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Geometry of $N = 1$ Dualities in Four Dimensions

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Abstract

We discuss how $N = 1$ dualities in four dimensions are geometrically realized by wrapping D-branes about 3-cycles of Calabi-Yau threefolds. In this setup the $N = 1$ dualities for SU, SO and USp gauge groups with fundamental fields get mapped to statements about the monodromy and relations among 3-cycles of Calabi-Yau threefolds. The connection between the theory and its dual requires passing through configurations which are T-dual to the well-known phenomenon of decay of BPS states in $N = 2$ field theories in four dimensions. We compare our approach to recent works based on configurations of D-branes in the presence of NS 5-branes and give simple classical geometric derivations of various exotic dynamics involving D-branes ending on NS-branes.

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1. Introduction

Many of the field theory dualities have now been embedded into string theory. The basic idea is to construct a local description of the field theory in a stringy setup. This local description can involve either purely geometric aspects of compactification manifold [1], a local geometry together with D-branes wrapped around cycles [2] or D-branes in the presence of NS 5-branes in a flat geometry [3], [4], [5]. In particular recently Elitzur, Giveon and Kutasov [5], following the approach of Hanany and Witten in constructing $N = 4$ theories in $d = 3$ [3], found a rather simple description of how Seiberg's $N = 1$ duality in four dimensions arises. They also suggested that their approach is T-dual to that of [2]. However their configuration of D-branes is more transparent and it allowed them in particular to see the appearance of the fundamental magnetic meson field in a simple way. In this paper we provide a local geometric description with wrapped D-branes in the spirit of [2] but in a somewhat simpler way, for which one can also follow the D-brane configurations in detail and see a particularly simple geometric realization of Seiberg's duality.

We will also discuss how the approach of [5] is related to the geometric description presented in this paper. The advantage of [3] and [5] is that the spacetime geometry is flat. On the other hand, the dilaton field is not constant in the presence of NS 5-branes, and in fact the string coupling constant blows up at cores of the branes. As noted in [3], this makes it difficult to analyse exactly what happens when D-branes end on NS 5-branes, which is a typical situation in their cases. For example, in their construction it was assumed that an open string stretched between two D-branes ending on opposite sides of an NS 5-brane gives a matter multiplet. In such a situation, however, the open string has to go through the strong coupling region inside the core of the NS 5-brane and the derivation of this statement would be beyond the reach of perturbative string theory. In [3], it was also suggested that when D and NS-branes cross each other a third brane should be created. This conjecture was motivated by comparison with field theory results and on the conservation of the NS-NS charge. We will show that these and other exotic dynamics involving D-branes ending on NS-branes have somewhat simpler counterparts in our construction and can be explained in purely classical geometric terms. Moreover the geometrical approach we follow easily extends to SO and USp gauge theories similar to [2].

2. Geometrical Setup

Compactification of type II strings on Calabi-Yau threefolds leads to $N = 2$ theories in $d = 4$. We will be interested in a local model for such a compactification which corresponds to a non-compact Calabi-Yau threefold. A canonical class of BPS states corresponds to Dirichlet p-branes wrapped around p-cycles of the Calabi-Yau. They preserve 1/2 of the supersymmetry, i.e. on their worldline we obtain the reduction of an $N = 1$ system from four dimensions to $(0 + 1)$-dimension. If we consider the spatial directions to be a T^3 and T-dualize (exchanging IIA and IIB strings) we end up with $(p + 3)$-branes partially wrapped around cycles of Calabi-Yau threefold, and at the same time filling the spacetime. The theory living on the (3+1)-part of the spacetime worldvolume of the $(p + 3)$-brane is an $N = 1$ theory in $d = 4$. In this way we have mapped BPS states of an $N = 2$ string theory to $N = 1$ field theories in four dimensions[1].

We would now like to explore some aspects of the resulting field theory in connection with the D-brane configurations. Let us consider type IIB on Calabi-Yau threefold, and consider some number of D3-branes wrapped around a set of three cycles C_i of Calabi-Yau threefold. Let ω denote the holomorphic 3-form of the Calabi-Yau. For a set of 3-cycles such a configuration can correspond to a BPS state only if [6]

$$|\sum_i \int_{C_i} \omega| = \sum_i |\int_{C_i} \omega| \tag{2.1}$$

i.e. the vectors $\int_{C_i} \omega$ in the complex plane are all parallel. If we consider T-dualizing the 3-space, we end up with type IIA theory with D6-branes wrapping around 3-cycles C_i of Calabi-Yau and filling the spacetime. Again the condition (2.1) is the condition corresponding to having an $N = 1$ supersymmetric field theory in $d = 4$. There are two natural classes of 3-cycles that appear in Calabi-Yau threefolds: (A) $S^2 \times S^1$ and (B) S^3. Moreover in a neighborhood of these cycles the Calabi-Yau threefold can be approximated by the cotangent space $T^*(S^2 \times S^1) = T^*S^2 \times S^1 \times R$ and T^*S^3. In case (A), the situation is locally the same as D-branes wrapped around $S^2 \times S^1$ in $K3 \times T^2$ compactification, where we view T^*S^2 as part of $K3$ and T^*S^1 as part of T^2. In this case the field theory in $d = 4$ will thus have $N = 2$ instead of $N = 1$. In fact if we consider N D-branes wrapped around such a cycle we end up getting an $N = 2$ system with $U(N)$ gauge symmetry and

[1] More generally this connection may provide an interesting link between black-hole dynamics in various dimensions and the T-dual field theories.

no matter [7]. In $N = 1$ terminology this is the same as an $U(N)$ gauge system with an adjoint chiral multiplet. If we wrap N D-branes around cycles of type (B) we end up with a pure $N = 1$ gauge system with gauge group $U(N)$. Note that in either case the bare gauge coupling constant is related to the volume of the three-cycle C_i by

$$\frac{1}{g^2} \sim V_{C_i}$$

which follows from the contribution to gauge coupling constant from 7 to 4 dimensions upon compactification on the 3-cycle. We have to note, however, this formula can get strong quantum corrections when V_{C_i} is small.

If a pair of wrapped cycles C_i, C_j intersect one another, the corresponding wrapped D-branes will be intersecting, in which case we can obtain extra massless matter from open strings ending on the pair of D-branes. For the intersection to be supersymmetric (and in particular to be compatible with (2.1)) we need that the number of local Dirichlet versus Neumann boundary conditions for the open string sector to be 0 mod 4, which in the present context means that the cycles C_i and C_j intersect on a circle. If we have N_i D-branes wrapped around C_i and N_j D-branes wrapped around C_j in such a situation the open string sector will give us a chiral matter of the type (N_i, N_j) (i.e. one $N = 1$ chiral multiplet in the fundamental representation in $U(N_i)$ times the conjugate representation in $U(N_j)$). We will refer to it as bifundamental. If C_i is of A type, we in addition will have a superpotential interaction of the form $qM\tilde{q}$ where $(q, \tilde{q})$ correspond to chiral matter matter and M is the adjoint matter of $U(N_i)$ coming from the D-branes wrapped around C_i. This follows from the fact that the theory as seen from the D-branes wrapped around C_i has an $N = 2$ supersymmetry.

We shall be interested in changing the complex moduli of Calabi-Yau threefold and following what happens to the wrapped cycles and discuss the corresponding field theory interpretation. In particular we shall consider a situation where cycles of both (A) and (B) type appear. Our local description of Calabi-Yau is that given in [8] which we now review. Consider local coordinates of the Calabi-Yau 3-fold given by $(x, y; x', y'; z)$ subject to two relations:

$$x^2 + y^2 = \prod_i (z - a_i)$$

$$x'^2 + y'^2 = \prod_j (z - b_j)$$

This geometry can be viewed more abstractly as a $C^* \times C^*$ bundle over the z-plane, viewing (x, y) and (x', y') as coordinates of the C^*'s; we can in turn view each C^* as approximating the structure of an elliptic curve near its degeneration. Note that the total space is non-singular if all a_i and b_j are distinct; the local degeneration of the fibers is an artifact of how we are slicing the total space. The space becomes singular if any pair of the a_i's or b_j's coincide where we get some vanishing cycles. Let us see how these cycles arise: To any pair of a_i's (and similarly b_j's) we can associate a 3-cycle of type (A) and to any (a_i, b_j) pair we can associate a 3-cycle of type (B). To see this note that for a fixed z, away from a_i and b_i there is a non-trivial S^1 in each of the C^*'s. For example the equation $x^2 + y^2 =$ const. defines a circle (note that if the constant is a positive real number this is realized by taking x and y real. Otherwise by an overall change of phase of x and y the situation reduces to the above case). Note that if we are at $z = a_i$ (or $z = b_j$) the corresponding circle vanishes. We consider 3-cycles which are a product of $S^1 \times S^1$ cycles over each point on the z-plane, together with segments in the z-plane ending on the a_i or b_j. If we go between two a_i's without going through b_j the corresponding three cycles sweep out an $S^2 \times S^1$ (and similarly if we go between any two b_j's). However if we go between a_i and b_j the three cycle we obtain is an S^3. To see this note that by continuous deformation the situation is the same as the case where a_i is close to b_j in which case locally the situation is the same as

$$x^2 + y^2 \sim z - a \qquad x'^2 + y'^2 \sim b - z$$

which implies that

$$x^2 + y^2 + x'^2 + y'^2 \sim b - a$$

which clearly describes an S^3 (with no loss of generality take b and a to be real and take (x, y) and (x', y') also to be real).

Connecting the pairs of a_i and b_j by paths in the z-plane we can associate 3-cycles to each path. Let us denote the 3-cycle connecting a_i to b_j by $[a_i, b_j]$. An important aspect of the above geometry that we shall use later is that the three cycle $[a_i, b_j] + [b_j, a_k] = [a_i, a_k]$. This in particular means that the sum of two 3-cycles of type (B) cycles can be a cycle of type (A). This is actually T-dual to the statement that a vector multiplet can decay to hypermultiplets in the $N = 2$ situation, as is well known in field theory [9] and its stringy realization [10].

Finally we wish to compute the integral of the holomorphic 3-form ω over the $S^1 \times S^1$ fiber over each point on z-plane and obtain a 1-form. This is similar to the situation studied in [10] and one finds (by a suitable choice for ω) we have

$$\int_{S^1 \times S^1} \omega = dz$$

This in particular means that if we wish to have the condition (2.1) satisfied, the beginning and end point cycles that we end upon must be in the same direction, i.e. $a_i - a_j, a_i - b_j, b_i - b_j$ must all be parallel if they correspond to the end points of the cycles which have D-branes wrapped around. Moreover to minimize the volume of the cycle we must take the image of D-branes on the z-plane to be straight lines.

3. Geometric realization of $N = 1$ dualities

Consider the geometric setup described in the previous section. Suppose we have two points a_1 and a_2 along the real part of z-plane where the first C^* degenerates and one point b, again on the real axis *between* a_1, a_2 where the second C^* degenerates. In particular along the real axis we have three ordered special points a_1, b, a_2. Suppose we wrap N_c D-branes around the S^3 cycle $[a_1, b]$ and N_f D-branes around the S^3 cycle $[b, a_2]$. Note that $[a_1, b]$ and $[b, a_2]$ meet along the circle on the first C^* at $z = b$. Thus from the considerations of the previous section it follows that the field theory we end up with is given by an $N = 1$ gauge theory $U(N_c) \times U(N_f)$ with chiral matter in $(N_c, N_f) \oplus$ c.c. representation. We will assume $N_f \geq N_c$. Note that the above system is the same as $N = 1$ supersymmetric QCD where we have also gauged the flavor group.

We now wish to change the moduli of Calabi-Yau and come to a configuration where the degeneration points are still along the real z-axis but the orders have been changed from (a_1, b, a_2) to (b, a_1, a_2). To do this we first push the point b up along the imaginary direction. It is now energetically preferable for the D-branes to reconnect so that N_c of them go directly between (a_1, a_2), by combining N_c pairs of S^3 cycles and converting them to N_c cycles of $S^2 \times S^1$ type, and $(N_f - N_c)$ of them go between (b, a_2). Now we push b along the negative real axis so that it has passed the x-coordinate of a_1 and then we bring it down to the real axis. At this point the $(N_f - N_c)$ D-branes which were going between (b, a_2) will decompose to $(N_f - N_c)$ branes between (b, a_1) and $(N_f - N_c)$ D-branes between (a_1, a_2). The N_c branes we previously had along (a_1, a_2) will recombine with the

new $(N_f - N_c)$ D-branes giving a total of N_f D-branes along (a_1, a_2) cycle. Thus the final configuration we end up with is a configuration of points ordered as (b, a_1, a_2) with $(N_f - N_c)$ D-branes wrapped around (b, a_1) and N_f D-branes wrapped around (a_1, a_2) cycle. The field theory we end up with is again easy to read off from the discussion of the previous section, namely

$$U(N_f - N_c) \times U(N_f)$$

with matter $q, \tilde{q}$ in $(N_f - N_c, N_f) \oplus$ c.c. representation and in addition, since the (a_1, a_2) system is an $N = 2$ system we have an extra adjoint field M which interacts with the above quarks in the usual form dictated by $N = 2$ supersymmetry, i.e. with a superpotential $qM\tilde{q}$. We have thus transformed the supersymmetric QCD with gauged flavor group to the Seiberg dual [11] where the flavor group continues to be gauged (note that on the magnetic dual side the flavor gauge system is an $N = 2$ system, as we have above).

One may ask what is the field theoretic meaning of the above operation. This is simply turning on the FI D-term for the $U(1)$ (common to the flavor and the color group). This breaks supersymmetry completely which is reflected by the fact that the intermediate D-brane configurations we were considering were not parallel. One may wonder if we can pass only through supersymmetric preserving configurations. This can be done in two ways. One way is to just pass the point b over a_1 along the real axis, in which case the conservation of D-brane charge will tell us how many D-branes we will have wrapped around each cycle after we pass through the singular configuration. Another, and perhaps a more satisfactory description is to take the point $a_2 \to \infty$, in which case the flavor gauge group coupling goes to zero and thus becomes just a global symmetry group throughout the above process. In this case the D-brane configurations will not break supersymmetry, because in this limit the lines on the z-plane are parallel, in accord with the fact that in this case the $U(1)$ FI D-term does not break supersymmetry[2].

[2] There is yet another way to turn on the FI D-term without breaking supersymmetry. To do this, we add one more point a_3 on the real axis to the right of a_2 and allow the first C^* to degenerate there also. We then wrap additional N_c D-branes on the $S^2 \times S^1$ cycle $[a_2, a_3]$. In this case, we can lift the S^3 cycle $[b, a_2]$ off the real axis together with $(N_f - N_c)$ D-branes on it while keeping $(a_2 - b)$ parallel to the real axis. Note that we can now do this without sending a_2 and a_3 to ∞. The field theory counterpart of this construction is to consider a theory with $U(N_c) \times U(N_f) \times U(N_c)$ gauge group. It is easy to see that this contains a FI-parameter corresponding to the lifting of the D-branes which does not break supersymmetry. We can push this line of argument further and consider N_f ordered points $a_2, ..., a_{N_f+1}$ to the right of b, with

Note that the same manipulations as above would have also worked if we had considered the $N = 2$ configuration with (a_1, a_2, a_3) in which case, in the limit we freeze the flavor group we would have connected the $N = 2$ system

$$U(N_c) \to U(N_f - N_c)$$

each with N_f fundamental hypermultiplets.

4. Generalizations to SO and USp theories

In this section we will generalize the above construction to the case of SO and USp gauge theories, very much in the spirit of the second reference in [2], and obtain the $N = 1$ dual pairs proposed in [11], [12], [13].

We start with the same setup as in the SU case and consider the double fibration

$$x^2 + y^2 = -(z - a)(z - a')$$

$$x'^2 + y'^2 = -z$$

where we take a, a' as real numbers with $a < 0$ and $a' > 0$. We thus have two S^3 cycles $[a, 0]$ and $[0, a']$. Note that the S^3 associated with $[a, 0]$ is realized by considering real values for x, y, x', y', z (because for $a < z < 0$ both $x^2 + y^2$ and $x'^2 + y'^2$ are positive), however the S^3 associated with $[0, a']$ is realized by purely imaginary values for x, y, x', y' but real value for z.

We wrap N_c D6-branes around $[a, 0]$ and N_f D6-branes around $[0, a']$. Now we orientifold the above configuration by combining the operation

$$(x, y, x', y', z) \to (x^*, y^*, x'^*, y'^*, z^*)$$

with exchange of left- and right-movers on the worldsheet. This is a symmetry of the above equation for a and a' real. Note that the fixed point space, i.e. the orientifold 6-space, is precisely the first S^3 associated with $[a, 0]$ times the uncompactified spacetime. Under this orientifolding the groups we started with $U(N_c) \times U(N_f)$ now change either

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$(N_f - n)$ D-brane wrapping on the cycle $[a_{n+1}, a_{n+2}]$ $(n = 1, ..., N_f - 1)$. By taking all $a_{n+1} \to \infty$, we recover the $U(N_c)$ gauge theory with N_f quarks, but this construction allows us to give a different mass parameter to each quark.

to $SO(N_c) \times USp(N_f)$ or $USp(N_c) \times SO(N_f)$, depending on the choice of the sign for the cross cap diagrams, with matter in bifundamentals as before. In the terminology of type I' theory it is natural to count the D-branes after orientifolding as $\frac{1}{2}N_c$ and $\frac{1}{2}N_f$ D6-branes respectively. Note that the reduction of the gauge factor associated with the $\frac{1}{2}N_f$ D6-branes arise because of the action of the orientifolding on the D6-branes wrapped around $[0, a']$ cycle; even though this cycle is not fixed under this transformation pointwise it is still mapped to itself.

Let us note that for the net D6-brane charge on the $[a, 0]$ cycle, in addition to the contribution from the physical D6 branes, there is a contribution from the orientifold plane. Since we have an orientifold 6-space this contribution is $\mp 16/2^3 = \mp 2$ (i.e. down from the case of orientifold 9-space by a factor of 2^3 arising from T-duality 3 times each of which splits it to two copies). The $\mp$ sign refers to the $SO(N_c)$ versus $USp(N_c)$ cases respectively. Thus the net $D6$-brane charge of the $[a, 0]$ cycle is $\frac{1}{2}N_c \mp 2$.The D6-brane charge of the $[0, a']$ cycle is $\frac{1}{2}N_f$ as there is no additional orientifold contribution to it.

Now we try to repeat the same process as in the $U(N_c)$ case. The main difference here is that we cannot lift the degeneration points off the real axis, as that is not consistent with the orientifolding operation. This is in accord with the field theory description where in this case we do not have the freedom to turn on a FI D-term. Instead we take a along the real axis from negative to positive values. After $a > 0$ the S^3 represented by the $[a, 0]$ for $a < 0$ now becomes an S^3 representing $[0, a]$ with purely imaginary values for x, y, x', y'. In particular the orientifolding operation has no fixed points anymore but the gauge group still continues to be SO or USp due to the action of the orientifold group on it. To find out how many D-branes we have wrapped around $[0, a]$ and $[a, a']$ we simply use charge conservation; this is an assumption which strictly speaking we cannot prove because we have passed through a strong coupling region, however the experience of the $U(N_c)$ case shows that it is reasonable. Taking into account the orientations of the D-branes we now should have $\frac{1}{2}N_f - (\frac{1}{2}N_c \mp 2)$ D6-brane charge on $[0, a]$ and $\frac{1}{2}N_f$ D6-branes on $[a, a']$. Noting that for $a > 0$ there is no orientifold plane all these charges should be accounted for by physical D-branes, and thus we obtain the dual groups $SO(N_f - N_c + 4) \times USp(N_f)$ or $USp(N_f - N_c - 4) \times SO(N_f)$ again with bifundamentals. Moreover, just as in the $U(N_c)$ case we will again obtain an $N = 2$ system in the flavor group, which implies that we have the fundamental magnetic meson with the right interactions.

5. Does this prove $N = 1$ duality?

In the $N = 1$ context the above process connects the electric gauge system with its magnetic dual. In what sense does this prove they are equivalent?[3] Note for example, in the context of heterotic string compactifications on T^2, the fact that we can continuously connect an $SU(3)$ gauge system with an $SU(2) \times SU(2)$ gauge system does not imply their equivalence. In fact as discussed above in the context of $N = 2$ systems we have connected a $U(N_c)$ system with an $U(N_f - N_c)$ system which clearly are inequivalent (for $N_f \neq 2N_c$) as they even have different dimensions for their Coulomb branch.

One hint of how one may try to understand in which cases we should expect an equivalence is that if in the process of exchange we had not pushed the middle point off the real axis and just gone along the negative real axis to the point where it would meet the first degeneration point, the original theory and the dual theory would meet and become the same theory at that point. This is the point where the gauge coupling constant in both theories are going to infinity. Now if we take into account the quantum corrections, and assuming both the original and the dual theories are asymptotically free (in the non-trivial SU part of the gauge group), taking the infrared limit on both theories will push both to the strong coupling regime where we can see how they can become equivalent. Of course this is a region where we should expect strong quantum corrections to the classical D-brane picture; however the above heuristic argument seems to at least give a conservative rationale to indicate in which cases the above interpolation between theories may imply infrared equivalence. Note that in the $N = 2$ case either the original or its dual are not in the asymptotically free regime, except for $N_f = 2N_c$ (in the SU case) where the above equivalence is the conjectured S-duality of $N = 2$ systems, the above connection would not necessarily imply their equivalence in the infrared, thus avoiding a contradiction. However in the $N = 1$ with $SU(N_c)$ gauge group for $\frac{3}{2}N_c < N_f < 3N_c$ since both the original and the dual theory are asymptotically free the above interpolation between theories suggests their infrared equivalence. It does not seem clear to us why in the regime $N_f < \frac{3}{2}N_c$ or $N_f > 3N_c$ where either the magnetic or the electric system is not in the asymptotically free region the above interpolation shows their infrared equivalence.

[3] We have benefited from discussions with N. Seiberg in preparation of this section.

6. Relation to other approaches

As noted before our approach is similar in spirit to that of [2]. In particular in the $N = 2$ situation it is identical to it by T-duality: If we consider type IIB compactified on $K3$ and consider N_c D7-branes wrapped around $K3$ and N_f D3-branes on it, the process considered in [2] consists of taking the volume of $K3$ to be small and using T-duality, exchanging 0- and 4-cycles on $K3$, to obtain the induced D-brane charges[4]. This however can be simplified by noting that the $SO(20,4;\mathbf{Z})$ T-duality on $K3$ [15] maps the above process into the classical monodromy of 2-cycles[5]. This also maps 3-brane and 7-brane configuration to 5-brane configurations wrapped around two-cycles of $K3$. By T-duality around one extra circle, this is exactly the configuration we have considered in the previous section in the $N = 2$ case.

However our construction of the $N = 1$ case seems more difficult to relate to [2], and in particular for the case of SU gauge groups it is more closely related to the recent construction of Elitzur, Giveon and Kutasov [5].

6.1. From Calabi-Yau to multiple brane configuration

The connection of our approach to that of [5] becomes apparent when we note that the A-type singularity on $K3$,

$$x^2 + y^2 = \prod_i (z - a_i) \tag{6.1}$$

is related, by T-duality, to a configuration of parallel NS 5-branes [16]. This can be shown by performing the T-duality on the elliptic fiber, along the natural S^1 action on C^*. In the original geometry (6.1), the elliptic fiber undergoes a monodromy transformation $\tau \to \tau + 1$ around each point $z = a_i$. After the T-duality, exchanging type IIA and type IIB, this becomes a unit integral shift in the NS-NS B-field on the fiber. Therefore the integral of $H = dB$ on a small circle around $z = a_i$ times the fiber gives 1, namely the region near a_i carries the minimum unit of the NS-NS charge. Note that the dilaton gets turned on in this process since the radius of S^1 on the fiber depends on the position z on the base. This shows that the T-duality replaces the degeneration of the fiber at each a_i by one NS 5-brane.

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[4] These aspects are studied further in [14].

[5] To be more specific, the T-duality which exchanges the 0 and the 4-cycles is conjugate, by the mirror symmetry, to the classical monodromy of $K3$.

We can also perform the T-duality on each C^* of the double elliptic fibration (which now takes type IIA or IIB back to itself),

$$\begin{aligned} x^2 + y^2 &= \prod_i (z - a_i) \\ x'^2 + y'^2 &= \prod_i (z - b_i), \end{aligned} \tag{6.2}$$

giving rise to two types of NS 5-branes, oriented differently. Let us choose coordinates so that NS 5-branes of the first type are parallel to the $x^0, ..., x^3, x^4, x^5$ plane, and NS 5-branes of the second type are stretched in the $x^0, ..., x^3, x^8, x^9$ directions. Since x^6, x^7 are common transverse directions to both types of NS 5-branes, we may regard (x^6, x^7) as real and imaginary parts of z in (6.2). Therefore $x^6 + ix^7 = b_i$ for a location of an NS 5-brane of the first type and $x^6 + ix^7 = a_i$ for the second type. The type II string on this geometry would give an $N = 2$ theory in four dimensions in the $x^0, ..., x^3$ directions. Following [5], we refer NS 5-branes of the first and second types as NS and NS'-branes respectively.

Let us consider D6-branes wrapping on the $S^2 \times S^1$ cycles $[a_i, a_j]$, $[b_i, b_j]$ or on the S^3 cycles $[a_i, b_j]$. Since these D6-branes locally look like $S^1 \times S^1$ on the fiber times line segments on the base z-plane, the T-duality on the fiber squeezes the $S^1 \times S^1$ directions on the branes and leaves them stretched on the line segments on the base. Thus the D6-branes turn into D4-branes connecting the NS 5-branes. Earlier in this paper, we found $a_i - a_j, b_i - b_j, a_i - b_j$ must all be parallel when there are D6-branes wrapping on the corresponding cycles. From the T-dual picture, the reason for this is that all the D4-branes have to be parallel in order to preserve the $N = 1$ supersymmetry. We choose coordinates so that this direction is parallel to the x^6 axis, i.e. $a_i - a_j, b_i - b_j, a_i - b_j$ are constrained to be real.

The geometric realization of the $N = 1$ $U(N_c) \times U(N_f)$ gauge theory with chiral matter in $(N_c, N_f) \oplus$ c.c. in the previous section is then mapped to the configuration of D-branes in the presence of the NS 5-branes. By reading from the right to left along the x^6 axis, an NS'-brane located at a_2 on the base is connected by N_f D4-branes to an NS-brane at b which is then connected by N_c D4-branes to another NS'-brane at a_1. One recognizes that this configuration is similar to that of [5] except that, in their case, the role of the right-most NS'-brane is played by N_f D6-brane stretched in the $x^0, ..., x^3, x^7, x^8, x^9$ directions.

Let us compare the two approaches. In [5], one has to make an assumption about a configuration which involves D4-branes ending on an NS-brane. For example, it is assumed that an open string stretched between two D4-branes attached on opposite sides ($x^6 < b$ and $b < x^6$) of the NS-brane gives the matter in (N_c, N_f). However, as noted in [3], such an open string has to go through the core of the NS-brane where the string coupling constant blows up, and it is difficult to see what exactly is happening there. This issue is avoided in our construction since the dilaton is constant. Moreover the total space of the elliptic fibration is non-singular even at $z = b$.

There are other interesting dynamical effects associated to the presence of NS 5-branes. It was suggested by Hanany and Witten [3] that, when the D6-brane crosses the NS-brane by cutting through it, an extra D4-brane should be created between the D6 and NS-branes. This conjecture was motivated by the consistency with field theory results and the conservation of the NS-NS charge. Similarly they argued that, if there are more than one D4-branes stretching between the D6 and NS-branes, the resulting configuration (called the s-configuration in [3]) should not have a supersymmetric ground state. We will show below that the corresponding statements in our setup can be explained by geometric terms.

6.2. Geometric derivation of the Hanany-Witten effect

What happens when a D6-brane stretched in the $x^0, ..., x^3, x^7, x^8, x^9$ directions crosses an NS-brane stretched in the $x^0, ..., x^3, x^4, x^5$ directions? According to Hanany and Witten, there must appear a D4-brane parallel to the $x^0, ..., x^3, x^6$ plane and connecting the D6 and NS-branes[6].

To understand its geometric meaning, let us perform the T-duality back to the double elliptic fibration of the Calabi-Yau manifold. Let us call the homology 1-cycles on the first elliptic fiber α_1 and β_1, and the cycles on the second fiber α_2 and β_2. We choose the basis of the cycles so that, after the T-duality, the α_2-cycle vanishes at $b = x^6 + ix^7$ where the NS-brane was located. The D6-brane in question is localized in the x^4, x^5 direction, i.e. on the first elliptic fiber, and is wrapping on the entire second fiber. It is also stretched along the x^7 direction. Therefore after the T-duality on α_1 and α_2, this D6-brane transforms

[6] To simplify our notations, we ignore the common $x^0, ..., x^3$ directions in the following discussions.

itself into another D6-brane now wrapping on the α_1 and β_2 cycles and stretched in the x^7 direction.

Since the geometry is asymptotically locally Euclidean, we can impose boundary conditions for large x^7 so that the D6-brane configuration looks asymptotically like $\alpha_1 \times \beta_2 \times$ (the x^7 direction). Let us move the D6-brane along the x^6 axis toward $z = b$ and see what happens. We should note that the local degeneration of the fiber at $z = b$ is an artifact of how we are slicing the total space, and there is no geometric singularity at $z = b$. Therefore we should be able to describe the passing of the D6-brane through $z = b$ by purely geometric language and the change of its shape should be smooth. Now let us push x^6 to the other side of b while keeping these boundary conditions at $x^7 \to \pm\infty$. Because of the monodromy $\beta_2 \to \beta_2 + \alpha_2$ around $z = b$, with an appropriate marking of the cycles on the fiber, a cross section of the D6-brane configuration for fixed x^7 right above $x^7 = 0$ is now $[\alpha_1 \times (\beta_2 + \alpha_2)]$ while a cross section right below $x^7 = 0$ remains $[\alpha_1 \times \beta_2]$. They do not match at $x^7 = 0$. The only thing that can happen is that the $x^7 > 0$ and $x^7 < 0$ portions of the D6-brane combine to create another D6-brane wrapping on $[\alpha_1 \times \alpha_2]$ at $x^7 = 0$ through a *pants-diagram.* The new D6-brane then can go from x^6 to b where α_2 is annihilated. This new portion of the D6-brane has topology of a solid torus whose boundary is $[\alpha_1 \times \alpha_2]$ at x^6 and the α_2-cycle is contractible inside of the solid torus. The boundary of this solid torus fills the mismatch of the $x^7 > 0$ and $x^7 < 0$ portions of the D6-brane, and the resulting configuration is supersymmetric and of minimal volume with respect to the boundary conditions at $x^7 \to \pm\infty$ given in the above. After performing the T-duality on α_1 and α_2, a portion of the D6-brane wrapping on the solid torus turns into a D4-brane on the line segment $[x^6, b]$. We see that this is exactly the configuration conjectured in [3], i.e. the D6-brane is now connected by a D4-brane to the NS-brane.

From this discussion, it is also clear why the configuration with more than one D4-branes going between the D6 and NS-branes (called the s-configuration in [3]) is not supersymmetric. The corresponding configuration in our setup would involve two portions of D6-branes whose cross sections for fixed x^7 are $[\alpha_1 \times \beta_2]$ at $x^7 < 0$ and $[\alpha_1 \times (\beta_2 + n\alpha_2)]$ at $x^7 > 0$ with $n > 1$. To tie them together at $x^7 = 0$, we need a solid torus whose boundary is $[\alpha_1 \times n\alpha_2]$. However we cannot set it in between x^6 and b without creating a curvature singularity at $z = b$.

Thus the entire construction of [3] and [5] is mapped into geometrical language we have been considering.

7. Comment on the instanton moduli space on the ALE space

We would like to comment on Kronheimer-Nakajima's construction [17] of the instanton moduli space on the ALE space since a geometric construction similar to those discussed in the above gives a natural D-brane interpretation of their result[7]. According to Kronheimer and Nakajima, the moduli space $\mathcal{M}_k(V)$ of instantons of degree k on a vector bundle[8] $V = \oplus \mathcal{R}_i^{\otimes v_i}$ with gauge group $U(V)$ is the largest Higgs branch of the $N = 4$ gauge theory in three dimensions with gauge group $\prod_{i=1}^n U(k)_i$ with v_i hypermultiplets in k of $U(k)_i$ and one bifundamental with respect to of $U(k)_i \otimes U(k)_{i+1}$.

This $N = 4$ gauge theory can be obtained by compactifying the type II string on $K3 \times T^3$ and wrapping D4 branes on 2-cycles on $K3$ localized at points on T^3. The relevant local model is again the elliptic fibration over the z-plane, but in order to reproduce the field content we compactify the real part of z on S^1 and pick n-points $a_1, ..., a_n$ on S^1 where the fiber degenerates. There are n S^2-cycles on this space, $[a_1, a_2]$, $[a_2, a_3]$, ... , $[a_n, a_1]$, and we wrap k D4-branes on each of the cycles. This gives the $\prod_{i=1}^n U(k)_i$ gauge group and the bifundamentals. To reproduce the v_i hypermultiplets, we wrap v_i D4-branes on a 2-cycle dual to $[a_i, a_{i+1}]$. The configuration space of the D4-branes wrapping the S^2-cycles gives the moduli space of the theory. In particular, their configuration on $K3$ parametrizes the hypermultiplet moduli space while their positions on T^3 span part of the vector multiplet moduli space.

We can now see that the largest Higgs branch of this theory is the instanton moduli space. To go to this Higgs branch, we move all the D4-brane to the same location on T^3 (this corresponds to moving to the origin of the Coulomb branch and turning off masses of the hypermultiplet fields.). We can then move the $n \times k$ D4-branes wrapping on the n S^2-cycles off toward the imaginary direction in z. The $n \times k$ D4-branes are then reconnected into k D4-branes wrapping on a cylinder $S^1 \times S^1$, where the first S^1 is on the fiber and the second S^1 is the real part of z. By the T-duality on this $S^1 \times S^1$, these D4-branes becomes D2-branes localized on $K3$. On the other hand, the v_i D4-branes wrapping on the dual 2-cycles become D6-brane wrapping on the entire $K3$. Since the configuration of the D2-branes parametrizes the hypermultiplet moduli space (the last sentence in the previous paragraph) and the D2-branes on the D6-branes are the same as instantons on the

[7] We would like to thank M. Douglas and N. Seiberg for discussion on this subject.

[8] $\mathcal{R}_i$ $(i = 1, ..., n)$ are particular line bundles over an ALE space of the A_{n-1} type associated to the different representation of $\mathbf{Z}_n$

D6-branes, it is clear that the largest Higgs branch of this theory is the instanton moduli space of degree k of rank $m = \sum_i v_i$ vector bundle. With some more work, one can show that the vector bundle is exactly $V = \oplus \mathcal{R}_i^{\otimes v_i}$.

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References

[1] S. Katz, A. Klemm and C. Vafa, "Geometric Engineering of Quantum Field Theories," hep-th/9609239

[2] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov, C. Vafa, "F-theory, Geometric Engineering and $N = 1$ Dualities," hep-th/9612052;
C. Vafa and B. Zwiebach,"$N = 1$ Dualities of SO and USp Gauge Theories and T-Duality of String," hep-th/9701015

[3] A. Hanany and E. Witten, "Type IIB Superstrings, BPS Monopoles, and Three-Dimensional Gauge Dynamics," hep-th/9611230.

[4] J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin, "Mirror Symmetry in Three-Dimensional Gauge Theories, $SL(2, Z)$ and D-Brane Moduli Space," hep-th/9612131, to appear in Nucl. Phys. B.

[5] S. Elitzur, A. Giveon and D. Kutasov, "Branes and $N = 1$ Duality in String Theory," hep-th/970214.

[6] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, "Duality Transformations in Supersymmetric Yang-Mills Theories Coupled to Supergravity," Nucl. Phys. B444 (1995) 92, hep-th/9502072

[7] M. Bershadsky, V. Sadov and C. Vafa, "D-Branes and Topological Field Theories," Nucl. Phys. B463 (1996) 420, hep-th/9511222.

[8] M. Bershadsky, V. Sadov and C. Vafa, "D-Strings on D-Manifolds," Nucl. Phys. B463 (1996) 398, hep-th/9510225.

[9] N. Seiberg and E. Witten,"Electric-Magnetic Duality, Monopole Condensation and Confinement in $N = 2$ Supersymmetric Yang-Mills Theory," Nucl. Phys. B426 (1994) 19, hep-th/9407087

[10] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, "Selfdual Strings and $N = 2$ Supersymmetric Field Theory," Nucl. Phys. B477 (1996) 746, hep-th/9604034.

[11] N. Seiberg, "Electric-Magnetic Duality in Supersymmetric Nonabelian Gauge Theories," Nucl. Phys. B435 (1995) 129, hep-th/9411149.

[12] K. Intriligator and N. Seiberg, "Duality, Monopoles, Dyons, Confinement and Oblique Confinement in Supersymmetric $SO(N_c)$ Gauge Theories," Nucl. Phys. B444 (1995) 125, hep-th/9503179.

[13] K. Intriligator and P. Pouliot,"Exact Superpotentials, Quantum Vacua and Duality in Supersymmetric $SP(N_c)$ Gauge Theories," Phys. Lett. B353 (1995) 471, hep-th/9505006.

[14] K. Hori and Y. Oz, "F-Theory, T-duality on $K3$ Surfaces and $N = 2$ Supersymmetric Gauge Theories in Four Dimensions," hep-th/9702173.

[15] P.S. Aspinwall and D.R. Morrison, "String Theory on K3 Surfaces," hep-th/9404151

[16] H. Ooguri and C. Vafa, "Two-Dimensional Black Hole and Singularities of CY Manifolds," Nucl. Phys. B463 (1996) 55, hep-th/9511164.

[17] P.B. Kronheimer and H. Nakajima, "Yang-Mills Instantons on ALE Gravitational Instantons," Math. Ann. 288 (1990) 263.

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