenomenon we present the following explicit example of instability.

Example: Consider the incompressible limit ($\gamma\to\infty$, $\beta\to 1$) in which ρ is a constant along streamlines, $\rho'=0$. Equation (2) implies that B is also constant on streamlines, and Bernoulli's law (4) reduces to $q^2/2+p/\rho=\text{const}(\psi)$. This merely serves to determine p while q decouples from ρ and is the only nonconstant entity along

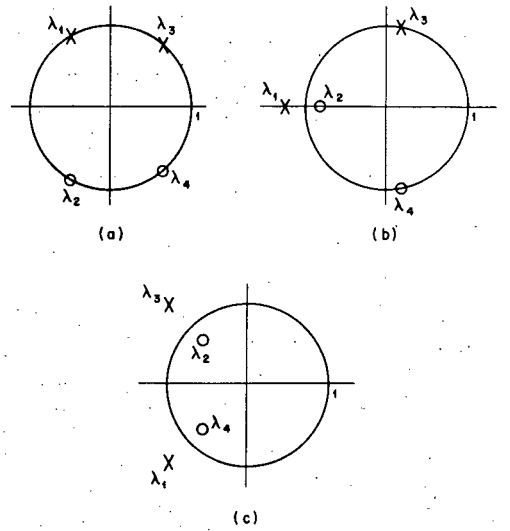

FIG. 1. Initial behavior of the Floquét multipliers as k increases from zero; $\lambda_{1,3}$ move counterclockwise, $\lambda_{2,4}$ move clockwise. (a) System is stable. (b) First unstable coupling, the complex ω's satisfy Re $\omega = \Omega/2$. (c) Second coupling, all eigenvalues are complex.

ψ. System (10) now decouples into two independent systems

$$y_1' = (kB/\rho)y_3\,, \quad y_3' = -kBy_1\,, \tag{20}$$

$$y_2' = (kB/\rho q^2)y_4\,, \quad y_4' = -kBq^2 y_2\,. \tag{21}$$

System (20) has constant coefficients and is readily seen to be stable. For the second system we define $w = qy_2$ and get

$$w'' + [k^2(B^2/\rho) - (q''/q)]w = 0\,. \tag{22}$$

This is Hill's equation which is known to have unstable bands in k. To be more specific, replace $-q''/q$ by $f(\tau)$ and k^2B^2/ρ by λ and consider (22) an eigenvalue problem for λ with periodic boundary conditions, and f assumed to be known. The eigenfunction corresponding to the lowest eigenvalue λ_0 is known[16] to be positive everywhere. Let q be that eiegnfunction ($q = |u_0|$ should be positive). It satisfies

$$q'' + [\lambda_0 + f(\tau)]q = 0\,, \tag{23}$$

or $-q''/q = \lambda_0 + f$. In (22) we have

$$w'' + [(k^2B^2/\rho) + \lambda_0 + f(\tau)]w = 0\,. \tag{24}$$

If now f is so chosen that (24) becomes Mathieu's equation, say $f = \cos 2\tau$, λ_0 serves as a shift in k such that $k^2 = 0$ corresponds to the boundary of the first stable region. Once $q(\tau)$ is found one can pick an analytic closed curve of length $L = \int_0^\tau q\,d\tau$, specify q appropriately on it, and then solve for a flow in the neighborhood of the curve. This will be a flow with unstable continuous spectrum. Notice that unbounded growth in τ of solutions to Eq. (22) is equivalent to the instability in τ of the continuum. Both are related to the existence of Floquét multipliers outside the unit circle.

In the example, it is clear that the origin of the instability is parametric. Namely, the rotation acts as a periodic driving force and couples with the plasma Alfvén frequency $kB/\sqrt{\rho}$. In the general case, coupling may occur also with the cusp frequency $k\sqrt{\beta}B/\sqrt{\rho}$ or with both frequencies. It should be remarked that each flux surface generates unstable bands in k. Consideration of all flux surfaces broadens those bands and, although in actual systems k may attain only discrete values, it is difficult to envision a case which will not be unstable. This instability in itself probably dooms ideas like the rotating Tormac[18] which rely on poloidal mass rotation to inhibit particle drifts. In the next section we show, however, that not all is lost, and that an important class of steady equilibria has stable continua.

VI. SOME STABLE CONTINUA

The condition of frozen-in magnetic fields which prevails for ideal plasmas, restricts the class of flows allowed in an equilibrium state: A flow parallel to the magnetic field leaves the field lines unchanged and therefore may exist. In fact, one can show that for axisymmetric equilibria with ergodic flux surfaces, the flow can be decomposed into a parallel flow plus rigid rotation of each flux surface. Here, we treat the case of parallel flow in incompressible plasmas without the use of any symmetry. Such an equilibrium is distinguished by the conditions

$$\operatorname{div}\mathbf{u} = 0\,, \quad \mathbf{u}\cdot\nabla\rho = 0\,, \tag{25}$$

$$\mathbf{u} = \lambda\mathbf{B}\,, \quad \mathbf{B}\cdot\nabla(\lambda\rho) = 0\,. \tag{26}$$

If flux surfaces are characterized by a flux function ψ, (26) implies

$$\lambda = \lambda(\psi)\,, \quad \rho = \rho(\psi)\,. \tag{27}$$

Bernoulli's law for this equilibrium reads

$$\tfrac{1}{2}|\mathbf{u}|^2 + p/\rho = H(\psi)\,. \tag{28}$$

We first show that in this case, unlike the general steady equilibrium case, the current $\mathbf{J} = \operatorname{curl}\mathbf{B}$ lies in the flux surface. Multiplying the momentum equation by $\mathbf{J}$ and making use of (28) yields

$$\mathbf{J}\cdot\nabla(\rho H) - \tfrac{1}{2}|\mathbf{B}|^2\mathbf{J}\cdot\nabla(\lambda^2\rho) = 0$$

from which we conclude that $\mathbf{J}\cdot\nabla\psi = 0$ as desired.

Denote by $\mathbf{v}, \mathbf{b}$ the perturbed velocity and magnetic field. According to the discussion in Sec. III, the continuum is determined by the following system of equations in each flux surface:

$$i\omega\rho\mathbf{v} + \rho\mathbf{u}\cdot\nabla\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{u} = \mathbf{B}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{B}\,, \tag{29a}$$

$$i\omega\mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} + \mathbf{v}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{b}\cdot\nabla\mathbf{u} = 0\,, \tag{29b}$$

where $\mathbf{v}$ is considered a two-dimensional vector with components in the flux surface, and (29a) equates only components in that surface while the normal component of the equation is ignored. The normal component of (29b) yields

$$i\omega(\nabla\psi\cdot\mathbf{b}) + \mathbf{u}\cdot\nabla(\nabla\psi\cdot\mathbf{b}) = 0\,. \tag{30}$$

This equation involves only the normal component of $\mathbf{b}$ and clearly yields a stable piece of the continuum. We ignore this part by taking $\mathbf{b}\cdot\nabla\psi = 0$ so that now both $\mathbf{v}$

and b are vectors in the flux surface.

Define the operators

$$Av = \mathbf{B}\cdot\nabla\mathbf{v}\,,\quad Hv = \mathbf{v}\cdot\nabla\mathbf{B}\,. \tag{31}$$

Equation (29) can now be written as

$$(A+H)(\mathbf{b}-\lambda\rho\mathbf{v}) = i\omega\rho\mathbf{v}\,,\quad (A-H)(\mathbf{v}-\lambda\mathbf{b}) = i\omega\mathbf{b}\,, \tag{32}$$

or, after defining $\beta = \mathbf{b}-\lambda\rho\mathbf{v}$, $\eta = \mathbf{v}-\lambda\mathbf{b}$, we obtain

$$\begin{bmatrix} 0 & A+H \\ A-H & 0 \end{bmatrix}\begin{bmatrix}\eta \\ \beta\end{bmatrix} = i\omega P\begin{bmatrix}\eta \\ \beta\end{bmatrix}, \tag{33}$$

where P is a symmetric matrix given by

$$P = \frac{1}{1-\lambda^2\rho}\begin{bmatrix}\rho & \lambda\rho \\ \lambda\rho & 1\end{bmatrix}. \tag{34}$$

Notice that $\lambda^2\rho$ is the Alfvén Mach number and P is (positive) definite if $\lambda^2\rho<1$, that is, for sub-Alfvénic flows. Equation (33) is a fourth-order eigenvalue problem in $\xi = \{\eta, \beta\}$. It is useful to introduce the inner product

$$(\xi,\eta) = \oint \eta^*\cdot\xi\,\frac{dS}{|\nabla V|}\,, \tag{35}$$

where dS is an area element, and V labels the flux surfaces and is equal to the volume enclosed by the surface. System (33) will be an anti-self-adjoint eigenvalue problem with real ω's if we can show that H is Hermitian and A is anti-Hermitian with respect to (35). The first assertion is derived from the simple relation

$$\mathbf{v}\cdot(\mathbf{u}\cdot\nabla\mathbf{B}) - \mathbf{u}\cdot(\mathbf{v}\cdot\nabla\mathbf{B}) = \mathbf{u}\times\mathbf{v}\cdot\operatorname{curl}\mathbf{B}\,.$$

The triple product vanishes because all three vectors lie in the magnetic surface. The second assertion follows from the relation

$$\mathbf{v}\cdot(\mathbf{B}\cdot\nabla\mathbf{u}) - \mathbf{u}\cdot(\mathbf{B}\cdot\nabla\mathbf{v}) = \mathbf{B}\cdot\nabla(\mathbf{u}\cdot\mathbf{v}) = \operatorname{div}(\mathbf{u}\cdot\mathbf{v}\mathbf{B})\,,$$

and

$$\oint \operatorname{div}(\mathbf{u}\cdot\mathbf{v}\mathbf{B})\,\frac{dS}{|\nabla V|} = \frac{d}{dV}\int_{V_0}^{V}\oint \operatorname{div}(\mathbf{u}\cdot\mathbf{v}\mathbf{B})\,\frac{dS}{|\nabla V'|}\,dV'$$

$$= \frac{d}{dV}\oint (\mathbf{u}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\,\frac{dS}{|\nabla V|}\,,$$

which involves the use of the divergence theorem. $\mathbf{n}$ is normal to the surface and the last integral clearly vanishes for all V, hence also its derivative. Thus, we have shown that the continuous spectrum is stable for a sub-Alfvénic, parallel, incompressible flow.

Essentially the same proof can be used to show that the continuum is stable in a general static equilibrium. In order to determine the continuum for this case, it is useful to replace the continuity equation by

$$p_t + \mathbf{u}\cdot\nabla p + \gamma p\operatorname{div}\mathbf{u} = 0\,, \tag{36}$$

and then add a multiple of (36) to the second equation in (1) (Faraday's law) so as to eliminate the div u term there. The fourth-order system which determines the Alfvén and cusp parts of the spectrum now has exactly the form of (33), with $\eta = \mathbf{v}$, $\beta = \mathbf{b}$, and

$$P\begin{bmatrix}\mathbf{v} \\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}\rho\mathbf{v} \\ \mathbf{b} + \dfrac{1}{\gamma p}(\mathbf{B}\cdot\mathbf{b})\mathbf{B}\end{bmatrix};$$

P is also positive definite. Indeed,

$$(P\xi,\xi) = \oint\left(\rho|\mathbf{v}|^2 + |\mathbf{b}|^2 + \frac{1}{\gamma p}|\mathbf{B}\cdot\mathbf{b}|^2\right)\frac{dS}{|\nabla V|}\,.$$

The proof is complete.

VII. CONCLUSION

We have discussed the stability of the Alfvén and cusp continuous spectra in magnetohydrodynamics for several systems, with stationary flow or with no flow. The static equilibrium always has a real (and thus exponentially stable) continuum, while the presence of flow may give rise to an instability. Two causes for this instability were indicated. In the case of the noncircular rotating θ pinch it was shown that the flow acts as a periodic driving force which may couple with the natural frequencies of the system, that is, the frequencies of the Alfvén and slow waves, and thus drive a parameteric instability much like a system described by the Mathieu equation.

A different cause of instability was indicated in the case of parallel flow. In the incompressible limit treated, stability was assured only for sub-Alfvénic flows. We interpret this restriction on the stable regime as heralding the development of shock waves due to "supersonic" speeds. It is known[12] that the equation determining the equilibrium state is elliptic when the flow is slow enough, but becomes hyperbolic, and admits discontinuous solutions, at the critical cusp speed $|\mathbf{u}|^2 = \beta|\mathbf{B}|^2/\rho$, with β defined in (11). In the incompressible case, $\beta\to 1$ and the critical speed equals the Alfvén speed. We expect the transition from a stable to an unstable continuous spectrum to coincide with the critical speed in the compressible case as well.

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[1] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London Ser. A 224, 1 (1958).
[2] E. Frieman and M. Rotenberg, Rev. Mod. Phys. 32, 898 (1960).
[3] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin, 1966).
[4] J. A. Tataronis and W. Grossmann, Z. Phys. 261, 203 (1973).
[5] J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. R. Soc. London Ser. A 365, 1 (1979).
[6] G. O. Spies, Phys. Fluids 21, 580 (1978).
[7] E. Hameiri, Phys. Fluids 22, 89 (1979).
[8] Y. P. Pao, Nucl. Fusion 18, 621 (1975).
[9] J. A. Tataronis (private communication).
[10] E. Hameiri and J. H. Hammer, Bull. Am. Phys. Soc. 23, 872 (1978).

[11]T. A. K. Hellsten and G. O. Spies, Phys. Fluids **22**, 743 (1979).
[12]H. Grad, Rev. Mod. Phys. **32**, 830 (1960).
[13]K. Hain and R. Lüst, Z. Naturforsch **13a**, 936 (1958).
[14]E. Hameiri (to be published).
[15]W. Grossmann, J. A. Tataronis, and H. Weitzner, Phys. Fluids **20**, 239 (1977).
[16]E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955), Chaps. 6–8.
[17]V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients* (Keter, Jerusalem, 1975), Vol. 1.
[18]J. H. Hammer and W. B. Kunkel, Phys. Fluids **19**, 438 (1976).

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