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Conformal Field Theory on Affine Lie Groups

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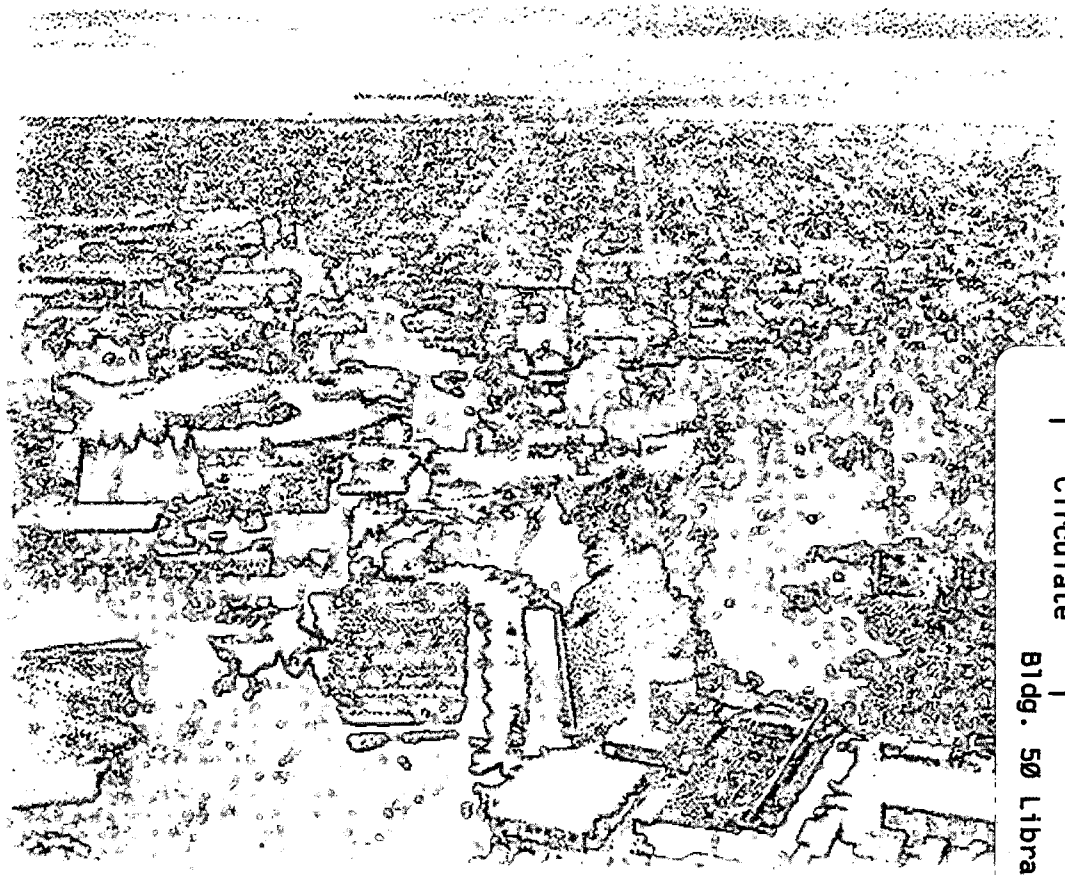
Ph.D. Thesis



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Conformal Field Theory on Affine Lie Groups*

by

Kenneth Sherman Clubok

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Committee in charge:

Professor Martin B. Halpern, Chair
Professor Hiroshi Ooguri
Professor Vaughan Jones

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Abstract

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Doctor of Philosophy in Physics
University of California, Berkeley
Professor Martin B. Halpern, Chair

Working directly on affine Lie groups, we construct several new formulations of the WZW model, the gauged WZW model, and the generic affine-Virasoro action. In one formulation each of these conformal field theories (CFTs) is expressed as a one-dimensional mechanical system whose variables are coordinates on the affine Lie group. When written in terms of the affine group element, this formulation exhibits a two-dimensional WZW term. In another formulation each CFT is written as a two-dimensional field theory, with a three-dimensional WZW term, whose fields are coordinates on the affine group. On the basis of these equivalent formulations, we develop a translation dictionary in which the new formulations on the affine Lie group are understood as mode formulations of the conventional formulations on the Lie group. Using this dictionary, we also express each CFT as a three-dimensional field theory on the Lie group with a four-dimensional WZW term.

Martin B. Halpern, Chair

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Chapter 1

Introduction

1.1 Overview

Affine Lie algebra, or current algebra on S^1, was discovered independently in mathematics [1, 2] and physics [3]. A large class of conformal field theories, including the affine-Sugawara constructions [3-7], the coset constructions [3, 4, 8], and the generic affine-Virasoro construction [9, 10] are constructed from the currents of the affine algebras. In these constructions, the energy-momentum stress tensor is expressed as a bilinear of the affine currents. Passage from the Hamiltonian formulation to the action formulation of these theories yields their world-sheet descriptions. In this way, the affine-Sugawara construction is equivalent to the WZW model [11, 12], the coset constructions are equivalent to the gauged WZW model [13-17], and the generic affine-Virasoro construction is equivalent to the generic affine-Virasoro action [18]. In all these world-sheet descriptions, the conformal field theories have been described in terms of variables on the Lie group manifold. See Ref. [19] for a more detailed history of affine Lie algebra and conformal field theory.

Affine Lie groups, which are generated by the affine algebras, are infinite dimensional generalizations of Lie groups. This thesis discusses equivalent reformulations of the WZW model, the gauged WZW model, and the generic affine-Virasoro action in terms of coordinates on the corresponding affine Lie groups. These reformulations of conformal field theory are based on recent work by Halpern and Sochen [20], who used the coordinates on the affine Lie group $\hat{G}$ to construct a new first-order differential representation of affine $g \times g$. In Ref. [21], that representation was used to construct several reformulations of the WZW model. This work repeats the treatment of Ref. [21] for the WZW model, and then goes on to generalize the treatment to the gauged WZW model and the generic affine-Virasoro action.

An outline of this thesis is as follows. The remainder of this chapter presents a brief review of affine Lie algebra and its role in conformal field theory. The following three chapters are devoted respectively to the affine-Sugawara constructions and the WZW model, the coset constructions and the gauged WZW model, and the generic affine-Virasoro construction and the generic affine-Virasoro action. In each of these chapters, the formulation on the Lie group is first discussed, followed by the translation onto the affine Lie group.

The overall picture which emerges is a quartet of equivalent formulations of each conformal field theory,

$$\begin{tikzcd}
\hat{g}(x(\tau)) \arrow[r, leftrightarrow, "\text{constraint}"] \arrow[d, leftrightarrow, "\text{modes}"'] & \hat{g}(x(\tau,\sigma)) \arrow[d, leftrightarrow, "\text{modes}"] \\
g(x(\tau,\sigma)) \arrow[r, leftrightarrow, "\text{constraint}"] & g(x(\tau,\sigma,\tilde{\sigma}))
\end{tikzcd} \tag{1.1.1}$$

where g is a Lie group element, $\hat{g}$ is an affine Lie group element, x is the target space coordinate on the group or affine group manifold, and τ, σ, and $\tilde{\sigma}$ are the appropriate base space coordinates for each formulation. This picture shows that the operation of moding is a partial transmutation between the base space and the target space. This picture is discussed in detail in Section 2.8.

As the picture makes clear, this work presents a plethora of formulations of conformal field theories. All told, there are twelve such formulations: four each for the WZW model, the gauged WZW model, and the generic affine-Virasoro action. (In addition, many of the formulations are expressed in several different ways.) Two of those formulations — the conventional formulations of the WZW model and the gauged WZW model — are not at all original to this work. Four of the formulations — the new formulations of the WZW model and the original formulation of the linearized generic affine-Virasoro action — were first presented in previous papers [21-23] on which I collaborated. The other six formulations are completely original to this thesis.

1.2 Preliminaries: Affine Lie algebra and conformal field theory

Mathematically an affine Lie algebra [1, 2, 3] $\hat{g}$ is the loop algebra of a finite dimensional Lie algebra g (maps from $S^1 \to g$) with a central extension. The generators of the affine algebra, called the affine currents, can be represented as $J_a(\sigma)$,

$a = 1 \ldots \dim g$ with σ the angle parametrizing S^1. The Fourier modes $J_a(m)$, $m \in \mathbb{Z}$, of the currents are often used as a convenient basis. See Ref. [1] for a detailed discussion of affine Lie algebras in mathematics.

The mode algebra of untwisted simple affine g is

$$[J_a(m), J_b(n)] = i f_{ab}{}^c J_c(m+n) + mk\eta_{ab}\delta_{m+n,0} \quad , \quad m, n \in \mathbb{Z} \tag{1.2.1}$$

where $a, b, c = 1 \ldots \dim g$, and $f_{ab}{}^c$ and η_{ab} are respectively the structure constants and Killing metric of g. In terms of the local currents $J_a(\sigma)$, the algebra is written

$$[J_a(\sigma), J_b(\sigma') = 2\pi i[f_{ab}{}^c J_c(\sigma)\delta(\sigma - \sigma') + k\eta_{ab}\partial_\sigma\delta(\sigma - \sigma')] \tag{1.2.2a}$$

$$J_a(\sigma) = \sum_m e^{-im\sigma} J_a(m) \tag{1.2.2b}$$

where $0 \leq \sigma < 2\pi$ For simplicity, the discussion in this thesis is restricted to simple compact g.

The currents of affine Lie algebras are used to generate conformal field theories via the affine-Virasoro construction [9, 10],

$$T = L^{ab} \underset{*}{\overset{*}{}} J_a J_b \underset{*}{\overset{*}{}} \tag{1.2.3a}$$

$$L^{ab} = 2kL^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{ce}{}^a f_{df}{}^b - L^{cd}f_{ce}{}^f f_{df}{}^{(a}L^{b)e} \tag{1.2.3b}$$

$$c = 2k\eta_{ab}L^{ab} \tag{1.2.3c}$$

where (1.2.3b) is the Virasoro master equation. The master equation is a large set of coupled quadratic equations for the inverse inertia tensor L^{ab}. For each solution L^{ab} of the master equation, one obtains a conformal field theory with central charge c given in (1.2.3c). Generically-irrational central charge, even on positive integer levels of the affine algebra, is an immediate consequence of the quadratic structure of the master equation. Some important properties of the master equation are as follows:

a) Affine-Sugawara construction [3-7]. The affine-Sugawara construction is the simplest solution of the master equation, with

$$L_g^{ab} = \frac{\eta^{ab}}{2k + Q_g} \quad , \quad c_g = \frac{2k \dim g}{2k + Q_g} \tag{1.2.4}$$

where η^{ab} is the inverse Killing metric, and $Q_g\eta_{ab} = -f_{ac}{}^d f_{bd}{}^c$.

b) K-conjugation covariance [3, 4, 8, 24, 9]. A central feature of the master equation is that its solutions come in K-conjugate pairs L and $\tilde{L}$, where

$$L^{ab} + \tilde{L}^{ab} = L_g^{ab} \quad , \quad c + \tilde{c} = c_g. \tag{1.2.5}$$

The corresponding K-conjugate stress tensors T and $\tilde{T}$,

$$T(z)\tilde{T}(w) = \text{reg.} \quad , \quad T(z) + \tilde{T}(z) = T_g(z) \tag{1.2.6}$$

commute and sum to the affine-Sugawara stress tensor T_g.

K-conjugation is used to generate additional solutions, as in the familiar case of the coset constructions [3, 4, 8] $T_h + T_{g/h} = T_g$. At the level of dynamics, K-conjugation also provides the minimal local symmetry of any ICFT, as discussed below.

c) Semi-classical solutions [25, 26]. On simple g, the generic solutions of the master equation live in level-families $L^{ab}(k)$ whose high-level forms are

$$L^{ab} = \frac{P^{ab}}{2k} + \mathcal{O}(k^{-2}) \quad , \quad \tilde{L}^{ab} = \frac{\tilde{P}^{ab}}{2k} + \mathcal{O}(k^{-2}) \tag{1.2.7a}$$

$$L^{ab} + \tilde{L}^{ab} = L_g^{ab} = \frac{\eta^{ab}}{2k} + \mathcal{O}(k^{-2}) \tag{1.2.7b}$$

$$c = \text{rank } P + \mathcal{O}(k^{-1}) \quad , \quad \tilde{c} = \text{rank } \tilde{P} + \mathcal{O}(k^{-1}) \quad , \quad c_g = \dim g + \mathcal{O}(k^{-1}) \tag{1.2.7c}$$

$$P^2 = P \quad , \quad \tilde{P}^2 = \tilde{P} \quad , \quad P\tilde{P} = 0 \quad , \quad P + \tilde{P} = 1 \tag{1.2.7d}$$

where P and $\tilde{P}$ are the high-level projectors of the L and the $\tilde{L}$ theory respectively. Here, high-level is taken to be synonymous with semi-classical because $1/k$ takes the role of $\hbar$.

The partial classification [27, 28] of ICFT by graph theory is based on these high-level forms, and it is believed [19] that the generic level-family is generically unitary on positive integer levels of $\hat{g}$.

d) Dynamics and gauge theories [18, 29, 22]. The basic Hamiltonian of the generic affine-Virasoro construction is

$$H_0 = L_0 + \bar{L}_0 = L^{ab}({}^{*}_{*} J_a J_b + \bar{J}_a \bar{J}_b {}^{*}_{*})_0 \tag{1.2.8}$$

where the barred currents $\bar{J}$ are right-mover copies of the left-mover currents J. In general, this basic Hamiltonian must be gauged, according to the local symmetry group of H_0.

For the g/h coset constructions the symmetry algebra of H_0 is affine H, which leads to a world-sheet description by spin-one (or Lie-algebra) gauged WZW models [13-17]. The coset constructions and gauged WZW models are discussed in Chapter 3.

In the space of all conformal field theories, the coset constructions are only special points of higher symmetry. For the generic affine-Virasoro construction L^{ab}, the

symmetry group of H_0 is Diff S_1 $\times$ Diff S_1, generated by the left- and right-mover stress tensors

$$\tilde{T} = \tilde{L}^{ab} \, {}^{*}_{*} J_a J_b \, {}^{*}_{*} \quad , \quad \bar{\tilde{T}} = \tilde{L}^{ab} \, {}^{*}_{*} \bar{J}_a \bar{J}_b \, {}^{*}_{*} \tag{1.2.9}$$

of the K-conjugate theory $\tilde{L}^{ab}$. As a consequence, the world-sheet action of the generic ICFT is a spin-two gauged WZW model [18, 22], as discussed in Chapter 4.

Chapter 2

Affine-Sugawara Constructions and the WZW Model

We begin with a description of the conventional WZW model. The purpose of this section is to illustrate the general approach to conformal field theory that is employed throughout this thesis. The strategy is to begin with a classical Hamiltonian constructed from a high-level solution to the Virasoro master equation. Using a canonical representation of the classical affine currents, this Hamiltonian is written in terms of canonical variables. It is then straightforward to make the transition to the action formulation of the theory.

After discussing the conventional formulation of the WZW model, we move to the formulation on the affine Lie group. This discussion is divided into three parts. In Part I, we discuss the first-order differential representation of affine $g \times g$ from Ref. [20]. Part II applies this representation to construct the formulations of the WZW model on the affine Lie groups. Part III presents a translation dictionary between the formulations on the Lie group and those on the affine group.

The affine-Sugawara constructions were first studied in [3-7], and the WZW action was first studied in [11, 12]. The steps shown in Section 2.1 are largely taken from Ref. [30]. The remainder of the chapter (Parts I-III), discussing the formulation of the WZW model on the affine group manifold and the relations among the various formulations, gives results that were presented in Ref. [21].

2.1 The WZW Model on the Group Manifold

We begin on the group manifold G with group element $g(x) \in G$ in matrix representation T,

$$[T_a, T_b] = i f_{ab}{}^c, \quad \mathrm{Tr}(T_a T_b) = \chi \eta_{ab}, \quad a, b, c = 1 \ldots \dim g \tag{2.1.1}$$

and a set of coordinates $x^i(\sigma, \tau), i = 1 \ldots \dim g$. We take the base space to be a cylindrical worldsheet, with $\tau \in \mathbb{R}$ and $0 \leq \sigma < 2\pi$. Introduce the left-invariant and right-invariant vielbeins $e_i, \bar{e}_i$ and the antisymmetric tensor field B_{ij} by

$$e_i \equiv -ig^{-1}\partial_i g = e_i{}^a T_a \quad \bar{e}_i \equiv -ig\partial_i g^{-1} = \bar{e}_i{}^a T_a \tag{2.1.2a}$$

$$\partial_{[i} e_{j]}{}^a = e_i{}^b e_j{}^c f_{bc}{}^a, \quad \partial_{[i} \bar{e}_{j]}{}^a = \bar{e}_i{}^b \bar{e}_j{}^c f_{bc}{}^a \tag{2.1.2b}$$

$$\frac{i}{\chi}\mathrm{Tr}(e_i[e_j, e_k]) = -\frac{i}{\chi}\mathrm{Tr}(\bar{e}_i[\bar{e}_j, \bar{e}_k]) = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}. \tag{2.1.2c}$$

We also introduce inverse vielbeins $e_a{}^i$ and $\bar{e}_a{}^i$. The left- and right-invariant vielbeins and inverse vielbeins are related by the adjoint action Ω of g,

$$g(x) T_a g^{-1}(x) = \Omega_a{}^b(x) T_b, \qquad \Omega(x) = g^{-1}(T^{adj}, x) \tag{2.1.3a}$$

$$\bar{e}_i{}^a = -e_i{}^b \Omega_b{}^a, \qquad \bar{e}_a{}^i = -(\Omega^{-1})_a{}^b e_b{}^i \tag{2.1.3b}$$

$$\Omega_a{}^c \eta_{cd} \Omega_b{}^d = \eta_{ab} \tag{2.1.3c}$$

where $g^{-1}(T^{adj}, x)$ is the group element in the adjoint representation.

In terms of these quantities, one can write the canonical representation[30]

$$E_a = 2\pi e_a{}^i p_i(B) + \frac{k}{2}\eta_{ab} e_i{}^b \partial_\sigma x^i, \quad \bar{E}_a = 2\pi \bar{e}_a{}^i p_i(B) - \frac{k}{2}\eta_{ab} \bar{e}_i{}^b \partial_\sigma x^i \tag{2.1.4a}$$

$$p_i(B) \equiv p_i - \frac{k}{4\pi} B_{ij} \partial_\sigma x^j \tag{2.1.4b}$$

of the affine currents, which satisfies the bracket (classical) form of affine $g \times g$,

$$\{x^i(\tau,\sigma), p_j(\tau,\sigma')\} = i\delta_j{}^i \delta(\sigma - \sigma'), \qquad \{x^i(\tau,\sigma), x^j(\tau,\sigma')\} = \{p_i(\tau,\sigma), p_j(\tau,\sigma')\} = 0 \tag{2.1.5a}$$

$$\{E_a(\tau,\sigma), E_b(\tau,\sigma')\} = 2\pi i[f_{ab}{}^c E_c(\tau,\sigma)\delta(\sigma - \sigma') + k\eta_{ab}\partial_\sigma \delta(\sigma - \sigma')] \tag{2.1.5b}$$

$$\{\bar{E}_a(\tau,\sigma), \bar{E}_b(\tau,\sigma')\} = 2\pi i[f_{ab}{}^c \bar{E}_c(\tau,\sigma)\delta(\sigma - \sigma') - k\eta_{ab}\partial_\sigma \delta(\sigma - \sigma')] \tag{2.1.5c}$$

$$\{E_a(\tau,\sigma), \bar{E}_b(\tau,\sigma')\} = 0 \tag{2.1.5d}$$

In terms of these currents, one constructs the classical affine-Sugawara Hamiltonian,

$$H_{WZW} = \int_0^{2\pi} d\sigma \mathcal{H}_{WZW} \tag{2.1.6a}$$

$$\begin{aligned} \mathcal{H}_{WZW} &= \frac{1}{2\pi} L^{ab}_{g,\infty}(E_a E_b + \bar{E}_a \bar{E}_b) && (2.1.6b) \\ &= \frac{2\pi}{k}\eta^{ab} e_a{}^i e_b{}^j p_i(B) p_j(B) + \frac{k}{8\pi}\eta_{ab} e_i{}^a e_j{}^b \partial_\sigma x^i \partial_\sigma x^j && (2.1.6c) \end{aligned}$$

$$L^{ab}_{g,\infty} = \frac{1}{2k}\eta^{ab} \tag{2.1.6d}$$

where $L^{ab}_{g,\infty}$ is the high-level form of the affine-Sugawara construction on g.

The corresponding WZW action [11, 12] is given by

$$S_{WZW} = \int d\tau d\sigma \mathcal{L}_{WZW} \tag{2.1.7a}$$

$$\mathcal{L}_{WZW} = \frac{k}{8\pi}\eta_{ab} e_i{}^a e_j{}^b (\partial_\tau x^i \partial_\tau x^j - \partial_\sigma x^i \partial_\sigma x^j) + \frac{k}{4\pi} B_{ij} \partial_\tau x^i \partial_\sigma x^j \tag{2.1.7b}$$

where the last term is called the WZW term.

The WZW term can be rewritten as a three-dimensional term, using the identity

$$\mathrm{Tr}(g^{-1}dg)^3 \equiv d^3\xi \varepsilon^{ABC} \mathrm{Tr}(g^{-1}\partial_A g g^{-1}\partial_B g g^{-1}\partial_C g) = \delta^3\xi \partial_A W^A \tag{2.1.8a}$$

$$W^A \equiv -\frac{3}{2}\chi \varepsilon^{ABC} \partial_B x^i \partial_C x^j B_{ij} \tag{2.1.8b}$$

$$A = (\tau, \sigma, \rho), \qquad d^3\xi = d\tau d\sigma d\rho, \qquad \varepsilon^{012} = +1 \tag{2.1.8c}$$

$$\int \mathrm{Tr}(g^{-1}dg)^3 = \int d\tau d\sigma W^\rho(\tau, \sigma, \rho = 1) = -3\chi \int d\tau d\sigma \partial_\tau x^i \partial_\sigma x^j B_{ij} \tag{2.1.8d}$$

where $0 \leq \rho \leq 1$ is the radial coordinate of the cylinder bounded by the worldsheet at radius $\rho = 1$.

Using (2.1.8), we obtain the g form of the conventional two-dimensional WZW action with a three-dimensional WZW term,

$$S_{WZW} = -\frac{k}{2\pi\chi}\int d\tau d\sigma \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial} g) - \frac{k}{12\pi\chi}\int \mathrm{Tr}(g^{-1}dg)^3 \tag{2.1.9}$$

where $\partial = (\partial_\tau + \partial_\sigma)/2$ and $\bar{\partial} = (\partial_\tau - \partial_\sigma)/2$.

Part I

Affine Lie groups and affine Lie derivatives

2.2 First order differential representation of affine $g \times g$

In this section, we review the first-order differential representation of affine Lie algebra recently given by Halpern and Sochen [20].

We begin with the current modes $J_a(m)$ of untwisted simple affine Lie g [1-3],

$$[J_a(m), J_b(n)] = i f_{ab}{}^c J_c(m+n) + mk\eta_{ab}\delta_{m+n,0} \tag{2.2.1a}$$

$$a, b = 1 \ldots \dim g, \qquad m, n \in \mathbb{Z} \tag{2.2.1b}$$

where k is the level of the affine algebra and $f_{ab}{}^c$ and η_{ab} are the structure constants and Killing metric of Lie g. It is convenient to write the affine algebra as an infinite dimensional Lie algebra,

$$[\mathcal{J}_L, \mathcal{J}_M] = i f_{LM}{}^N \mathcal{J}_N \tag{2.2.2a}$$

$$\mathcal{J}_L = (J_a(m), k), \qquad L = (am, y_*) \tag{2.2.2b}$$

$$f_{am,bn}{}^{cp} = f_{ab}{}^c \delta_{m+n,p}, \qquad f_{am,bn}{}^{y_*} = -im\eta_{ab}\delta_{m+n,0} \tag{2.2.2c}$$

where L is the general tangent-space index. Here the central element k is included as a generator, and the non-zero structure constants $f_{LM}{}^N$ are given in (2.2.2c). The summation convention in (2.2.2a) is generally assumed throughout this paper. The adjoint matrix representation of the affine algebra,

$$(\hat{T}_L^{adj})_M{}^N = -i f_{LM}{}^N, \qquad [\hat{T}_L^{adj}, \hat{T}_M^{adj}] = i f_{LM}{}^N \hat{T}_N^{adj} \tag{2.2.3}$$

is constructed from the structure constants as usual.

We also introduce the object $\hat{\eta}_{LM}$

$$\hat{\eta}_{am,bn} = \eta_{ab}\delta_{m+n,0}, \qquad \hat{\eta}_{y_*,L} = \hat{\eta}_{L,y_*} = 0 \tag{2.2.4}$$

which will be called the rescaled Killing metric of the affine algebra. The relation of the rescaled Killing metric to the formal Killing metric of the affine algebra will be discussed in Section 2.7, which also discusses other formal issues of this nature.

The rescaled Killing metric can be used to obtain the totally antisymmetric structure constants of the affine algebra,

$$f_{LMN} \equiv f_{LM}{}^{P}\hat{\eta}_{PN} \tag{2.2.5a}$$

$$f_{am,bn,cp} = f_{abc}\delta_{m+n+p,0} \tag{2.2.5b}$$

$$(\hat{T}_L^{adj})_{MN} = -(\hat{T}_L^{adj})_{NM} \tag{2.2.5c}$$

whose non-zero components are given in (2.2.5b). The rescaled Killing metric arises frequently in the development below, although we often find it convenient to simplify mode sums by using the Kronecker delta.

We consider next the affine Lie group $\hat{G}$ generated by the affine algebra, whose arbitrary element $\hat{\gamma} \in \hat{G}$ can be written

$$\hat{\gamma}(\mathcal{J}, x, y) = e^{iyk}\hat{g}(J, x). \tag{2.2.6}$$

Here, y and $x^{i\mu}, i = 1 \ldots \dim g, \mu \in \mathbb{Z}$ are the coordinates on the affine group manifold, and $\hat{g}$ is the reduced affine group element. For simplicity, we assume that $\hat{G}$ is simply connected and we limit ourselves in this paper to the β-family of bases

$$\hat{g}(J, x) = \exp(i\beta^{am}(x)J_a(m)) \tag{2.2.7}$$

for the reduced affine group element. We also assume that the tangent-space coordinates β^{am} are invertible functions of x so that $x^{i\mu}(\beta)$ is well defined. An example of this family of bases is the standard basis,

$$\beta^{am}(x) = x^{i\mu}e_{i\mu}{}^{am}(0) \tag{2.2.8}$$

where $e_{i\mu}{}^{am}(0)$ is the left-invariant vielbein on $\hat{G}$ (defined below) at the origin.

The left- and right-invariant vielbeins $e_\Lambda{}^L(x)$ and $\bar{e}_\Lambda{}^L(x)$ on $\hat{G}$ are defined as follows,

$$e_\Lambda = -i\hat{\gamma}^{-1}\partial_\Lambda\hat{\gamma} = e_\Lambda{}^L\mathcal{J}_L, \qquad \bar{e}_\Lambda = -i\hat{\gamma}\partial_\Lambda\hat{\gamma}^{-1} = \bar{e}_\Lambda{}^L\mathcal{J}_L \tag{2.2.9a}$$

$$L = (am, y_*), \qquad \Lambda = (i\mu, y) \tag{2.2.9b}$$

where Λ is the general Einstein index. Note that we have distinguished the Einstein index y from the tangent space index y_*, both of which are associated to the coordinate y. The inverse vielbeins may be used to construct the left- and right-invariant affine Lie derivatives $\mathcal{E}$ and $\bar{\mathcal{E}}$,

$$\mathcal{E}_L = -ie_L{}^\Lambda\partial_\Lambda, \qquad \bar{\mathcal{E}}_L = -i\bar{e}_L{}^\Lambda\partial_\Lambda \tag{2.2.10a}$$

$$\mathcal{E}_L\hat{\gamma} = \hat{\gamma}\mathcal{J}_L, \qquad \bar{\mathcal{E}}_L\hat{\gamma} = -\mathcal{J}_L\hat{\gamma} \tag{2.2.10b}$$

$$\mathcal{E}_L = (\mathcal{E}_a(m), \mathcal{E}_{y_*}), \qquad \bar{\mathcal{E}}_L = (\bar{\mathcal{E}}_a(m), \bar{\mathcal{E}}_{y_*}). \tag{2.2.10c}$$

The relations in (2.2.10b) guarantee that the affine Lie derivatives $\mathcal{E}_L$ and $\bar{\mathcal{E}}_L$ satisfy two commuting affine algebras with central elements $\mathcal{E}_{y_*}$ and $\bar{\mathcal{E}}_{y_*}$.

From (2.2.6) and (2.2.9), one finds that the vielbeins satisfy the relations,

$$e_{i\mu} = -i\hat{g}^{-1}\partial_{i\mu}\hat{g} = e_{i\mu}{}^{am}J_a(m) + e_{i\mu}{}^{y_*}k \tag{2.2.11a}$$

$$\bar{e}_{i\mu} = -i\hat{g}\partial_{i\mu}\hat{g}^{-1} = \bar{e}_{i\mu}{}^{am}J_a(m) + \bar{e}_{i\mu}{}^{y_*}k \tag{2.2.11b}$$

$$e_y{}^L = -\bar{e}_y{}^L = \delta_y{}^L, \qquad e_{i\mu}{}^L, \bar{e}_{i\mu}{}^L \text{ are independent of } y \tag{2.2.11c}$$

$$e_{y_*}{}^\Lambda = -\bar{e}_{y_*}{}^\Lambda = \delta_{y_*}{}^\Lambda, \qquad e_{am}{}^\Lambda, \bar{e}_{am}{}^\Lambda \text{ are independent of } y \tag{2.2.11d}$$

$$e_{am}{}^{i\mu}e_{i\mu}{}^{bn} = \bar{e}_{am}{}^{i\mu}\bar{e}_{i\mu}{}^{bn} = \delta_{am}{}^{bn}, \qquad e_{i\mu}{}^{am}e_{am}{}^{j\nu} = \bar{e}_{i\mu}{}^{am}\bar{e}_{am}{}^{j\nu} = \delta_{i\mu}{}^{j\nu} \tag{2.2.11e}$$

$$e_{am}{}^y = -e_{am}{}^{i\mu}e_{i\mu}{}^{y_*}, \qquad \bar{e}_{am}{}^y = \bar{e}_{am}{}^{i\mu}\bar{e}_{i\mu}{}^{y_*} \tag{2.2.11f}$$

and (2.2.11c) also implies that $\bar{\mathcal{E}}_{y_*} = -\mathcal{E}_{y_*} = i\partial_{y_*}$.

The induced action of the affine Lie derivatives on the reduced group element $\hat{g}$ is described by the reduced affine Lie derivatives E and $\bar{E}$ [20],

$$E_a(m) = -ie_{am}{}^{i\mu}\mathcal{D}_{i\mu}, \qquad \bar{E}_a(m) = -i\bar{e}_{am}{}^{i\mu}\bar{\mathcal{D}}_{i\mu} \tag{2.2.12a}$$

$$\mathcal{D}_{i\mu} \equiv \partial_{i\mu} - ike_{i\mu}{}^{y_*}, \qquad \bar{\mathcal{D}}_{i\mu} \equiv \partial_{i\mu} + ik\bar{e}_{i\mu}{}^{y_*} \tag{2.2.12b}$$

$$E_a(m)\hat{g} = \hat{g}J_a(m), \qquad \bar{E}_a(m)\hat{g} = -J_a(m)\hat{g} \tag{2.2.12c}$$

where $\mathcal{D}$ and $\bar{\mathcal{D}}$ are called the covariant derivatives. It follows from (2.2.12c) that the reduced affine Lie derivatives satisfy two commuting copies of the affine algebra,

$$[E_a(m), E_b(n)] = if_{ab}{}^cE_c(m+n) + mk\eta_{ab}\delta_{m+n,0} \tag{2.2.13a}$$

$$[\bar{E}_a(m), \bar{E}_b(n)] = if_{ab}{}^c\bar{E}_c(m+n) - mk\eta_{ab}\delta_{m+n,0} \tag{2.2.13b}$$

$$[E_a(m), \bar{E}_b(n)] = 0 \tag{2.2.13c}$$

at level k and $-k$ respectively. Although we will not need to do so, one may obtain two commuting copies of the affine algebra at the same level k by defining $\bar{E}'_a(m) \equiv \bar{E}_a(-m)$.

The reduced affine Lie derivatives (2.2.12a) are a first-order differential representation of the currents of affine $g \times g$. Other first-order differential representations of affine Lie algebra are known, such as the coadjoint orbit representations in Refs. [31] and [32], but these provide only a single chiral copy of the algebra.

Although the construction above guarantees that the reduced affine Lie derivatives (2.2.12) satisfy the algebra (2.2.13) of affine $g \times g$, it is useful to have the machinery to check these relations directly.

We need in particular the Cartan-Maurer and inverse Cartan-Maurer identities for the left-invariant vielbein,

$$\partial_\Lambda e_\Gamma{}^L - \partial_\Gamma e_\Lambda{}^L = e_\Lambda{}^M e_\Gamma{}^N f_{MN}{}^L \tag{2.2.14a}$$

$$e_L{}^\Lambda \partial_\Lambda e_M{}^\Gamma - e_M{}^\Lambda \partial_\Lambda e_L{}^\Gamma = f_{ML}{}^N e_N{}^\Gamma \tag{2.2.14b}$$

which follow from (2.2.11a). The same relations with $e \to \bar{e}$ hold for the right-invariant vielbein.

These identities are sufficient to compute the curvature of the covariant derivatives in (2.2.12b),

$$[\mathcal{D}_{i\mu}, \mathcal{D}_{j\nu}] = k \sum_n n e_{i\mu}{}^{a,-n} \eta_{ab} e_{j\nu}{}^{bn}, \qquad [\bar{\mathcal{D}}_{i\mu}, \bar{\mathcal{D}}_{j\nu}] = -k \sum_n n \bar{e}_{i\mu}{}^{a,-n} \eta_{ab} \bar{e}_{j\nu}{}^{bn} \tag{2.2.15}$$

and also to check that the reduced affine Lie derivatives satisfy the affine Lie algebras (2.2.13a,b).

To check the commutator of E and $\bar{E}$, we first introduce the adjoint action $\hat{\Omega}$ of $\hat{g}$, which satisfies

$$\hat{g} \mathcal{J}_L \hat{g}^{-1} = \hat{\Omega}_L{}^M \mathcal{J}_M, \qquad \hat{\Omega}(x) = \hat{g}^{-1}(\hat{T}^{adj}, x), \qquad \hat{\Omega}_{y_*}{}^L = \delta_{y_*}{}^L \tag{2.2.16a}$$

$$e_{am}{}^{i\mu} \partial_{i\mu} \hat{\Omega}_L{}^M = -f_{am,L}{}^N \hat{\Omega}_N{}^M, \qquad \bar{e}_{am}{}^{i\mu} \partial_{i\mu} \hat{\Omega}_L{}^M = \hat{\Omega}_L{}^N f_{am,N}{}^M \tag{2.2.16b}$$

$$\hat{\Omega}_{am}{}^{cp} \hat{\eta}_{cp,dq} \hat{\Omega}_{bn}{}^{dq} = \hat{\eta}_{am,bn} \tag{2.2.16c}$$

where the matrix-valued group element $\hat{g}(\hat{T}^{adj}, x)$ is $\hat{g}(J, x)$ with $J \to \hat{T}^{adj}$ (see eq.(2.2.3)). The quantity $\hat{\eta}$ is the rescaled Killing metric (2.2.4), and the pseudo-orthogonality relation (2.2.16c) follows from the antisymmetry of $\hat{T}^{adj}$.

We also need the relations between the right- and left-invariant quantities,

$$\bar{e}_\Lambda{}^L = -e_\Lambda{}^M \hat{\Omega}_M{}^L, \qquad \bar{e}_L{}^\Lambda = -(\hat{\Omega}^{-1})_L{}^M e_M{}^\Lambda \tag{2.2.17a}$$

$$\bar{E}_a(m) = (\hat{\Omega}^{-1})_{am}{}^{bn} (\hat{\Omega}_{bn}{}^{y_*} k - E_b(n)) \tag{2.2.17b}$$

which follow from eqs.(2.2.16a) and (2.2.9). Using (2.2.17b), (2.2.13a), and (2.2.16b), it is straightforward to verify that $\bar{E}$ commutes with E.

2.3 The antisymmetric tensor field on the affine group

In this section, we find an antisymmetric tensor field $\hat{B}_{i\mu,j\nu} = -\hat{B}_{j\nu,i\mu}$ [21] on the affine Lie group $\hat{G}$, which, as we shall see, is analogous to the antisymmetric tensor field $B_{ij} = -B_{ji}$ of the conventional WZW model on G.

We begin by choosing a β-basis

$$\hat{g}(J,x) = \exp\left(i\beta^{am}(x)J_a(m)\right) \tag{2.3.1}$$

for the reduced affine group element $\hat{g}$. In any such basis, the vielbein has the explicit form,

$$e_{i\mu}{}^{L}(x) = \partial_{i\mu}\beta^{am}(x)M(x)_{am}{}^{L}, \qquad e_{y}{}^{L} = \delta_{y_*}{}^{L} \tag{2.3.2a}$$

$$M(x) \equiv \frac{1-\hat{\Omega}^{-1}}{\log\hat{\Omega}} \tag{2.3.2b}$$

where $\hat{\Omega}$ is the adjoint action in (2.2.16). This form generalizes the result given for the standard basis $\beta^{am} = x^{i\mu}e_{i\mu}{}^{am}(0)$ in Ref. [20]. We may eliminate β^{am} in (2.3.2) to find the relation between e^{y_*} and $\hat{\Omega}^{y_*}$,

$$e_{i\mu}{}^{y_*} = \frac{1}{2}\left(\hat{B}_{i\mu,j\nu}e_{a,-m}{}^{j\nu}\eta^{ab} - e_{i\mu}{}^{bm}\right)\hat{\Omega}_{bm}{}^{y_*} \tag{2.3.3a}$$

$$\hat{B}_{i\mu,j\nu} \equiv e_{i\mu}{}^{am}\mathcal{N}_{am}{}^{bn}\hat{\eta}_{bn,cp}e_{j\nu}{}^{cp} \tag{2.3.3b}$$

$$\mathcal{N}(x) \equiv \frac{(\hat{\Omega}^{-1}-\hat{\Omega}) + 2\log\hat{\Omega}}{(\hat{\Omega}-1)(\hat{\Omega}^{-1}-1)} \tag{2.3.3c}$$

where η^{ab} is the inverse Killing metric of Lie g and $\hat{B}_{i\mu,j\nu}$ is the desired tensor field on $\hat{G}$.

We know that $\hat{\Omega}$ in (2.2.16c) is pseudo-orthogonal, so the matrix $\mathcal{N}\hat{\eta}$ in (2.3.3b) is antisymmetric. It follows that $\hat{B}_{i\mu,j\nu}$ is antisymmetric,

$$\hat{B}_{i\mu,j\nu} = -\hat{B}_{j\nu,i\mu}. \tag{2.3.4}$$

A useful property of the antisymmetric tensor field is

$$\partial_{i\mu}(\hat{B}_{j\nu,k\rho}e_{a,-m}{}^{k\rho}\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}) - (i\mu \leftrightarrow j\nu) = e_{i\mu}{}^{bn}e_{j\nu}{}^{am}f_{am,bn}{}^{cp}\hat{\Omega}_{cp}{}^{y_*} \tag{2.3.5}$$

which follows from (2.3.3a) and the Cartan-Maurer identity (2.2.14a) for $e_{i\mu}{}^{y_*}$.

With (2.2.17a) and (2.3.3a), we may rewrite the reduced affine Lie derivatives (2.2.12) in the $\hat{B}$-form,

$$E_a(m) = -ie_{am}{}^{i\mu}\mathcal{D}_{i\mu}(\hat{B}) + \frac{1}{2}k\hat{\Omega}_{am}{}^{y_*}, \qquad \bar{E}_a(m) = -i\bar{e}_{am}{}^{i\mu}\mathcal{D}_{i\mu}(\hat{B}) - \frac{1}{2}k(\hat{\Omega}^{-1})_{am}{}^{y_*} \tag{2.3.6a}$$

$$\mathcal{D}_{i\mu}(\hat{B}) \equiv \partial_{i\mu} - \frac{i}{2}k\hat{B}_{i\mu,j\nu}e_{a,-m}{}^{j\nu}\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}. \tag{2.3.6b}$$

Using the identity (2.3.5) and the steps of the previous section, it is straightforward to check explicitly that the $\hat{B}$-form of the reduced affine Lie derivatives satisfies the algebra (2.2.13) of affine $g \times g$.

In Section 2.7, we will find that $\hat{B}_{i\mu,j\nu}$ satisfies other identities which show further analogy with the conventional WZW tensor field B_{ij} on G (see also Section 2.9). Further discussion of the operator currents (2.3.6) is given in Section 2.10.

2.4 Bracket representation of affine $g \times g$

To construct classical dynamics on the affine Lie group, we need the Poisson bracket representation which corresponds to the first-order differential representation (2.2.12) or (2.3.6) of affine $g \times g$. The bracket representation may be obtained by the usual prescription,

$$\partial_{i\mu} \to i p_{i\mu} \tag{2.4.1a}$$

$$\{x^{i\mu}, p_{j\nu}\} = i\delta_{j\nu}{}^{i\mu}, \qquad \{x^{i\mu}, x^{j\nu}\} = \{p_{i\mu}, p_{j\nu}\} = 0 \tag{2.4.1b}$$

where $\{A, B\}$ is Poisson bracket and $p_{i\mu}$ are (classical) canonical momenta.

Using the substitution (2.4.1a) in the reduced affine Lie derivatives (2.2.12), we find one form of the classical current modes

$$E_a(m) = e_{am}{}^{i\mu} p^-_{i\mu}, \qquad \bar{E}_a(m) = \bar{e}_{am}{}^{i\mu} p^+_{i\mu} \tag{2.4.2a}$$

$$p^-_{i\mu} \equiv p_{i\mu} - k e_{i\mu}{}^{y_*}, \qquad p^+_{i\mu} \equiv p_{i\mu} + k \bar{e}_{i\mu}{}^{y_*}. \tag{2.4.2b}$$

Similarly, the equivalent $\hat{B}$-form of the classical current modes,

$$E_a(m) = e_{am}{}^{i\mu} p_{i\mu}(\hat{B}) + \frac{1}{2} k \hat{\Omega}_{am}{}^{y_*}, \qquad \bar{E}_a(m) = \bar{e}_{am}{}^{i\mu} p_{i\mu}(\hat{B}) - \frac{1}{2} k (\hat{\Omega}^{-1})_{am}{}^{y_*} \tag{2.4.3a}$$

$$p_{i\mu}(\hat{B}) \equiv p_{i\mu} - \frac{1}{2} k \hat{B}_{i\mu, j\nu} e_{a,-m}{}^{j\nu} \eta^{ab} \hat{\Omega}_{bm}{}^{y_*} \tag{2.4.3b}$$

is obtained by the same substitution in the operator $\hat{B}$-form (2.3.6). Following the steps of Section 2.2, the bracket algebra of affine $g \times g$,

$$\{E_a(m), E_b(n)\} = i f_{ab}{}^c E_c(m+n) + m k \eta_{ab} \delta_{m+n,0} \tag{2.4.4a}$$

$$\{\bar{E}_a(m), \bar{E}_b(n)\} = i f_{ab}{}^c \bar{E}_c(m+n) - m k \eta_{ab} \delta_{m+n,0} \tag{2.4.4b}$$

$$\{E_a(m), \bar{E}_b(n)\} = 0 \tag{2.4.4c}$$

is easily verified for both forms of the classical currents.

In what follows, we interchangeably use the terms classical affine Lie derivatives or classical currents to refer to $E_a(m)$ and $\bar{E}_a(m)$ in (2.4.2,3).

Part II

Actions on the affine Lie group $\hat{G}$

2.5 WZW as a mechanical system on the affine group

In this section we construct a natural action for classical mechanics on the affine Lie group $\hat{G}$. As is clear from its construction, this mechanical action must be equivalent to the WZW action on the corresponding Lie group G. The equivalence is studied explicitly in Part III.

The operator WZW Hamiltonian $H = L_g^{ab} \,{}^*_*J_a J_b + \bar{J}_a \bar{J}_b \,{}^*_{*0}$ sums the zero modes of left- and right-mover affine-Sugawara constructions. When written in terms of the reduced affine Lie derivatives (2.2.12), the coordinate-space form $H = L_g^{ab} \,{}^*_*E_a E_b + \bar{E}_a \bar{E}_b \,{}^*_{*0}$ of this Hamiltonian is a natural Laplacian [20] on the affine Lie group.

To construct the corresponding action on the affine Lie group, we begin with the standard classical WZW Hamiltonian written in terms of the classical affine Lie derivatives (2.4.2,3),

$$H = \frac{1}{2k}\hat{\eta}^{am,bn}\left(E_a(m)E_b(n) + \bar{E}_a(m)\bar{E}_b(n)\right) \tag{2.5.1a}$$

$$= \frac{1}{2k}\eta^{ab}\sum_m \left(E_a(-m)E_b(m) + \bar{E}_a(-m)\bar{E}_b(m)\right) \tag{2.5.1b}$$

$$\hat{\eta}^{am,bn} \equiv \eta^{ab}\delta_{m+n,0} \tag{2.5.1c}$$

$$\partial_\tau A = i\{H, A\} \tag{2.5.1d}$$

where $\hat{\eta}^{am,bn}$ in (2.5.1c) is the inverse of the rescaled Killing metric $\hat{\eta}_{am,bn}$ in (2.2.4). As seen in (2.5.1a), this Hamiltonian is the natural generalization to affine $g \times g$ of the Casimir operator on Lie g.

The time dependence of the classical currents

$$\partial_\tau E_a(m,\tau) = -imE_a(m,\tau), \qquad \partial_\tau \bar{E}_a(m,\tau) = im\bar{E}_a(m,\tau) \tag{2.5.2a}$$

$$E_a(m,\tau) = e^{-im\tau}E_a(m), \qquad \bar{E}_a(m,\tau) = e^{im\tau}\bar{E}_a(m) \tag{2.5.2b}$$

follows immediately from (2.5.1d) and the current algebra (2.4.4).

Using the explicit form (2.4.2) of the classical affine Lie derivatives, we obtain an explicit form of the Hamiltonian

$$H = \frac{1}{k}\eta^{ab} e_{a,-m}{}^{i\mu}(x) e_{bm}{}^{j\nu}(x) p^-_{i\mu} p^-_{j\nu} + \left(\frac{k}{2}\hat{\Omega}_{a,-m}{}^{y_*}(x) - e_{a,-m}{}^{i\mu}(x) p^-_{i\mu}\right)\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}(x) \tag{2.5.3}$$

where $p^-_{i\mu}$ is defined in (2.4.2b). This Hamiltonian describes a classical mechanics whose coordinates x are coordinates on the affine Lie group. The equivalent $\hat{B}$-form of the Hamiltonian is

$$H = \frac{1}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p_{i\mu}(\hat{B}) p_{j\nu}(\hat{B}) + \frac{k}{4}\eta^{ab}\hat{\Omega}_{a,-m}{}^{y_*}\hat{\Omega}_{b,m}{}^{y_*} \tag{2.5.4}$$

where we have suppressed the x-dependence of all quantities, and $p_{i\mu}(\hat{B})$ is defined in (2.4.3b). The Hamiltonian equations of motion

$$\partial_\tau x^{i\mu} = \frac{2}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p^-_{j\nu} - \eta^{ab} e_{a,-m}{}^{i\mu}\hat{\Omega}_{bm}{}^{y_*} \tag{2.5.5a}$$

$$= \frac{2}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p_{j\nu}(\hat{B}) \tag{2.5.5b}$$

$$\partial_\tau p_{i\mu} = -\partial_{i\mu} H \tag{2.5.5c}$$

follow from eqs.(2.5.3,4).

Coordinate space

We turn now to the coordinate-space formulation of the mechanical theory on $\hat{G}$. To begin, one uses eq.(2.4.2) or (2.4.3) and the equations of motion (2.5.5) to obtain

$$E_a(m) = \frac{k}{2}(\eta_{ab} e_{i\mu}{}^{b,-m}\partial_\tau x^{i\mu} + \hat{\Omega}_{am}{}^{y_*}), \qquad \bar{E}_a(m) = \frac{k}{2}(\eta_{ab}\bar{e}_{i\mu}{}^{b,-m}\partial_\tau x^{i\mu} - (\hat{\Omega}^{-1})_{am}{}^{y_*}) \tag{2.5.6}$$

for the coordinate-space form of the currents.

Similarly, the mechanical action on the affine Lie group [21]

$$S_M = \int d\tau L_M, \qquad L_M = \partial_\tau x^{i\mu} p_{i\mu} - H \tag{2.5.7a}$$

$$L_M = \frac{k}{4}\eta_{ab} e_{i\mu}{}^{a,-m} e_{j\nu}{}^{bm}\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \frac{k}{4}\eta^{ab}\hat{\Omega}_{a,-m}{}^{y_*}\hat{\Omega}_{bm}{}^{y_*} + k\partial_\tau x^{i\mu}\left(e_{i\mu}{}^{y_*} + \frac{1}{2}e_{i\mu}{}^{am}\hat{\Omega}_{am}{}^{y_*}\right) \tag{2.5.7b}$$

is obtained from the Hamiltonian (2.5.3) in the usual way. The equivalent $\hat{B}$-form of the Lagrangian is [21]

$$L_M = \frac{k}{4}\eta_{ab} e_{i\mu}{}^{a,-m} e_{j\nu}{}^{bm}\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \frac{k}{4}\left(\hat{\Omega}_{am}{}^{y_*} + 2\hat{B}_{i\mu,j\nu}\partial_\tau x^{j\nu} e_{am}{}^{i\mu}\right)\eta^{ab}\hat{\Omega}_{b,-m}{}^{y_*}. \tag{2.5.8}$$

It is straightforward to check that the associated Lagrange equations of motion of either form of the action reproduce the Hamiltonian equations of motion (2.5.5). A third form of the mechanical action, in terms of the reduced affine group element $\hat{g}$, is given in Section 2.11.2.

This action is one of the central results of this chapter. Although the classical mechanics (2.5.7) or (2.5.8) on $\hat{G}$ shows no spatial coordinate σ, this formulation must be equivalent to the conventional WZW model on G because the Hamiltonians of the two formulations are isomorphic. This equivalence is studied in detail in Part III, where we will see that this action is in fact a mode formulation of WZW.

2.6 WZW as a field theory on the affine group

In the last section, we expressed the WZW model as a mechanical system on the affine Lie group,

$$\hat{g}(x(\tau)): \quad \mathbb{R} \mapsto \hat{G}. \tag{2.6.1}$$

In this section, we show that the model can also be expressed as a two-dimensional field theory

$$\hat{g}(x(\tau,\sigma)): \quad \mathbb{R} \times S^1 \mapsto \hat{G} \tag{2.6.2}$$

on the affine Lie group.

Mathematically, our task is to extend the base space from the line (τ) to the cylinder (τ,σ), thereby promoting the mechanical variables $x^{i\mu}(\tau)$ to σ-dependent fields $x^{i\mu}(\tau,\sigma)$. The reason that we can make such an equivalent field-theoretic formulation is that the Hamiltonian of the system

$$H = \frac{1}{2k}\eta^{ab}\sum_m \left(E_a(-m)E_b(m) + \bar{E}_a(-m)\bar{E}_b(m)\right) \tag{2.6.3a}$$

$$= \frac{1}{k}\eta^{ab}e_{a,-m}{}^{i\mu}e_{bm}{}^{j\nu}p_{i\mu}(\hat{B})p_{j\nu}(\hat{B}) + \frac{k}{4}\eta^{ab}\hat{\Omega}_{a,-m}{}^{y_*}\hat{\Omega}_{b,m}{}^{y_*} \tag{2.6.3b}$$

$$\partial_\tau A = i\{H, A\} \tag{2.6.3c}$$

admits a commuting quantity P

$$P = \frac{1}{2k}\eta^{ab}\sum_m \left(E_a(-m)E_b(m) - \bar{E}_a(-m)\bar{E}_b(m)\right) \tag{2.6.4a}$$

$$= \hat{\Omega}_{a,-m}{}^{y_*}\eta^{ab}e_{bm}{}^{i\mu}p_{i\mu} \tag{2.6.4b}$$

$$\{H, P\} = 0 \tag{2.6.4c}$$

$$\partial_\sigma A \equiv i\{P, A\} \tag{2.6.4d}$$

which we may interpret, according to (2.6.4d), as the generator of spatial translations. This form of P is the standard WZW momentum, written in terms of the classical affine Lie derivatives (2.4.3), and eq.(2.6.4c) follows immediately from the current algebra (2.4.4).

The spacetime dependence of any observable A is determined by this system, and, in particular, the relations

$$\partial_\sigma H = \partial_\tau P = \partial_\tau H = \partial_\sigma P = 0 \tag{2.6.5}$$

follow from eq.(2.6.4c).

In this framework, all fields are now σ-dependent, and the bracket relations of the earlier sections, e.g.

$$\{x^{i\mu}(\sigma), p_{j\nu}(\sigma)\} = i\delta_{j\nu}{}^{i\mu}, \qquad \{x^{i\mu}(\sigma), x^{j\nu}(\sigma)\} = \{p_{i\mu}(\sigma), p_{j\nu}(\sigma)\} = 0 \tag{2.6.6a}$$

$$\{E_a(m,\sigma), E_b(n,\sigma)\} = if_{ab}{}^c E_c(m+n,\sigma) + mk\eta_{ab}\delta_{m+n,0} \tag{2.6.6b}$$

should be read at equal σ. Moreover, all field products should be read at equal σ, for example

$$P = \frac{1}{2k}\eta^{ab}\sum_m \Big(E_a(-m,\sigma)E_b(m,\sigma) - \bar{E}_a(-m,\sigma)\bar{E}_b(m,\sigma)\Big) \tag{2.6.7a}$$

$$= \hat{\Omega}_{a,-m}{}^{y_*}(x(\sigma))\eta^{ab}e_{bm}{}^{i\mu}(x(\sigma))p_{i\mu}(\sigma) \tag{2.6.7b}$$

and similarly for the Hamiltonian.

The spatial dependence of the affine Lie derivatives follows immediately from the current algebra,

$$\partial_\sigma E_a(m,\tau,\sigma) = -imE_a(m,\tau,\sigma), \qquad \partial_\sigma \bar{E}_a(m,\tau,\sigma) = -im\bar{E}_a(m,\tau,\sigma) \tag{2.6.8a}$$

$$E_a(m,\tau,\sigma) = e^{-im\sigma}E_a(m,\tau), \qquad \bar{E}_a(m,\tau,\sigma) = e^{-im\sigma}\bar{E}_a(m,\tau). \tag{2.6.8b}$$

Combining this with the known time dependence (2.5.2), we find that the affine Lie derivatives are chiral,

$$E_a(m,\tau,\sigma) = e^{-im(\tau+\sigma)}E_a(m), \qquad \bar{E}_a(m,\tau,\sigma) = e^{im(\tau-\sigma)}\bar{E}_a(m). \tag{2.6.9}$$

The usual local chiral currents are then identified as

$$E_a(\tau,\sigma) \equiv \sum_m E_a(m,\tau,\sigma) = \sum_m e^{-im(\tau+\sigma)}E_a(m) \tag{2.6.10a}$$

$$\bar{E}_a(\tau,\sigma) \equiv \sum_m \bar{E}_a(m,\tau,\sigma) = \sum_m e^{im(\tau-\sigma)}\bar{E}_a(m) \tag{2.6.10b}$$

and the usual local current algebra

$$\{E_a(\tau,\sigma), E_b(\tau,\sigma')\} = 2\pi i[f_{ab}{}^c E_c(\tau,\sigma)\delta(\sigma-\sigma') + k\eta_{ab}\partial_\sigma\delta(\sigma-\sigma')] \tag{2.6.11a}$$

$$\{\bar{E}_a(\tau,\sigma), \bar{E}_b(\tau,\sigma')\} = 2\pi i[f_{ab}{}^c \bar{E}_c(\tau,\sigma)\delta(\sigma-\sigma') - k\eta_{ab}\partial_\sigma\delta(\sigma-\sigma')] \tag{2.6.11b}$$

$$\{E_a(\tau,\sigma), \bar{E}_b(\tau,\sigma')\} = 0 \tag{2.6.11c}$$

follows from the mode algebra (2.4.4). Further discussion of these local currents is given in Section 2.10.

The spacetime derivatives of the canonical variables,

$$\partial_\tau x^{i\mu} = \frac{2}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p_{j\nu}(\hat{B}) \tag{2.6.12a}$$

$$\partial_\sigma x^{i\mu} = \eta^{ab} e_{a,-m}{}^{i\mu} \hat{\Omega}_{bm}{}^{y_*} \tag{2.6.12b}$$

$$\partial_\tau p_{i\mu} = -\partial_{i\mu} H \tag{2.6.12c}$$

$$\partial_\sigma p_{i\mu} = -p_{j\nu}\partial_{i\mu}(e_{a,-m}{}^{j\nu}\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}) \tag{2.6.12d}$$

also follow from the equal-σ brackets (2.6.6a) and the explicit forms of H and P.

Spatial constraints

The procedure described above to extend the base space via commuting P operators is quite general. One may guarantee that the extended theory (on the extended base space) is equivalent to the theory on the unextended base space by considering the spatial derivative relations (such as (2.6.12b,d)) to be constraints on the canonical variables. In a functional formulation, this prescription corresponds to the functional identity

$$\int \left(\prod_\tau dx(\tau)\right) F[x(\tau)] = \int \left(\prod_{\tau,\sigma} dx(\tau,\sigma)\right) \det\left(\frac{\delta\mathcal{C}}{\delta x}\right) F[x(\tau,\sigma)]\delta[\mathcal{C}(x(\tau,\sigma))] \tag{2.6.13a}$$

$$\mathcal{C} \equiv \partial_\sigma x(\tau,\sigma) - i\{P, x(\tau,\sigma)\} \tag{2.6.13b}$$

and similarly for the canonical momenta. The functional delta function in (2.6.13a) enforces the spatial constraints in the extended theory, and it follows that the spatial constraints can be implemented as usual in a Dirac formulation of the theory.

In the case at hand, this prescription guarantees equivalence of the field-theoretic formulation on $\hat{G}$ with the mechanical formulation (2.5.8) on $\hat{G}$, and hence with the conventional WZW model itself.

To be more explicit about the equivalence with the mechanical formulation, we first rewrite the constraint (2.6.12b) in terms of the tangent-space fields β^{am},

$$\partial_\sigma \beta^{am} - im\beta^{am} = (\partial_\sigma x^{i\mu} - \eta^{bc} e_{b,-n}{}^{i\mu} \hat{\Omega}_{cn}{}^{y_*}) \partial_{i\mu} \beta^{am} \tag{2.6.14a}$$

$$= 0 \tag{2.6.14b}$$

$$\beta^{am}(x(\tau,\sigma)) = e^{im\sigma} \beta^{am}(x(\tau)) \tag{2.6.14c}$$

whose simple σ-dependence is given in (2.6.14c). The identity (2.6.14a) is obtained by using eq.(2.6.12b) and the explicit forms of the vielbein and the adjoint action given in Appendix B. It follows by chain rule from (2.6.14a) that the measure factor in (2.6.13a) is effectively constant

$$\det\left(\frac{\delta \mathcal{C}}{\delta x}\right) = \text{field-independent} \tag{2.6.15}$$

and can be ignored. It also follows from (2.6.14c) that averages $\langle\ \rangle$ in the mechanical theory ($\hat{G}_M$) and the field theory ($\hat{G}_F$) on $\hat{G}$ are simply related, so long as the formal functional measures (see Section 2.12) of $\hat{G}_M$ and $\hat{G}_F$ are suitably adjusted. Specifically, one has [21]

$$\langle \mathcal{F}[\beta^{am}(x(\tau,\sigma))] \rangle_{\hat{G}_F} = \langle \mathcal{F}[\beta^{am}(x(\tau)) e^{im\sigma}] \rangle_{\hat{G}_M} \tag{2.6.16}$$

for any function $\mathcal{F}$, which includes averages over products of the affine group elements.

There is another formulation of the theory, on the constrained subspace, in which the spatial derivative relations are identities (as in conventional formulations). As a first step in this formulation, use the constraint (2.6.12b) to reexpress the Hamiltonian and the momentum [21]

$$H = \frac{1}{k} \eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p_{i\mu}(\hat{B}) p_{j\nu}(\hat{B}) + \frac{k}{4} \eta_{ab} e_{i\mu}{}^{a,-m} e_{j\nu}{}^{bm} \partial_\sigma x^{i\mu} \partial_\sigma x^{j\nu} \tag{2.6.17a}$$

$$P = p_{i\mu} \partial_\sigma x^{i\mu} \tag{2.6.17b}$$

$$p_{i\mu}(\hat{B}) = p_{i\mu} - \frac{k}{2} \hat{B}_{i\mu,j\nu} \partial_\sigma x^{j\nu} \tag{2.6.17c}$$

in terms of $\partial_\sigma x$. This form of the system is complete with the canonical brackets (2.6.6a) and the constraints (2.6.12b,d), or with the canonical brackets and an auxiliary set of equal-σ brackets which must be computed from the constraints. As examples of the auxiliary set, we have

$$\{x^{i\mu}(\sigma), \partial_\sigma x^{j\nu}(\sigma)\} = \{x^{i\mu}(\sigma), \eta^{ab} e_{a,-m}{}^{i\mu}(x(\sigma)) \hat{\Omega}_{bm}{}^{y_*}(x(\sigma))\} = 0 \tag{2.6.18a}$$

$$\begin{aligned}\{p_{i\mu}(\sigma), \partial_\sigma x^{j\nu}(\sigma)\} &= \{p_{i\mu}(\sigma), \eta^{ab} e_{a,-m}{}^{j\nu}(x(\sigma))\hat{\Omega}_{bm}{}^{y_*}(x(\sigma))\} \\ &= -i\eta^{ab}\partial_{i\mu}(e_{a,-m}{}^{j\nu}(x(\sigma))\hat{\Omega}_{bm}{}^{y_*}(x(\sigma))) \end{aligned} \tag{2.6.18b}$$

$$\begin{aligned}\{p_{i\mu}(\sigma), \partial_\sigma p_{j\nu}(\sigma)\} &= -\{p_{i\mu}(\sigma), p_{k\rho}(\sigma)\partial_{j\nu}(e_{a,-m}{}^{k\rho}(x(\sigma))\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}(x(\sigma)))\} \\ &= ip_{k\rho}(\sigma)\partial_{i\mu}\partial_{j\nu}(e_{a,-m}{}^{k\rho}(x(\sigma))\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}(x(\sigma))). \end{aligned} \tag{2.6.18c}$$

Using these auxiliary brackets, one now obtains the conventional identity

$$\partial_\sigma x^{i\mu} \equiv i\{P, x^{i\mu}\} = i\partial_\sigma x^{j\nu}\{p_{j\nu}, x^{i\mu}\} = \partial_\sigma x^{i\mu} \tag{2.6.19}$$

as expected on the constrained subspace.

Density formulation

The Hamiltonian systems above are unconventional formulations of a field theory because they are not written in terms of spatial densities. Since H and P are independent of σ, however, we can define the densities as proportional to the Hamiltonian and momentum,

$$\mathcal{H} \equiv \frac{H}{2\pi}, \qquad H = \int d\sigma \mathcal{H} \tag{2.6.20a}$$

$$\mathcal{P} \equiv \frac{P}{2\pi}, \qquad P = \int d\sigma \mathcal{P}. \tag{2.6.20b}$$

These σ-independent densities may be used for either dynamical system (2.6.3,4) or (2.6.17), but the constraints (or the auxiliary brackets) must be included in the latter case. Because the densities are σ-independent, one has the bracket equations of motion

$$\begin{aligned}\partial_\tau A(\sigma) &= i\{H, A(\sigma)\} = \int d\sigma'\{\mathcal{H}(\sigma'), A(\sigma)\} = i\int d\sigma'\{\mathcal{H}(\sigma), A(\sigma)\} \\ &= 2\pi i\{\mathcal{H}(\sigma), A(\sigma)\} \end{aligned} \tag{2.6.21}$$

and similarly for $\partial_\sigma A = i\{P, A\}$. Any particular bracket equation of motion can then be computed in either formulation using only equal-σ brackets, and these results agree with (2.6.12).

Coordinate space

We turn now to the coordinate-space formulation of the theory. As a first step, we use the phase-space currents (2.4.3) and eqs.(2.6.12a,b) in the form

$$p_{i\mu}(\hat{B}) = \frac{k}{2}\eta_{ab} e_{i\mu}{}^{a,-m} e_{j\nu}{}^{bn}\partial_\tau x^{j\nu}, \qquad \hat{\Omega}_{am}{}^{y_*} = \eta_{ab} e_{i\mu}{}^{b,-m}\partial_\sigma x^{i\mu} \tag{2.6.22}$$

to obtain the simple coordinate-space form of the chiral current modes,

$$E_a(m,\tau,\sigma) = k\eta_{ab}e_{i\mu}{}^{b,-m}(x(\tau,\sigma))\partial x^{i\mu}(\tau,\sigma) \tag{2.6.23a}$$

$$\bar{E}_a(m,\tau,\sigma)) = k\eta_{ab}\bar{e}_{i\mu}{}^{b,-m}(x(\tau,\sigma))\bar{\partial} x^{i\mu}(\tau,\sigma) \tag{2.6.23b}$$

$$\bar{\partial}E_a(m,\tau,\sigma) = \partial\bar{E}_a(m,\tau,\sigma) = 0 \tag{2.6.23c}$$

$$\partial = \frac{1}{2}(\partial_\tau + \partial_\sigma), \qquad \bar{\partial} = \frac{1}{2}(\partial_\tau - \partial_\sigma). \tag{2.6.23d}$$

This form of the current modes bears a strong resemblance to the usual coordinate-space currents $k\eta_{ab}e_i{}^b\partial x^i$, $k\eta_{ab}\bar{e}_i{}^b\bar{\partial}x^i$ of the conventional WZW model on G.

Following the usual canonical density formulation, we also obtain the action of the two-dimensional field theory on $\hat{G}$ [21],

$$S_{\hat{F}} = \int d\tau d\sigma \mathcal{L}_{\hat{F}}, \qquad \mathcal{L}_{\hat{F}} = p_{i\mu}\partial_\tau x^{i\mu} - \mathcal{H}(\lambda) \tag{2.6.24a}$$

$$\begin{aligned}\mathcal{L}_{\hat{F}} &= \frac{k}{8\pi}\eta_{ab}e_{i\mu}{}^{a,-m}e_{j\nu}{}^{bm}(\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \partial_\sigma x^{i\mu}\partial_\sigma x^{j\nu}) + \frac{k}{4\pi}\hat{B}_{i\mu,j\nu}\partial_\tau x^{i\mu}\partial_\sigma x^{j\nu} \\ &\quad + k\lambda_{i\mu}(\partial_\sigma x^{i\mu} - \eta^{ab}e_{a,-m}{}^{i\mu}\hat{\Omega}_{bm}{}^{y_*})\end{aligned} \tag{2.6.24b}$$

where $\mathcal{H}(\lambda)$ and $\mathcal{L}_{\hat{F}}$ include the constraint (2.6.12b) with a Lagrange multiplier $\lambda_{i\mu}(\tau,\sigma)$. Sections 2.7.2 and 2.12 give alternate forms of this action in terms of the affine group element $\hat{g}$.

It is straightforward to check that the Lagrange equations of motion of this system are equivalent to the Hamiltonian equations (2.6.12a,c) and the $\partial_\sigma x$ constraint (2.6.12b). One does not need to include the $\partial_\sigma p$ constraint (2.6.12d) explicitly in the action formulation; it is implied by the $\partial_\sigma x$ constraint, the $\partial_\tau x$ equation of motion, and the fact $\partial_\tau\partial_\sigma x = \partial_\sigma\partial_\tau x$.

On the constrained subspace it is also straightforward to show that $\partial_\sigma\mathcal{L}_{\hat{F}} = 0$, so the Lagrangian is proportional to its density $L_{\hat{F}} = \int d\sigma\mathcal{L}_{\hat{F}} = 2\pi\mathcal{L}_{\hat{F}}$, in parallel with the Hamiltonian. One can then check backwards that

$$L_{\hat{F}} = L_M, \qquad S_{\hat{F}} = S_M \tag{2.6.25}$$

on the constrained subspace, where L_M is the mechanical Lagrangian on $\hat{G}$ in (2.5.8).

The action (2.6.24) is another central result of this chapter. We remark that it bears a strong resemblance to the sigma model form (2.1.7) of the conventional WZW action, except that this action involves fields on the affine Lie group, and there is an additional term to enforce the constraint. In the following section we discuss rewriting this action as a function of the affine group element $\hat{g} \in \hat{G}$, in analogy to the $g \in G$ formulation (2.1.9) of the conventional WZW action.

2.7 Trace formulation on the affine group

2.7.1 Rescaled Killing metric and rescaled traces

We begin by investigating the Killing metric on the affine group $\hat{G}$. Recall first the definition of the Killing metric η_{ab} on Lie G,

$$-f_{ac}{}^{d} f_{bd}{}^{c} = \mathrm{Tr}(T_a^{adj} T_b^{adj}) = Q_\psi \eta_{ab}, \qquad Q_\psi = \psi^2 \tilde{h} \tag{2.7.1}$$

where $f_{ab}{}^{c}$, $\tilde{h}$ and ψ are respectively the structure constants, the dual Coxeter number and the highest root of Lie g.

In the same way, the formal Killing metric η_{MN} on $\hat{G}$ is defined by the relation

$$-f_{ML}{}^{P} f_{NP}{}^{L} = \mathrm{Tr}(\hat{T}_M^{adj} \hat{T}_N^{adj}) = \hat{Q} \eta_{MN} \tag{2.7.2}$$

where $f_{LM}{}^{N}$ are the affine structure constants and $\hat{T}^{adj}$ is the adjoint matrix representation of the affine algebra in (2.2.3). In (2.7.2), the formal trace Tr is in fact a sum over the reduced carrier space am because $(\hat{T}_M^{adj})_{y_*}{}^{N} = 0$. By explicit computation, one finds that

$$\mathrm{Tr}(\hat{T}_{y_*}^{adj} \hat{T}_L^{adj}) = \mathrm{Tr}(\hat{T}_L^{adj} \hat{T}_{y_*}^{adj}) = 0 \tag{2.7.3a}$$

$$\mathrm{Tr}(\hat{T}_{a_1 m_1}^{adj} \hat{T}_{a_2 m_2}^{adj}) = \hat{\mathrm{Tr}}(\hat{T}_{a_1 m_1}^{adj} \hat{T}_{a_2 m_2}^{adj}) \left(\sum_{m \in \mathbb{Z}} \right) \tag{2.7.3b}$$

$$\hat{\mathrm{Tr}}(\hat{T}_{am}^{adj} \hat{T}_{bn}^{adj}) \equiv \delta_{m+n,0} \mathrm{Tr}(T_a^{adj} T_b^{adj}) = Q_\psi \eta_{ab} \delta_{m+n,0} \tag{2.7.3c}$$

so the formal traces and the product $\hat{Q}\eta$ have a divergent factor, which is the sum over modes in (2.7.3b). The rescaled trace $\hat{\mathrm{Tr}}$ in (2.7.3c) and the corresponding rescaled Killing metric $\hat{\eta}_{LM}$

$$\hat{\mathrm{Tr}}(\hat{T}_L^{adj} \hat{T}_M^{adj}) = Q_\psi \hat{\eta}_{LM} \tag{2.7.4a}$$

$$\hat{\eta}_{am,bn} = \eta_{ab} \delta_{m+n,0}, \qquad \hat{\eta}_{y_* L} = \hat{\eta}_{L y_*} = 0 \tag{2.7.4b}$$

are finite, however, and the rescaled Killing metric, introduced in Section 2.2, has been used many times in the development of the previous sections.

In fact, the rescaled trace and rescaled Killing metric continue to be sufficient for our purposes. We recall from Section 2.2 that the rescaled Killing metric can be used to lower indices and, in particular, one finds the completely antisymmetric structure constants,

$$f_{am,bn,cp} = f_{am,bn}{}^{dq} \hat{\eta}_{dq,cp} = f_{abc} \delta_{m+n+p,0} \tag{2.7.5}$$

where the structure constants $f_{am,bn}{}^{dq}$ are given in (2.2.2c). Because it vanishes on the y_* subspace, the rescaled metric is not invertible on the full space (am, y_*). However, an inverse exists on the am subspace,

$$\hat{\eta}^{am,bn} = \eta^{ab}\delta_{m+n,0} \tag{2.7.6a}$$

$$H = \frac{1}{2k}\hat{\eta}^{am,bn}(E_a(m)E_b(n) + \bar{E}_a(m)\bar{E}_b(n)) \tag{2.7.6b}$$

and, following its natural occurrence in the WZW Hamiltonian (2.7.6b), this inverse has been used many times in the development of the previous sections.

The divergent factor in (2.7.3b) is not unique to the trace of the adjoint representation $\hat{T}^{adj}$. Indeed, the same factor will recur in the traces over any other matrix representation which is faithful to the mode structure of the affine algebra.

A large class of such representations can easily be constructed. For each matrix irrep $(T_a)_I{}^J, I, J = 1 \ldots \dim T$ of Lie g,

$$[T_a, T_b] = if_{ab}{}^c T_c, \qquad \mathrm{Tr}(T_a T_b) = \chi(T)\eta_{ab} \tag{2.7.7}$$

one has the corresponding matrix representation $\hat{T}(T)$ of affine g,

$$(\hat{T}_{am}(T))_{In}{}^{Jp} \equiv \delta_{m+n,p}(T_a)_I{}^J, \qquad (\hat{T}_{y_*}(T))_{In}{}^{Jp} \equiv 0 \tag{2.7.8a}$$

$$[\hat{T}_L(T), \hat{T}_M(T)] = if_{LM}{}^N \hat{T}_N(T) \tag{2.7.8b}$$

where the carrier space of $\hat{T}(T)$ is (Im). When one takes a naive trace of a product of N of these matrices, one finds

$$\mathrm{Tr}(\hat{T}_{a_1 m_1}(T)\cdots\hat{T}_{a_N m_N}(T)) = \hat{\mathrm{Tr}}(\hat{T}_{a_1 m_1}(T)\cdots\hat{T}_{a_N m_N}(T))\left(\sum_{m\in\mathbb{Z}}\right) \tag{2.7.9a}$$

$$\begin{aligned}\hat{\mathrm{Tr}}(\hat{T}_{a_1 m_1}(T)\cdots\hat{T}_{a_N m_N}(T)) &\equiv \sum_I (\hat{T}_{a_1 m_1}(T)\cdots\hat{T}_{a_N m_N})_{Im}{}^{Im}, \qquad \forall m \in \mathbb{Z} \\ &= \delta_{m_1+\ldots+m_N,0}\mathrm{Tr}(T_{a_1}\cdots T_{a_N}) \end{aligned} \tag{2.7.9b}$$

$$\hat{\mathrm{Tr}}(\hat{T}_{am}(T)\hat{T}_{bn}(T)) = \chi(T)\hat{\eta}_{am,bn} \tag{2.7.9c}$$

in parallel to eq.(2.7.4). Again, the rescaled traces in (2.7.9b) are finite and sufficient for our purposes. We will also need the more general relation

$$\hat{\mathrm{Tr}}(\mathcal{F}(\hat{T}(T)) = \sum_I (\mathcal{F}(\hat{T}(T))_{Im}{}^{Im}, \qquad \forall m \in \mathbb{Z} \tag{2.7.10}$$

which holds for any matrix-valued power series $\mathcal{F}$.

Note that we now have two affine representations, $\hat{T}^{adj}$ and $\hat{T}(T^{adj})$, corresponding to the adjoint representation of Lie g. These two representations are closely related

$$(\hat{T}^{adj}_{am})_{bn}{}^{cp} = (\hat{T}_{am}(T^{adj}))_{bn}{}^{cp} = -if_{am,bn}{}^{cp} \tag{2.7.11a}$$

$$(\hat{T}^{adj}_{am})_{bn}{}^{y_*} = -m\hat{\eta}_{am,bn} \tag{2.7.11b}$$

although $\hat{T}^{adj}$ has the extra dimension y_* in its carrier space. The traces of the two representations are however the same

$$\hat{\mathrm{Tr}}(\hat{T}^{adj}_{a_1m_1}\cdots\hat{T}^{adj}_{a_Nm_N}) = \hat{\mathrm{Tr}}(\hat{T}_{a_1m_1}(T^{adj})\cdots\hat{T}_{a_Nm_N}(T^{adj})) \tag{2.7.12a}$$

$$\hat{\mathrm{Tr}}(\hat{T}^{adj}_{am}\hat{T}^{adj}_{bn}) = \hat{\mathrm{Tr}}(\hat{T}_{am}(T^{adj})\hat{T}_{bn}(T^{adj})) = \chi(T^{adj})\hat{\eta}_{am,bn}, \quad \chi(T^{adj}) = Q_\psi \tag{2.7.12b}$$

because $(T^{adj}_M)_{y_*}{}^N = 0$. In what follows, we use the unified notation $\hat{T}$ to denote any one of the representations $\hat{T}(T)$ or $\hat{T}^{adj}$.

In matrix representation $\hat{T}$, the group element $\hat{\gamma} \in \hat{G}$ equals the reduced group element $\hat{g}$,

$$\hat{\gamma}(\hat{T}, x, y) = \hat{g}(\hat{T}, x) = \exp(\beta^{am}(x)\hat{T}_{am}) \tag{2.7.13}$$

because $\hat{T}_{y_*} = 0$ replaces the level in these representations. Then one obtains $\hat{g}(\hat{T})$ analogues of the basic relations which we obtained for $\hat{g}(J)$ in Section 2.2. For example, one has the $\hat{g}(\hat{T})$ analogue of eqs.(2.2.11a,b),

$$e_{i\mu}(\hat{T}) = -i\hat{g}^{-1}(\hat{T})\partial_{i\mu}\hat{g}(\hat{T}) = e_{i\mu}{}^{am}\hat{T}_{am} \tag{2.7.14a}$$

$$\bar{e}_{i\mu}(\hat{T}) = -i\hat{g}(\hat{T})\partial_{i\mu}\hat{g}^{-1}(\hat{T}) = \bar{e}_{i\mu}{}^{am}\hat{T}_{am} \tag{2.7.14b}$$

where the vielbeins $e_{i\mu}{}^{am}(x), \bar{e}_{i\mu}{}^{am}(x)$, being representation independent, are the same as those above.

Many, but not all, of these relations can be obtained, as in (2.7.14), by the map $J \to \hat{T}$ and $k \to 0$. An important exception involves the operator form (2.2.12) of the reduced affine Lie derivatives, which satisfy

$$E_a(m)\hat{g}(\hat{T}) = \hat{g}(\hat{T})(\hat{T}_{am} + ke_{am}{}^y), \qquad \bar{E}_a(m)\hat{g}(\hat{T}) = -(\hat{T}_{am} - k\bar{e}_{am}{}^y)\hat{g}(\hat{T}). \tag{2.7.15}$$

This differs in form from the relation (2.2.12c) because the reduced affine Lie derivatives are independent of representation and explicitly dependent on the level.

2.7.2 WZW in terms of $\hat{g}(x(\tau,\sigma)) \in \hat{G}$

The two-dimensional field theory on $\hat{G}$ studied in Section 2.6,

$$S_{\hat{F}} = \int d\tau d\sigma \mathcal{L}_{\hat{F}} \tag{2.7.16a}$$

$$\begin{aligned}\mathcal{L}_{\hat{F}} &= \frac{k}{8\pi}\eta_{ab}e_{i\mu}{}^{a,-m}e_{j\nu}{}^{bm}(\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \partial_\sigma x^{i\mu}\partial_\sigma x^{j\nu}) + \frac{k}{4\pi}\hat{B}_{i\mu,j\nu}\partial_\tau x^{i\mu}\partial_\sigma x^{j\nu} \\ &\quad + k\lambda_{i\mu}(\partial_\sigma x^{i\mu} - \eta^{ab}e_{a,-m}{}^{i\mu}\hat{\Omega}_{bm}{}^{y_*})\end{aligned} \tag{2.7.16b}$$

is expressed in terms of coordinates on the affine Lie group. We turn now to rewriting this theory in terms of rescaled traces of functions of the reduced affine group element $\hat{g}(\hat{T}, x(\tau,\sigma))$, using the results of Sections 2.6 and 2.7.1.

Note first that the classical coordinate-space current modes (2.6.23a,b) can be written in matrix form,

$$E_a(m,\tau,\sigma)\eta^{ab}\hat{T}_{b,-m} = -ik\hat{g}^{-1}(\hat{T},x(\tau,\sigma))\partial\hat{g}(\hat{T},x(\tau,\sigma)) \tag{2.7.17a}$$

$$\bar{E}_a(m,\tau,\sigma)\eta^{ab}\hat{T}_{b,-m} = -ik\hat{g}(\hat{T},x(\tau,\sigma))\bar{\partial}\hat{g}^{-1}(\hat{T},x(\tau,\sigma)) \tag{2.7.17b}$$

using (2.7.14). Correspondingly, the current modes can be written as rescaled traces,

$$E_a(m,\tau,\sigma) = -\frac{i}{\chi(T)}k\hat{\mathrm{Tr}}\left(\hat{T}_{am}\hat{g}^{-1}(\hat{T},x(\tau,\sigma))\partial\hat{g}(\hat{T},x(\tau,\sigma))\right) \tag{2.7.18a}$$

$$\bar{E}_a(m,\tau,\sigma) = -\frac{i}{\chi(T)}k\hat{\mathrm{Tr}}\left(\hat{T}_{am}\hat{g}(\hat{T},x(\tau,\sigma))\bar{\partial}\hat{g}^{-1}(\hat{T},x(\tau,\sigma))\right) \tag{2.7.18b}$$

in any matrix representation $\hat{T}$, using (2.7.9c). For brevity below, we write $\hat{g}$ for $\hat{g}(\hat{T},x(\tau,\sigma))$ and χ for $\chi(T)$.

Following the example of the currents, the kinetic terms in the action (2.7.16) can be written

$$\frac{k}{8\pi}\eta_{ab}e_{i\mu}{}^{a,-m}e_{j\nu}{}^{bm}(\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \partial_\sigma x^{i\mu}\partial_\sigma x^{j\nu}) = -\frac{k}{2\pi\chi}\hat{\mathrm{Tr}}\left(\hat{g}^{-1}\partial\hat{g}\hat{g}^{-1}\bar{\partial}\hat{g}\right) \tag{2.7.19}$$

using eq.(2.7.14).

We next consider the $\hat{B}$ term in the action. Note first that the antisymmetric tensor field $\hat{B}$, defined in (2.3.3), can be written as a rescaled trace

$$\hat{B}_{i\mu,j\nu}(x(\tau,\sigma)) = \frac{i}{\chi}\int_0^1 dt\,\hat{\mathrm{Tr}}\left(\hat{H}\partial_{[i\mu}e^{-it\hat{H}}\partial_{j\nu]}e^{it\hat{H}}\right), \qquad \hat{H} = \beta^{am}(x(\tau,\sigma))\hat{T}_{am} \tag{2.7.20a}$$

$$X_{[i\mu}Y_{j\nu]} \equiv X_{i\mu}Y_{j\nu} - X_{j\nu}Y_{i\mu} \tag{2.7.20b}$$

in any β-basis. The equality of (2.3.3) and (2.7.20a) follows by integration over the

parameter t. Using the integral representation (2.7.20a), one also verifies the cyclic identity

$$\partial_{i\mu}\hat{B}_{j\nu,k\rho} + \partial_{j\nu}\hat{B}_{k\rho,i\mu} + \partial_{k\rho}\hat{B}_{i\mu,j\nu} = \frac{i}{\chi}\hat{\mathrm{Tr}}(e_{i\mu}[e_{j\nu}, e_{k\rho}]). \tag{2.7.21}$$

Both identities (2.7.20a) and (2.7.21) are analogous to standard relations (see Appendix A) satisfied by the conventional antisymmetric tensor field B_{ij} on Lie g.

The rest of the discussion on $\hat{G}$ parallels the usual development of the WZW term on G. The cyclic identity can be used to integrate the following three-form on $\hat{G}$,

$$\hat{\mathrm{Tr}}(\hat{g}^{-1}d\hat{g})^3 \equiv d^3\xi\, \varepsilon^{ABC}\hat{\mathrm{Tr}}(\hat{g}^{-1}\partial_A\hat{g}\hat{g}^{-1}\partial_B\hat{g}\hat{g}^{-1}\partial_C\hat{g}) = d^3\xi\, \partial_A\mathcal{W}^A \tag{2.7.22a}$$

$$\mathcal{W}^A \equiv -\frac{3}{2}\chi\varepsilon^{ABC}\partial_B x^{i\mu}\partial_C x^{j\nu}\hat{B}_{i\mu,j\nu} \tag{2.7.22b}$$

$$A = (\tau, \sigma, \rho), \qquad d^3\xi = d\tau d\sigma d\rho, \qquad \varepsilon^{012} = +1 \tag{2.7.22c}$$

$$\int \hat{\mathrm{Tr}}(\hat{g}^{-1}d\hat{g})^3 = \int d\tau d\sigma \mathcal{W}^\rho(\tau, \sigma, \rho = 1) = -3\chi \int d\tau d\sigma \partial_\tau x^{i\mu}\partial_\sigma x^{j\nu}\hat{B}_{i\mu,j\nu} \tag{2.7.22d}$$

where $0 \leq \rho \leq 1$ is the radial coordinate of the usual WZW cylinder, and the two-dimensional $\hat{g}$ in (2.7.17-19) is the boundary value of this $\hat{g}$ at radius $\rho = 1$.

Using (2.7.22), we obtain our first $\hat{g}$ form of the two-dimensional field theory on $\hat{G}$ [21],

$$\begin{aligned} S_{\hat{F}} &= -\frac{k}{2\pi\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}\left(\hat{g}^{-1}\partial\hat{g}\hat{g}^{-1}\bar{\partial}\hat{g}\right) - \frac{k}{12\pi\chi}\int \hat{\mathrm{Tr}}(\hat{g}^{-1}d\hat{g})^3 \\ &\quad + \frac{k}{\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}(\hat{T}_{am}\ln\hat{g})(i\partial_\sigma + m)\lambda^{am} \end{aligned} \tag{2.7.23}$$

where the second term is a three-dimensional WZW term on the affine group. The last term in (2.7.23) is the constraint term, whose form follows from eq.(2.6.14). The multiplier λ^{am} is defined by the invertible relation $\lambda_{i\mu} = \partial_{i\mu}\beta^{am}\hat{\eta}_{am,bn}\lambda^{bn}$ and a sum on $m \in \mathbb{Z}$ is understood in this term.

The first two terms of the action (2.7.23), without the constraint term, were considered as a theory in Ref. [33], and the theory was found to have an infinite degeneracy in that case. In the formulation (2.7.23), it is the role of the constraint to remove that degeneracy and to implement the classical equivalence with the conventional formulation of the WZW model on the Lie group. In Section 2.12, we find a more elegant form of the constraint term which is associated to the formal quantum equivalence of this formulation with the conventional formulation.

Part III

Translation dictionary: $\hat{G} \leftrightarrow G$

2.8 Strategy: Partial transmutation of target and base spaces

In Part II, we found two actions, (2.5.7,8) and (2.7.23), on the affine Lie group $\hat{G}$ which must be equivalent to the conventional formulation of the WZW model on the Lie group G. This equivalence is clear because the Hamiltonians of the theories are isomorphic.

In this part, we find the explicit translation dictionary between the formulations on $\hat{G}$ and G, which shows that our new actions on $\hat{G}$ are mode formulations of WZW.

The overall picture which emerges is a quartet of equivalent formulations of WZW theory,

$$\begin{array}{ccc} \hat{g}(x(\tau)) & \xleftrightarrow{\text{constraint}} & \hat{g}(x(\tau,\sigma)) \\ \text{modes}\Big\updownarrow & & \Big\updownarrow\text{modes} \\ g(x(\tau,\sigma)) & \xleftrightarrow{\text{constraint}} & g(x(\tau,\sigma,\tilde{\sigma})) \end{array} \tag{2.8.1}$$

where $\hat{g} \in \hat{G}$ and $g \in G$. The conventional formulation of WZW on G is in the lower left. The top line of the picture describes the two formulations on $\hat{G}$ in Part II, whose equivalence to each other was discussed in Section 2.6. More generally, the horizontal direction of the picture shows the use of constraints to change the dimension of the base space (see also Section 2.13). The left column describes the equivalence (which is studied in Sections 2.9-2.11) of the mechanical formulation of WZW on $\hat{G}$ with the conventional formulation on G.

The central relation underlying the equivalence in the left column is the mode identity

$$\beta^a(x^i(\tau,\sigma)) = \sum_m e^{im\sigma}\beta^{am}(x^{i\mu}(\tau)) \tag{2.8.2}$$

where $x^i(\tau,\sigma)$ is the local WZW coordinate on G, and the tangent-space coordinates β^a and β^{am} appear in the group elements of G and $\hat{G}$ as

$$g(T, x(\tau,\sigma)) = e^{i\beta^a(x(\tau,\sigma))T_a}, \qquad \hat{g}(\hat{T}, x(\tau)) = e^{i\beta^{am}(x(\tau))\hat{T}_{am}} . \tag{2.8.3}$$

As in Section 2.7, $\hat{T}$ is any of the affine matrix representations $\hat{T}(T)$ or $\hat{T}^{adj}$, and T is the corresponding matrix representation of Lie g.

The mode identity (2.8.2) emphasizes the fact that the operation of moding is a partial transmutation

$$x^{i\mu}(\tau) \leftrightarrow x^{i}(\tau,\sigma) \tag{2.8.4a}$$

$$\hat{g}(x(\tau)): \ \mathbb{R} \mapsto \hat{G}, \qquad g(x(\tau,\sigma)): \ \mathbb{R}\times S^1 \mapsto G \tag{2.8.4b}$$

between the target space and the base space. More generally, such partial transmutations operate in the vertical direction of the picture, and the transmutation in the right column allows us to express the field-theoretic formulation on $\hat{G}$ as a three-dimensional field theory on G (see Section 2.13).

2.9 Modes on $\hat{G}$ and local fields on G

In this section, we develop the translation dictionary which underlies the equivalence

$$\begin{array}{ccccc} \hat{g}(x(\tau)) & \cdot & \cdot & \cdot & \hat{g}(x(\tau,\sigma)) \\ \text{modes}\Big\updownarrow & & & & \vdots \\ g(x(\tau,\sigma)) & \cdot & \cdot & \cdot & g(x(\tau,\sigma,\tilde{\sigma})) \end{array} \tag{2.9.1}$$

described by the left column in the picture (2.8.1). In Section 2.13, the dictionary will also be applied to the right column of the picture.

As a conceptual orientation, we recall first that the operator current modes (2.3.6),

$$E_a(m) = -ie_{am}{}^{i\mu}\mathcal{D}_{i\mu}(\hat{B}) + \frac{1}{2}k\hat{\Omega}_{am}{}^{y_*}, \qquad \bar{E}_a(m) = -i\bar{e}_{am}{}^{i\mu}\mathcal{D}_{i\mu}(\hat{B}) - \frac{1}{2}k(\hat{\Omega}^{-1})_{am}{}^{y_*} \tag{2.9.2a}$$

$$\mathcal{D}_{i\mu}(\hat{B}) = \partial_{i\mu} - \frac{i}{2}k\hat{B}_{i\mu,j\nu}e_{a,-m}{}^{j\nu}\eta^{ab}\hat{\Omega}_{bm}{}^{y_*} \tag{2.9.2b}$$

and the corresponding mechanical Lagrangian (2.5.8) on $\hat{G}$,

$$L_M = \frac{k}{4}\eta_{ab}e_{i\mu}{}^{a,-m}e_{j\nu}{}^{bm}\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \frac{k}{4}\left(\hat{\Omega}_{am}{}^{y_*} + 2\hat{B}_{i\mu,j\nu}\partial_\tau x^{j\nu}e_{am}{}^{i\mu}\right)\eta^{ab}\hat{\Omega}_{b,-m}{}^{y_*} \tag{2.9.3}$$

make no reference to any spatial variable σ. In what follows, we will define σ-dependent local fields on G whose σ-independent modes are the quantities on $\hat{G}$.

Except where it is relevant, we will suppress the time dependence of all quantities. We will discuss the translation dictionary first for the quantum system, returning later to indicate the simple changes necessary for the corresponding classical results.

We begin with the canonical operator system

$$[x^{i\mu}, p_{j\nu}] = i\delta_{j\nu}{}^{i\mu}, \qquad p_{i\mu} = -i\partial_{i\mu} \tag{2.9.4}$$

where $x^{i\mu}$ are the Einstein coordinates on $\hat{G}$. It is useful to define the corresponding canonical tangent-space system,

$$p_{am} \equiv -i\frac{\partial}{\partial\beta^{am}} \equiv -i\partial_{am} = (\partial_{am}x^{i\mu})p_{i\mu} \tag{2.9.5a}$$

$$[\beta^{am}, p_{bn}] = i\delta_{bn}{}^{am}, \qquad [\beta^{am}, \beta^{bn}] = [p_{am}, p_{bn}] = 0 \tag{2.9.5b}$$

where β^{am} is the tangent-space coordinate on $\hat{G}$ in (2.8.3) and $\partial_{am}x^{i\mu}(\beta)$ is the inverse of the matrix $\partial_{i\mu}\beta^{am}(x)$.

We then define the local WZW fields in terms of a periodic coordinate $0 \leq \sigma < 2\pi$,

$$\beta^a(x(\sigma)) \equiv \sum_m e^{im\sigma}\beta^{am}(x) \tag{2.9.6a}$$

$$p_a(\sigma) \equiv \frac{1}{2\pi}\sum_m e^{-im\sigma}p_{am} \tag{2.9.6b}$$

where $x^i(\sigma), i = 1\ldots\dim g$ are the local Einstein coordinates on G and $\beta^a(x(\sigma))$ are the tangent-space coordinates on G in (2.8.3). Note that in this moding upper and lower tangent-space indices are associated to $e^{im\sigma}$ and $e^{-im\sigma}$ respectively.

As we will see below, the tangent-space mode identities (2.9.6) are correct because:

- The mode identities guarantee the same simple $e^{\pm im\sigma}$ moding for all objects with tangent-space or carrier-space indices. The moding of objects with Einstein indices is more involved, as discussed below.
- The mode identities guarantee equivalence of the mechanical system (2.9.3) on $\hat{G}$ with the conventional WZW model on G. We mention in particular (see Section 2.11) that conventional WZW averages on G

$$\langle\mathcal{F}[\beta^a(x(\tau,\sigma))]\rangle_G = \langle\mathcal{F}[\sum_m e^{im\sigma}\beta^{am}(x(\tau))]\rangle_{\hat{G}_M} \tag{2.9.7}$$

can be computed for any $\mathcal{F}$ as shown, using the mechanical formulation ($\hat{G}_M$) on $\hat{G}$.

The relation (2.9.6a) allows independent choices of bases on $\hat{G}$ and on G, which control the moding of the Einstein coordinates. For example, one may choose the

standard bases

$$\beta^a(x(\sigma)) = x^i(\sigma)e_i{}^a(0), \qquad \beta^{am}(x) = x^{i\mu}e_{i\mu}{}^{am}(0) \tag{2.9.8}$$

where $e_i{}^a(0)$ and $e_{i\mu}{}^{am}(0)$ are the vielbeins at the origin on G and $\hat{G}$ respectively. Then one obtains the mode relation of the Einstein coordinates,

$$x^i(\sigma) = x^{j\nu}(\sum_m e_{j\nu}{}^{am}(0)e^{im\sigma})e_a{}^i(0) \tag{2.9.9}$$

and more complicated relations are generally obtained for other basis choices.

It is straightforward to check that the local tangent-space fields in (2.9.6) are canonical

$$[\beta^a(x(\sigma)), p_b(\sigma')] = i\delta_b{}^a\delta(\sigma - \sigma'), \qquad [\beta^a(x(\sigma)), \beta^b(x(\sigma'))] = [p_a(\sigma), p_b(\sigma')] = 0 \tag{2.9.10}$$

so $p_a(\sigma)$ is the functional derivative $-i\delta/\delta\beta^a(x(\sigma))$.

Although their moding may be complicated, the Einstein coordinates are periodic functions of σ, and one can also find the corresponding local canonical Einstein system,

$$p_i(\sigma) \equiv \partial_i\beta^a(x(\sigma))p_a(\sigma) \tag{2.9.11a}$$

$$[x^i(\sigma), p_j(\sigma')] = i\delta_j{}^i\delta(\sigma - \sigma'), \qquad [x^i(\sigma), x^j(\sigma')] = [p_i(\sigma), p_j(\sigma')] = 0 \tag{2.9.11b}$$

by using chain rule from the local tangent-space system (2.9.10). It follows that the local Einstein momenta are functional derivatives

$$\begin{aligned} p_i(\sigma) &= \frac{1}{2\pi}\partial_i\beta^a(x(\sigma))\sum_m e^{-im\sigma}(\partial_{am}x^{i\mu}(\beta))i\partial_{i\mu} \\ &= -i\frac{\delta}{\delta x^i(\sigma)} \end{aligned} \tag{2.9.12}$$

with respect to the local Einstein coordinates.

Group elements

We consider next the the group elements $\hat{g} \in \hat{G}$ and $g \in G$ in (2.8.3), whose mode relations have the form [21]

$$\sum_m e^{i(n-m)\sigma}\hat{g}(\hat{T}, x)_{Im}{}^{Jn} = g(T, x(\sigma))_I{}^J, \qquad \forall n \in \mathbb{Z} \tag{2.9.13a}$$

$$\hat{g}(\hat{T}, x)_{Im}{}^{Jn} = \frac{1}{2\pi}\int d\sigma e^{i(m-n)\sigma}g(T, x(\sigma))_I{}^J \tag{2.9.13b}$$

where $I, J = 1 \ldots \dim T$. The inverse relation (2.9.13b) follows from (2.9.13a). We

emphasize that these relations define the domain of the affine Lie group which is in one-to-one correspondence with the Lie group.

The relation (2.9.13a) is proven separately for each order in $\hat{H}$ and H, where

$$\hat{H} \equiv \beta^{am}(x)\hat{T}_{am}, \qquad \hat{g}(\hat{T}, x) = e^{i\hat{H}} \tag{2.9.14a}$$

$$H(\sigma) \equiv \beta^{a}(x(\sigma))T_{a}, \qquad g(T, x(\sigma)) = e^{iH(\sigma)}. \tag{2.9.14b}$$

As an illustration, we discuss the lowest orders explicitly. To zeroth order, the relation (2.9.13a) is an identity because

$$(\mathbb{1})_{Im}{}^{Jn} = \delta_I{}^J \delta_m{}^n, \qquad (\mathbb{1})_I{}^J = \delta_I{}^J. \tag{2.9.15}$$

The first order computation is

$$\begin{aligned}
\sum_m e^{i(n-m)\sigma} \hat{H}_{Im}{}^{Jn} &= \sum_{p,m} e^{i(n-m)\sigma} \beta^{ap}(x)(\hat{T}_{ap})_{Im}{}^{Jn} \\
&= \sum_{p,m} e^{i(n-m)\sigma} \beta^{ap}(x)(T_a)_I{}^J \delta_{p+m,n} \\
&= \sum_p e^{ip\sigma} \beta^{ap}(x)(T_a)_I{}^J \\
&= \beta^a(x(\sigma))(T_a)_I{}^J \\
&= H(\sigma)_I{}^J
\end{aligned} \tag{2.9.16}$$

and higher orders are easily checked following similar steps.

The form of the result (2.9.13a) is somewhat surprising, because one might have expected a double sum over m and n. In fact this form is natural because, in their carrier space indices, the affine matrix representations $\hat{T}_{Im}{}^{Jn}$ (see (2.7.8)) and the affine group elements $\hat{g}_{Im}{}^{Jn}$ are functions only of $m - n$.

In the same way, one establishes the general relations

$$\sum_m e^{i(n-m)\sigma} (\mathcal{F}(\hat{H}))_{Im}{}^{Jn} = (\mathcal{F}(H(\sigma)))_I{}^J, \qquad \forall n \in \mathbb{Z} \tag{2.9.17a}$$

$$\mathcal{F}(\hat{H}))_{Im}{}^{Jn} = \frac{1}{2\pi} \int d\sigma e^{i(m-n)\sigma} (\mathcal{F}(H(\sigma)))_I{}^J \tag{2.9.17b}$$

$$\hat{\mathrm{Tr}}(\mathcal{F}(\hat{H})) = \frac{1}{2\pi} \int d\sigma \mathrm{Tr}(\mathcal{F}(H(\sigma))) \tag{2.9.17c}$$

for all power series $\mathcal{F}(H)$, where Tr is trace on Lie g and the rescaled traces $\hat{\mathrm{Tr}}$ on $\hat{G}$ are defined in Section 2.7.

Antisymmetric tensor fields

The general relation (2.9.17a) is sufficient to obtain the mode relations of any object with two free indices, such as the group elements in (2.9.13). As a second example, we consider the mode relation between the antisymmetric tensor field $\hat{B}_{i\mu,j\nu}$ on $\hat{G}$ and the standard antisymmetric tensor field B_{ij} on G. Since these objects have Einstein indices, we employ the transition functions $\partial_{am}x^{i\mu}(\beta)$ and $\partial_a x^i(\beta(\sigma))$ to generate tangent-space structures with simple moding,

$$\partial_{am}x^{i\mu}(\beta)\hat{B}_{i\mu,j\nu}(x)\partial_{bn}x^{j\nu}(\beta) = (N(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn} \tag{2.9.18a}$$

$$\partial_a x^i(\beta(\sigma))B_{ij}(x(\sigma))\partial_b x^j(\beta(\sigma)) = (N(H^{adj}(\sigma)))_a{}^c\eta_{cb} \tag{2.9.18b}$$

$$\sum_m e^{i(n-m)\sigma}(N(\hat{H}^{adj}))_{am}{}^{bn} = (N(H^{adj}(\sigma)))_a{}^b, \qquad \forall n \in \mathbb{Z} \tag{2.9.18c}$$

$$\hat{H}^{adj} \equiv \beta^{am}(x)\hat{T}^{adj}_{am}, \qquad H^{adj}(\sigma) \equiv \beta^a(x(\sigma))T^{adj}_a \tag{2.9.18d}$$

where the function $N(H)$, which is the same for both $\hat{G}$ and G, is given explicitly in Appendices A and B. The mode relation (2.9.18c) is a special case of (2.9.17a), and, using this relation, we find that

$$\sum_m e^{-i(m+n)\sigma}\partial_{am}x^{i\mu}(\beta)\hat{B}_{i\mu,j\nu}(x)\partial_{bn}x^{j\nu}(\beta) = \partial_a x^i(\beta(\sigma))B_{ij}(x(\sigma))\partial_b x^j(\beta(\sigma)). \tag{2.9.19}$$

This mode relation may also be written

$$B_{ij}(x(\sigma)) = \partial_i\beta^a(x(\sigma))\left(\sum_m e^{-i(m+n)\sigma}\partial_{am}x^{k\mu}(\beta)\hat{B}_{k\mu,l\nu}(x)\partial_{bn}x^{l\nu}(\beta)\right)\partial_j\beta^b(x(\sigma)) \tag{2.9.20}$$

because the transition functions are invertible. In both eqs.(2.9.19) and (2.9.20), $n \in \mathbb{Z}$ is arbitrary.

Vielbeins

As another example with two indices, we follow similar steps to obtain the mode relations of the vielbeins on $\hat{G}$ and G,

$$\begin{aligned}\sum_m e^{i(n-m)\sigma}\partial_{am}x^{i\mu}(\beta)e_{i\mu}{}^{bn}(x) &= \sum_m e^{i(n-m)\sigma}(M(\hat{H}^{adj}))_{am}{}^{bn} && (2.9.21a)\\ &= (M(H^{adj}(\sigma)))_a{}^b && (2.9.21b)\\ &= \partial_a x^i(\beta(\sigma))e_i{}^b(x(\sigma)) && (2.9.21c)\end{aligned}$$

$$e_i{}^b(x(\sigma)) = \partial_i\beta^a(x(\sigma))\sum_m e^{i(n-m)\sigma}\partial_{am}x^{j\nu}(\beta)e_{j\nu}{}^{bn}(x), \qquad \forall n \in \mathbb{Z} \tag{2.9.21d}$$

where the function $M(H)$ is given explicitly in Appendices A and B. We also find the corresponding results

$$e_b{}^i(x(\sigma)) = \sum_m e^{-i(n-m)\sigma} e_{bn}{}^{j\nu}(x)\partial_{j\nu}\beta^{am}(x)\partial_a x^i(\beta(\sigma)) \tag{2.9.22a}$$

$$\bar{e}_i{}^b(x(\sigma)) = \partial_i\beta^a(x(\sigma))\sum_m e^{i(n-m)\sigma}\partial_{am}x^{j\nu}(\beta)\bar{e}_{j\nu}{}^{bn}(x) \tag{2.9.22b}$$

$$\bar{e}_b{}^i(x(\sigma)) = \sum_m e^{-i(n-m)\sigma}\bar{e}_{bn}{}^{j\nu}(x)\partial_{j\nu}\beta^{am}(x)\partial_a x^i(\beta(\sigma)), \qquad \forall n \in \mathbb{Z} \tag{2.9.22c}$$

for the other vielbein and the inverse vielbeins.

Other mode relations

We turn now to mode relations for general objects with fewer than two free indices. These relations will be important for the currents and the action, discussed in Sections 2.10 and 2.11. In what follows, we limit ourselves to the adjoint representations.

A general class of one-index tangent-space objects is formed by contraction of the tangent-space coordinates or tangent-space momenta with the general two-index structure $\mathcal{F}(H)$ in (2.9.17a). For example, one has

$$\begin{aligned}\beta^a(x(\sigma))(\mathcal{F}(H^{adj}(\sigma)))_a{}^b &= \sum_m e^{im\sigma}\beta^{am}(x)\sum_n e^{i(n-m)\sigma}(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{bn} \\ &= \sum_n e^{in\sigma}\beta^{am}(x)(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{bn}.\end{aligned} \tag{2.9.23}$$

Similarly, one finds the relations

$$\sum_n e^{in\sigma}\partial_\tau\beta^{am}(x)(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{bn} = \partial_\tau\beta^a(x(\sigma))(\mathcal{F}(H^{adj}(\sigma)))_a{}^b \tag{2.9.24a}$$

$$\sum_n e^{in\sigma}im\beta^{am}(x)(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{bn} = \partial_\sigma\beta^a(x(\sigma))(\mathcal{F}(H^{adj}(\sigma)))_a{}^b \tag{2.9.24b}$$

$$\sum_m e^{-im\sigma}(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{bn}p_{bn} = 2\pi(\mathcal{F}(H^{adj}(\sigma)))_a{}^b p_b(\sigma) \tag{2.9.24c}$$

by using $\partial_\tau\beta$, $\partial_\sigma\beta$ or p instead of β in (2.9.23).

As an application of (2.9.24b), we find the relations

$$\sum_m e^{-im\sigma}\hat{g}(\hat{T}^{adj},x)_{am}{}^{y_*} = \partial_\sigma x^i(\sigma)\bar{e}_i{}^b(x(\sigma))\eta_{ba} \tag{2.9.25a}$$

$$\sum_m e^{-im\sigma}\hat{\Omega}_{am}{}^{y_*}(x) = \partial_\sigma x^i(\sigma)e_i{}^b(x(\sigma))\eta_{ba} \tag{2.9.25b}$$

where we have also used the explicit β-basis forms of the adjoint action $\hat{\Omega} = \hat{g}^{-1}(\hat{T}^{adj})$

and the vielbeins given in the appendices. Taken with eq.(2.9.13), these results complete the mode relations of the affine group elements.

A large class of mode relations for objects with no free indices is similarly constructed. We give here only some representative results,

$$\beta^{am}(x)(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn}\beta^{bn}(x) = \frac{1}{2\pi}\int d\sigma\beta^a(x(\sigma))(\mathcal{F}(H^{adj}(\sigma)))_a{}^c\eta_{cb}\beta^b(x(\sigma)) \tag{2.9.26a}$$

$$\partial_\tau\beta^{am}(x)(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn}\partial_\tau\beta^{bn}(x) = \frac{1}{2\pi}\int d\sigma\partial_\tau\beta^a(x(\sigma))(\mathcal{F}(H^{adj}(\sigma)))_a{}^c\eta_{cb}\partial_\tau\beta^b(x(\sigma)) \tag{2.9.26b}$$

$$\partial_\tau\beta^{am}(x)(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn}in\beta^{bn}(x) = \frac{1}{2\pi}\int d\sigma\partial_\tau\beta^a(x(\sigma))(\mathcal{F}(H^{adj}(\sigma)))_a{}^c\eta_{cb}\partial_\sigma\beta^b(x(\sigma)) \tag{2.9.26c}$$

$$m\beta^{am}(x)(\mathcal{F}(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn}n\beta^{bn}(x) = -\frac{1}{2\pi}\int d\sigma\partial_\sigma\beta^a(x(\sigma))(\mathcal{F}(H^{adj}(\sigma)))_a{}^c\eta_{cb}\partial_\sigma\beta^b(x(\sigma)) \tag{2.9.26d}$$

$$im\beta^{am}(x)(\mathcal{F}_1(\hat{H}^{adj}))_{am}{}^{dq}\beta^{bn}(x)(\mathcal{F}_2(\hat{H}^{adj}))_{bn}{}^{er}\partial_\tau\beta^{cp}(x)(\mathcal{F}_3(\hat{H}^{adj}))_{cp}{}^{fs}f_{dq,er,fs} = \frac{1}{2\pi}\int d\sigma\partial_\sigma\beta^a(x(\sigma))(\mathcal{F}_1(H^{adj}(\sigma))_a{}^d\beta^b(x(\sigma)) \times (\mathcal{F}_2(H^{adj}(\sigma))_b{}^e\partial_\tau\beta^c(x(\sigma))(\mathcal{F}_3(H^{adj}(\sigma)))_c{}^f f_{def} \tag{2.9.26e}$$

although other relations of this type are easily obtained to describe contraction of $\mathcal{F}(H)$ with tangent-space momenta.

As anticipated, the translation dictionary of this section conforms to the rule that upper and lower (affine) tangent space and carrier space indices are associated to $e^{im\sigma}$ and $e^{-im\sigma}$ respectively. This rule holds as well when (affine) tangent-space indices are raised and lowered with the rescaled Killing metric.

The translation dictionary has an isomorphic classical form which is obtained by replacing the quantum operators $p_{i\mu}$, p_{am}, $p_i(\sigma)$, and $p_a(\sigma)$ with their corresponding classical momenta.

In the following sections, we apply the translation dictionary to the central structures of the theory, that is the currents and the action.

2.10 Local fields and local currents

The operator current modes (2.9.2)

$$E_a(m) = -ie_{am}{}^{i\mu}\mathcal{D}_{i\mu}(\hat{B}) + \frac{1}{2}k\hat{\Omega}_{am}{}^{y_*}, \qquad \bar{E}_a(m) = -i\bar{e}_{am}{}^{i\mu}\mathcal{D}_{i\mu}(\hat{B}) - \frac{1}{2}k(\hat{\Omega}^{-1})_{am}{}^{y_*} \tag{2.10.1a}$$

$$\mathcal{D}_{i\mu}(\hat{B}) \equiv \partial_{i\mu} - \frac{i}{2}k\hat{B}_{i\mu,j\nu}e_{a,-m}{}^{j\nu}\eta^{ab}\hat{\Omega}_{bm}{}^{y_*} \tag{2.10.1b}$$

are functions of the coordinates on the affine Lie group, and derivatives with respect to these coordinates. In this section, we use the translation dictionary above to express the local operator currents,

$$E_a(\sigma) = \sum_m e^{-im\sigma}E_a(m), \qquad \bar{E}_a(\sigma) = \sum_m e^{-im\sigma}\bar{E}_a(m) \tag{2.10.2}$$

as functions of the local fields on the Lie group, and functional derivatives with respect to these fields. This section is strictly current-algebraic and does not depend on any particular dynamics*.

For the left-invariant currents, follow the steps

$$\begin{aligned}
E_a(\sigma) &= \sum_m e^{-im\sigma}(e_{am}{}^{i\mu}(x)p_{i\mu}(\hat{B}) + \frac{1}{2}k\hat{\Omega}_{am}{}^{y_*}(x)) && (2.10.3a)\\
&= \sum_m e^{-im\sigma}((C(\hat{H}^{adj}))_{am}{}^{bn}p_{bn} - ikn\beta^{bn}(x)(D(\hat{H}^{adj}))_{bn}{}^{cp}\hat{\eta}_{cp,am} && (2.10.3b)\\
&= 2\pi(C(H^{adj}(\sigma)))_a{}^b p_b(\sigma) - k\partial_\sigma\beta^c(x(\sigma))(D(H^{adj}(\sigma)))_c{}^b\eta_{ba} && (2.10.3c)\\
&= 2\pi e_a{}^i(x(\sigma))p_i(B,\sigma) + \frac{k}{2}\eta_{ab}e_i{}^b(x(\sigma))\partial_\sigma x^i(\sigma) && (2.10.3d)
\end{aligned}$$

$$p_{i\mu}(\hat{B}) \equiv -i\mathcal{D}_{i\mu}(\hat{B}) = p_{i\mu} - \frac{k}{2}\hat{B}_{i\mu,j\nu}(x)e_{a,-m}{}^{j\nu}(x)\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}(x) \tag{2.10.3e}$$

$$p_i(B,\sigma) \equiv p_i(\sigma) - \frac{k}{4\pi}B_{ij}(x(\sigma))\partial_\sigma x^j(\sigma) \tag{2.10.3f}$$

where the operator momenta $p_{i\mu}$, p_{am}, $p_i(\sigma)$, and $p_a(\sigma)$ are defined in Section 2.9. To obtain (2.10.3b), we used the explicit β-basis form of the reduced affine Lie derivatives given in Appendix B. The functions $C(H)$ and $D(H)$ are also given in this Appendix. Eq.(2.10.3c) follows from the mode relations (2.9.24b) and (2.9.24c). Finally to obtain

*In the context of the mechanical model on $\hat{G}$, the classical analogues of the current modes (2.10.1) were denoted by $E_a(m,\tau)$ and $\bar{E}_a(m,\tau)$ in eq.(2.5.2b). In the context of the field theory on $\hat{G}$, the classical analogues of the local currents (2.10.2) were denoted by $E_a(\tau,\sigma)$ and $\bar{E}_a(\tau,\sigma)$ in eq.(2.6.10).

(2.10.3d), we reorganized (2.10.3c) using the explicit β-basis forms (see Appendix A) of the vielbein $e_i{}^a$ and the antisymmetric tensor field B_{ij} on G.

Following similar steps for $\bar{E}$, we summarize the results for the local operator currents of affine $g \times g$,

$$E_a(\sigma) = -2\pi i e_a(x(\sigma))^i \mathcal{D}_i(B,\sigma) + \frac{k}{2}\eta_{ab} e_i{}^b(x(\sigma))\partial_\sigma x^i(\sigma) \tag{2.10.4a}$$

$$\bar{E}_a(\sigma) = -2\pi i \bar{e}_a{}^i(x(\sigma))\mathcal{D}_i(B,\sigma) - \frac{k}{2}\eta_{ab}\bar{e}_i{}^b(x(\sigma))\partial_\sigma x^i(\sigma) \tag{2.10.4b}$$

$$\mathcal{D}_i(B,\sigma) \equiv ip_i(B,\sigma) = \frac{\delta}{\delta x^i(\sigma)} - \frac{i}{4\pi}kB_{ij}(x(\sigma))\partial_\sigma x^j(\sigma) \tag{2.10.4c}$$

where we have used the fact that the local Einstein momentum $p_i(\sigma)$ in (2.9.12) is a functional derivative.

The classical version of this result is Bowcock's canonical representation [30] of affine $g \times g$,

$$E_a(\sigma) = 2\pi e_a{}^i(x(\sigma))p_i(B,\sigma) + \frac{k}{2}\eta_{ab} e_i{}^b(x(\sigma))\partial_\sigma x^i(\sigma) \tag{2.10.5a}$$

$$\bar{E}_a(\sigma) = 2\pi \bar{e}_a{}^i(x(\sigma))p_i(B,\sigma) - \frac{k}{2}\eta_{ab}\bar{e}_i{}^b(x(\sigma))\partial_\sigma x^i(\sigma) \tag{2.10.5b}$$

$$p_i(B,\sigma) = p_i(\sigma) - \frac{k}{4\pi}B_{ij}(x(\sigma))\partial_\sigma x^j(\sigma) \tag{2.10.5c}$$

where $p_i(\sigma)$ are classical canonical momenta.

2.11 Mechanics on $\hat{G}$ and WZW on G

2.11.1 Classical equivalence

In this section, we use the translation dictionary of Section 2.9 to show that the mechanical Lagrangian L_M on $\hat{G}$ in (2.9.3) is equal to the conventional WZW Lagrangian,

$$L_M = \frac{k}{4}\eta_{ab} e_{i\mu}{}^{a,-m} e_{j\nu}{}^{bm}\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \frac{k}{4}\left(\hat{\Omega}_{am}{}^{y_*} + 2\hat{B}_{i\mu,j\nu}\partial_\tau x^{j\nu} e_{am}{}^{i\mu}\right)\eta^{ab}\hat{\Omega}_{b,-m}{}^{y_*} \tag{2.11.1a}$$

$$= \frac{k}{8\pi}\int d\sigma \left[\eta_{ab} e_i{}^a e_j{}^b(\partial_\tau x^i \partial_\tau x^j - \partial_\sigma x^i \partial_\sigma x^j) + 2B_{ij}\partial_\tau x^i \partial_\sigma x^j\right] \equiv L_{WZW} \tag{2.11.1b}$$

where L_{WZW} in (2.11.1b) is the usual sigma model form of WZW on G.

To see this equality, follow the steps,

$$
\begin{aligned}
L_M &= \frac{k}{4}(\partial_\tau\beta^{am}\partial_\tau\beta^{bn} + m\beta^{am}n\beta^{bn})(F(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn} \\
&\quad + \frac{k}{2}\partial_\tau\beta^{am}(G(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn}in\beta^{bn} && (2.11.2a) \\
&= \frac{k}{8\pi}\int d\sigma\left[(\partial_\tau\beta^a\partial_\tau\beta^b - \partial_\sigma\beta^a\partial_\sigma\beta^b)(F(H^{adj}))_a{}^c\eta_{cb} + 2\partial_\tau\beta^a(G(H^{adj}))_a{}^c\eta_{cb}\partial_\sigma\beta^b\right] \\
& && (2.11.2b) \\
&= \frac{k}{8\pi}\int d\sigma\left[\eta_{ab}e_i{}^a e_j{}^b(\partial_\tau x^i\partial_\tau x^j - \partial_\sigma x^i\partial_\sigma x^j) + 2B_{ij}\partial_\tau x^i\partial_\sigma x^j\right] && (2.11.2c) \\
&= L_{WZW}. && (2.11.2d)
\end{aligned}
$$

To obtain (2.11.2a) from (2.11.1a), we used the explicit β-basis form of the mechanical action given in Appendix B. This Appendix also gives the functions $F(H)$ and $G(H)$. The form in (2.11.2b) then follows from the mode relations (2.9.26b,c,d). Finally, one notices that (2.11.2b) is the β-basis form (see Appendix A) of the conventional WZW Lagrangian on G. In this computation, the $(\hat{\Omega}^{y_*})^2$ term becomes the $(\partial_\sigma x)^2$ term (as can also be seen from the relation (2.9.25b)) and the $\hat{B}$ term becomes the B term.

2.11.2 WZW in terms of $\hat{g}(x(\tau)) \in \hat{G}$

We can also use the translation dictionary to obtain the $\hat{g}$ form of the mechanical action [21],

$$
\begin{aligned}
S_M &= -\frac{k}{4\chi}\int d\tau\hat{\mathrm{Tr}}(\hat{g}^{-1}\partial_\tau\hat{g}\hat{g}^{-1}\partial_\tau\hat{g} - \hat{g}^{-1}\hat{g}'\hat{g}^{-1}\hat{g}') \\
&\quad + \frac{k}{2\chi}\int d\tau\int_0^1 d\rho\varepsilon^{AB}\hat{\mathrm{Tr}}(\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\partial_A\hat{g}\hat{g}^{-1}\partial_B\hat{g}) && (2.11.3a)
\end{aligned}
$$

$$
(\hat{g}')_{Im}{}^{Jn} \equiv i(n-m)\hat{g}_{Im}{}^{Jn}, \qquad A = (\tau,\rho), \qquad \varepsilon^{01} = +1 \qquad (2.11.3b)
$$

where $\hat{g}(\hat{T}, x(\tau)) \in \hat{G}$ is the reduced affine group element in affine matrix representation $\hat{T}$. This form of the mechanical action follows directly from the conventional WZW action

$$
\begin{aligned}
S_{WZW} &= -\frac{k}{2\pi\chi}\int d\tau d\sigma\mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial}g) - \frac{k}{12\pi\chi}\int \mathrm{Tr}(g^{-1}dg)^3 \\
&= S_M && (2.11.4)
\end{aligned}
$$

using[†] the $g \leftrightarrow \hat{g}$ mode relation (2.9.13) and the trace relation (2.9.17c).

The elegant action (2.11.3) is another central result of this chapter. This form of the mechanical action shows a two-dimensional WZW term on $\hat{G}$, which is equal, under the translation dictionary, to the conventional three-dimensional WZW term on G.

One can say more about the structure of the two-dimensional WZW term. Comparing (2.11.1a) and (2.11.3a), we find the identity

$$\int d\tau \int_0^1 d\rho \varepsilon^{AB} \hat{\mathrm{Tr}}(\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\partial_A \hat{g}\hat{g}^{-1}\partial_B \hat{g}) = \chi \int d\tau \partial_\tau x^{i\mu} \hat{B}_{i\mu,j\nu} e_{am}{}^{j\nu} \eta^{ab} \hat{\Omega}_{b,-m}{}^{y_*} \quad (2.11.5)$$

between the two-dimensional and the one-dimensional forms of the WZW term on $\hat{G}$. This identity also follows from the divergence relation

$$\varepsilon^{AB} \hat{\mathrm{Tr}}(\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\partial_A \hat{g}\hat{g}^{-1}\partial_B \hat{g}) = \partial_A W^A \quad (2.11.6a)$$

$$W^A \equiv -\chi \varepsilon^{AB} \partial_B x^{i\mu} \hat{B}_{i\mu,j\nu} e_{am}{}^{j\nu} \eta^{ab} \hat{\Omega}_{b,-m}{}^{y_*} \quad (2.11.6b)$$

$$\begin{aligned} \int d\tau \int_0^1 d\rho \varepsilon^{AB} \hat{\mathrm{Tr}}(\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\partial_A \hat{g}\hat{g}^{-1}\partial_B \hat{g}) &= \int d\tau W^\rho(\tau, \rho = 1) \\ &= \chi \int d\tau \partial_\tau x^{i\mu} \hat{B}_{i\mu,j\nu} e_{am}{}^{j\nu} \eta^{ab} \hat{\Omega}_{b,-m}{}^{y_*} \quad (2.11.6c) \end{aligned}$$

whose structure parallels that of the WZW term on $\hat{G}$ in (2.7.22) and the conventional WZW term on G. For a direct proof of the divergence relation, follow the steps,

$$\begin{aligned} \varepsilon^{AB} \hat{\mathrm{Tr}}(\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\partial_A \hat{g}\hat{g}^{-1}\partial_B \hat{g}) &= \frac{1}{2\pi} \int d\sigma \mathrm{Tr}(g^{-1}\partial_\sigma g[g^{-1}\partial_\tau g, g^{-1}\partial_\rho g]) \quad (2.11.7a) \\ &= \frac{\chi}{2\pi} \int d\sigma f_{ab}{}^c \partial_\tau x^i \partial_\rho x^j \partial_\sigma x^k e_i{}^a e_j{}^b e_k{}^d \eta_{dc} \quad (2.11.7b) \\ &= \chi f_{am,bn}{}^{cp} \partial_\tau x^{i\mu} \partial_\rho x^{j\nu} e_{i\mu}{}^{am} e_{j\nu}{}^{bn} \hat{\Omega}_{cp}{}^{y_*} \quad (2.11.7c) \\ &= -\chi \varepsilon^{AB} \partial_B x^{i\mu} \partial_A (\hat{B}_{i\mu,j\nu} e_{am}{}^{j\nu} \eta^{ab} \hat{\Omega}_{b,-m}{}^{y_*}) \quad (2.11.7d) \\ &= \partial_A W^A. \quad (2.11.7e) \end{aligned}$$

To obtain (2.11.7a) one uses the $g \leftrightarrow \hat{g}$ mode relation (2.9.13) and the trace relation (2.9.17c), as above. The form in (2.11.7b) follows by evaluation of the trace. To obtain (2.11.7c), one returns to $\hat{G}$ by the mode relation (2.9.26e) (or the vielbein

[†]For the WZW terms, the mode identity is $\beta^a(\tau, \sigma, \rho) = \sum_m e^{im\sigma} \beta^{am}(\tau, \rho)$. Regularity of the conventional WZW term requires $\partial_\sigma \beta^a(\rho = 0) = 0$ along the axis of the cylinder, which implies that $m\beta^{am}(\rho = 0) = 0$ on $\hat{G}$. It follows that $W^A(\rho = 0) = 0$, where W^A is defined in (2.11.6b). This is why there is no boundary term at $\rho = 0$ in (2.11.6c).

relation (2.9.21)) and the explicit β-basis form of the adjoint action $\hat{\Omega}$ in Appendix B. The $\hat{B}$ form in (2.11.7d) is then obtained from (2.3.5).

We have also worked out the variation of the mechanical action (2.11.3), using

$$\delta(\hat{g}')_{Im}{}^{Jn} = i(n-m)(\delta\hat{g})_{Im}{}^{Jn}. \tag{2.11.8}$$

For this computation, it is useful to define the prime operation on any matrix-valued function,

$$(\mathcal{F}')_{Im}{}^{Jn} \equiv i(n-m)\mathcal{F}_{Im}{}^{Jn} \tag{2.11.9}$$

which includes (2.11.3b) as a special case. The prime operation satisfies a Leibnitz rule and an identity which is analogous to integration by parts,

$$(\mathcal{F}_1\mathcal{F}_2)' = \mathcal{F}_1'\mathcal{F}_2 + \mathcal{F}_1\mathcal{F}_2' \tag{2.11.10a}$$

$$\hat{\mathrm{Tr}}(\mathcal{F}_1'\mathcal{F}_2) = -\hat{\mathrm{Tr}}(\mathcal{F}_1\mathcal{F}_2'). \tag{2.11.10b}$$

Then we find that the variation of the WZW term is a total derivative,

$$\delta\left(\varepsilon^{AB}\hat{\mathrm{Tr}}(\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\partial_A\hat{g}\hat{g}^{-1}\partial_B\hat{g})\right) = \partial_A\left(\varepsilon^{AB}\hat{\mathrm{Tr}}(\hat{g}^{-1}\delta\hat{g}[\hat{g}^{-1}\partial_B\hat{g},\hat{g}^{-1}\hat{g}'])\right) \tag{2.11.11}$$

and the resulting equation of motion of the mechanical system is

$$\partial_\tau(\hat{g}^{-1}\partial_\tau\hat{g} + \hat{g}^{-1}\hat{g}') - (\hat{g}^{-1}\partial_\tau\hat{g} + \hat{g}^{-1}\hat{g}')' = 0. \tag{2.11.12}$$

Under the translation dictionary, the prime operation becomes the derivative with respect to σ, and the equation of motion (2.11.12) becomes the usual equation of motion $\bar{\partial}(g^{-1}\partial g) = 0$ of the conventional WZW formulation on G.

2.11.3 Formal Haar measure on $\hat{G}$ and formal quantum equivalence

In this section, we give the relation between the formal Haar measure on the affine group and the Haar measure on the Lie group. We use this relation to establish the formal quantum equivalence of the mechanical formulation on $\hat{G}$ with the conventional WZW formulation on G. The statements of this section hold only up to irrelevant constants.

The formal Haar measure $d\hat{g}$ on $\hat{G}$ is equal, under the translation dictionary, to the spatial product of Haar measures dg on G,

$$d\hat{g}(x) = \prod_\sigma dg(x(\sigma)) \tag{2.11.13a}$$

$$d\hat{g}(x) \equiv \left(\prod_{i,\mu} dx^{i\mu}\right)\sqrt{\det \hat{G}(x)}, \qquad dg(x(\sigma)) \equiv \left(\prod_i dx^i(\sigma)\right)\sqrt{\det G(x(\sigma))} \tag{2.11.13b}$$

$$\hat{G}_{i\mu,j\nu} \equiv e_{i\mu}{}^{am}\hat{\eta}_{am,bn}e_{j\nu}{}^{bn}, \qquad G_{ij} \equiv e_i{}^a\eta_{ab}e_j{}^b \tag{2.11.13c}$$

where $\hat{G}_{i\mu,j\nu}$ and G_{ij} are the target-space metrics on $\hat{G}$ and G.

Our proof of (2.11.13a) goes through tangent-space variables as usual,

$$d\hat{g}(x) = \left(\prod_{a,m} d\beta^{am}(x)\right) e^{\frac{1}{2}(\hat{\mathrm{Tr}}\ln\hat{\mathcal{G}})(\sum_m)} \tag{2.11.14a}$$

$$= \left(\prod_{a,\sigma} d\beta^a(x(\sigma))\right) e^{\frac{1}{2}\delta(0)\int d\sigma \mathrm{Tr}\ln\mathcal{G}} \tag{2.11.14b}$$

$$= \prod_\sigma dg(x(\sigma)) \tag{2.11.14c}$$

$$\hat{\mathcal{G}}_{am}{}^{bn} \equiv \partial_{am}x^{i\mu}\hat{G}_{i\mu,j\nu}\partial_{cp}x^{j\nu}\hat{\eta}^{cp,bn}, \qquad \mathcal{G}_a{}^b \equiv \partial_a x^i G_{ij}\partial_c x^j \eta^{cb} \tag{2.11.14d}$$

where $\delta(0) = (1/2\pi)\sum_m$ is the periodic delta function $\delta(\sigma)$ at $\sigma = 0$.

Taken with the action equality $S_M = S_{WZW}$ in (2.11.4), this result shows the formal quantum equivalence of the mechanical formulation on $\hat{G}$ with the conventional formulation of WZW on G [21],

$$\int(\mathcal{D}_M\hat{g})\, e^{iS_M} = \int(\mathcal{D}g)\, e^{iS_{WZW}} \tag{2.11.15a}$$

$$\mathcal{D}_M\hat{g} \equiv \prod_\tau d\hat{g}(x(\tau)), \qquad \mathcal{D}g \equiv \prod_{\tau,\sigma} dg(x(\tau,\sigma)) \tag{2.11.15b}$$

$$\mathcal{D}_M\hat{g} = \mathcal{D}g \tag{2.11.15c}$$

where $\mathcal{D}g$ in (2.11.15b) is the formal functional measure of the conventional formulation. It follows from (2.11.15) and (2.9.6a) that conventional WZW averages on G

$$\langle\mathcal{F}[\beta^a(x(\tau,\sigma))]\rangle_G = \langle\mathcal{F}[\sum_m e^{im\sigma}\beta^{am}(x(\tau))]\rangle_{\hat{G}_M} \tag{2.11.16}$$

can be formally computed for any $\mathcal{F}$ as shown [21], using the mechanical formulation $(\hat{G}_M)$ on $\hat{G}$.

As seen above, the formal measures $d\hat{g}(x) = \prod_\sigma dg(x(\sigma))$ and $\mathcal{D}_M\hat{g} = \mathcal{D}g$ have closely related formal divergences. It is an important open problem to find suitably regularized forms of these measures.

2.12 Field theory on $\hat{G}$ and formal quantum equivalence

In this section we use the formal Haar measure on $\hat{G}$ to discuss the formal quantum equivalence

$$\begin{array}{ccc} \hat{g}(x(\tau)) & \xleftrightarrow{\text{constraint}} & \hat{g}(x(\tau,\sigma)) \\ \vdots & & \vdots \\ g(x(\tau,\sigma)) & \cdots & g(x(\tau,\sigma,\tilde{\sigma})) \end{array} \tag{2.12.1}$$

of the field theory on $\hat{G}$ with the mechanical system on $\hat{G}$ and the conventional formulation of WZW on G. These equivalences were discussed at the classical level in Section 2.6. Our conclusion is that we may take the spacetime product of Haar measures on $\hat{G}$ as the functional measure for the field theory on $\hat{G}$, and this measure dictates a more elegant form for the constraint term of this formulation. Again, the measure relations of this section hold only up to irrelevant constants.

For the discussion here, it is convenient to write the action (2.7.23) of the field theory on $\hat{G}$ as

$$S_{\hat{F}} = S_0 + S_{\mathcal{C}} \tag{2.12.2a}$$

$$S_0 = -\frac{k}{2\pi\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}\left(\hat{g}^{-1}\partial\hat{g}\hat{g}^{-1}\bar{\partial}\hat{g}\right) - \frac{k}{12\pi\chi}\int \hat{\mathrm{Tr}}(\hat{g}^{-1}d\hat{g})^3 \tag{2.12.2b}$$

$$S_{\mathcal{C}} = \frac{k}{\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}(\hat{T}_{am}\ln\hat{g})(i\partial_\sigma + m)\lambda^{am} \tag{2.12.2c}$$

where $S_{\mathcal{C}}$ is the constraint term.

Starting from the partition function (2.11.15a) of the mechanical formulation, we can use the constraint identities (2.6.13) and (2.6.15) to derive the partition function of the field theory on $\hat{G}$. The result is

$$\begin{aligned} \int(\mathcal{D}_M\hat{g})\, e^{iS_M} &= \int\left(\prod_{\tau,\sigma,i,\mu} dx^{i\mu}(\tau,\sigma)\right)\left(\prod_\tau \sqrt{\det \hat{G}(x(\tau,\sigma_0)}\right) e^{iS_0} \\ &\quad \times\delta[\partial_\sigma x^{i\mu} - \eta^{ab}e_{am}{}^{i\mu}\hat{\Omega}_{b,-m}{}^{y_*}] \end{aligned} \tag{2.12.3}$$

where $\hat{G}$ is defined in (2.11.13c) and we used the fact (see eq.(2.6.25)) that $S_0 = S_M$ on the constrained subspace. The functional measure in this relation contains an unaesthetic product of affine Haar measures at a fixed reference point σ_0 in σ.

We can obtain a more elegant form of the functional measure by changing variables to the reduced affine group element $\hat{g}$. One begins with the solution of the

constraint

$$\beta^{am}(x(\tau,\sigma)) = e^{im\sigma}\beta^{am}(x(\tau)) \tag{2.12.4}$$

given in (2.6.14). Following steps parallel to those used in proving eq.(2.9.13a), we find the σ-dependence of $\hat{g}$

$$\hat{g}(\hat{T}, x(\tau,\sigma))_{Im}{}^{Jn} = e^{i\beta^{am}(x(\tau,\sigma))\hat{T}_{am}} = e^{i(n-m)\sigma}\hat{g}(\hat{T}, x(\tau))_{Im}{}^{Jn} \tag{2.12.5a}$$

$$\partial_\sigma \hat{g} = \hat{g}' \tag{2.12.5b}$$

$$(\hat{g}')_{Im}{}^{Jn} \equiv i(n-m)\hat{g}_{Im}{}^{Jn} \tag{2.12.5c}$$

on the constrained subspace.

One may then prove the following relation

$$\hat{g}^{-1}(\partial_\sigma \hat{g} - \hat{g}') = i(\partial_\sigma x^{i\mu} - \eta^{ab} e_{am}{}^{i\mu}\hat{\Omega}_{b,-m}{}^{y_*})e_{i\mu}{}^{cp}\hat{T}_{cp} \tag{2.12.6}$$

which holds on or off the constrained subspace, and which gives the $\hat{g}$ form of the constraint. This relation can be proven order by order in β^{am}, or by the following simple argument. Using only chain rule and the vielbein relation (2.7.14), one finds that $\hat{g}(x(\sigma))$ satisfies

$$\hat{g}^{-1}\partial_\sigma \hat{g} = i\partial_\sigma x^{i\mu} e_{i\mu}{}^{am}\hat{T}_{am} \tag{2.12.7}$$

on or off the constrained subspace. Because both sides of (2.12.6) vanish on the constrained subspace, it then follows that

$$\hat{g}^{-1}\hat{g}' = i\eta^{ab}\hat{\Omega}_{a,-m}{}^{y_*}\hat{T}_{bm} \tag{2.12.8}$$

on the constrained subspace. But neither side of (2.12.8) has any sigma derivatives, so this relation, and hence (2.12.6), must be true on or off the constrained subspace.

Using the relation (2.12.6) in the partition function (2.12.3), we find that

$$\int (\mathcal{D}_M \hat{g})\, e^{iS_M} = \int (\mathcal{D}_F \hat{g})\, e^{iS_0}\delta[\hat{g}^{-1}(\partial_\sigma \hat{g} - \hat{g}')] = \int (\mathcal{D}\lambda \mathcal{D}_F \hat{g})\, e^{iS_F} \tag{2.12.9a}$$

$$\mathcal{D}_F \hat{g} \equiv \prod_{\tau,\sigma} d\hat{g}(x(\tau,\sigma)) \tag{2.12.9b}$$

where the $\hat{g}$ form of the constraint appears in the functional delta function, and $\mathcal{D}_F\hat{g}$ is the spacetime product of Haar measures on $\hat{G}$. The improved action S_F of the field theory on $\hat{G}$ is [21]

$$\begin{aligned} S_F &= -\frac{k}{2\pi\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}\left(\hat{g}^{-1}\partial\hat{g}\hat{g}^{-1}\bar{\partial}\hat{g}\right) - \frac{k}{12\pi\chi}\int \hat{\mathrm{Tr}}(\hat{g}^{-1}d\hat{g})^3 \\ &\quad + \frac{k}{\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}(\lambda\hat{g}^{-1}(\partial_\sigma\hat{g} - \hat{g}')) \end{aligned} \tag{2.12.10}$$

where $\hat{g}(\hat{T}, x(\tau,\sigma)) \in \hat{G}$ is the reduced affine group element and $\hat{g}'$ is defined in eq.(2.12.5c). Because of the formal quantum equivalence (2.12.9a), this form of the field theory on $\hat{G}$ is another central result of the chapter. Of course, the improved action S_F and the action $S_{\hat{F}} = S_0 + S_C$ in (2.12.2) are classically equivalent, differing only in the form of the constraint.

Summary of the quantum equivalences

Collecting the results (2.11.15a) and (2.12.9a), we have the formal quantum equivalences [21],

$$\int (\mathcal{D}g)\, e^{iS_{WZW}} = \int (\mathcal{D}_M \hat{g})\, e^{iS_M} = \int (\mathcal{D}\lambda \mathcal{D}_F \hat{g})\, e^{iS_F} \tag{2.12.11}$$

among all three formulations of WZW theory. Similarly, the results (2.6.16) and (2.11.16) may be combined to obtain the relations [21]

$$\langle \mathcal{F}[\beta^a(x^i(\tau,\sigma))]\rangle_G = \langle \mathcal{F}[\sum_m e^{im\sigma}\beta^{am}(x^{i\mu}(\tau))]\rangle_{\hat{G}_M} = \langle \mathcal{F}[\sum_m \beta^{am}(x^{i\mu}(\tau,\sigma))]\rangle_{\hat{G}_F} \tag{2.12.12}$$

among the formal averages of all three formulations.

2.13 WZW as a field theory in three dimensions

We finally turn to WZW as a three-dimensional field theory on G, a formulation which we will derive from the two-dimensional field theory on $\hat{G}$

$$\begin{array}{ccc} \hat{g}(x(\tau)) & \cdots & \hat{g}(x(\tau,\sigma)) \\ \vdots & & \updownarrow \text{modes} \\ g(x(\tau,\sigma)) & \cdots & g(x(\tau,\sigma,\tilde{\sigma})) \end{array} \tag{2.13.1}$$

by another application of the translation dictionary of Section 2.9. In this case, the partial transmutation of the base and target space is

$$x^{i\mu}(\tau,\sigma) \leftrightarrow x^i(\tau,\sigma,\tilde{\sigma}) \tag{2.13.2a}$$

$$\hat{g}(x(\tau,\sigma)) : \ \mathbb{R}\times S^1 \mapsto \hat{G}, \qquad g(x(\tau,\sigma,\tilde{\sigma})) : \ \mathbb{R}\times T^2 \mapsto G \tag{2.13.2b}$$

where $0 \le \tilde{\sigma} < 2\pi$ is an additional periodic coordinate.

We begin with the form of the two-dimensional field-theory on $\hat{G}$,

$$\begin{aligned} S_F &= -\frac{k}{2\pi\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}\left(\hat{g}^{-1}\partial\hat{g}\hat{g}^{-1}\bar{\partial}\hat{g}\right) - \frac{k}{12\pi\chi}\int \hat{\mathrm{Tr}}(\hat{g}^{-1}d\hat{g})^3 \\ &\quad + \frac{k}{\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}(\lambda\hat{g}^{-1}(\partial_\sigma\hat{g} - \hat{g}')) \end{aligned} \tag{2.13.3}$$

obtained in Section 2.12. Next, we define fields on G which are local in $\tilde{\sigma}$,

$$\beta^a(x(\tau,\sigma,\tilde{\sigma})) \equiv \sum_m e^{im\tilde{\sigma}}\beta^{am}(x(\tau,\sigma)), \qquad (\lambda(\tau,\sigma,\tilde{\sigma}))_I{}^J \equiv \sum_m e^{i(n-m)\tilde{\sigma}}(\lambda(\tau,\sigma))_{Im}{}^{Jn} \tag{2.13.4a}$$

$$g(T, x(\tau,\sigma,\tilde{\sigma})) \equiv e^{i\beta^a(x(\tau,\sigma,\tilde{\sigma}))T_a} \in G. \tag{2.13.4b}$$

Then the action (2.13.3) can be reexpressed as a three-dimensional field theory on G [21],

$$\begin{aligned} S_3 &= -\frac{k}{4\pi^2\chi}\int d\tau d\sigma d\tilde{\sigma}\mathrm{Tr}\left(g^{-1}\partial g g^{-1}\bar{\partial}g\right) - \frac{k}{24\pi^2\chi}\int \mathrm{Tr}(g^{-1}dg)^3\wedge d\tilde{\sigma} \\ &\quad + \frac{k}{2\pi\chi}\int d\tau d\sigma d\tilde{\sigma}\mathrm{Tr}(\lambda g^{-1}(\partial_\sigma - \partial_{\tilde{\sigma}})g) \end{aligned} \tag{2.13.5a}$$

$$\mathrm{Tr}(g^{-1}dg)^3\wedge d\tilde{\sigma} = d\tau d\sigma d\rho d\tilde{\sigma}\varepsilon^{ABC}\mathrm{Tr}(g^{-1}\partial_A g g^{-1}\partial_B g g^{-1}\partial_C g) \tag{2.13.5b}$$

$$A = (\tau,\sigma,\rho), \qquad \varepsilon^{012} = +1 \tag{2.13.5c}$$

with a four-dimensional WZW term. The first two terms of (2.13.5a) follow from (2.13.3) in a single step using the trace identity (2.9.17c), and the form of the constraint term follows from (2.9.17c) and (2.13.4a).

The three-dimensional action (2.13.5) completes the quartet of formulations of WZW theory announced in the introduction and shown in eq.(2.8.1).

It is clear from the derivation above that the three-dimensional form of WZW theory is equivalent to the other three formulations. Instead of developing translation dictionaries between this formulation and the formulations on $\hat{G}$, we confine ourselves here to showing a direct equivalence with the conventional WZW formulation on G,

$$\begin{array}{ccc} \hat{g}(x(\tau)) & \cdots & \hat{g}(x(\tau,\sigma)) \\ \vdots & & \vdots \\ g(x(\tau,\sigma)) & \xleftrightarrow{\text{constraint}} & g(x(\tau,\sigma,\tilde{\sigma})) \end{array} \tag{2.13.6}$$

in parallel with the demonstration at the end of Section 2.6.

To see this equivalence, we start with the three-dimensional formulation and solve its constraint,

$$(\partial_\sigma - \partial_{\tilde{\sigma}})g(x(\tau,\sigma,\tilde{\sigma})) = 0 \tag{2.13.7}$$

which is the equation of motion of the multiplier λ in (2.13.5a). This constraint can be simplified to

$$(\partial_\sigma - \partial_{\tilde{\sigma}})x^i(\tau,\sigma,\tilde{\sigma}) = 0 \tag{2.13.8}$$

and one also finds that

$$\beta(x(\tau,\sigma,\tilde{\sigma})) = e^{im(\sigma+\tilde{\sigma})}\beta^{am}(x(\tau)) \tag{2.13.9}$$

by using and eqs.(2.13.4a) and (2.13.8).

Then it is convenient to define new variables on T^2,

$$\sigma^+ \equiv \sigma + \tilde{\sigma}, \qquad \sigma^- \equiv \sigma - \tilde{\sigma}, \qquad 0 \leq \sigma^- < 2\pi \tag{2.13.10}$$

so that $x^i = x^i(\tau,\sigma^+)$. It follows from periodicity on T^2 that, as in (2.13.9), all the quantities of the theory on the constrained subspace are periodic functions $f_{2\pi}(x(\tau,\sigma^+))$ of σ^+ with period 2π, and moreover,

$$\int_0^{2\pi} d\sigma \int_0^{2\pi} d\tilde{\sigma} f_{2\pi}(x(\tau,\sigma^+)) = \int_0^{2\pi} d\sigma^- \int_0^{2\pi} d\sigma^+ f_{2\pi}(x(\tau,\sigma^+)). \tag{2.13.11}$$

Therefore, on the constrained subspace, the three-dimensional action (2.13.5a) has the form

$$S = -\frac{k}{2\pi\chi}\int d\tau d\sigma^+ \mathrm{Tr}\left(g^{-1}\partial g g^{-1}\bar{\partial} g\right) - \frac{k}{12\pi\chi}\int \mathrm{Tr}(g^{-1}dg)^3 \tag{2.13.12}$$

after doing the integration over σ^-. The three-form in (2.13.12) now contains a factor $d\tau d\sigma^+ d\rho$ so, with the identification $\sigma_{WZW} \equiv \sigma^+$, this is the conventional WZW action on G.

Chapter 3

Coset Constructions and the Gauged WZW Model

The coset constructions are a relatively simple solution to the master equation, formed from the affine-Sugawara construction and K-conjugation. This chapter treats them in the same way that the affine-Sugawara construction was treated in the previous chapter. We begin with the Hamiltonian of the coset construction and show how to reach the gauged WZW action. As in the last chapter, we begin on the Lie group manifold for orientation and then repeat the process on the affine Lie group manifold.

The coset constructions were first studied in Ref. [3, 4, 8], and the gauged WZW model was first studied in Ref. [13-17]. The treatment on the affine Lie group is original to this work, while the treatment on the Lie group is essentially the reverse of that in Ref. [30].

3.1 Gauged WZW on the Lie group

We first assume that the group G has a subgroup H, such that the quotient space G/H is a reductive coset space. In terms of the group's structure constants and Killing metric, this means that we can pick a basis such that

$$f_{AB}{}^{I} = f_{AI}{}^{B} = 0, \qquad \eta_{AI} = 0 \tag{3.1.1}$$

where $A, B \in h$ and $I \in g/h$. The classical basic Hamiltonian of the coset construction is then

$$H_0 = \int_0^{2\pi} \mathcal{H}_0, \qquad \mathcal{H}_0 = \frac{1}{2\pi} L^{ab}_{g/h,\infty}(E_a E_b + \bar{E}_a \bar{E}_b) \tag{3.1.2}$$

where $L^{ab}_{g/h,\infty} = P^{ab}_{g/h}/2k$ is the high-level form of the coset construction inverse inertial stress tensor. Here, $P^{ab}_{g/h} = \eta^{ab}_g - \eta^{ab}_h$ is a projector onto the g/h coset space.

Because of the relations in (3.1.1), the left- and right-invariant h affine currents have vanishing Poisson brackets

$$\{E_a, H_0\} = \{\bar{E}_a, H_0\} = 0, \qquad a \in h \tag{3.1.3}$$

with the basic Hamiltonian (3.1.2). We may consistently take the linear combination $E_a + \bar{E}_a$ to be a first-class constraint,

$$E_a + \bar{E}_a \approx 0, \qquad a \in h \tag{3.1.4}$$

because its bracket with itself vanishes

$$\{E_a(\sigma) + \bar{E}_a(\sigma), E_b(\sigma') + \bar{E}_b(\sigma')\} = 2\pi i f_{ab}{}^c[E_c(\sigma) + \bar{E}_c(\sigma)]\delta(\sigma - \sigma') \approx 0, \qquad a \in h \tag{3.1.5}$$

on the constrained subspace. (As discussed in Ref. [30], the left- and right-invariant h affine currents may also be individually set to constants, but this represents a gauge choice.)

Following Dirac's prescription, we gauge the constraint (3.1.4) by altering the Hamiltonian (3.1.2)

$$\mathcal{H}' = \mathcal{H}_0 + \Delta\mathcal{H}_1 \tag{3.1.6a}$$

$$\Delta\mathcal{H}_1 = \frac{1}{2\pi}\mathcal{A}_0^a(E_a + \bar{E}_a) \tag{3.1.6b}$$

where $\mathcal{A}_0^a$, $a \in h$ is a set of dim h Lagrange multipliers which enforce the constraint. It turns out that these form the time component of the gauge field of the gauged WZW model.

To achieve the full standard gauged WZW Hamiltonian, we add two more terms to the Hamiltonian (3.1.6)

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}_1 + \Delta\mathcal{H}_2 + \Delta\mathcal{H}_3 \tag{3.1.7a}$$

$$\Delta\mathcal{H}_2 = \frac{1}{4\pi}L_{h,\infty}^{ab}(E_a + \bar{E}_a)(E_b + \bar{E}_b) \tag{3.1.7b}$$

$$\Delta\mathcal{H}_3 = \frac{1}{4\pi}L_{h,\infty}^{ab}(2k\eta_{ac}\mathcal{A}_1^c - E_a + \bar{E}_a)(2k\eta_{bd}\mathcal{A}_1^d - E_b + \bar{E}_b) \tag{3.1.7c}$$

where $L_{h,\infty}^{ab} = P_h^{ab}/2k$ is the high-level form of inverse inertia tensor for the affine-Sugawara construction on the h subalgebra. The term $\Delta\mathcal{H}_2$ is quadratic in the constraint, and so will not affect the equations of motion on the constrained subspace. The auxiliary field $\mathcal{A}_1^a$, $a \in h$ is included as a perfect Gaussian in $\Delta\mathcal{H}_3$, and so can be immediately integrated out without affecting the averages of the other fields. The field $\mathcal{A}_1^a$ turns out to form the space component of the gauge field of the gauged WZW model.

Using the canonical representation (2.1.4) of the affine currents, the Hamiltonian (3.1.7) can be expressed in terms of canonical variables

$$\mathcal{H} = \mathcal{H}_{WZW} + \frac{1}{2\pi}\left[2\bar{\mathcal{A}}^a E_a + 2\mathcal{A}^a \bar{E}_a + k\eta_{ab}\mathcal{A}_1^a\mathcal{A}_1^b\right] \tag{3.1.8a}$$

$$\begin{aligned} &= \frac{2\pi}{k}\eta^{ab}e_a{}^i e_b{}^j p_i(B)p_j(B) + \frac{k}{8\pi}\eta_{ab}e_i{}^a e_j{}^b\partial_\sigma x^i\partial_\sigma x^j \\ &\quad +2\bar{\mathcal{A}}^a\left(e_a{}^i p_i(B) + \frac{k}{4\pi}\eta_{ab}e_i{}^b\partial_\sigma x^i\right) + 2\mathcal{A}^a\left(\bar{e}_a{}^i p_i(B) - \frac{k}{4\pi}\eta_{ab}\bar{e}_i{}^b\partial_\sigma x^i\right) \\ &\quad +\frac{k}{2\pi}\eta_{ab}\mathcal{A}_1^a\mathcal{A}_1^b \end{aligned} \tag{3.1.8b}$$

where $\mathcal{A}^a \equiv (\mathcal{A}_0^a + \mathcal{A}_1^a)/2$ and $\bar{\mathcal{A}}^a \equiv (\mathcal{A}_0^a - \mathcal{A}_1^a)/2$. This form of the gauged WZW Hamiltonian was first given in Ref. [30].

The Hamiltonian equation of motion

$$\partial_\tau x^i = \frac{4\pi}{k}\eta^{ab}e_a{}^i e_b{}^j p_j(B) + 2\bar{\mathcal{A}}^a e_a{}^i + 2\mathcal{A}^a\bar{e}_a{}^i \tag{3.1.9}$$

is easily obtained from (3.1.8). Eliminating p in favor of $\partial_\tau x$, one finds the gauged WZW action

$$S_G = \int d\tau d\sigma \mathcal{L}_G, \qquad \mathcal{L}_G = \mathcal{L}_{WZW} + \Delta\mathcal{L}_G \tag{3.1.10a}$$

$$\begin{aligned} \Delta\mathcal{L}_G &= -\frac{k}{2\pi}(\partial_\tau x^i + \partial_\sigma x^i)e_i{}^a\eta_{ab}\bar{\mathcal{A}}^b - \frac{k}{2\pi}(\partial_\tau x^i - \partial_\sigma x^i)\bar{e}_i{}^a\eta_{ab}\mathcal{A}^b \\ &\quad +\frac{k}{\pi}\eta_{ab}\mathcal{A}^a\bar{\mathcal{A}}^b - \frac{k}{\pi}\bar{\mathcal{A}}^a\Omega_a{}^c\eta_{cb}\mathcal{A}^b \end{aligned} \tag{3.1.10b}$$

where $\mathcal{L}_{WZW}$ is the conventional WZW Lagrange density in eq. (2.1.7b). In terms of the group variable g, the gauged WZW action is written [13-17]

$$S_G = S_{WZW} + \Delta S_G \tag{3.1.11a}$$

$$\Delta S_G = \frac{k}{\pi\chi}\int d\tau d\sigma \mathrm{Tr}(i\bar{\mathcal{A}}g^{-1}\partial g - i\mathcal{A}\bar{\partial}gg^{-1} + \mathcal{A}\bar{\mathcal{A}} - \mathcal{A}g\bar{\mathcal{A}}g^{-1}) \tag{3.1.11b}$$

where $\mathcal{A} \equiv \mathcal{A}^a T_a$ and $\bar{\mathcal{A}} \equiv \bar{\mathcal{A}}^a T_a$. This is the familiar form of the gauged WZW action. It is invariant under an affine H gauge transformation,

$$\delta g = i[g, \kappa] \tag{3.1.12a}$$

$$\delta\mathcal{A} = \partial\kappa + i[\mathcal{A}, \kappa], \qquad \delta\bar{\mathcal{A}} = \bar{\partial}\kappa + i[\bar{\mathcal{A}}, \kappa] \tag{3.1.12b}$$

$$\kappa \equiv \kappa^a(\sigma, \tau)T_a, \qquad a \in h \tag{3.1.12c}$$

where κ is the affine H gauge parameter.

3.2 Gauged WZW on the affine Lie group

We again begin with the classical basic Hamiltonian of the coset construction, this time in terms of the classical affine Lie derivatives of (2.4.2) and (2.4.3),

$$H_0 = L^{ab}_{g/h,\infty} \sum_m (E_a(-m)E_b(m) + \bar{E}_a(-m)\bar{E}_b(m)). \tag{3.2.1}$$

This Hamiltonian is to be interpreted as that of a classical mechanics, in the same way as (2.5.1a). The linear combinations of affine h Lie derivatives $E_a(m) + \bar{E}_a(m), a \in h$ may be taken as a set of first-class constraints

$$\{E_a(m) + \bar{E}_a(m), H_0\} = 0, \qquad a \in h \tag{3.2.2a}$$

$$\{E_a(m) + \bar{E}_a(m), E_b(n) + \bar{E}_b(n)\} = i f_{ab}{}^c [E_c(m+n) + \bar{E}_c(m+n)] \approx 0 \tag{3.2.2b}$$

because of the vanishing of their brackets with the basic Hamiltonian and with each other.

Following the logic of the previous section, we add three terms to the basic Hamiltonian,

$$H = H_0 + \Delta H_1 + \Delta H_2 + \Delta H_3 \tag{3.2.3a}$$

$$\Delta H_1 = \mathcal{A}_0^{am}(E_a(m) + \bar{E}_a(m)) \tag{3.2.3b}$$

$$\Delta H_1 = \frac{1}{2} L^{ab}_{h,\infty}(E_a(-m) + \bar{E}_a(-m))(E_b(m) + \bar{E}_b(m)) \tag{3.2.3c}$$

$$\Delta H_2 = \frac{1}{2} L^{ab}_{h,\infty}(2k\eta_{ac}\mathcal{A}_1^{cm} - E_a(-m) + \bar{E}_a(-m))(2k\eta_{bd}\mathcal{A}_1^{d,-m} - E_b(m) + \bar{E}_b(m)) \tag{3.2.3d}$$

where $\mathcal{A}_0^{am}$ and $\mathcal{A}_1^{am}, a \in h, m \in \mathbb{Z}$ are the components of the affine H gauge field. Since we are now dealing with a mechanical model (with a one-dimensional base space), we of course cannot identify $\mathcal{A}_0$ and $\mathcal{A}_1$ as time and space components of the gauge field. However, they still do perform a role analogous to that of their counterparts in the conventional gauged WZW model. As in the conventional gauged WZW model, the term ΔH_1 enforces the constraint, the term ΔH_2 is quadratic in the constraint, and so will not affect the equations of motion on the constrained subspace, and the term ΔH_3 can be eliminated by integrating out $\mathcal{A}_1$.

Writing the mechanical Hamiltonian H in terms of the classical reduced affine Lie derivatives (2.4.2,3), we have

$$\begin{aligned} H &= H_{WZW} + 2\bar{\mathcal{A}}^{am}E_a(m) + 2\mathcal{A}^{am}\bar{E}_a(m) + k\eta_{ab}\mathcal{A}_1^{a,-m}\mathcal{A}_1^{bm} \qquad (3.2.4a) \\ &= \frac{1}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p^-_{i\mu} p^-_{j\nu} + \left(\frac{k}{2}\hat{\Omega}_{a,-m}{}^{y_*} - e_{a,-m}{}^{i\mu} p^-_{i\mu}\right) \eta^{ab} \hat{\Omega}_{bm}{}^{y_*} \end{aligned}$$

$$+2\bar{\mathcal{A}}^{am} e_{am}{}^{i\mu} p^-_{i\mu} + 2\mathcal{A}^{am} \bar{e}_{am}{}^{i\mu} p^+_{i\mu} + k\eta_{ab}\mathcal{A}_1^{a,-m}\mathcal{A}_1^{bm} \tag{3.2.4b}$$

$$\begin{aligned} = & \frac{1}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p_{i\mu}(\hat{B}) p_{j\nu}(\hat{B}) + \frac{k}{4}\eta^{ab}\hat{\Omega}_{a,-m}{}^{y_*}\hat{\Omega}_{b,m}{}^{y_*} \\ & +\bar{\mathcal{A}}^{am}(2e_{am}{}^{i\mu} p_{i\mu}(\hat{B}) + k\hat{\Omega}_{am}{}^{y_*}) + \mathcal{A}^{am}(2\bar{e}_{am}{}^{i\mu} p_{i\mu}(\hat{B}) - k(\hat{\Omega}^{-1})_{am}{}^{y_*}) \\ & +k\eta_{ab}\mathcal{A}_1^{a,-m}\mathcal{A}_1^{bm} \end{aligned} \tag{3.2.4c}$$

where $\mathcal{A}^{am} \equiv (\mathcal{A}_0^{am} + \mathcal{A}_1^{am})/2$ and $\bar{\mathcal{A}}^{am} \equiv (\mathcal{A}_0^{am} - \mathcal{A}_1^{am})/2$. The Hamiltonian equation of motion for $\partial_\tau x$ is then

$$\partial_\tau x^{i\mu} = \frac{2}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p^-_{j\nu} - \eta^{ab} e_{a,-m}{}^{i\mu}\hat{\Omega}_{bm}{}^{y_*} + 2\bar{\mathcal{A}}^{am} e_{am}{}^{i\mu} + 2\mathcal{A}^{am}\bar{e}_{am}{}^{i\mu} \tag{3.2.5a}$$

$$= \frac{2}{k}\eta^{ab} e_{a,-m}{}^{i\mu} e_{bm}{}^{j\nu} p_{j\nu}(\hat{B}) + 2\bar{\mathcal{A}}^{am} e_{am}{}^{i\mu} + 2\mathcal{A}^{am}\bar{e}_{am}{}^{i\mu}. \tag{3.2.5b}$$

Eliminating $p_{i\mu}$ in favor of $\partial_\tau x^{i\mu}$, we find the mechanical formulation of the gauged WZW action on the affine Lie group,

$$S_{GM} = \int d\tau L_{GM}, \qquad L_{GM} = L_M + \Delta L_{GM} \tag{3.2.6a}$$

$$\begin{aligned} \Delta L_{GM} = & -k(\partial_\tau x^{i\mu} e_{i\mu}{}^{b,-m}\eta_{ba} + \hat{\Omega}_{am}{}^{y_*})\bar{\mathcal{A}}^{am} - k(\partial_\tau x^{i\mu}\bar{e}_{i\mu}{}^{b,-m}\eta_{ba} - (\hat{\Omega}^{-1})_{am}{}^{y_*})\mathcal{A}^{am} \\ & +2k\eta_{ab}\mathcal{A}^{a,-m}\bar{\mathcal{A}}^{bm} - 2k\bar{\mathcal{A}}^{am}\hat{\Omega}_{am}{}^{c,-n}\eta_{cb}\mathcal{A}^{bn} \end{aligned} \tag{3.2.6b}$$

where L_M is the mechanical WZW Lagrangian in eq. (2.5.7b) or (2.5.8). Using the translation dictionary of Section 2.9 and the mode identities

$$\mathcal{A}^a(\tau,\sigma) = \sum_m e^{im\sigma}\mathcal{A}^{am}(\tau), \qquad \bar{\mathcal{A}}^a(\tau,\sigma) = \sum_m e^{im\sigma}\bar{\mathcal{A}}^{am}(\tau) \tag{3.2.7}$$

this action is easily checked to be equivalent to the conventional gauged WZW action (3.1.10). Following the method in Section 2.11.2, we may also obtain the $\hat{g}$ form of the mechanical gauged WZW action,

$$S_{GM} = S_M + \Delta S_{GM} \tag{3.2.8a}$$

$$\Delta S_{GM} = \frac{k}{\chi}\int d\tau \hat{\mathrm{Tr}}(i\hat{\bar{\mathcal{A}}}\hat{g}^{-1}(\partial_\tau\hat{g} + \hat{g}') - i\hat{\mathcal{A}}(\partial_\tau\hat{g} - \hat{g}')\hat{g}^{-1} + 2\hat{\mathcal{A}}\hat{\bar{\mathcal{A}}} - 2\hat{\mathcal{A}}\hat{g}\hat{\bar{\mathcal{A}}}\hat{g}^{-1}) \tag{3.2.8b}$$

$$(\hat{g}')_{Im}{}^{Jn} \equiv i(n-m)\hat{g}_{Im}{}^{Jn} \tag{3.2.8c}$$

where $\hat{\mathcal{A}} \equiv \mathcal{A}^{am}\hat{T}_{am}$ and $\hat{\bar{\mathcal{A}}} \equiv \bar{\mathcal{A}}^{am}\hat{T}_{am}$, S_M is the mechanical WZW action in

eq. (2.11.3), and $\hat{g}(\hat{T}, x(\tau)) \in \hat{G}$ is the reduced affine group element in affine matrix representation $\hat{T}$ (see Section 2.7).

Like the conventional gauged WZW model, the mechanical action (3.2.8) is invariant under an affine H gauge transformation,

$$\delta\hat{g} = i[\hat{g}, \hat{\kappa}] \tag{3.2.9a}$$

$$\delta\hat{\mathcal{A}} = \frac{1}{2}(\partial_\tau\hat{\kappa} + \hat{\kappa}') + i[\hat{\mathcal{A}}, \lambda], \qquad \delta\hat{\bar{\mathcal{A}}} = \frac{1}{2}(\partial_\tau\hat{\kappa} - \hat{\kappa}') + i[\hat{\bar{\mathcal{A}}}, \lambda] \tag{3.2.9b}$$

$$\hat{\kappa} \equiv \kappa^{am}(\tau)\hat{T}_{am}, \qquad a \in h, \quad m \in \mathbb{Z} \tag{3.2.9c}$$

where $\hat{\kappa}$ is the affine H gauge parameter, and $\hat{\kappa}'$ is defined in analogy to $\hat{g}'$ in (3.2.8c). The parameter $\hat{\kappa}$ is related to the parameter in eq. (3.1.12)

$$\kappa^a(\sigma, \tau) = \sum_m e^{im\sigma}\kappa^{am}(\tau) \tag{3.2.10}$$

under the translation dictionary of Section 2.9. In order to verify the affine H gauge invariance, we obtain the term-by-term results,

$$\delta S_M = -\frac{ki}{2\chi}\int d\tau \hat{\mathrm{Tr}}\left(\hat{g}^{-1}(\partial_\tau\hat{g} + \hat{g}')(\partial_\tau\hat{\kappa} - \hat{\kappa}') - (\partial_\tau\hat{g} - \hat{g}')\hat{g}^{-1}(\partial_\tau\hat{\kappa} + \hat{\kappa}')\right) \tag{3.2.11a}$$

$$\begin{aligned}\delta&\left(\frac{k}{\chi}\int d\tau\hat{\mathrm{Tr}}\left(i\hat{\bar{\mathcal{A}}}\hat{g}^{-1}(\partial_\tau\hat{g} + \hat{g}')\right)\right)\\ &= \frac{k}{2\chi}\int d\tau\hat{\mathrm{Tr}}\left(i\hat{g}^{-1}(\partial_\tau g + \hat{g}')(\partial_\tau\hat{\kappa} - \hat{\kappa}') - 2\bar{\mathcal{A}}(\partial_\tau\hat{\kappa} + \hat{\kappa}') + 2\bar{\mathcal{A}}\hat{g}^{-1}(\partial_\tau\hat{\kappa} + \hat{\kappa}')\hat{g}\right)\end{aligned} \tag{3.2.11b}$$

$$\begin{aligned}\delta&\left(\frac{k}{\chi}\int d\tau\hat{\mathrm{Tr}}\left(-i\hat{\mathcal{A}}(\partial_\tau\hat{g} - \hat{g}')\hat{g}^{-1}\right)\right)\\ &= \frac{k}{2\chi}\int d\tau\hat{\mathrm{Tr}}\left(-i(\partial_\tau\hat{g} - \hat{g}')\hat{g}^{-1}(\partial_\tau\hat{\kappa} + \hat{\kappa}') - 2\mathcal{A}(\partial_\tau\hat{\kappa} - \hat{\kappa}') + 2\mathcal{A}\hat{g}(\partial_\tau\hat{\kappa} - \hat{\kappa}')\hat{g}^{-1}\right)\end{aligned} \tag{3.2.11c}$$

$$\delta\left(\frac{k}{\chi}\int d\tau\hat{\mathrm{Tr}}(2\hat{\mathcal{A}}\hat{\bar{\mathcal{A}}})\right) = \frac{k}{\chi}\int d\tau\hat{\mathrm{Tr}}\left(\mathcal{A}(\partial_\tau\hat{\kappa} - \hat{\kappa}') + \bar{\mathcal{A}}(\partial_\tau\hat{\kappa} + \hat{\kappa}')\right) \tag{3.2.11d}$$

$$\delta\left(\frac{k}{\chi}\int d\tau\hat{\mathrm{Tr}}(-2\hat{\mathcal{A}}\hat{g}\hat{\bar{\mathcal{A}}}\hat{g}^{-1})\right) = -\frac{k}{\chi}\int d\tau\hat{\mathrm{Tr}}\left(\bar{\mathcal{A}}\hat{g}^{-1}(\partial_\tau\hat{\kappa} + \hat{\kappa}')\hat{g} + \mathcal{A}(\hat{g}(\partial_\tau\hat{\kappa} - \hat{\kappa}')\hat{g}^{-1}\right) \tag{3.2.11e}$$

whose sum δS_{GM} is zero on inspection.

3.3 Constrained formulations of gauged WZW

So far, we have described the formulations of the gauged WZW model depicted in the left-hand column of the picture

$$\begin{array}{ccc} \hat{g}(x(\tau)) & \xleftrightarrow{\text{constraint}} & \hat{g}(x(\tau,\sigma)) \\ \text{modes}\Big\updownarrow & & \Big\updownarrow\text{modes} \\ g(x(\tau,\sigma)) & \xleftrightarrow{\text{constraint}} & g(x(\tau,\sigma,\tilde{\sigma})) \end{array} \tag{3.3.1}$$

that was introduced in Section 2.8. For completeness, we now give the results for the constrained formulations in the right-hand column of the picture.

3.3.1 Gauged WZW as a field theory on the affine group

To express the gauged WZW model as a two-dimensional field theory on the affine Lie group, we follow the steps in Section 2.6. We begin by finding an operator P_G to use as the generator of spatial translations.

A reasonable candidate is the momentum operator P_{WZW} from eq. (2.6.4),

$$P_{WZW} = \frac{1}{2k}\eta^{ab}\sum_m \Big(E_a(-m)E_b(m) - \bar{E}_a(-m)\bar{E}_b(m)\Big) \tag{3.3.2a}$$

$$\{H, P_{WZW}\} = 2m\bar{\mathcal{A}}^{am}E_a(m) + 2m\mathcal{A}^{am}\bar{E}_a(m) \tag{3.3.2b}$$

but, as shown in (3.3.2b), this object does not commute with the gauged WZW Hamiltonian (3.2.4a). We correct this by adding two terms to the candidate momentum operator,

$$P_G = P_{WZW} + im\mathcal{A}^{am}p^{\mathcal{A}}_{am} + im\bar{\mathcal{A}}^{am}p^{\bar{\mathcal{A}}}_{am} \tag{3.3.3}$$

where $p^{\mathcal{A}}_{am}$ and $p^{\bar{\mathcal{A}}}_{am}$, $a \in h$, $m \in \mathbb{Z}$ are momenta* canonically conjugate to the gauge fields $\mathcal{A}^{am}$ and $\bar{\mathcal{A}}^{am}$,

$$\{\mathcal{A}^{am}, p^{\mathcal{A}}_{bn}\} = i\delta_{bn}{}^{am}, \qquad \{\bar{\mathcal{A}}^{am}, p^{\bar{\mathcal{A}}}_{bn}\} = i\delta_{bn}{}^{am}. \tag{3.3.4}$$

The revised momentum operator in (3.3.3) then commutes with the Hamiltonian as required,

$$\{H, P_G\} = 0 \tag{3.3.5a}$$

*In a complete description of the dynamics of the canonical momenta $p^{\mathcal{A}}$ and $p^{\bar{\mathcal{A}}}$, one sets them equal to zero in a system of second-class constraints, introducing Dirac brackets. This is not necessary for our purposes here, but the complete treatment on the Lie group is given in Ref. [30].

$$\partial_\sigma A \equiv i\{P_G, A\} \tag{3.3.5b}$$

and so may be used to define spatial traslations, according to (3.3.5b).

We may now find the spatial derivatives

$$\partial_\sigma x^{i\mu} = \eta^{ab} e_{a,-m}{}^{i\mu} \hat{\Omega}_{bm}{}^{y_*} \tag{3.3.6a}$$

$$\partial_\sigma \mathcal{A}^{am} = im\mathcal{A}^{am}, \qquad \partial_\sigma \bar{\mathcal{A}}^{am} = im\bar{\mathcal{A}}^{am} \tag{3.3.6b}$$

of the affine group coordinate x and the gauge fields $\mathcal{A}$ and $\bar{\mathcal{A}}$. In analogy to the procedure in Section 2.6, the equation (3.3.6b) is interpreted as a constraint on the gauge fields.

From this point, it is a simple matter to follow the steps of Section 2.6, obtaining the action of the two-dimensional field theory on $\hat{G}$,

$$S_{G\hat{F}} = \int d\tau d\sigma \mathcal{L}_{G\hat{F}}, \qquad \mathcal{L}_{G\hat{F}} = \mathcal{L}_{\hat{F}} + \Delta\mathcal{L}_{G\hat{F}} \tag{3.3.7a}$$

$$\begin{aligned}\Delta\mathcal{L}_{G\hat{F}} \;=\; & -\frac{k}{2\pi}(\partial_\tau x^{i\mu} + \partial_\sigma x^{i\mu}) e_{i\mu}{}^{a,-m}\eta_{ab}\bar{\mathcal{A}}^{bm} - \frac{k}{2\pi}(\partial_\tau x^{i\mu} - \partial_\sigma x^{i\mu})\bar{e}_{i\mu}{}^{a,-m}\eta_{ab}\mathcal{A}^{bm} \\ & + \frac{k}{\pi}\eta_{ab}\mathcal{A}^{a,-m}\bar{\mathcal{A}}^{bm} - \frac{k}{\pi}\bar{\mathcal{A}}^{am}\Omega_{am}{}^{c,-n}\eta_{cb}\mathcal{A}^{bn} \\ & + k\lambda^{\mathcal{A}}_{am}(\partial_\sigma \mathcal{A}^{am} - im\mathcal{A}^{am}) + k\bar{\lambda}^{\mathcal{A}}_{am}(\partial_\sigma \bar{\mathcal{A}}^{am} - im\bar{\mathcal{A}}^{am})\end{aligned} \tag{3.3.7b}$$

where $\mathcal{L}_{\hat{F}}$ is the Lagrange density of the constrained two-dimensional WZW model on $\hat{G}$ in eq. (2.6.24), and $\lambda^{\mathcal{A}}$ and $\bar{\lambda}^{\mathcal{A}}$ are Lagrange multipliers that enforce the constraint (3.3.6b).

To express this formulation in terms of the affine group variable $\hat{g}$, we follow the steps of Sections 2.7 and 2.12 to obtain the action,

$$S_{GF} = S_F + \Delta S_{GF} \tag{3.3.8a}$$

$$\begin{aligned}\Delta S_{GF} \;=\; & \frac{k}{\pi\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}(i\hat{\bar{\mathcal{A}}}\hat{g}^{-1}\partial\hat{g} - i\hat{\mathcal{A}}\bar{\partial}\hat{g}\hat{g}^{-1} + \hat{\mathcal{A}}\hat{\bar{\mathcal{A}}} - \hat{\mathcal{A}}\hat{g}\hat{\bar{\mathcal{A}}}\hat{g}^{-1}) \\ & + \frac{k}{\chi}\int d\tau d\sigma \hat{\mathrm{Tr}}\left(\lambda^{\mathcal{A}}(\partial_\sigma\hat{\mathcal{A}} - \hat{\mathcal{A}}') + \bar{\lambda}^{\mathcal{A}}(\partial_\sigma\hat{\bar{\mathcal{A}}} - \hat{\bar{\mathcal{A}}}')\right)\end{aligned} \tag{3.3.8b}$$

where $\lambda^{\mathcal{A}} \equiv \lambda^{\mathcal{A}}_{a,-m}\eta^{ab}\hat{T}_{am}$, $\bar{\lambda}^{\mathcal{A}} \equiv \bar{\lambda}^{\mathcal{A}}_{a,-m}\eta^{ab}\hat{T}_{am}$, $\hat{\mathcal{A}}'$ and $\hat{\bar{\mathcal{A}}}'$ are defined in analogy to $\hat{g}'$ in (3.2.8c), and S_F is given in eq. (2.12.10). In this form, the affine H gauge transformation is

$$\delta\hat{g} = i[\hat{g}, \hat{\kappa}] \tag{3.3.9a}$$

$$\delta\hat{\mathcal{A}} = \partial\hat{\kappa} + i[\hat{\mathcal{A}}, \hat{\kappa}], \qquad \delta\hat{\bar{\mathcal{A}}} = \bar{\partial}\hat{\kappa} + i[\hat{\bar{\mathcal{A}}}, \hat{\kappa}] \tag{3.3.9b}$$

$$\hat{\kappa} \equiv \kappa^{am}(\sigma,\tau)\hat{T}_{am}, \qquad a \in h, \quad m \in \mathbb{Z} \tag{3.3.9c}$$

where the gauge parameter κ is constrained so that $\partial_\sigma \kappa^{am} = im\kappa^{am}$. Notice that the affine H gauge transformation in this formulation bears a striking resemblance to the transformation in the conventional formulation (3.1.12).

3.3.2 Gauged WZW as a field theory in three dimensions

Following the steps of Section 2.13, we define fields that are local in a third coordinate $\tilde{\sigma}$,

$$\mathcal{A}^a(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{im\tilde{\sigma}}\mathcal{A}^{am}(\tau,\sigma), \qquad \bar{\mathcal{A}}^a(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{im\tilde{\sigma}}\bar{\mathcal{A}}^{am}(\tau,\sigma) \tag{3.3.10a}$$

$$(\lambda^{\mathcal{A}}(\tau,\sigma,\tilde{\sigma}))_I{}^J \equiv \sum_m e^{i(n-m)\tilde{\sigma}}(\lambda^{\mathcal{A}}(\tau,\sigma,\tilde{\sigma}))_{Im}{}^{Jn} \tag{3.3.10b}$$

$$(\bar{\lambda}^{\mathcal{A}}(\tau,\sigma,\tilde{\sigma}))_I{}^J \equiv \sum_m e^{i(n-m)\tilde{\sigma}}(\bar{\lambda}^{\mathcal{A}}(\tau,\sigma,\tilde{\sigma}))_{Im}{}^{Jn} \tag{3.3.10c}$$

to supplement the definitions in (2.13.4). Then we may obtain the formulation of the gauged WZW model as a three-dimensional field theory on the Lie group,

$$S_{G3} = S_3 + \Delta S_{G3} \tag{3.3.11a}$$

$$\begin{aligned}\Delta S_{G3} &= \frac{k}{2\pi^2\chi}\int d\tau d\sigma d\tilde{\sigma}\mathrm{Tr}(i\bar{\mathcal{A}}g^{-1}\partial g - i\mathcal{A}\bar{\partial}gg^{-1} + \mathcal{A}\bar{\mathcal{A}} - \mathcal{A}g\bar{\mathcal{A}}g^{-1}) \\ &\quad + \frac{k}{2\pi\chi}\int d\tau d\sigma d\tilde{\sigma}\mathrm{Tr}(\lambda^{\mathcal{A}}(\partial_\sigma - \partial_{\tilde{\sigma}})\mathcal{A}) + \frac{k}{2\pi\chi}\int d\tau d\sigma d\tilde{\sigma}\mathrm{Tr}(\bar{\lambda}^{\mathcal{A}}(\partial_\sigma - \partial_{\tilde{\sigma}})\bar{\mathcal{A}})\end{aligned} \tag{3.3.11b}$$

where S_3 is given in eq. (2.13.5). The affine H gauge transformation in this formulation bears the same form as that in the convertional formulation (3.1.12), except that the gauge parameter $\kappa^a(\tau,\sigma,\tilde{\sigma})$ now depends on the third coordinate, but is constrained so that $(\partial_\sigma - \partial_{\tilde{\sigma}})\kappa = 0$.

Chapter 4

The Generic Affine-Virasoro Constructions and Action

Among the solutions of the Virasoro master equation, the affine-Sugawara constructions and the coset constructions represent special points of particularly high symmetry. The generic solution to the master equation possesses no special symmetry, other than that guaranteed by K-conjugation. As is well known [18, 22], that symmetry is given by the diffeomorphism group, embedded within affine $G \times G$.

In this chapter, we apply the methods of the last two chapters to the generic affine-Virasoro construction. As before, we work from the Hamiltonian to the action, and we consider first the case on the Lie group manifold, and then on the affine Lie group manifold.

The generic affine-Virasoro construction was first studied in Ref. [9, 10]. The non-linear form of the generic affine-Virasoro action was given in Ref. [18]. Here, we focus on the linearized form of the action, which was given on the Lie group in Ref. [22]. The treatment on the affine Lie group is original to this work.

4.1 The generic affine-Virasoro action on the Lie group

Given a high-level solution of the Virasoro master equation

$$L_{\infty}^{ab} = P^{ab}/2k \tag{4.1.1}$$

the classical basic Hamiltonian of the generic affine-Virasoro construction is [18],

$$H_0 = \int_0^{2\pi} d\sigma \mathcal{H}, \tag{4.1.2a}$$

$$\mathcal{H}_0 = \frac{1}{2\pi} L_\infty^{ab}(E_a E_b + \bar{E}_a \bar{E}_b). \tag{4.1.2b}$$

This basic Hamiltonian has vanishing Poisson brackets with the operators K and $\bar{K}$ of the K-conjugate theory,

$$K \equiv \frac{1}{2\pi} \tilde{L}_\infty^{ab} E_a E_b, \qquad \bar{K} \equiv \frac{1}{2\pi} \tilde{L}_\infty^{ab} \bar{E}_a \bar{E}_b \tag{4.1.3a}$$

$$\{H_0, K(\sigma)\} = \{H_0, \bar{K}(\sigma)\} = 0 \tag{4.1.3b}$$

where $\tilde{L}_\infty^{ab} = \tilde{P}^{ab}/2k = \eta^{ab}/2k - L_\infty^{ab}$ is the high-level K-conjugate solution of the master equation corresponding to L_∞^{ab}. These operators generate the algebra of Diff $S_1 \times$ Diff S_1,

$$\{K(\sigma), K(\sigma')\} = i[K(\sigma) + K(\sigma')]\partial_\sigma \delta(\sigma - \sigma') \tag{4.1.4a}$$

$$\{\bar{K}(\sigma), \bar{K}(\sigma')\} = -i[\bar{K}(\sigma) + \bar{K}(\sigma')]\partial_\sigma \delta(\sigma - \sigma') \tag{4.1.4b}$$

$$\{K(\sigma), \bar{K}(\sigma')\} = 0 \tag{4.1.4c}$$

and so may be taken as a set of first-class constraints,

$$K \approx 0, \qquad \bar{K} \approx 0 \tag{4.1.5}$$

because their brackets with each other vanish on the constrained subspace.

As in the last chapter, we follow Dirac's prescription, and gauge the constraint (4.1.5) by altering the Hamiltonian [18]

$$\mathcal{H}' = \mathcal{H}_0 + \Delta\mathcal{H}_1 \tag{4.1.6a}$$

$$\Delta\mathcal{H}_1 = \frac{1}{2\pi}(v \tilde{L}_\infty^{ab} E_a E_b + \bar{v} \tilde{L}_\infty^{ab} \bar{E}_a \bar{E}_b) \tag{4.1.6b}$$

where v and $\bar{v}$ are Lagrange multipliers which enforce the constraints. They may be identified with a metric on the world-sheet [18].

This Hamiltonian may be directly translated into an action, leading to the non-linear affine-Virasoro action given in Ref. [18]. Here, we will instead follow the example of the last chapter, introducing auxiliary fields which will lead to relatively simple equations of motion, and eventually the linearized affine-Virasoro action of Ref. [22].

We begin by defining two matrices,

$$W_a{}^b \equiv \delta_a{}^b + \alpha \tilde{P}_a{}^b, \qquad (W^{-1})_a{}^b = \delta_a{}^b + \frac{1}{2}(v-1)\tilde{P}_a{}^b \tag{4.1.7a}$$

$$\bar{W}_a{}^b \equiv \delta_a{}^b + \bar{\alpha} \tilde{P}_a{}^b, \qquad (\bar{W}^{-1})_a{}^b = \delta_a{}^b + \frac{1}{2}(\bar{v}-1)\tilde{P}_a{}^b \tag{4.1.7b}$$

$$\alpha \equiv \frac{1-v}{1+v}, \qquad \bar{\alpha} \equiv \frac{1-\bar{v}}{1+\bar{v}}. \tag{4.1.7c}$$

Using these matrices, we add two terms to the Hamiltonian (4.1.6),

$$\mathcal{H} = \mathcal{H}' + \Delta\mathcal{H}_2 \tag{4.1.8a}$$

$$\begin{aligned}\Delta\mathcal{H}_2 &= -\frac{1}{2\pi k}\left(k\mathcal{B}_a - (W^{-1})_a{}^d E_d\right)\eta^{ac}W_c{}^b\left(k\mathcal{B}_b - (W^{-1})_b{}^d E_d\right) \\ &\quad -\frac{1}{2\pi k}\left(k\bar{\mathcal{B}}_a - (\bar{W}^{-1})_a{}^d \bar{E}_d\right)\eta^{ac}\bar{W}_c{}^b\left(k\bar{\mathcal{B}}_b - (\bar{W}^{-1})_b{}^d \bar{E}_d\right). \end{aligned} \tag{4.1.8b}$$

Here, $\mathcal{B}_a$ and $\bar{\mathcal{B}}_a$, $a = 1, \ldots, \dim g$ are a set of auxiliary fields, called the auxiliary connections for reasons discussed below. Since $\Delta\mathcal{H}_2$ is quadratic in $\mathcal{B}$ and $\bar{\mathcal{B}}$, the auxiliary connections are easily integrated out, showing that the theory is unchanged for averages of functions of x and p.

For passage to the action, it is convenient to rearrange H as follows,

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_{WZW} \\ &\quad -\frac{\alpha k^2}{\pi}\tilde{L}^{ab}_{\infty}\mathcal{B}_a\mathcal{B}_b - \frac{1}{2\pi k}(k\mathcal{B}_a - E_a)\eta^{ab}(k\mathcal{B}_b - E_b) \\ &\quad -\frac{\bar{\alpha} k^2}{\pi}\tilde{L}^{ab}_{\infty}\bar{\mathcal{B}}_a\bar{\mathcal{B}}_b - \frac{1}{2\pi k}(k\bar{\mathcal{B}}_a - \bar{E}_a)\eta^{ab}(k\bar{\mathcal{B}}_b - \bar{E}_b)\end{aligned} \tag{4.1.9a}$$

$$\dot{x}^i = -\frac{4\pi}{k}e_a{}^i\eta^{ab}e_b{}^j p_j(B) + 2e_a{}^i\eta^{ab}(\mathcal{B}_b - \Omega_b{}^c\bar{\mathcal{B}}_c). \tag{4.1.9b}$$

Because of the simple equation of motion in (4.1.9b), we obtain the linearized form [22]

$$S_{AV} = \int d\tau d\sigma \mathcal{L}_{AV} \tag{4.1.10a}$$

$$\begin{aligned}\mathcal{L}_{AV} &= -\frac{k}{8\pi}\eta_{ab}e_i{}^a e_j{}^b(\partial_\tau x^i\partial_\tau x^j - \partial_\sigma x^i\partial_\sigma x^j) + \frac{k}{4\pi}B_{ij}\partial_\tau x^i\partial_\sigma x^j \\ &\quad +\frac{\alpha k^2}{\pi}\tilde{L}^{ab}_{\infty}\mathcal{B}_a\mathcal{B}_b + \frac{k}{2\pi}(\partial_\tau x^i - \partial_\sigma x^i)e_i{}^a\mathcal{B}_a \\ &\quad +\frac{\bar{\alpha} k^2}{\pi}\tilde{L}^{ab}_{\infty}\bar{\mathcal{B}}_a\bar{\mathcal{B}}_b + \frac{k}{2\pi}(\partial_\tau x^i + \partial_\sigma x^i)\bar{e}_i{}^a\bar{\mathcal{B}}_a \\ &\quad +\frac{k}{\pi}\eta^{ab}\mathcal{B}_a\Omega_b{}^c\bar{\mathcal{B}}_c\end{aligned} \tag{4.1.10b}$$

of the affine-Virasoro action. Note that the first two terms in $\mathcal{L}$ almost comprise the WZW Lagrange density $\mathcal{L}_{WZW}$ in eq. (2.1.7), but the kinetic energy term has the wrong sign. This sign may be corrected via a redefinition of the auxiliary fields (see Ref. [22]), but this is not necessary for our purposes. As discussed in Ref. [22], the

auxiliary connections $\mathcal{B}, \bar{\mathcal{B}}$ may be integrated out of the linearized action (4.1.10) to recover the non-linear form of the action of Ref. [18].

In terms of the group variable g, the linearized affine-Virasoro action is written [19, 23]

$$S_{AV} = S_{WZW} + \Delta S_{AV} \tag{4.1.11a}$$

$$\begin{aligned} \Delta S_{AV} &= \int d\tau \left(\frac{k^2 \alpha}{\pi \chi^2} \tilde{L}^{ab}_{\infty} \mathrm{Tr}(T_a \mathcal{B}) \mathrm{Tr}(T_b \mathcal{B}) + \frac{k^2 \bar{\alpha}}{\pi \chi^2} \tilde{L}^{ab}_{\infty} \mathrm{Tr}(T_a \bar{\mathcal{B}}) \mathrm{Tr}(T_b \bar{\mathcal{B}}) \right. \\ &\quad \left. - \frac{k}{\pi \chi} \mathrm{Tr}(\bar{D}_{\mathcal{B}} g D_{\mathcal{B}} g^{-1}) \right) \end{aligned} \tag{4.1.11b}$$

$$\mathcal{B} \equiv \mathcal{B}_a \eta^{ab} T_b, \qquad \bar{\mathcal{B}} \equiv \bar{\mathcal{B}}_a \eta^{ab} T_b, \qquad D_{\mathcal{B}} \equiv \partial + i\mathcal{B}, \qquad \bar{D}_{\mathcal{B}} \equiv \bar{\partial} + i\bar{\mathcal{B}}. \tag{4.1.11c}$$

where S_{WZW} is the WZW action (2.1.9), and we have defined the covariant derivatives $D_{\mathcal{B}}$ and $\bar{D}_{\mathcal{B}}$.

In this form, the theory is clearly seen as a Diff S_2-gauged WZW model, which bears an intriguing resemblance to the form of the usual (Lie algebra) gauged WZW model (3.1.11). As we shall see in the following section, this resemblance is due to the Diff $S_2(K)$ transformations of the auxiliary connections.

4.2 Gauge invariance of the generic affine-Virasoro action

As mentioned above, the generic affine-Virasoro action is invariant under the group of diffeomorphisms Diff $S_2(K)$ generated by the K-conjugate stress tensors. For the form of the action (4.1.10), the transformations are

$$\delta x^i = \kappa^a e_a{}^i + \bar{\kappa}^a \bar{e}_a{}^i \tag{4.2.1a}$$

$$\delta \alpha = -\bar{\partial}\xi + \xi \overleftrightarrow{\partial} \alpha, \qquad \delta \bar{\alpha} = -\partial \bar{\xi} + \bar{\xi} \overleftrightarrow{\bar{\partial}} \bar{\alpha} \tag{4.2.1b}$$

$$\delta \mathcal{B}_a = f_{ab}{}^c \kappa^b \mathcal{B}_c + \eta_{ab} \partial \kappa^b, \quad \delta \bar{\mathcal{B}}_a = f_{ab}{}^c \bar{\kappa}^b \bar{\mathcal{B}}_c + \eta_{ab} \bar{\partial} \bar{\kappa}^b \tag{4.2.1c}$$

$$\kappa^a \equiv 2k\xi \tilde{L}^{ab}_{\infty} \mathcal{B}_b, \quad \bar{\kappa}^a \equiv 2k\bar{\xi} \tilde{L}^{ab}_{\infty} \bar{\mathcal{B}}_b \tag{4.2.1d}$$

where $\xi(\tau, \sigma)$ and $\bar{\xi}(\tau, \sigma)$ are the diffeomorphism parameters. For the group element form (4.1.11) of the action, the transformations

$$\delta g = gi\kappa - i\bar{\kappa} g \tag{4.2.2a}$$

$$\delta \mathcal{B} = \partial \kappa + i[\mathcal{B}, \kappa], \quad \delta \bar{\mathcal{B}} = \bar{\partial} \bar{\kappa} + i[\bar{\mathcal{B}}, \bar{\kappa}] \tag{4.2.2b}$$

$$\kappa \equiv \kappa^a T_a, \qquad \bar{\kappa} \equiv \bar{\kappa}^a T_a \tag{4.2.2c}$$

replace those for x and $\mathcal{B}$ in (4.2.1).

Following Refs. [18, 22], we remark on the embedding of Diff $S_2(K)$ in (classical) affine $G \times G$. The transformation of the group element g in eq.(4.2.2a) shows that infinitesimal Diff $S_2(K)$ transformations are particular transformations in (classical) affine $G \times G$, with the field-dependent local Lie $g \times$ Lie g gauge parameters $\kappa, \bar{\kappa}$ in eq. (4.2.1d). In this sense, Diff $S_2(K)$ is embedded locally in (classical) affine $G \times G$. Moreover, the result (4.2.2b) shows that the auxiliary connections $\mathcal{B}, \bar{\mathcal{B}}$ transform under Diff $S_2(K)$ as local Lie $g \times$ Lie g gauge fields, or connections, with the same gauge parameters $\kappa, \bar{\kappa}$. The Diff $S_2(K)$ transformations of the auxiliary connections are also responsible for the intriguing resemblance of the linearized action (4.1.11) to the form of the usual (Lie algebra) gauged WZW model in (3.1.11). The embedding of Diff S_2 in the affine algebra is the underlying geometry of the generic affine-Virasoro construction, and this geometry will continue to play a central role in the other formulations of the action below.

Because verification of the diffeomorphism invariance of the linearized actions involves considerable algebra, we give the term-by-term results for the g form of the final action in (4.1.11),

$$\delta S_{WZW} = -\frac{ik}{\pi\chi}\int d\tau d\sigma\, \mathrm{Tr}(g^{-1}\partial g\bar{\partial}\kappa - \bar{\partial}g g^{-1}\partial\bar{\kappa}) \tag{4.2.3a}$$

$$\delta\left(-\frac{k}{\pi\chi}\int d\tau d\sigma \mathrm{Tr}(\bar{D}_{\mathcal{B}} g D_{\mathcal{B}} g^{-1})\right) = -\frac{ik}{\pi\chi}\int d\tau d\sigma \mathrm{Tr}(g\bar{\partial}\kappa D_{\mathcal{B}} g^{-1} + g^{-1}\partial\bar{\kappa}\bar{D}_{\mathcal{B}} g) \tag{4.2.3b}$$

$$\delta\left(\frac{k^2}{\pi\chi^2}\int d\tau d\sigma \alpha \tilde{L}^{ab}_{\infty}\mathrm{Tr}(T_a\mathcal{B})\mathrm{Tr}(T_b\mathcal{B})\right) = -\frac{k}{\pi\chi}\int d\tau d\sigma \mathrm{Tr}(\mathcal{B}\bar{\partial}\kappa) \tag{4.2.3c}$$

$$\delta\left(\frac{k^2}{\pi\chi^2}\int d\tau d\sigma \bar{\alpha} \tilde{L}^{ab}_{\infty}\mathrm{Tr}(T_a\bar{\mathcal{B}})\mathrm{Tr}(T_b\bar{\mathcal{B}})\right) = -\frac{k}{\pi\chi}\int d\tau d\sigma \mathrm{Tr}(\bar{\mathcal{B}}\partial\bar{\kappa}) \tag{4.2.3d}$$

whose sum δS_{AV} is zero on inspection.

It should also be emphasized that the non-linear and linear forms of the affine-Virasoro action are conformal and Diff $S_2(K)$-invariant because L^{ab}_{∞} and $\tilde{L}^{ab}_{\infty}$ are solutions of the high-level Virasoro master equation. For example, temporarily assuming arbitrary $\tilde{L}^{ab}_{\infty}$, we obtain the extra terms in the $\bar{\kappa}$ variation of S_{AV},

$$\begin{aligned}\delta S_{AV} &= \frac{-2k^2}{\pi\chi^2}\int d\tau d\sigma\, \bar{\alpha}\mathrm{tr}(\bar{\mathcal{B}}T_a)(\tilde{L}^{ab}_{\infty} - 2k\tilde{L}^{ac}_{\infty}\eta_{cd}\tilde{L}^{db}_{\infty})\partial\left[\bar{\xi}\mathrm{tr}(\bar{\mathcal{B}}T_b)\right] \\ &= 0\end{aligned} \tag{4.2.4}$$

which vanish for the explicit solutions $\tilde{L}^{ab}_{\infty}$ in (4.1.1).

4.3 The generic action on the affine Lie group

We again begin with the classical basic Hamiltonian of the generic affine-Virasoro construction, this time in terms of the classical affine Lie derivatives of (2.4.2,3),

$$H_0 = L_\infty^{ab} \sum_m (E_a(-m)E_b(m) + \bar{E}_a(-m)\bar{E}_b(m)). \tag{4.3.1}$$

This Hamiltonian is to be interpreted as that of a classical mechanics, in the same way as (2.5.1a) and (3.2.1).

The operator modes K_m and $\bar{K}_m$ of the K-conjugate theory may be taken as a set of first-class constraints,

$$K_m \equiv \frac{1}{2\pi}\tilde{L}_\infty^{ab} E_a(-n)E_b(m+n) \approx 0, \qquad \bar{K}_m \equiv \frac{1}{2\pi}\tilde{L}_\infty^{ab} \bar{E}_a(-n)\bar{E}_b(m+n) \approx 0 \tag{4.3.2a}$$

$$\{H_0, K_m\} = \{H_0, \bar{K}_m\} = 0 \tag{4.3.2b}$$

$$\{K_m, K_n\} = (m-n)K_{m+n} \approx 0, \qquad \{\bar{K}_m, \bar{K}_n\} = -(m-n)\bar{K}_{m+n} \approx 0 \tag{4.3.2c}$$

$$\{K_m, \bar{K}_n\} = 0 \tag{4.3.2d}$$

because they generate the mode algebra of Diff $S_1 \times$ Diff S_1 in (4.3.2c,d).

As in Section 4.1, we gauge the constraint (4.3.2a) by altering the Hamiltonian (4.3.1)

$$H' = H_0 + \Delta\mathcal{H}_1 \tag{4.3.3a}$$

$$\Delta H_1 = v^m \tilde{L}_\infty^{ab} E_a(-n)E_b(m+n) + \bar{v}^m \tilde{L}_\infty^{ab} \bar{E}_a(-n)\bar{E}_b(m+n) \tag{4.3.3b}$$

where v^m and $\bar{v}^m$, $m \in \mathbb{Z}$ are Lagrange multipliers which enforce the constraints.

In order to introduce auxiliary fields, some preliminary steps are necessary. We first define matrices $\hat{v}$ and $\hat{\bar{v}}$

$$\hat{v}_m{}^n \equiv v^p \delta_{p,n-m}, \qquad \hat{\bar{v}}_m{}^n \equiv \bar{v}^p \delta_{p,n-m} \tag{4.3.4}$$

which are functions of $n-m$ only. We then construct matrices $\hat{\alpha}$ and $\hat{\bar{\alpha}}$

$$\hat{\alpha} \equiv \frac{1-\hat{v}}{1+\hat{v}}, \qquad \hat{\bar{\alpha}} \equiv \frac{1-\hat{\bar{v}}}{1+\hat{\bar{v}}}. \tag{4.3.5}$$

Because of the delta-function forms of the matrices in (4.3.4), any power series function $\mathcal{F}(\hat{v})_m{}^n$ or $\mathcal{F}(\hat{\bar{v}})_m{}^n$ of those matrices is guaranteed to be a function of $n-m$ only. We can therefore decompose the $\hat{\alpha}$ and $\hat{\bar{\alpha}}$ matrices,

$$\hat{\alpha}_m{}^n = \alpha^p \delta_{p,n-m}, \qquad \hat{\bar{\alpha}}_m{}^n = \bar{\alpha}^p \delta_{p,n-m} \tag{4.3.6}$$

where α^m and $\bar{\alpha}^m$, $m \in \mathbb{Z}$ are a new set of fields that may be used to replace v^m and $\bar{v}^m$.

We may now define two matrices,

$$\hat{W}_{am}{}^{bn} \equiv (\delta_{am}{}^{bn} + \hat{\alpha}_m{}^n)\tilde{P}_a{}^b, \qquad (\hat{W}^{-1})_{am}{}^{bn} = \delta_{am}{}^{bn} + \frac{1}{2}(\hat{v}_m{}^n - \delta_m{}^n)\tilde{P}_a{}^b \tag{4.3.7a}$$

$$\hat{\bar{W}}_{am}{}^{bn} \equiv (\delta_{am}{}^{bn} + \hat{\bar{\alpha}}_m{}^n)\tilde{P}_a{}^b, \qquad (\hat{\bar{W}}^{-1})_{am}{}^{bn} = \delta_{am}{}^{bn} + \frac{1}{2}(\hat{\bar{v}}_m{}^n - \delta_m{}^n)\tilde{P}_a{}^b \tag{4.3.7b}$$

which will play the the same role as those we defined in (4.1.7c) on the Lie group. Using these matrices, we add two terms to the Hamiltonian (4.3.3),

$$H = H' + \Delta H_2 \tag{4.3.8a}$$

$$\begin{aligned}\Delta H_2 \;=\; & -\frac{1}{k}\left(k\mathcal{B}_{a,-m} - (\hat{W}^{-1})_{a,-m}{}^{dp}E_d(p)\right)\eta^{ac}\hat{W}_{cm}{}^{bn}\left(k\mathcal{B}_{bn} - (\hat{W}^{-1})_{bn}{}^{dp}E_d(p)\right)\\ & -\frac{1}{k}\left(k\bar{\mathcal{B}}_{a,-m} - (\hat{\bar{W}}^{-1})_{a,-m}{}^{dp}\bar{E}_d(p)\right)\eta^{ac}\hat{\bar{W}}_{cm}{}^{bn}\left(k\bar{\mathcal{B}}_{bn} - (\hat{\bar{W}}^{-1})_{bn}{}^{dp}\bar{E}_d(p)\right)\end{aligned} \tag{4.3.8b}$$

where $\mathcal{B}_{am}$ and $\bar{\mathcal{B}}_{am}$, $a = 1\ldots\dim g$, $m \in \mathbb{Z}$ are a set of auxiliary fields, called the auxiliary connections on the affine group. As in Section 4.1, since the action is quadratic in $\mathcal{B}$ and $\bar{\mathcal{B}}$, the auxiliary connections are easily integrated out, showing that the theory is once again unchanged for averages of functions of x and p.

For passage to the action, it is convenient to rearrange H as follows,

$$\begin{aligned}H \;=\; & H_{WZW}\\ & -2k^2\hat{\alpha}_{-m}{}^n\tilde{L}^{ab}\mathcal{B}_{am}\mathcal{B}_{bn} - \frac{1}{k}\left(k\mathcal{B}_{a,-m}) - E_a(-m)\right)\eta^{ab}\left(k\mathcal{B}_{bm} - E_b(m)\right)\\ & -2k^2\hat{\bar{\alpha}}_{-m}{}^n\tilde{L}^{ab}\bar{\mathcal{B}}_{am}\bar{\mathcal{B}}_{bn} - \frac{1}{k}\left(k\bar{\mathcal{B}}_{a,-m}) - \bar{E}_a(-m)\right)\eta^{ab}\left(k\bar{\mathcal{B}}_{bm} - \bar{E}_b(m)\right)\end{aligned} \tag{4.3.9a}$$

$$\begin{aligned}\partial_\tau x^{i\mu} \;=\; & -\frac{2}{k}e_{a,-m}{}^{i\mu}\eta^{ab}e_{bm}{}^{j\nu}p^-_{j\nu} + \eta^{ab}e_{a,-m}{}^{i\mu}\hat{\Omega}_{bm}{}^{y*} + 2e_{a,-m}{}^{i\mu}\eta^{ab}(\mathcal{B}_{bm} - \hat{\Omega}_{bm}{}^{cn}\bar{\mathcal{B}}_{cn})\\ =\; & -\frac{2}{k}e^{i\mu}_{a,-m}\eta^{ab}e_{bm}{}^{j\nu}p_{j\nu}(\hat{B}) + 2e_{a,-m}{}^{i\mu}\eta^{ab}(\mathcal{B}_{bm} - \hat{\Omega}_{bm}{}^{cn}\bar{\mathcal{B}}_{cn}).\end{aligned} \tag{4.3.9b}$$

The Hamiltonian equations of motion (4.3.9b) may easily be inverted to find the linearized action on the affine Lie group,

$$S_{AVM} = \int d\tau L_{AVM} \tag{4.3.10a}$$

$$
\begin{aligned}
L_{AVM} &= -\frac{k}{4}\eta_{ab}e_{i\mu}{}^{a,-m}e_{j\nu}{}^{bm}\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} + \frac{k}{4}\eta^{ab}\hat{\Omega}_{a,-m}{}^{y_*}\hat{\Omega}_{bm}{}^{y_*} \\
&\quad +k\partial_\tau x^{i\mu}\left(e_{i\mu}{}^{y_*} + \frac{1}{2}e_{i\mu}{}^{am}\hat{\Omega}_{am}{}^{y_*}\right) \\
&\quad +2k^2\hat{\alpha}_{-m}{}^{n}\tilde{L}^{ab}\mathcal{B}_{am}\mathcal{B}_{bn} + k(\partial_\tau x^{i\mu}e_{i\mu}{}^{am} - \hat{\Omega}_{b,-m}{}^{y_*}\eta^{ab})\mathcal{B}_{am} \\
&\quad +2k^2\hat{\bar{\alpha}}_{-m}{}^{n}\tilde{L}^{ab}\bar{\mathcal{B}}_{am}\bar{\mathcal{B}}_{bn} + k(\partial_\tau x^{i\mu}\bar{e}_{i\mu}{}^{am} + (\hat{\Omega}^{-1})_{b,-m}{}^{y_*}\eta^{ab})\bar{\mathcal{B}}_{am} \\
&\quad +2k\eta^{ab}\mathcal{B}_{a,-m}\hat{\Omega}_{bm}{}^{cn}\bar{\mathcal{B}}_{cn} \qquad (4.3.10b) \\
&= -\frac{k}{4}\eta_{ab}e_{i\mu}{}^{a,-m}e_{j\nu}{}^{bm}\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} + \frac{k}{4}\left(\hat{\Omega}_{am}{}^{y_*} - 2\hat{B}_{i\mu,j\nu}\partial_\tau x^{j\nu}e_{am}{}^{i\mu}\right)\eta^{ab}\hat{\Omega}_{b,-m}{}^{y_*} \\
&\quad +2k^2\hat{\alpha}_{-m}{}^{n}\tilde{L}^{ab}\mathcal{B}_{am}\mathcal{B}_{bn} + k(\partial_\tau x^{i\mu}e_{i\mu}{}^{am} - \hat{\Omega}_{b,-m}{}^{y_*}\eta^{ab})\mathcal{B}_{am} \\
&\quad +2k^2\hat{\bar{\alpha}}_{-m}{}^{n}\tilde{L}^{ab}\bar{\mathcal{B}}_{am}\bar{\mathcal{B}}_{bn} + k(\partial_\tau x^{i\mu}\bar{e}_{i\mu}{}^{am} + (\hat{\Omega}^{-1})_{b,-m}{}^{y_*}\eta^{ab})\bar{\mathcal{B}}_{am} \\
&\quad +2k\eta^{ab}\mathcal{B}_{a,-m}\hat{\Omega}_{bm}{}^{cn}\bar{\mathcal{B}}_{cn}. \qquad (4.3.10c)
\end{aligned}
$$

Using the translation dictionary of Section 2.9 and the mode identities

$$
\mathcal{B}^a(\tau,\sigma) = \sum_m e^{im\sigma}\mathcal{B}^{am}(\tau), \qquad \bar{\mathcal{B}}^a(\tau,\sigma) = \sum_m e^{im\sigma}\bar{\mathcal{B}}^{am}(\tau) \qquad (4.3.11a)
$$

$$
\alpha(\tau,\sigma) = \sum_m e^{i(n-m)\sigma}\hat{\alpha}_m{}^n(\tau) = \sum_m e^{im\sigma}\alpha^m(\tau) \qquad (4.3.11b)
$$

$$
\bar{\alpha}(\tau,\sigma) = \sum_m e^{i(n-m)\sigma}\hat{\bar{\alpha}}_m{}^n(\tau) = \sum_m e^{im\sigma}\bar{\alpha}^m(\tau) \qquad (4.3.11c)
$$

the action 4.3.10 is easily checked to be equivalent to the linearized affine-Virasoro action (4.1.10) on the Lie group. Following the method in Section 2.11.2, we may also obtain the $\hat{g}$ form of the mechanical affine-Virasoro action,

$$
S_{AVM} = S_M + \Delta S_{AVM} \qquad (4.3.12a)
$$

$$
\begin{aligned}
\Delta S_{AVM} &= \int d\tau\left(\frac{2k^2}{\chi^2}\hat{\alpha}_{-m}{}^{n}\tilde{L}^{ab}_{\infty}\hat{\mathrm{Tr}}(\hat{T}_{am}\hat{\mathcal{B}})\hat{\mathrm{Tr}}(\hat{T}_{bn}\hat{\mathcal{B}}) + \frac{2k^2}{\chi^2}\hat{\alpha}_{-m}{}^{n}\tilde{L}^{ab}_{\infty}\hat{\mathrm{Tr}}(\hat{T}_{am}\hat{\bar{\mathcal{B}}})\hat{\mathrm{Tr}}(\hat{T}_{bn}\hat{\bar{\mathcal{B}}})\right. \\
&\quad \left. -\frac{k}{2\chi}\hat{\mathrm{Tr}}\left((\partial_\tau\hat{g} - \hat{g}' + 2i\hat{\bar{\mathcal{B}}}\hat{g})(\partial_\tau\hat{g}^{-1} + (\hat{g}^{-1})' + 2i\hat{\mathcal{B}}\hat{g}^{-1})\right)\right) \qquad (4.3.12b)
\end{aligned}
$$

$$
\hat{\mathcal{B}} \equiv \mathcal{B}_{a,-m}\eta^{ab}\hat{T}_{bm}, \qquad \hat{\bar{\mathcal{B}}} \equiv \bar{\mathcal{B}}_{a,-m}\eta^{ab}\hat{T}_{bm} \qquad (4.3.12c)
$$

where S_M is the mechanical form of the WZW action (2.11.3), and $\hat{g}(\hat{T}, x(\tau)) \in \hat{G}$ is the reduced affine group element in affine matrix representation $\hat{T}$ (see Section 2.7).

4.4 Gauge invariance of the mechanical affine-Virasoro action

Under Diff $S_2(K)$, the action (4.3.10) is invariant under the transformation

$$\delta x^{i\mu} = \kappa^{am} e_{am}{}^{i\mu} + \bar{\kappa}^{am} \bar{e}_{am}{}^{i\mu} \tag{4.4.1a}$$

$$\delta\alpha^m = -\partial_\tau \xi^m + im\xi^m + \xi^{-n} \overleftrightarrow{\partial_t}\, \alpha^{m+n} + i(m+2n)\xi^{-n}\alpha^{m+n} \tag{4.4.1b}$$

$$\delta\bar{\alpha}^m = -\partial_\tau \bar{\xi}^m - im\bar{\xi}^m + \bar{\xi}^{-n} \overleftrightarrow{\partial_t}\, \bar{\alpha}^{m+n} - i(m+2n)\bar{\xi}^{-n}\bar{\alpha}^{m+n} \tag{4.4.1c}$$

$$\delta\mathcal{B}_{am} = f_{am,bn}{}^{cp}\kappa^{bn}\mathcal{B}_{cp} + \eta_{ab}(\partial_\tau - im)\kappa^{b,-m}, \quad \delta\bar{\mathcal{B}}_{am} = f_{am,bn}{}^{cp}\bar{\kappa}^{bn}\bar{\mathcal{B}}_{cp} + \eta_{ab}(\partial_\tau + im)\bar{\kappa}^{b,-m} \tag{4.4.1d}$$

$$\kappa^{am} \equiv 2k\xi^{-n}\tilde{L}^{ab}_{\infty}\mathcal{B}_{b,n-m}, \qquad \bar{\kappa}^{am} \equiv 2k\bar{\xi}^{-n}\tilde{L}^{ab}_{\infty}\bar{\mathcal{B}}_{b,n-m} \tag{4.4.1e}$$

where $\xi^m(\tau)$ and $\bar{\xi}^m(\tau)$, $m \in \mathbb{Z}$ are the diffeomorphism parameters. For the $\hat{g}$ form (4.3.12) of the action, the transformations

$$\delta\hat{g} = \hat{g}i\hat{\kappa} - i\hat{\bar{\kappa}}\hat{g} \tag{4.4.2a}$$

$$\delta\hat{\mathcal{B}} = \partial_\tau\hat{\kappa} + \hat{\kappa}' + i[\hat{\mathcal{B}}, \hat{\kappa}], \qquad \delta\hat{\bar{\mathcal{B}}} = \partial_\tau\hat{\bar{\kappa}} - \hat{\bar{\kappa}}' + i[\hat{\bar{\mathcal{B}}}, \hat{\bar{\kappa}}] \tag{4.4.2b}$$

$$\hat{\kappa} \equiv \kappa^{am}\hat{T}_{am}, \qquad \hat{\bar{\kappa}} \equiv \bar{\kappa}^{am}\hat{T}_{am} \tag{4.4.2c}$$

replace those for x and $\mathcal{B}$ in (4.4.1). As was the case for the affine Virasoro action on the Lie group (see Section 4.1), Diff $S_2(K)$ is embedded in (classical) affine $G \times G$. The transformation of the affine group element $\hat{g}$ in (4.4.2a) is again that of a transformation in affine $G \times G$, restricted according to eq. (4.4.1e). The auxiliary connections $\hat{\mathcal{B}}, \hat{\bar{\mathcal{B}}}$ on the affine group transform as connections. The gauge parameters $\kappa, \bar{\kappa}$ and $\xi, \bar{\xi}$ above are related to those in Section 4.2

$$\xi(\tau,\sigma) = \sum_m e^{im\sigma}\xi^m(\tau), \qquad \bar{\xi}(\tau,\sigma) = \sum_m e^{im\sigma}\bar{\xi}^m(\tau) \tag{4.4.3a}$$

$$\kappa^a(\tau,\sigma) = \sum_m e^{im\sigma}\kappa^{am}(\tau), \qquad \bar{\kappa}^a(\tau,\sigma) = \sum_m e^{im\sigma}\bar{\kappa}^{am}(\tau) \tag{4.4.3b}$$

under the translation dictionary of Section 2.9.

In order to verify the diffeomorphism invariance of the action (4.3.12), we give the term-by-term results,

$$\delta S_M = -\frac{ik}{2\chi}\int d\tau \hat{\mathrm{Tr}}\left(\hat{g}^{-1}(\partial_\tau\hat{g} + \hat{g}')(\partial_\tau\hat{\kappa} - \hat{\kappa}') - (\partial_\tau\hat{g} - \hat{g}')\hat{g}^{-1}(\partial_\tau\hat{\bar{\kappa}} + \hat{\bar{\kappa}}')\right) \tag{4.4.4a}$$

$$\delta\left(-\frac{k}{2\chi}\int d\tau \hat{\mathrm{Tr}}\left((\partial_\tau\hat{g} - \hat{g}' + 2i\hat{\bar{\mathcal{B}}}\hat{g})(\partial_\tau\hat{g}^{-1} + (\hat{g}^{-1})' + 2i\hat{\mathcal{B}}\hat{g}^{-1})\right)\right)$$

$$= -\frac{ik}{2\chi}\int d\tau \hat{\mathrm{Tr}}\Big(\hat{g}(\partial_\tau\hat{\kappa}-\hat{\kappa}')(\partial_\tau\hat{g}^{-1}+(\hat{g}^{-1})'+2i\hat{\mathcal{B}}\hat{g}^{-1})$$

$$+\hat{g}^{-1}(\partial_\tau\hat{\bar{\kappa}}+\hat{\bar{\kappa}}')(\partial_\tau\hat{g}-\hat{g}'+2i\hat{\bar{\mathcal{B}}}\hat{g})\Big) \tag{4.4.4b}$$

$$\delta\left(\frac{2k^2}{\chi^2}\int d\tau\hat{\alpha}_{-m}{}^n\tilde{L}^{ab}_\infty\hat{\mathrm{Tr}}(\hat{T}_{am}\hat{\mathcal{B}})\hat{\mathrm{Tr}}(\hat{T}_{bn}\hat{\mathcal{B}})\right) = -\frac{2k}{\chi}\int d\tau\hat{\mathrm{Tr}}\left(\hat{\mathcal{B}}(\partial_\tau\hat{\kappa}-\hat{\kappa}')\right) \tag{4.4.4c}$$

$$\delta\left(\frac{2k^2}{\chi^2}\int d\tau\hat{\alpha}_{-m}{}^n\tilde{L}^{ab}_\infty\hat{\mathrm{Tr}}(\hat{T}_{am}\hat{\bar{\mathcal{B}}})\hat{\mathrm{Tr}}(\hat{T}_{bn}\hat{\bar{\mathcal{B}}})\right) = -\frac{2k}{\chi}\int d\tau\hat{\mathrm{Tr}}\left(\hat{\bar{\mathcal{B}}}(\partial_\tau\hat{\bar{\kappa}}+\hat{\bar{\kappa}}')\right) \tag{4.4.4d}$$

whose sum δS_{AVM} is zero on inspection.

4.5 Constrained formulations of the affine-Virasoro action

So far, we have described the formulations of the affine-Virasoro action depicted in the left-hand column of the picture

$$\begin{array}{ccc}
\hat{g}(x(\tau)) & \xleftrightarrow{\text{constraint}} & \hat{g}(x(\tau,\sigma)) \\
\text{modes}\Big\updownarrow & & \Big\updownarrow\text{modes} \\
g(x(\tau,\sigma)) & \xleftrightarrow{\text{constraint}} & g(x(\tau,\sigma,\tilde{\sigma}))
\end{array} \tag{4.5.1}$$

that was introduced in Section 2.8. For completeness, we now give the results for the constrained formulations in the right-hand column of the picture.

4.5.1 The affine-Virasoro action as a field theory on the affine group

To express the generic affine-Virasoro action as a two-dimensional field theory on the affine Lie group, we follow the steps in Sections 2.6 and 3.3.1. We begin by finding an operator P_{AV} to use as the generator of spatial translations.

Following the logic of Section 3.3.1, it is clear that that appropriate momentum operator is

$$P_{AV} = P_{WZW} - im\mathcal{B}_{am}p^{am}_{\mathcal{B}} - im\bar{\mathcal{B}}_{am}p^{am}_{\bar{\mathcal{B}}} + im\alpha^m p^\alpha_m + im\bar{\alpha}^m p^{\bar{\alpha}}_m \tag{4.5.2}$$

where P_{WZW} is the WZW momentum operator from eq. (2.6.4), $p^{am}_{\mathcal{B}}$ and $p^{am}_{\bar{\mathcal{B}}}$, $a = 1\ldots\dim g$, $m \in \mathbb{Z}$ are momenta canonically conjugate to the auxiliary connections

$\mathcal{B}_{am}$ and $\bar{\mathcal{B}}_{am}$, and p^{α}_{am} and $p^{\bar{\alpha}}$, $m \in \mathbb{Z}$ are momenta canonically conjugate to the gauge fields α and $\bar{\alpha}$

$$\{\mathcal{B}_{am}, p^{bn}_{\mathcal{B}}\} = i\delta_{am}{}^{bn}, \qquad \{\bar{\mathcal{B}}_{am}, p^{bn}_{\bar{\mathcal{B}}}\} = i\delta_{am}{}^{bn} \tag{4.5.3a}$$

$$\{\alpha^m, p^{\alpha}_n\} = i\delta_n{}^m, \qquad \{\bar{\alpha}^m, p^{\bar{\alpha}}_n\} = i\delta_n{}^m. \tag{4.5.3b}$$

As expected, the momentum operator (4.5.2) commutes with the Hamiltonian (4.3.9a),

$$\{H, P_{AV}\} = 0 \tag{4.5.4a}$$

$$\partial_\sigma A \equiv i\{P_{AV}, A\} \tag{4.5.4b}$$

and so may be used to define spatial translations, according to (4.5.4b).

We may now find the spatial derivatives,

$$\partial_\sigma x^{i\mu} = \eta^{ab} e_{a,-m}{}^{i\mu} \hat{\Omega}_{bm}{}^{y_*} \tag{4.5.5a}$$

$$\partial_\sigma \mathcal{B}_{am} = -im\mathcal{B}_{am}, \qquad \partial_\sigma \bar{\mathcal{B}}_{am} = -im\bar{\mathcal{B}}_{am} \tag{4.5.5b}$$

$$\partial_\sigma \alpha^m = im\alpha^m, \qquad \partial_\sigma \bar{\alpha}^m = im\bar{\alpha}^m \tag{4.5.5c}$$

of the affine group coordinate x, the auxiliary connections $\mathcal{B}$ and $\bar{\mathcal{B}}$, and the gauge fields α and $\bar{\alpha}$. In analogy to the procedure in Sections 2.6 and 3.3.1, the equations (4.5.5b) and (4.5.5c) are interpreted as constraints on the auxiliary connections and the gauge fields.

From this point, it is a simple matter to follow the steps of Section 2.6, obtaining the action of the two-dimensional field theory on $\hat{G}$,

$$S_{AV\hat{F}} = \int d\tau d\sigma \mathcal{L}_{AV\hat{F}} \tag{4.5.6a}$$

$$\begin{aligned}
\mathcal{L}_{AV\hat{F}} \;=\; & -\frac{k}{8\pi}\eta_{ab} e_{i\mu}{}^{a,-m} e_{j\nu}{}^{bm}(\partial_\tau x^{i\nu}\partial_\tau x^{j\nu} - \partial_\sigma x^{i\mu}\partial_\sigma x^{j\nu}) + \frac{k}{4\pi} B_{i\mu,j\nu}\partial_\tau x^{i\mu}\partial_\sigma x^{j\nu} \\
& + \frac{k^2}{\pi}\hat{\alpha}_{-m}{}^n \tilde{L}^{ab}_{\infty}\mathcal{B}_{am}\mathcal{B}_{bn} + \frac{k}{2\pi}(\partial_\tau x^{i\mu} - \partial_\sigma x^{i\mu}) e_{i\mu}{}^{am}\mathcal{B}_{am} \\
& + \frac{k^2}{\pi}\hat{\alpha}_{-m}{}^n \tilde{L}^{ab}_{\infty}\bar{\mathcal{B}}_{am}\bar{\mathcal{B}}_{bn} + \frac{k}{2\pi}(\partial_\tau x^{i\mu} + \partial_\sigma x^{i\mu}) \bar{e}_{i\mu}{}^{am}\bar{\mathcal{B}}_{am} \\
& + \frac{k}{\pi}\eta^{ab}\mathcal{B}_{a,-m}\hat{\Omega}_{bm}{}^{cn}\bar{\mathcal{B}}_{cn} + k\lambda_{i\mu}(\partial_\sigma x^{i\mu} - \eta^{ab} e_{a,-m}{}^{i\mu}\hat{\Omega}_{bm}{}^{y_*}) \\
& + k\lambda^{am}_{\mathcal{B}}(\partial_\sigma \mathcal{B}_{am} + im\mathcal{B}_{am}) + k\bar{\lambda}^{am}_{\mathcal{B}}(\partial_\sigma \bar{\mathcal{B}}_{am} + im\bar{\mathcal{B}}_{am}) \\
& + k\lambda^{\alpha}_m(\partial_\sigma \alpha^m - im\alpha^m) + k\bar{\lambda}^{\alpha}_m(\partial_\sigma \bar{\alpha}^m - im\bar{\alpha}^m)
\end{aligned} \tag{4.5.6b}$$

where λ, $\lambda_{\mathcal{B}}$, $\lambda_{\bar{\mathcal{B}}}$, λ^{α}, and $\lambda^{\bar{\alpha}}$ are Lagrange multipliers that enforce the various constraints.

To express this formulation in terms of the affine group variable $\hat{g}$, we follow the steps of Sections 2.7 and 2.12 to obtain the action,

$$S_{AVF} = S_F + \Delta S_{AVF} \tag{4.5.7a}$$

$$\begin{aligned}\Delta S_{AVF} \;=\; & \int d\tau \left(\frac{k^2}{\pi\chi^2} \hat{\alpha}_{-m}{}^{n} \tilde{L}^{ab}_{\infty} \hat{\mathrm{Tr}}(\hat{T}_{am}\hat{\mathcal{B}}) \hat{\mathrm{Tr}}(\hat{T}_{bn}\hat{\mathcal{B}}) + \frac{k^2}{\pi\chi^2} \hat{\bar{\alpha}}_{-m}{}^{n} \tilde{L}^{ab}_{\infty} \hat{\mathrm{Tr}}(\hat{T}_{am}\hat{\bar{\mathcal{B}}}) \mathrm{Tr}(\hat{T}_{bn}\hat{\bar{\mathcal{B}}}) \right. \\ & \left. - \frac{k}{\pi\chi} \hat{\mathrm{Tr}}(\bar{D}_{\hat{\mathcal{B}}} g D_{\hat{\mathcal{B}}} g^{-1}) \right) \\ & + \frac{k}{\chi} \int d\tau d\sigma \hat{\mathrm{Tr}} \left(\lambda_{\mathcal{B}} (\partial_\sigma \hat{\mathcal{B}} - \hat{\mathcal{B}}') + \bar{\lambda}_{\mathcal{B}} (\partial_\sigma \hat{\bar{\mathcal{B}}} - \hat{\bar{\mathcal{B}}}') \right) \\ & + \frac{k}{\chi} \int d\tau d\sigma \left(\lambda^{\alpha}_{m} (\partial_\sigma \alpha^m - im\alpha^m) + \bar{\lambda}^{\alpha}_{m} (\partial_\sigma \bar{\alpha}^m - im\bar{\alpha}^m) \right)\end{aligned} \tag{4.5.7b}$$

$$D_{\hat{\mathcal{B}}} \equiv \partial + i\hat{\mathcal{B}}, \qquad \bar{D}_{\hat{\mathcal{B}}} \equiv \bar{\partial} + i\hat{\bar{\mathcal{B}}} \tag{4.5.7c}$$

where $\lambda_{\mathcal{B}} \equiv \lambda^{am}_{\mathcal{B}} \hat{T}_{am}$, $\bar{\lambda}_{\mathcal{B}} \equiv \bar{\lambda}^{am}_{\mathcal{B}} \hat{T}_{am}$, $\hat{\mathcal{B}}'$ and $\hat{\bar{\mathcal{B}}}'$ are defined in analogy to $\hat{g}'$ in (2.11.3b), and S_F is given in eq. (2.12.10). In this form, the Diff S_2 gauge transformation is

$$\delta\hat{g} = \hat{g} i\hat{\kappa} - i\hat{\bar{\kappa}}\hat{g} \tag{4.5.8a}$$

$$\delta\alpha^m = -\bar{\partial}\xi^m + \xi^{-n} \overleftrightarrow{\partial}\, \alpha^{m+n}, \qquad \delta\bar{\alpha}^m = -\partial\bar{\xi}^m + \bar{\xi}^{-n} \overleftrightarrow{\bar{\partial}}\, \bar{\alpha}^{m+n} \tag{4.5.8b}$$

$$\delta\hat{\mathcal{B}} = \partial\hat{\kappa} + i[\mathcal{B}, \hat{\kappa}], \qquad \delta\hat{\bar{\mathcal{B}}} = \bar{\partial}\hat{\bar{\kappa}} + i[\hat{\bar{\mathcal{B}}}, \hat{\bar{\kappa}}] \tag{4.5.8c}$$

where the Diff S_2 parameters $\xi, \bar{\xi}$ are constrained so that $\partial_\sigma \xi^m = im\xi^m$ and $\partial_\sigma \bar{\xi}^m = im\xi^m$. Notice that the Diff S_2 gauge transformation in this formulation bears a striking resemblance to the transformation in the formulation on the Lie group (4.2.2).

4.5.2 The affine-Virasoro action as a field theory in three dimensions

Following the steps of Section 2.13, we define fields that are local in a third coordinate $\tilde{\sigma}$,

$$\mathcal{B}^a(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{-im\tilde{\sigma}} \mathcal{B}_{am}(\tau,\sigma), \qquad \bar{\mathcal{B}}^a(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{-im\tilde{\sigma}} \bar{\mathcal{B}}_{am}(\tau,\sigma) \tag{4.5.9a}$$

$$\alpha(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{im\tilde{\sigma}} \alpha^m(\tau,\sigma), \qquad \bar{\alpha}(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{im\tilde{\sigma}} \bar{\alpha}^m(\tau,\sigma) \tag{4.5.9b}$$

$$(\lambda_{\mathcal{B}}(\tau,\sigma,\tilde{\sigma}))_I{}^J \equiv \sum_m e^{i(n-m)\tilde{\sigma}}(\lambda_{\mathcal{B}}(\tau,\sigma))_{Im}{}^{Jn} \tag{4.5.9c}$$

$$(\bar{\lambda}_{\mathcal{B}}(\tau,\sigma,\tilde{\sigma}))_I{}^J \equiv \sum_m e^{i(n-m)\tilde{\sigma}}(\bar{\lambda}_{\mathcal{B}}(\tau,\sigma))_{Im}{}^{Jn} \tag{4.5.9d}$$

$$\lambda^\alpha(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{-im\tilde{\sigma}}\lambda^\alpha_m(\tau,\sigma), \qquad \bar{\lambda}^\alpha(\tau,\sigma,\tilde{\sigma}) \equiv \sum_m e^{-im\tilde{\sigma}}\bar{\lambda}^\alpha_m(\tau,\sigma) \tag{4.5.9e}$$

to supplement the definitions in (2.13.4). Then we may obtain the formulation of the gauged WZW model as a three-dimensional field theory on the Lie group,

$$S_{AV3} = S_3 + \Delta S_{AV3} \tag{4.5.10a}$$

$$\begin{aligned}
\Delta S_{AV3} \;=\; & \int d\tau d\sigma d\tilde{\sigma}\left(\frac{k^2\alpha}{2\pi^2\chi^2}\tilde{L}^{ab}_\infty \mathrm{Tr}(T_a\mathcal{B})\mathrm{Tr}(T_b\mathcal{B}) + \frac{k^2\bar{\alpha}}{2\pi^2\chi^2}\tilde{L}^{ab}_\infty \mathrm{Tr}(T_a\bar{\mathcal{B}})\mathrm{Tr}(T_b\bar{\mathcal{B}})\right. \\
& \left. -\frac{k}{2\pi^2\chi}\mathrm{Tr}(\bar{D}_{\mathcal{B}}gD_{\mathcal{B}}g^{-1})\right) \\
& +\frac{k}{2\pi\chi}\int d\tau d\sigma d\tilde{\sigma}\mathrm{Tr}\left(\lambda_{\mathcal{B}}(\partial_\sigma - \partial_{\tilde{\sigma}})\mathcal{B} + \bar{\lambda}_{\mathcal{B}}(\partial_\sigma - \partial_{\tilde{\sigma}})\bar{\mathcal{B}}\right) \\
& +\frac{k}{2\pi\chi}\int d\tau d\sigma d\tilde{\sigma}\left(\lambda^\alpha(\partial_\sigma - \partial_{\tilde{\sigma}})\alpha + \bar{\lambda}^\alpha(\partial_\sigma - \partial_{\tilde{\sigma}})\bar{\alpha}\right)
\end{aligned} \tag{4.5.10b}$$

where S_3 is given in eq. (2.13.5). The Diff S_2 gauge transformation in this formulation bears the same form as that in the two-dimensional formulation (4.2.2) on the Lie group, except that the gauge parameters $\xi(\tau,\sigma,\tilde{\sigma}), \bar{\xi}(\tau,\sigma,\tilde{\sigma})$ now depend on the third coordinate, but are constrained so that $(\partial_\sigma - \partial_{\tilde{\sigma}})\xi = (\partial_\sigma - \partial_{\tilde{\sigma}})\bar{\xi} = 0$.

Chapter 5

Conclusion

This thesis has generalized a procedure, first discussed in Ref. [21] for the WZW model, for generating new formulations of conformal field theories (CFTs). If a conformal field theory's Hamiltonian is conventionally constructed on the Lie group with local affine currents, then it may be transcribed onto the affine Lie group by means of the classical affine Lie derivatives. Once the Hamiltonian is written in terms of canonical variables on the affine group, it is a straightforward matter to find the action. In this way, a two-dimensional field theory on the Lie group with a three-dimensional WZW term may be converted into a one-dimensional mechanical model on the affine group, with a two-dimensional WZW term.

Alternatively, one may preserve the dimensionality of the base space by introducing a momentum operator P which will serve as the generator of spatial translations. This momentum operator must, of course, commute with the Hamiltonian. In this formulation, the conformal field theory takes the form of a two-dimensional field theory on the affine group with a three-dimensional WZW term, but the spatial dependence of the variables is constrained by the introduction of the momentum operator.

There is also a fourth formulation of conformal field theory, which makes use of similar spatial constraints. In this formulation, the theory appears as a three-dimensional field theory on the Lie group with a four-dimensional WZW term.

We have also shown how each of these formulations may be expressed either in terms of the coordinates on the appropriate group manifold, or in terms of the appropriate group elements. In addition, we have provided a translation dictionary that shows the equivalences among the various formulations. Under the translation dictionary, the new formulations on the affine group are understood as mode formulations of the formulations on the Lie group.

We have shown how to apply the above prescription to three specific theories:

the WZW model, the gauged WZW model, and the generic affine-Virasoro action. But the procedure is quite general, and can be applied at the Hamiltonian level for any theory whose Hamiltonian on the Lie group is written in terms of affine currents. At the action level, it is likely that one may also apply the translation dictionary to convert any theory from the Lie group to the affine Lie group. For example, one such application could be the generalized Thirring models of Refs. [34-36].

The potential utility of these new formulations lies in giving the researcher more points of view with which to examine a particular model. For some uses, the new formulations may provide more insight or functionality. For example, one might try to use some sort of mode cut-off on the affine group as a regulator. Or one might better understand the nature of the WZW term from the way in which it naturally arises within geometric structures on the affine group.

Whatever the practical uses of these new formulations, it seems only natural to formulate conformal field theories on the affine Lie group manifold, especially considering the intimate connection between CFTs and affine algebras.

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Appendix A: Identities on G

We list below some useful identities on the Lie group G, including in particular the explicit β-basis forms of various quantities which are central to the proofs of Sections 2.9, 2.10 and 2.11. Here, $J_a, a = 1 \ldots \dim g$ are the generators of Lie g, and $x^i, i = 1 \ldots \dim g$ are Einstein coordinates on the group manifold.

A. Group element and adjoint action.

$$g(J,x) = e^{i\beta^a(x)J_a} \tag{A.1a}$$

$$gJ_ag^{-1} = \Omega_a{}^bJ_b, \qquad \Omega_a{}^c\eta_{cd}\Omega_b{}^d = \eta_{ab} \tag{A.1b}$$

$$\Omega_a{}^b = \left(e^{-iH^{adj}}\right)_a{}^b, \qquad H^{adj} = \beta^aT_a^{adj}. \tag{A.1c}$$

The quantities η_{ab} and T_a^{adj} are the Killing metric and adjoint representation of Lie g.

B. Vielbeins and inverse vielbeins.

$$e_i = -ig^{-1}\partial_ig = e_i{}^aJ_a, \qquad \bar{e}_i = -ig\partial_ig^{-1} = \bar{e}_i{}^aJ_a \tag{A.2a}$$

$$\partial_ie_j{}^a - \partial_je_i{}^a = e_i{}^be_j{}^cf_{bc}{}^a, \qquad e_a{}^i\partial_ie_b{}^j - e_b{}^i\partial_ie_a{}^j = f_{ba}{}^ce_c{}^j \tag{A.2b}$$

$$\bar{e}_i{}^a = -e_i{}^b\Omega_b{}^a, \qquad \bar{e}_a{}^i = -(\Omega^{-1})_a{}^be_b{}^i \tag{A.2c}$$

$$e_i{}^a = \partial_i\beta^b(M(H^{adj}))_b{}^a, \qquad M(H) = \frac{e^{iH}-1}{iH}. \tag{A.2d}$$

The Cartan-Maurer and inverse Cartan-Maurer relations in (A.2b) hold also for $e \to \bar{e}$.

C. Antisymmetric tensor field.

$$B_{ij} = \partial_i\beta^b\partial_j\beta^a\eta_{bc}(N(H^{adj}))_a{}^c, \qquad N(H) = \frac{(e^{iH}-e^{-iH})-2iH}{(iH)^2} \tag{A.3a}$$

$$B_{ij} = \frac{i}{\chi}\int_0^1 dt\,\mathrm{Tr}\left(H\partial_{[i}e^{-itH}\partial_{j]}e^{itH}\right), \qquad H = \beta^aT_a \tag{A.3b}$$

$$\partial_iB_{jk} + \partial_jB_{ki} + \partial_kB_{ij} = \frac{i}{\chi}\mathrm{Tr}(e_i[e_j,e_k]), \qquad \mathrm{Tr}(T_aT_b) = \chi\eta_{ab}. \tag{A.3c}$$

T_a is any matrix irrep of Lie g.

D. Local currents on G.

$$E_a(\sigma) = 2\pi e_a{}^ip_i(B) + \frac{k}{2}\eta_{ab}e_i{}^b\partial_\sigma x^i = 2\pi(C(H^{adj}))_a{}^bp_b - k\partial_\sigma\beta^c(D(H^{adj}))_c{}^b\eta_{ba} \tag{A.4a}$$

$$\bar{E}_a(\sigma) = 2\pi\bar{e}_a{}^ip_i(B) - \frac{k}{2}\eta_{ab}\bar{e}_i{}^b\partial_\sigma x^i = 2\pi(\bar{C}(H^{adj}))_a{}^bp_b + k\partial_\sigma\beta^c(\bar{D}(H^{adj}))_c{}^b\eta_{ba} \tag{A.4b}$$

$$p_i(B) = p_i - \frac{k}{4\pi} B_{ij}\partial_\sigma x^j, \qquad p_i = \partial_i \beta^a p_a \tag{A.4c}$$

$$C(H) = \frac{iH}{e^{iH}-1}, \qquad D(H) = \frac{e^{-iH}-1+iH}{iH(e^{-iH}-1)} \tag{A.4d}$$

$$\bar{C}(H) = \frac{iH}{e^{-iH}-1}, \qquad \bar{D}(H) = \frac{e^{iH}-1-iH}{iH(e^{iH}-1)}. \tag{A.4e}$$

The canonical momenta $p_i(\sigma)$ and $p_a(\sigma)$, which can be classical or quantum, are defined in Section 2.9.

E. Conventional WZW Lagrangian on G

$$L_{WZW} = \frac{k}{8\pi}\int d\sigma \left[\eta_{ab} e_i{}^a e_j{}^b (\partial_\tau x^i \partial_\tau x^j - \partial_\sigma x^i \partial_\sigma x^j) + 2B_{ij}\partial_\tau x^i \partial_\sigma x^j\right] \tag{A.5a}$$

$$= \frac{k}{8\pi}\int d\sigma \Big((\partial_\tau \beta^a \partial_\tau \beta^b - \partial_\sigma \beta^a \partial_\sigma \beta^b)(F(H^{adj}))_a{}^c \eta_{cb} + 2\partial_\tau \beta^a (G(H^{adj}))_a{}^c \eta_{cb} \partial_\sigma \beta^b \Big) \tag{A.5b}$$

$$F(H) = \frac{2-e^{iH}-e^{-iH}}{H^2}, \qquad G(H) = N(H) = \frac{(e^{iH}-e^{-iH})-2iH}{(iH)^2}. \tag{A.5c}$$

The g form of the conventional WZW action is given in (2.1.9).

Appendix B: Identities on $\hat{G}$

We list some useful identities on the affine Lie group $\hat{G}$ (analogous to those on G in Appendix A) including in particular the explicit β-basis forms of various quantities which are central to the proofs of Section 2.9, 2.10 and 2.11. Here $J_a(m), a = 1\ldots\dim g, m \in \mathbb{Z}$ are the current modes and $x^{i\mu}, i = 1\ldots\dim g, \mu \in \mathbb{Z}$ are the coordinates on the reduced affine group manifold. Einstein and tangent-space indices are $\Lambda, \Gamma = (i\mu, y)$ and $L, M = (am, y_*)$ respectively, and $\mathcal{J}_L = (J_a(m), k)$ are the generators of the affine group.

A. Reduced group element and adjoint action.

$$\hat{g}(J,x) = e^{i\beta^{am}(x)J_a(m)} \tag{B.1a}$$

$$\hat{g}\mathcal{J}_L\hat{g}^{-1} = \hat{\Omega}_L{}^M \mathcal{J}_M, \qquad \hat{\Omega}_{am}{}^{cp}\hat{\eta}_{cp,dq}\hat{\Omega}_{bn}{}^{dq} = \hat{\eta}_{am,bn} \tag{B.1b}$$

$$\hat{\Omega}_L{}^M = \left(e^{-i\hat{H}^{adj}}\right)_L{}^M, \qquad \hat{H}^{adj} = \beta^{am}\hat{T}^{adj}_{am} \tag{B.1c}$$

$$\hat{\Omega}_{y_*}{}^L = \delta_{y_*}{}^L, \qquad \hat{\Omega}_{am}{}^{y_*} = \left(\frac{e^{-i\hat{H}^{adj}}-1}{\hat{H}^{adj}}\right)_{am}{}^{bn} (\hat{H}^{adj})_{bn}{}^{y_*}. \tag{B.1d}$$

The quantities $\hat{\eta}_{am,bn}$ and $\hat{T}^{adj}$ are the rescaled Killing metric (see Section 2.7) and the adjoint representation of the affine algebra.

B. Vielbeins and inverse vielbeins.

$$e_{i\mu} = -i\hat{g}^{-1}\partial_{i\mu}\hat{g} = e_{i\mu}{}^{L}\mathcal{J}_L, \qquad \bar{e}_{i\mu} = -i\hat{g}\partial_{i\mu}\hat{g}^{-1} = \bar{e}_{i\mu}{}^{L}\mathcal{J}_L \tag{B.2a}$$

$$\partial_{i\mu}e_{j\nu}{}^{L} - \partial_{j\nu}e_{i\mu}{}^{L} = e_{i\mu}{}^{M}e_{j\nu}{}^{N}f_{MN}{}^{L}, \qquad e_L{}^{i\mu}\partial_{i\mu}e_M{}^{j\nu} - e_M{}^{i\mu}\partial_{i\mu}e_L{}^{i\mu} = f_{ML}{}^{N}e_N{}^{j\nu} \tag{B.2b}$$

$$\bar{e}_{i\mu}{}^{L} = -e_{i\mu}{}^{M}\hat{\Omega}_M{}^{L}, \qquad \bar{e}_L{}^{i\mu} = -(\hat{\Omega}^{-1})_L{}^{M}e_M{}^{i\mu} \tag{B.2c}$$

$$e_{i\mu}{}^{L}(x) = \partial_{i\mu}\beta^{am}(x)(M(\hat{H}^{adj}))_{am}{}^{L}, \qquad M(H) = \frac{e^{iH}-1}{iH}. \tag{B.2d}$$

The Cartan-Maurer and inverse Cartan-Maurer relations in (B.2b) hold also for $e \to \bar{e}$.

C. Antisymmetric tensor field.

$$\hat{B}_{i\mu;j\nu} = \partial_{i\mu}\beta^{bn}\partial_{j\nu}\beta^{am}\eta_{bc}(N(\hat{H}^{adj}))_{am}{}^{c,-n}, \qquad N(H) = \frac{e^{iH}-e^{-iH}-2iH}{(iH)^2} \tag{B.3a}$$

$$e_{i\mu}{}^{y_*} = \frac{1}{2}\left(\hat{B}_{i\mu,j\nu}e_{a,-m}{}^{j\nu}\eta^{ab} - e_{i\mu}{}^{bm}\right)\hat{\Omega}_{bm}{}^{y_*} \tag{B.3b}$$

$$\hat{B}_{i\mu,j\nu} = \frac{i}{\chi}\int_0^1 dt\,\hat{\mathrm{Tr}}\left(\hat{H}\partial_{[i\mu}e^{-it\hat{H}}\partial_{j\nu]}e^{it\hat{H}}\right), \qquad \hat{H} = \beta^{am}\hat{T}_{am} \tag{B.3c}$$

$$\partial_{i\mu}\hat{B}_{j\nu,k\rho} + \partial_{j\nu}\hat{B}_{k\rho,i\mu} + \partial_{k\rho}\hat{B}_{i\mu,j\nu} = \frac{i}{\chi}\hat{\mathrm{Tr}}(e_{i\mu}[e_{j\nu}, e_{k\rho}]) \tag{B.3d}$$

$$\partial_{i\mu}(\hat{B}_{j\nu,k\rho}e_{a,-m}{}^{k\rho}\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}) - (i\mu \leftrightarrow j\nu) = e_{i\mu}{}^{bn}e_{j\nu}{}^{am}f_{am,bn}{}^{cp}\hat{\Omega}_{cp}{}^{y_*}. \tag{B.3e}$$

$\hat{T}$ and $\hat{\mathrm{Tr}}$ are the matrix representations of the affine algebra and the reduced traces discussed in Section 2.7.

D. Affine Lie derivatives.

$$E_a(m) = e_{am}{}^{i\mu}p_{i\mu}(\hat{B}) + \frac{1}{2}k\hat{\Omega}_{am}{}^{y_*} = (C(\hat{H}^{adj}))_{am}{}^{bn}p_{bn} - ikn\beta^{bn}(D(\hat{H}^{adj}))_{bn}{}^{cp}\hat{\eta}_{cp,am} \tag{B.4a}$$

$$\bar{E}_a(m) = \bar{e}_{am}{}^{i\mu}p_{i\mu}(\hat{B}) - \frac{1}{2}k\hat{\Omega}_{am}{}^{y_*} = (\bar{C}(\hat{H}^{adj}))_{am}{}^{bn}p_{bn} + ikn\beta^{bn}(\bar{D}(\hat{H}^{adj}))_{bn}{}^{cp}\hat{\eta}_{cp,am} \tag{B.4b}$$

$$p_{i\mu}(\hat{B}) \equiv p_{i\mu} - \frac{1}{2}k\hat{B}_{i\mu,j\nu}e_{a,-m}{}^{j\nu}\eta^{ab}\hat{\Omega}_{bm}{}^{y_*}, \qquad p_{i\mu} = \partial_{i\mu}\beta^{am}p_{am} \tag{B.4c}$$

$$C(H) = \frac{iH}{e^{iH}-1}, \qquad D(H) = \frac{e^{-iH}-1+iH}{iH(e^{-iH}-1)} \tag{B.4d}$$

$$\bar{C}(H) = \frac{iH}{e^{-iH}-1}, \qquad \bar{D}(H) = \frac{e^{iH}-1-iH}{iH(e^{iH}-1)}. \tag{B.4e}$$

The canonical momenta $p_{i\mu}$ and p_{am}, which can be classical or quantum, are defined in Sections 2.4 and 2.9.

E. Mechanical WZW Lagrangian on $\hat{G}$.

$$L_M = \frac{k}{4}\eta_{ab}e_{i\mu}{}^{a,-m}e_{j\nu}{}^{bm}\partial_\tau x^{i\mu}\partial_\tau x^{j\nu} - \frac{k}{4}\left(\hat{\Omega}_{am}{}^{y_*} + 2\hat{B}_{i\mu,j\nu}\partial_\tau x^{j\nu}e_{am}{}^{i\mu}\right)\eta^{ab}\hat{\Omega}_{b,-m}{}^{y_*} \tag{B.5a}$$

$$= \frac{k}{4}(\partial_\tau\beta^{am}\partial_\tau\beta^{bn} + m\beta^{am}n\beta^{bn})(F(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn}$$
$$+\frac{k}{2}\partial_\tau\beta^{am}(G(\hat{H}^{adj}))_{am}{}^{cp}\hat{\eta}_{cp,bn}in\beta^{bn} \tag{B.5b}$$

$$F(H) = \frac{2 - e^{iH} - e^{-iH}}{H^2}, \qquad G(H) = N(H) = \frac{(e^{iH} - e^{-iH}) - 2iH}{(iH)^2}. \tag{B.5c}$$

The purely group-theoretic forms of the mechanical action on $\hat{G}$ are given in eqs.(2.5.7) and (2.11.3).

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