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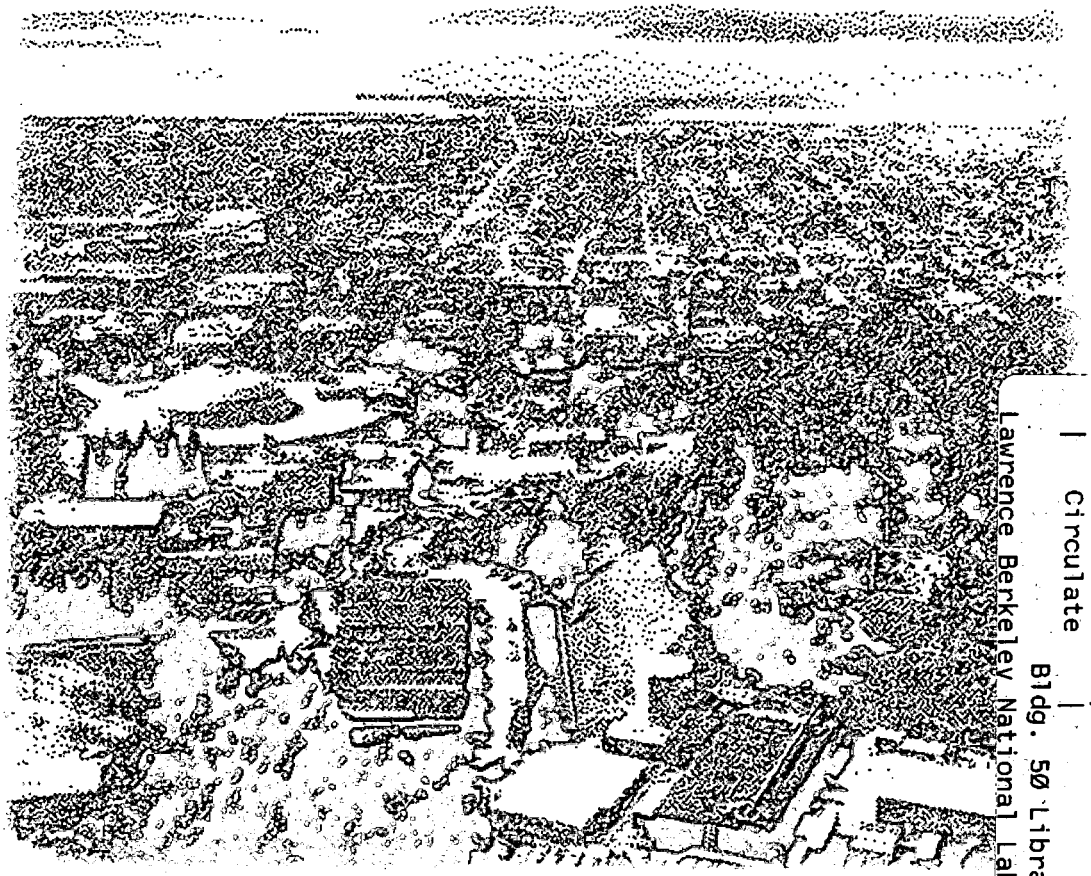

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ON MODULI SPACES OF FLAT CONNECTIONS WITH NON-SIMPLY CONNECTED STRUCTURE GROUP

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Abstract.
We consider the moduli space of flat G-bundles over the two-dimensional torus, where G is a real, compact, simple Lie group which is not simply connected. We show that the connected components that describe topologically non-trivial bundles are isomorphic as symplectic spaces to moduli spaces of topologically trivial bundles with a different structure group. Some physical applications of this isomorphism which allows to trade topological non-triviality for a change of the gauge group are sketched.

1 Introduction

In this letter we present an isomorphism between two different moduli spaces of gauge equivalent classes of flat connections on principal bundles over the two-dimensional torus Σ. Moduli spaces of flat connections over complex curves have been the subject of intensive investigations, since they play a key role both in Chern–Simons theory in three dimensions and in two-dimensional conformal field theory. In Chern–Simons theory with structure group G the moduli space $\mathcal{M}_G$ parametrizes the space of inequivalent classical solutions. Holomorphic quantization allows to associate a finite-dimensional complex vector space to $\mathcal{M}_G$, the space of conformal blocks [1, 10]. Appropriate sesquilinear combinations of elements of these spaces describe the correlation functions of the WZW model based on G. The latter constitute an important

subclass of two-dimensional conformal field theories; they also serve as the building blocks of many other conformal field theories, e.g. via the coset construction.

In the present note, we consider the case when the structure group G is a compact, connected, finite-dimensional Lie group, which is not simply connected. The corresponding moduli spaces arise naturally, e.g. in the description of WZW models or Chern–Simons theories based on these groups. Another important application of these spaces is the resolution of field identification fixed points in conformal field theories [6]. In the algebraic approach, the solution of this problem has given rise to a surprisingly rich structure, both in the case of coset conformal field theories [6] and integer spin simple current modular invariants [7]. It has been argued [8] that in order to describe the resolution of field identification fixed points in these models in a Lagrangean framework, non-simply connected structure groups G have to be considered. The result of this letter is therefore a first step towards a geometric understanding of the results of [6, 7].

If the structure group G is non-simply connected, the moduli space $\mathcal{M}_G$ consists of different connected components, which typically have different dimensions. Writing G as the quotient of $\tilde{G}$, the universal covering group of G, by a subgroup Z of the center of $\tilde{G}$,

$$G \cong \tilde{G}/Z\,, \tag{1.1}$$

the connected components of $\mathcal{M}_G$ are labeled by the finite abelian group Z:

$$\mathcal{M}_G = \bigcup_{\omega\in Z} \mathcal{M}_G^\omega\,. \tag{1.2}$$

If $\omega \in Z$ is not the identity, $\mathcal{M}_G^\omega$ is said to describe a topologically non-trivial sector of the theory.

The main result of this note is that the moduli space $\mathcal{M}_G^\omega$ describing a topologically non-trivial sector is isomorphic to the moduli space for some *other* Lie group G^ω, which describes the topologically *trivial* sector:

$$\mathcal{M}_G^\omega \cong \mathcal{M}_{G^\omega}^1\,. \tag{1.3}$$

The Lie group G^ω is again simple, finite-dimensional and compact. This isomorphism allows us to trade topological non-triviality for some other structure group and to reduce calculations to calculations in the topologically trivial sector only. The moduli spaces $\mathcal{M}_G$ can be obtained as a symplectic quotient of the infinite-dimensional symplectic space of all gauge potentials; as a consequence, a smooth dense open subset of them is a symplectic manifold with a symplectic form Ω. The symplectic structure Ω plays an important role, in particular for holomorphic quantization; we will see that the isomorphism (1.3) respects Ω.

Before we describe how the Lie group G^ω is obtained from ω and G, it is helpful to discuss the implications of this result for the quantized Chern–Simons theory. To apply the method of holomorphic quantization to the spaces $\mathcal{M}_G^1$ of topologically trivial connections one picks a complex structure on the torus Σ, parametrized by some complex number τ in the complex upper half plane. This turns Σ into a complex surface Σ_τ, and also induces a complex structure on $\mathcal{M}_G$. Next, one chooses a holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}_G^1$ such that its curvature is given by $2\pi\mathrm{i}\Omega$, where Ω is the symplectic form on $\mathcal{M}_G^1$. After fixing a positive integral value,

the level k (which plays the role of a coupling constant for the field theory), the quantization $\mathcal{B}_G^{1,k}$ of $\mathcal{M}_G^1$ is obtained as the finite-dimensional vector space of holomorphic sections of the k-th tensor power of $\mathcal{L}$:

$$\mathcal{B}_G^{1,k} := H^0(\mathcal{M}_G^1, \mathcal{L}^{\otimes k}) . \tag{1.4}$$

If the underlying Riemann surface is a torus Σ_τ, there is a distinguished basis for $\mathcal{B}_G^{1,k}$: denote the Lie algebra of G by $\bar{\mathfrak{g}}$, and consider the untwisted affine Lie algebra $\mathfrak{g} = \bar{\mathfrak{g}}^{(1)}$ based on $\bar{\mathfrak{g}}$. For fixed level k, there are finitely many unitarizable irreducible highest weight representations $\mathcal{H}_\Lambda$ of $\mathfrak{g}$. The character

$$\chi_\Lambda(\tau, h) := \mathrm{Tr}_{\mathcal{H}_\Lambda} e^{2\pi i \tau (L_0 - c/24)} e^{ih} , \tag{1.5}$$

of any irreducible representation is a function depending on τ and a variable h that takes values in the Cartan subalgebra $\bar{\mathfrak{g}}_0$ of $\bar{\mathfrak{g}}$. (After a choice of basis in $\bar{\mathfrak{g}}_0$, h can be described by its components, the so-called Cartan angles.) The characters of all irreducible representations at level k form a basis of the space $\mathcal{B}_G^{1,k}$ [1,3].

Let us now turn to the case of our interest, when G is not simply connected. The center of the universal covering group $\tilde{G}$ can be identified with a subgroup of the symmetries of the Dynkin diagram of the affine Lie algebra $\mathfrak{g}$. Any symmetry $\dot{\omega}$ of order N of the Dynkin diagram of $\mathfrak{g}$ induces an automorphism ω of the affine Lie algebra which acts like $\omega(E_\pm^i) := E_\pm^{\dot{\omega} i}$ on the step operators corresponding to the simple roots and $\omega(H^i) := H^{\dot{\omega} i}$ for the generators of the Cartan subalgebra. (The action on a full basis of the centrally extended loop algebra can be found in section 6 of [5].) The automorphism ω preserves the triangular decomposition of $\mathfrak{g}$ and in particular the Cartan subalgebra; hence the dual map $\omega^\star$ restricts to an isomorphism of the weight space $\mathfrak{g}_0^\star$ of $\mathfrak{g}$.

The automorphism ω gives rise [5] to 'twisted intertwiner maps', i.e. linear maps between irreducible highest weight representations of $\mathfrak{g}$

$$\tau_\omega : \quad \mathcal{H}_\Lambda \to \mathcal{H}_{\omega^\star(\Lambda)} \tag{1.6}$$

which obey

$$\tau_\omega \, x = \omega(x) \, \tau_\omega \quad \text{for all } x \in \mathfrak{g} \tag{1.7}$$

and which map the highest weight vector of $\mathcal{H}_\Lambda$ to the highest weight vector of $\mathcal{H}_{\omega^\star(\Lambda)}$. Those weights for which $\omega^\star(\Lambda) = \Lambda$, so-called *fixed points* of $\omega^\star$, are of particular interest: in this case τ_ω is an endomorphism, and one can insert τ_ω in the trace (1.5) to obtain a new set of functions on the Cartan subalgebra, the so-called *twining characters*

$$\chi_\Lambda^\omega(\tau, h) := \mathrm{Tr}_{\mathcal{H}_\Lambda} \tau_\omega e^{2\pi i \tau (L_0 - c/24)} e^{ih} . \tag{1.8}$$

The twining characters are dominated by the ordinary characters, and hence they converge wherever the ordinary characters converge. We will be interested in those symmetries of $\mathfrak{g}$ which are associated to elements of the center of $\tilde{G}$; these describe [9] the action of a so-called simple current in the corresponding WZW theory. The twining characters are close relatives of the index in a supersymmetric theory, where $(-1)^F$, F the fermion number, plays the role of τ_ω. Indeed, in any rational superconformal field theory the supercurrent is a simple current.

It was shown in [5] (see also [4]) that the twining character is identical, in a sense to be made precise later, to the character of some other Lie algebra, the *orbit Lie algebra* $\breve{\mathbf{g}}$. (The orbit Lie algebra depends on both $\mathbf{g}$ and ω; for the ease of notation, we do not mark the dependence on ω explicitly.) The Dynkin diagram of this Lie algebra is obtained by folding the Dynkin diagram of $\mathbf{g}$ according to the symmetry $\dot{\omega}$. More explicitly, the Cartan matrix of $\breve{\mathbf{g}}$ is defined as follows: Denote the Cartan matrix of $\mathbf{g}$ by $A = (a_{ij})_{i,j\in I}$, where the index set is $I = \{0, 1, \ldots, \operatorname{rank}\bar{\mathbf{g}}\}$. The symmetry $\dot{\omega}$ of the Dynkin diagram organizes I into orbits of different length N_i; we choose a set $\hat{I}$ of representatives in I from each $\dot{\omega}$-orbit. The Cartan matrix of the orbit Lie algebra is then labelled by the subset $\breve{I}$ of the set of orbits $\hat{I}$:

$$\breve{I} := \{i \in \hat{I} \mid \sum_{l=0}^{N_i-1} a_{i,\dot{\omega}^l i} > 0\}\,. \tag{1.9}$$

For any orbit we denote by s_i the number

$$s_i := \begin{cases} a_{ii}/\sum_{l=0}^{N_i-1} a_{i,\dot{\omega}^l i}\,, & \text{if } i \in \breve{I} \text{ and } a_{ii} \neq 0\,, \\ 1\,, & \text{otherwise}\,, \end{cases} \tag{1.10}$$

which is either 1 or 2. The elements of the Cartan matrix $\breve{A} = (\breve{a}_{ij})_{i,j\in\breve{I}}$ of the orbit Lie algebra $\breve{\mathbf{g}}$ are then given by

$$\breve{a}_{ij} := s_j \sum_{l=0}^{N_j-1} a_{i,\dot{\omega}^l j}\,. \tag{1.11}$$

Note that $\breve{I}$ can be the empty set, in which case the orbit Lie algebra is the trivial Lie algebra. One can show that the orbit Lie algebra of an affine Lie algebra is again an affine Lie algebra, unless it is trivial. [1]

We emphasize that the orbit Lie algebra $\breve{\mathbf{g}}$ is not constructed as a subalgebra of $\mathbf{g}$; in particular, the orbit Lie algebra is in general not isomorphic to the subalgebra of $\mathbf{g}$ that is fixed under ω. There is however a natural map P_ω from the subspace $\mathbf{g}_0^{(0)}$ of the Cartan subalgebra that is fixed under ω to the Cartan subalgebra $\breve{\mathbf{g}}_0$ of the orbit Lie algebra [5]. It is a bijection and the invariant bilinear forms on $\mathbf{g}_0^{(0)}$ and $\breve{\mathbf{g}}_0$ are related by

$$(h \mid h') = \frac{1}{N}\,(\mathrm{P}_\omega(h) \mid \mathrm{P}_\omega(h'))\,, \tag{1.12}$$

for all $h, h' \in \hat{\mathbf{g}}_0^{(0)}$. (Recall that N is the order of ω.) The dual relation for weights reads

$$(\lambda \mid \mu) = N \cdot (\mathrm{P}_\omega^{\star-1}(\lambda) \mid \mathrm{P}_\omega^{\star-1}(\mu))\,, \tag{1.13}$$

where we have assumed that the invariant bilinear form on weight space is normalized such that the highest root of the horizontal subalgebra has length squared 2. With this notation the statement that the twining characters are given by the characters of the orbit Lie algebras can be made precise [4, 5]:

$$\chi_\Lambda^{[\omega]}(\tau, h) = \breve{\chi}_{\mathrm{P}_\omega^{\star-1}(\Lambda)}(\tau, \mathrm{P}_\omega(h))\,. \tag{1.14}$$

[1] All results on orbit Lie algebras and twining characters are valid in a much more general context: they hold for arbitrary generalized Kac–Moody algebras [4].

Our results show that this theorem and the isomorphism (1.3) are closely related by the procedure of holomorphic quantization.

The key property of the orbit Lie algebra $\check{\mathbf{g}}$ is as follows: the dual map $\omega^\star$ acts as a linear map on the weight space of $\mathbf{g}$, on which also the Weyl group W of $\mathbf{g}$ acts. It can be shown [4] that the subgroup $\hat{W}$ of W that consists of all elements of W that commute with $\omega^\star$ is isomorphic to the Weyl group $\check{W}$ of the orbit Lie algebra $\check{\mathbf{g}}$. This fact enters crucially in the proof of (1.14); it will also be used in the present letter. Indeed, the simply connected Lie group G^ω appearing in (1.3) is just the simply connected compact Lie group whose Lie algebra is the horizontal subalgebra of $\check{\mathbf{g}}$.

The rest of this letter is organized as follows: in Section 2 we derive an explicit description of the moduli space $\mathcal{M}_G^\omega$, which is used to set up a map realizing the isomorphism (1.3). In Section 3 we check that this map preserves the symplectic structure and derive a condition on the level which is necessary for the existence of a quantization. In the last section we comment on applications of our result and present the conclusions.

2 The isomorphism

In this section we will derive an explicit description of $\mathcal{M}_G^\omega$. To this end we use the description of moduli spaces of flat connections in terms of monodromies around non-trivial cycles: the group G acts on the space of all group homomorphisms from the fundamental group $\pi_1(\Sigma)$ to G by conjugation. This action is just the action of gauge symmetries on the monodromies; the moduli space is then isomorphic to the quotient $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

The fundamental group of the torus is $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$, and for simply connected G we have to classify all solutions of the equation

$$g_a g_b (g_a)^{-1} (g_b)^{-1} = \mathbf{1}\,. \tag{2.1}$$

for $g_a, g_b \in G$, up to a simultaneous conjugation of g_a and g_b. In the case of non-simply connected G we prefer to work with elements of the universal covering group $\tilde{G}$ rather than with elements of $G \cong \tilde{G}/Z$. Thus we have to find all solutions (g_a, g_b), with $g_a, g_b \in \tilde{G}$, of the equation

$$g_a g_b (g_a)^{-1} (g_b)^{-1} = \omega\,, \tag{2.2}$$

where $\omega \in Z$ labels the topological sector. Again we have to identify solutions that are related by a simultaneous conjugation with some element of $\tilde{G}$.

In the topologically trivial sector equation (2.2) tells us that g_a and g_b commute. For any two commuting elements of the real compact Lie group $\tilde{G}$ there is a maximal torus containing both elements. The maximal torus is isomorphic to the Cartan subalgebra divided by the coroot lattice $\bar{L}^\vee$; the intersection of the orbits of conjugation with a maximal torus T are just the orbits of the Weyl group $\bar{W}_T$. As a consequence, the moduli space in the topologically trivial sector is

$$\mathcal{M}_G^1 \cong (\bar{\mathbf{g}}_0/\bar{L}^\vee \times \bar{\mathbf{g}}_0/\bar{L}^\vee)/\bar{W}_T\,, \tag{2.3}$$

where $\bar{W}_T$ acts diagonally.

The analogous analysis in the case of topologically non-trivial sectors is more involved; for the ease of the reader we present the result immediately. Consider the untwisted affine Lie algebra $\mathbf{g} = \bar{\mathbf{g}}^{(1)}$ with horizontal subalgebra $\bar{\mathbf{g}}$. To any element ω of the center of $\tilde{G}$ is associated a diagram automorphism of $\mathbf{g}$ corresponding to a simple current. (These correspond to the symmetries of the Dynkin diagram of $\mathbf{g}$ which are not already symmetries of the Dynkin diagram of $\bar{\mathbf{g}}$.) Upon identifying the Lie algebra h_T of T (which is a Cartan subalgebra for the Lie algebra $\bar{\mathbf{g}}$ of $\tilde{G}$) and the horizontal projection of the Cartan subalgebra of $\mathbf{g}^{(1)}$, ω gives rise to an affine map on h_T. This map can be expressed in terms of an element w_0 of the Weyl group $\bar{W}_T$ of $\bar{\mathbf{g}}$ and a shift by an element $p^\vee$ of the coweight lattice

$$\bar{\omega}(h) = p^\vee + w_0(h) = p^\vee + a_{w_0} h a_{w_0}^{-1} \quad . \tag{2.4}$$

Here we have chosen a group element a_{w_0} in $\tilde{G}$ to implement the action of the Weyl group element w_0; the element a_{w_0} is only determined up to an element of the maximal torus T, and we will have to fix some convenient choice for a_{w_0}. The map $\bar{\omega}$ leaves the (horizontal projection of the) fundamental affine Weyl chamber invariant. We will show that any solution of equation (2.2) is conjugate to a solution of the form

$$g_a = \exp(\mathrm{i}h) a_{w_0} \qquad \text{and} \qquad g_b = \exp(\mathrm{i}(h_0 + h')) \tag{2.5}$$

where h_0, h and h' are elements of the Cartan subalgebra h_T that obey $w_0(h) = h$, $w_0(h') = h'$ and $\bar{\omega}(h_0) = h_0$, respectively. Using the map P_ω the elements h and h' can be identified with elements of the Cartan subalgebra of the orbit Lie algebra $\breve{\mathbf{g}}$. We are interested in group elements and therefore any two solutions for which h and h' differ by elements of the coroot lattice should be identified. We will see that the elements $\beta^\vee$ of the coroot lattice of $\bar{\mathbf{g}}$ respecting the conditions $w_0(\beta^\vee) = \beta^\vee$ are in one-to-one correspondence to elements of the coroot lattice of $\bar{\breve{\mathbf{g}}}$. Next, we also have to take into account the effect of simultaneous conjugation with elements of $\tilde{G}$ that preserve the conditions on h and h'. We will see that this is described by the diagonal action of the subgroup $\hat{\bar{W}}$ of the Weyl group $\bar{W}$ of $\bar{\mathbf{g}}$ that commutes with $\bar{\omega}$. This subgroup, however, is isomorphic to the Weyl group of the horizontal subalgebra of the orbit Lie algebra, and comparing with (2.3) we obtain the isomorphism (1.3).

In order to prove that any solution of (2.2) is indeed conjugate to (2.5), we fix a maximal torus T of $\tilde{G}$ that contains g_b (for any element of a compact real Lie group such a torus exists); we can then write

$$g_b = \exp(\mathrm{i}h'') \tag{2.6}$$

with $h'' \in \mathsf{h}_T$. The element ω of the center can be written as $\omega = \exp(-\mathrm{i}p^\vee)$, where $p^\vee \in \mathsf{h}_T$ is an element of the co-weight lattice $\overline{L}^{\mathrm{w}\vee}$ of $\tilde{G}$ relative to T. Moreover, by adding elements of the co-root lattice and after choosing the convention for dividing the roots into positive and negative roots appropriately, we can assume that h'' is an element of the (closure of the) fundamental affine Weyl chamber. Without loss of generality we can write g_a as

$$g_a = r a_{w_0} \, , \tag{2.7}$$

where r is some element in $\tilde{G}$. At the present stage, we will fix some arbitrary choice for a_{w_0}; later on, we will determine a canonical choice for a_{w_0}.

Equation (2.2) then becomes

$$g_b \exp(-\mathrm{i}p^\vee) = r a_{w_0} \exp(\mathrm{i}h'') a_{w_0}^{-1} r^{-1} = r \exp(\mathrm{i}w_0(h'')) r^{-1} = r \exp(\bar{\omega}(\mathrm{i}h'')) \exp(-\mathrm{i}p^\vee) r^{-1} . \quad (2.8)$$

The element $\omega = \exp(-\mathrm{i}p^\vee)$ is in the center of $\tilde{G}$, equation (2.8) is therefore equivalent to

$$\exp(\bar{\omega}(\mathrm{i}h'')) = r^{-1} \exp(\mathrm{i}h'') r . \quad (2.9)$$

We now observe that $\bar{\omega}$ preserves the Cartan subalgebra of $\bar{\mathfrak{g}}$, hence the left hand side is in the maximal torus again. Since the orbits of conjugation on the maximal torus are the orbits of the Weyl group $\bar{W}_T$, we find that the right hand side is equal to $\exp(\mathrm{i}w(h''))$ with w a suitable element of the Weyl group $\bar{W}_T$ of $\bar{\mathfrak{g}}$. However, $\bar{\omega}$ preserves the fundamental affine Weyl chamber, and the only Weyl group element that does the same is the identity. From this we learn that $\bar{\omega}$ leaves h'' fixed, $\bar{\omega}(h'') = h''$, and as a consequence, (2.9) shows that r and $g_b = \exp(\mathrm{i}h'')$ commute. Now any two commuting elements of the real compact Lie group $\tilde{G}$ are contained in some maximal torus $\tilde{T}$, and since all maximal tori of $\tilde{G}$ are conjugate, we can find $g \in \tilde{G}$ such that

$$\tilde{T} = g^{-1} T g . \quad (2.10)$$

Denoting by $\mathsf{h}_{\tilde{T}}$ the Cartan subalgebra of $\bar{\mathfrak{g}}$ that is the Lie algebra of $\tilde{T}$, we have

$$g_b = g \exp(\mathrm{i}\eta'') g^{-1} \quad \text{and} \quad r = g \exp(\mathrm{i}\eta) g^{-1} \quad (2.11)$$

with $\eta, \eta'' \in \mathsf{h}_{\tilde{T}}$. This allows us to rewrite g_a as

$$g_a = r a_{w_0} = g(\mathrm{e}^{\mathrm{i}\eta} g^{-1} a_{w_0} g) g^{-1} . \quad (2.12)$$

Notice that $\tilde{a}_{w_0} := g^{-1} a_{w_0} g$ represents the element of the Weyl group $\bar{W}_{\tilde{T}}$ for the new maximal torus $\tilde{T}$ that corresponds to the same abstract Weyl group element as the one in $\bar{W}_T$ described by a_{w_0}.

Since we are interested in solutions of (2.2) only up to conjugation, we can drop the tildes and find that any solution of (2.2) is conjugate to

$$g_a = \exp(\mathrm{i}h) a_{w_0} \quad \text{and} \quad g_b = \exp(\mathrm{i}h''), \quad (2.13)$$

where $\exp(\mathrm{i}h)$ and $\exp(\mathrm{i}h'')$ are elements of the same maximal torus T of $\tilde{G}$, and h' is in the fundamental Weyl chamber. By conjugation with g, $\bar{\omega}$ gives also rise to an analogous affine map on the Lie algebra of the new maximal torus; h'' is invariant under the analogue of $\bar{\omega}$ on the new maximal torus.

The space $\mathcal{A}$ of $\bar{\omega}$-invariant elements of the Cartan subalgebra is an affine space relative to the vector space $\mathcal{F} := \ker(1 - w_0)$. The map P_ω gives an isomorphism between the affine space $\mathcal{A}$ and the horizontal projection of the weight space of the orbit Lie algebra; moreover, it also provides us with a distinguished base point in $\mathcal{A}$: $h_0 := \mathrm{P}_\omega^{-1}(0)$; we write

$$g_b = \exp(h_0 + h') , \quad \text{where} \quad h' \in \mathcal{F} = \ker(1 - w_0) . \quad (2.14)$$

We now have now to give a more detailed description of the element $h \in \bar{\mathfrak{g}}_0$ in (2.13): we remark that simultaneous conjugation of g_a and g_b with an element $\exp(\mathrm{i}\tilde{h})$ of the maximal torus does not change g_b. The action on g_a can be computed as follows:

$$\begin{aligned} g_a = \exp(\mathrm{i}h)a_{w_0} &\mapsto \exp(\mathrm{i}\tilde{h})\exp(\mathrm{i}h)a_{w_0}\exp(-\mathrm{i}\tilde{h}) \\ &= \exp(\mathrm{i}(h+\tilde{h}-w_0(\tilde{h})))a_{w_0}\,. \end{aligned} \tag{2.15}$$

Hence we always change h by a conjugation to $h \mapsto h + \tilde{h} - w_0(\tilde{h})$ and obtain an equivalent solution, i.e. we are free to add elements of the subspace range$(1-w_0)$. Using the fact that w_0 is an orthogonal transformation, this subspace can be expressed as

$$\mathrm{range}(1-w) = (\ker(1-w_0^t))^\perp = (\ker(1-w_0^{-1}))^\perp = (\ker(1-w_0))^\perp\,. \tag{2.16}$$

Hence we can assume, after conjugating g_a and g_b simultaneously with a suitable element of the form $\exp(\mathrm{i}\tilde{h})$, that both h and h' are in the kernel $\mathcal{F}$.

We have shown that the solutions of (2.4) are all conjugated to a solution of the form

$$g_a = \exp(\mathrm{i}h)a_{w_0} \quad \text{and} \quad g_b = \exp(\mathrm{i}(h_0+h')) = \exp(\mathrm{i}h_0)\exp(\mathrm{i}h') \tag{2.17}$$

where h and h' are in $\mathcal{F}$, and $h_0 + h'$ is in the (horizontal projection of the) fundamental affine Weyl chamber. Conversely, it is easy to check that any such pair of elements gives indeed a solution of (2.2).

At this point it seems as if there were an asymmetry between g_a and g_b. However, the situation is indeed symmetric: both h and h' are in $\mathcal{F}$, and hence a_{w_0} commutes with $\exp(\mathrm{i}h)$ and $\exp(\mathrm{i}h')$. This shows that we can find a second maximal torus $\tilde{T}$, which contains a_{w_0}, $\exp(\mathrm{i}h)$ and $\exp(\mathrm{i}h')$, but not $\exp(\mathrm{i}h_0)$. (Note that h and h' are fixed under w_0 and therefore so-called singular elements of the Cartan subalgebra; hence their exponentials can be indeed contained in two different maximal tori.)

The trivial rewriting of (2.2)

$$g_b g_a (g_b)^{-1}(g_a)^{-1} = \omega^{-1}\,, \tag{2.18}$$

allows us to change the roles of g_a and g_b in the above considerations, provided we replace w_0 by w_0^{-1} and $\bar{\omega}$ by $\bar{\omega}^{-1}$. In particular, we can write $g_a = \exp(\mathrm{i}h)a_{w_0}$ as the exponential of some element η of the Cartan subalgebra $\tilde{\bar{\mathfrak{g}}}_0$ belonging to $\tilde{T}$, $g_a = \exp(\mathrm{i}\eta)$, where η is invariant under the map $\tilde{\bar{\omega}}$ acting on $\tilde{\bar{\mathfrak{g}}}_0$. Again, there is a natural base point for this affine space: $\eta_0 := \tilde{\mathrm{P}}_\omega^{-1}(0)$, where $\tilde{\mathrm{P}}_\omega^{-1}$ is the analogue of P_ω^{-1} for $\tilde{T}$. Now recall that the element a_{w_0} implementing the Weyl group transformations is only specified up to an element of the maximal torus T. The group element $\exp(\mathrm{i}\eta_0)$ differs from a_{w_0} only by an element of the form $\exp(\mathrm{i}\tilde{h})$ with $\tilde{h} \in \mathcal{F}$, which is an element in the intersection of T and $\tilde{T}$. Hence we are free to replace a_{w_0} by $\exp(\mathrm{i}\eta_0)$ for our considerations.

Having found two distinguished base points, we can now describe any solution of the form (2.17) in a natural way in terms of the orbit Lie algebra: use the isomorphisms P_ω and $\tilde{\mathrm{P}}_\omega$ to associate to it the pair $(\mathrm{P}_\omega(h_0+h), \tilde{\mathrm{P}}_\omega(\eta_0+h'))$ in the weight space $\check{\mathcal{A}} \times \check{\mathcal{A}}$ of the orbit Lie algebra.

We are only interested in group elements rather than Lie algebra elements, and we should identify solution of (2.17) for which h and h' differ by elements of the coroot-lattice. However, we must preserve the condition that h and h' are fixed under w_0; so we are only allowed to add elements $\beta^\vee$ of the coroot lattice that are fixed under w_0 themselves. Then the translation by $\beta^\vee$ is an element of the affine Weyl group of $\mathbf{g}$ that commutes with $\bar{\omega}$; hence it is in the subgroup $\hat{W}$ and corresponds to an element in the affine Weyl group of the orbit Lie algebra which is a translation by a coroot of $\bar{\tilde{\mathbf{g}}}$. This shows that, after applying P_ω and $\tilde{\mathrm{P}}_\omega$, we simply have to project modulo the coroot lattice of the horizontal subalgebra of the orbit Lie algebra $\check{\mathbf{g}}$:

$$\check{\mathcal{A}} \times \check{\mathcal{A}} \to \check{\mathcal{A}}/\check{L}^\vee \times \check{\mathcal{A}}/\check{L}^\vee . \tag{2.19}$$

The only freedom we are left with now is simultaneous conjugation of g_a and g_b with a group element $g \in \tilde{G}$, such that the relations in (2.17) are preserved: then along with $g_b = \exp(\mathrm{i}(h_0 + h')$ we have $g g_b g^{-1} = \exp(\mathrm{i}\tilde{h}')$. The two elements $h_0 + h'$ and $\tilde{h}'$ are related by some element of the Weyl group $\bar{W}$; moreover, both $\tilde{h}'$ and $h_0 + h'$ are fixed under $\bar{\omega}$. It was shown in the proof of proposition 3.3. in [4] that then the Weyl group element $\hat{w}$ relating $h_0 + h'$ and $\tilde{h}'$ can be chosen in $\hat{\bar{W}}$:

$$g \exp(\mathrm{i}(h_0 + h'))g^{-1} = \exp(\mathrm{i}\hat{w}(h_0 + h')) . \tag{2.20}$$

The isomorphism P_ω intertwines the action of $\hat{\bar{W}}$ and the one of the affine Weyl group of the orbit Lie algebra. Hence $\hat{\bar{W}}$ leaves the base point fixed and therefore

$$g \exp(\mathrm{i}(h_0 + h'))g^{-1} = \exp(\mathrm{i}(h_0 + \hat{w}h') . \tag{2.21}$$

Analogous considerations can be applied to g_a, using this time the other maximal torus $\tilde{T}$; we reach the analogous conclusions. Combining the two results, we find that the remaining redundancies are taken into account by the diagonal action of the Weyl group $\hat{\bar{W}}$:

$$\mathcal{M}_G^\omega \cong (\check{\mathcal{A}}/\check{L}^\vee \times \check{\mathcal{A}}/\check{L}^\vee)/\hat{\bar{W}} . \tag{2.22}$$

Comparing this explicit description of the moduli space $\mathcal{M}_G^\omega$ with the standard description (2.3) of the moduli space of the orbit theory in the topologically trivial sector, we obtain the isomorphism (1.3).

3 The symplectic structure

In this section we want to extend the isomorphism (1.3) to include also the symplectic structure on a smooth open subset of $\mathcal{M}_G^\omega$. To this end we construct explicitly a topologically non-trivial G-bundle and a connection on it with the appropriate monodromies. We fix from now on a complex structure on the torus, which is parametrized by a complex number $\tau = \tau_1 + \mathrm{i}\tau_2$ with positive imaginary part $\tau_2 > 0$.

The corresponding torus Σ_τ can be obtained as the quotient of the complex plane by the following action of $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$:

$$R(m,n)(z) := z + m + n\tau \text{ , where } \quad m,n \in \mathbf{Z} . \tag{3.1}$$

The principal G-bundle is obtained by extending this action to an action on the trivial principal G-bundle $\mathbf{C} \times G$ over G: we fix an element h_0 with $\bar{\omega}(h_0) = h_0$ and set:

$$R_G(m,n)(z,g) := (z + m + n\tau,\, \mathrm{e}^{\mathrm{i}h_0 n} a_{w_0}^m g)\,. \tag{3.2}$$

Here g is an element of the non-simply connected group G, and we have written for simplicity a_{w_0} for the projection to G of the element $a_{w_0} \in \tilde{G}$ of the universal covering group we considered in the previous section. This defines indeed an action of $\mathbf{Z}^2$, as can be seen as follows: the equation $w_0(h_0) = h_0 - p^\vee$ implies that one has in $\tilde{G}$

$$\mathrm{e}^{\mathrm{i}h_0 n} a_{w_0}^m \mathrm{e}^{\mathrm{i}h_0 n'} a_{w_0}^{m'} = \mathrm{e}^{-\mathrm{i}n'mp^\vee} \mathrm{e}^{\mathrm{i}h_0(n+n')} a_{w_0}^{m+m'}\,. \tag{3.3}$$

The projection of the first element to the non-simply connected group G is trivial, and hence we have indeed an action of $\mathbf{Z}^2$. $(\mathbf{C} \times G)\,/\,\mathbf{Z}^2$ is a topologically non-trivial principal G-bundle over Σ_τ.

Let us now fix $h, h' \in \bar{\mathfrak{g}}_0$ such that $w_0(h) = h$ and $w_0(h') = h'$. Introduce the element

$$u := h' + \tau h \tag{3.4}$$

of the complexification of $\bar{\mathfrak{g}}_0$; the connection

$$A(z) := \frac{1}{2\tau_2}(-\bar{u}\mathrm{d}z + u\mathrm{d}\bar{z}) \tag{3.5}$$

on $\mathbf{C} \times G$ is then invariant under the induced action of $R_G(m,n)$: To see this, we remark that $a_{w_0}^{-1} u a_{w_0} = u$ and $\exp(-\mathrm{i}h_0)\, u \exp(\mathrm{i}h_0) = u$. Hence the induced action of $R(1,n)$ on the connection (3.5) is

$$R(1,n)A(z) = a_{w_0}^{-1} \mathrm{e}^{-\mathrm{i}h_0 n} A(z - 1 - b\tau) \mathrm{e}^{\mathrm{i}h_0 n} a_{w_0} = A(z)\,. \tag{3.6}$$

The connection (3.5) therefore gives rise to a connection on the principal G-bundle we constructed; this connection is flat. Let us now verify that this connection reproduces the monodromies g_a and g_b. We parametrize the first homology cycle C_a by $z(t) = t$ with $0 \le t \le 1$ and see that

$$\int_{C_a} A = \int_0^1 \mathrm{d}t A(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}) = \frac{1}{2\tau_2}(u - \bar{u}) = \mathrm{i}h\,. \tag{3.7}$$

Taking into account the additional twist by a_{w_0} in (3.2), the monodromy around C_a is indeed $g_a = \exp(\mathrm{i}h) a_{w_0}$. The second generator C_b of one-cycles can be parametrized as $z(t) = \tau t$; we find that

$$\int_{C_b} A = \int_0^1 A(\tau \frac{\partial}{\partial z} + \bar{\tau} \frac{\partial}{\partial \bar{z}}) \;= \frac{1}{2\tau_2}(\bar{\tau} u - \tau \bar{u}) = \mathrm{i}h'\,. \tag{3.8}$$

Taking again into account the additional twist, we see that the monodromy around C_b has the correct value $g_b = \exp(\mathrm{i}(h' + h_0))$ as well.

The symplectic structure is defined on the tangent space, which consists of $\bar{\mathfrak{g}}$-valued one-forms, which we can assume to be of the following form: define $\delta u := \delta h' + \tau \delta h$ as a complex

linear combination of two elements δh and $\delta h'$ of the Cartan subalgebra of the compact real form. An arbitrary element of the tangent space is then of the form

$$\delta A = \frac{1}{2\tau_2}(-\delta\bar{u}\mathrm{d}z + \delta u\mathrm{d}\bar{z}), \tag{3.9}$$

and the symplectic form is given by

$$\Omega_{\mathbf{g}}(\delta A_1, \delta A_2) := \frac{1}{2}\int \mathrm{d}^2 z\, \kappa_{\mathbf{g}}(\delta A_1 \wedge \delta A_2) \tag{3.10}$$

where $\kappa_{\bar{\mathbf{g}}}(\cdot,\cdot)$ is the Killing form on the Lie algebra $\bar{\mathbf{g}}$. Again, we adhere to the convention that Killing forms are normalized such that the highest $\bar{\mathbf{g}}$-root has length squared two. A standard calculation gives

$$\Omega_{\mathbf{g}}(\delta A_1, \delta A_2) = -\frac{\mathrm{i}}{2}(\kappa_{\bar{\mathbf{g}}}(\delta h_1', \delta h_2) - \kappa_{\bar{\mathbf{g}}}(\delta h_1, \delta h_2')), \tag{3.11}$$

which shows that the symplectic form is real and independent of the complex structure, which is parametrized by τ.

The comparison of this symplectic form with the one on $\mathcal{M}_{G^\omega}$ therefore reduces to a comparison of the Killing forms on $\mathbf{g}$ and $\check{\mathbf{g}}$. The relation (1.12) shows that they just differ by a factor of N, where N is the order of ω in Z. Taking into account the level $k \in \mathbf{Z}_{>0}$, we see that

$$k\Omega_{\mathbf{g}}(\delta A_1, \delta A_2) = \frac{k}{N}\Omega_{\check{\mathbf{g}}}(\delta \check{A}_1, \delta \check{A}_2). \tag{3.12}$$

This shows that upon expressing the symplectic form in terms of quantities in the orbit Lie algebra the level is divided by the order of the automorphism; this is exactly the relation between the levels of the Lie algebra and its orbit Lie algebra that was derived in [5]. An important consequence is that $k\Omega_G$ is an element of the *integral* cohomology only if the level is a multiple of N. Only in this case the moduli space can be quantized: this is the geometric counterpart of the fact that fixed points only occur at levels which are multiples of the order of the automorphism.

4 Applications and Conclusion

In this letter we have proven an isomorphism which, in physical terms, allows to trade topological non-triviality for a different gauge group. This result has several applications: the moduli spaces $\mathcal{M}_G^\omega$ appear naturally in the description of Chern–Simons theories or WZW-models on non-simply connected group manifolds. For the latter theories (indeed, for any integer spin simple current extension [9] of a conformal field theory) a formula for the modular matrix S was derived in [7]. The isomorphism (1.3) will be one ingredient to a rigorous proof of this formula. This formula in turn give a Verlinde formula for the dimension of the space of conformal blocks with a non-simply connected structure group, a problem that recently also has received attention in algebraic geometry [2].

Another application of the isomorphism (1.3) are coset conformal field theories, in the description as gauged WZW theories: it has been argued [8] that in these theories one actually has

to gauge a non-simply connected group. This observation has lead to the conjecture that the contributions from the topologically non-trivial sectors account for the resolution of field identification fixed points. This resolution can be written [6] in terms of quantities of the orbit Lie algebras. The isomorphism (1.3) therefore lends evidence to the conjecture relating orbit theories and topologically non-trivial sectors; it is also a first step towards a better understanding of coset conformal field theories in the Lagrangean framework.

We finally mention that our results only concern principal bundles over a two-dimensional torus; it would be interesting to unravel the implications of the structures we found for Riemann surfaces of higher genus.

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