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March 1997
Submitted to
*Physics Letters B*

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This work was supported in part by Grant-in-Aid for Scientific Research on Priority Area 213 "Infinite Analysis," Japan Ministry of Education, the National Science Foundation under Grant PHY-951497, and the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Recycled Paper

UT-769, YITP-97-8
LBNL-40098
March, 1997

# Quantum Cohomology and Virasoro Algebra

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**Abstract**

We propose that the Virasoro algebra controls quantum cohomologies of general Fano manifolds $M$ ($c_1(M) > 0$) and determines their partition functions at all genera. We construct Virasoro operators in the case of complex projective spaces and show that they reproduce the results of Kontsevich-Manin, Getzler etc. on the genus-0,1 instanton numbers. We also construct Virasoro operators for a wider class of Fano varieties. The central charge of the algebra is equal to $\chi(M)$, the Euler characteristic of the manifold $M$.

As is well known, the quantum cohomology of a (symplectic) manifold $M$ is described by the topological $\sigma$-model with $M$ being its target space [1]. The partition function of the topological $\sigma$-model is given by the sum over holomorphic maps from a Riemann surface $\Sigma_g$ to $M$. When the degree $d$ of the map is zero (constant map), correlation functions of the $\sigma$-model reproduces the classical intersection numbers among homology cycles of $M$. When the degree is non-zero, however, the theory describes the quantum modification of the classical geometry due to the presence of the world-sheet instantons. In the following we consider the case where the topological $\sigma$-model is coupled to two-dimensional gravity so that we consider Riemann surfaces of an arbitrary genus and incorporate the gravitational descendants.

Recently there have been extensive studies of topological $\sigma$ models coupled to two-dimensional gravity (Gromov-Witten invariants) [2, 3, 4, 6, 7, 8, 9, 10]. In ref.[11] superpotentials for a large class of Fano manifolds $M$ (complex projective spaces, Grassmannians, rational surfaces etc.) have been constructed so that they reproduce Gromov-Witten invariants by means of residue integrals (see [12] for a related analysis using oscillatory integrals). Existence of the superpotentials indicates the mirror phenomenon for Fano varieties analogous to the one in the case of Calabi-Yau manifolds. These results, however, have so far been limited to the genus-0 case and no general principle is known to control quantum cohomology at higher genera (recursion relation for genus-1 instanton numbers has recently been obtained by Getzler [13]).

In this article we would like to propose a new powerful algebraic machinery for organizing the quantum cohomology theory at all genus: we propose that the Virasoro algebra controls the quantum cohomology and determines their partition functions at an arbitrary genus. We first discuss the Virasoro generators in the case of $CP^N$ manifolds and show that the Virasoro conditions eq.(1) correctly reproduce the results on instanton numbers of projective spaces obtained by [2, 3, 4, 5, 13] using the associativity of quantum cohomology ring. We then present the general form of the Virasoro operators for a wider class of Fano varieties. It turns out that the central charge of the Virasoro algebra equals the Euler characteristic of the manifold $M$.

Our conditions take the form

$$L_n Z = 0, \qquad n = -1, 0, 1, 2, \cdots \tag{1}$$

where $Z$ is the partition function related to the free energy of the theory as

$$Z = \exp\left(\sum_{g=0} \lambda^{2g-2} F_g\right) \tag{2}$$

$\lambda$ is the genus expansion parameter. Operators $L_n$ form the Virasoro algebra $[L_n, L_m] = (n-m)L_{n+m}$.

Cohomology classes of the complex projective space $M = CP^N$ is given by $\{1, \omega, \omega^2, \cdots, \omega^N\}$ where $\omega$ is the Kähler class. Corresponding fields are denoted as $\{\mathcal{O}_\alpha, \alpha = 0, 1, 2, \cdots, N\}$ and their coupling parameters are given by $\{t^\alpha\}$. Gravitational descendants of $\mathcal{O}_\alpha$ are denoted as $\sigma_n(\mathcal{O}_\alpha)$, $n = 1, 2, \cdots$ and their parameters by $\{t_n^\alpha\}$. Virasoro operators for $CP^N$ are then defined by

$$L_{-1} = \sum_{\alpha=0}^{N} \sum_{m=1}^{\infty} m t_m^\alpha \partial_{m-1,\alpha} + \frac{1}{2\lambda^2} \sum_{\alpha=0}^{N} t^\alpha t_\alpha, \tag{3}$$

$$L_0 = \sum_{\alpha=0}^{N} \sum_{m=0}^{\infty} (b_\alpha + m) t_m^\alpha \partial_{m,\alpha} + (N+1) \sum_{\alpha=0}^{N-1} \sum_{m=0}^{\infty} m t_m^\alpha \partial_{m-1,\alpha+1} + \frac{1}{2\lambda^2} \sum_{\alpha=0}^{N-1} (N+1) t^\alpha t_{\alpha+1}$$
$$- \frac{1}{48}(N-1)(N+1)(N+3), \tag{4}$$

$$L_n = \sum_{m=0}^{\infty} \sum_{\alpha=0}^{N} \sum_{j=0}^{N-\alpha} (N+1)^j C_\alpha^{(j)}(m,n) t_m^\alpha \partial_{m+n-j,\alpha+j} \qquad n \geq 1 \tag{5}$$
$$+ \frac{\lambda^2}{2} \sum_{\alpha=0}^{N} \sum_{j=0}^{N-\alpha} \sum_{m=0}^{n-j-1} (N+1)^j D_\alpha^{(j)}(m,n) \partial_m^\alpha \partial_{n-m-j-1,\alpha+j} + \frac{1}{2\lambda^2} \sum_{\alpha=0}^{N-n-1} (N+1)^{n+1} t^\alpha t_{\alpha+n+1},$$

where

$$C_\alpha^{(j)}(m,n) \equiv \frac{(b_\alpha + m)(b_\alpha + m + 1) \cdots (b_\alpha + m + n)}{(m+1)(m+2) \cdots (m+n-j)}$$
$$\times \sum_{m \leq \ell_1 < \ell_2 < \cdots < \ell_j \leq m+n} \left( \prod_{i=1}^{j} \frac{1}{b_\alpha + \ell_i} \right), \tag{6}$$

and

$$D_\alpha^{(j)}(m,n) \equiv \frac{b^\alpha (b^\alpha + 1) \cdots (b^\alpha + m) b_\alpha (b_\alpha + 1) \cdots (b_\alpha + n - m - 1)}{m!(n-m-j-1)!}$$
$$\times \sum_{-m-1 \leq \ell_1 < \ell_2 < \cdots < \ell_j \leq n-m-1} \left( \prod_{i=1}^{j} \frac{1}{b_\alpha + \ell_i} \right). \tag{7}$$

Here $\partial_{n,\alpha} = \partial / \partial t_n^\alpha$ and $b_\alpha$ is related to the degree $q_\alpha$ of the cohomology class $\omega_\alpha$ as

$$b_\alpha \equiv q_\alpha - \frac{1}{2}(N-1), \qquad \omega_\alpha \in H^{2q_\alpha}(M). \tag{8}$$

The indices are raised and lowered by the metric

$$\eta^{\alpha\beta} = \delta_{\alpha+\beta,N} \tag{9}$$

as usual. Thus $q^\alpha = N - q_\alpha$, $b_\alpha + b^\alpha = 1$.

Note that the factor $N+1$ which appears in the right-hand-side of eqs.(4),(5) is the magnitude of the first Chern class of $CP^N, c_1(CP^N) =$

$(N+1)\omega$. Thus the terms $j \neq 0$ represent the broken scale invariance of the $\sigma$-model with the target manifold $M$ which is a Fano variety. If one considers a fictitious manifold with a vanishing first Chern class with dimension $k/k+2$, then it turns out that the above Virasoro operators reduce to the well-known expressions in the theory of two-dimensional gravity and KP hierarchy [14]. Thus eqs.(3-5) may be regarded as the generalization of Virasoro conditions of 2-dimensional gravity to the case of the topological $\sigma$-model. We also note that when $N = 1$ the above expressions reduce to those already obtained in [15].

Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad n, m \geq -1, \tag{10}$$

can be verified by making use of the identities

$$\begin{aligned}
&\sum_{j=0}^{i} C_\alpha^{(i-j)}(k,n) C_{\alpha+i-j}^{(j)}(k+n-i+j,m) \\
&= (b_\alpha + k + n) C_\alpha^{(i)}(k, n+m) + C_\alpha^{(i-1)}(k, n+m), \qquad (11) \\
&\sum_{j=0}^{i} D_\alpha^{(i-j)}(k,n) C_{\alpha+i-j}^{(j)}(n-k-i+j-1,m) \\
&= (b_\alpha + n - k - 1) D_\alpha^{(i)}(k, n+m) + D_\alpha^{(i-1)}(k, n+m) \qquad (12)
\end{aligned}$$

We have made an extensive check of the Virasoro conditions eq.(1). For the sake of illustration we consider the case of $CP^2$ and $L_1 Z = 0$ equation. We denote the three primary fields corresponding to the cohomology classes $1, \omega, \omega^2$ as $P, Q, R$ and denote their couplings as $t^P, t^Q, t^R$. Genus-0 free energy has an expansion

$$F_0 = \frac{1}{2} t^P (t^Q)^2 + \frac{1}{2}(t^P)^2 t^R + f(t^Q, t^R), \tag{13}$$

$$f(t^Q, t^R) = \sum_{d=1} N_d^{(0)} \frac{(t^R)^{3d-1}}{(3d-1)!} \exp(d\, t^Q) \tag{14}$$

where $N_d^{(0)}$ is the number of genus-0 instantons of degree $d$ (number of degree $d$ rational curves passing through $3d-1$ points).

In the small phase space with $t_n^\alpha = 0$ for all $\alpha$ and $n \geq 1$ except for $t_1^P = -1$, $L_1 Z = 0$ equation reads as

$$\begin{aligned}
&-\frac{3}{8}\langle \sigma_2(P)\rangle_0 + \frac{15}{4} t^R \langle \sigma_1(R)\rangle_0 + \frac{3}{4} t^Q \langle \sigma_1(Q)\rangle_0 - \frac{1}{4} t^P \langle \sigma_1(P)\rangle_0 - 6\langle \sigma_1(Q)\rangle_0 \\
&+6 t^Q \langle R\rangle_0 - 9\langle R\rangle_0 - \frac{3}{4}\langle P\rangle_0 \langle R\rangle_0 + \frac{1}{8}\langle Q\rangle_0 \langle Q\rangle_0 + \frac{9}{2}(t^P)^2 = 0. \qquad (15)
\end{aligned}$$

$\langle\cdots\rangle_0$ denotes the genus-0 correlation function. We take the second derivative of (15) in $t^Q$ and set $t^P = t^Q = 0$. We obtain

$$-\frac{3}{8}\langle\sigma_2(P)QQ\rangle_0 + \frac{15}{4}t^R\langle\sigma_1(R)QQ\rangle_0 + \frac{3}{2}\langle\sigma_1(Q)Q\rangle_0 - 6\langle\sigma_1(Q)QQ\rangle_0 \quad (16)$$
$$+12\langle RQ\rangle_0 - 9\langle RQQ\rangle_0 - \frac{3}{4}\langle PQQ\rangle_0\langle R\rangle_0 - \frac{3}{2}\langle PQ\rangle_0\langle RQ\rangle_0 - \frac{3}{4}\langle P\rangle_0\langle QQR\rangle_0$$
$$+\frac{1}{4}\langle QQ\rangle_0\langle QQ\rangle_0 + \frac{1}{4}\langle QQQ\rangle_0\langle Q\rangle_0 = 0.$$

In order to eliminate descendant fields in (16) we use the topological recursion relation (TRR) at genus-0 [16],

$$\langle\sigma_n(\mathcal{O}_\alpha)XY\rangle_0 = n\langle\sigma_{n-1}(\mathcal{O}_\alpha)\mathcal{O}_\beta\rangle_0\langle\mathcal{O}^\beta XY\rangle_0 \quad (17)$$

($X, Y$ are arbitrary fields) and also the "flow equations"

$$\langle\sigma_1(P)P\rangle_0 = uw + \frac{1}{2}v^2, \qquad \langle\sigma_1(P)Q\rangle_0 = vw - f_v + uf_{uv} + vf_{vv} \quad (18)$$
$$\langle\sigma_1(P)R\rangle_0 = \frac{1}{2}w^2 + uf_{uu} + vf_{uv} - f_u, \quad \langle\sigma_1(Q)Q\rangle_0 = vf_{uv} + \frac{1}{2}(w + f_{vv})^2$$

where $u, v, w$ are defined as

$$u = \langle PP\rangle_0 = t^R, \;\; v = \langle PQ\rangle_0 = t^Q, \;\; w = \langle PR\rangle_0 = t^P \quad (19)$$

and $f_u = \partial f/\partial u$ etc. Flow equations are derived using TRR. Then eq.(16) is rewritten as

$$3t^R\langle RR\rangle_0 + 6\langle QR\rangle_0 - 9\langle QQR\rangle_0 + \langle Q\rangle_0\langle QQQ\rangle_0 + 3t^R\langle QR\rangle_0\langle QQQ\rangle_0$$
$$-6\langle QQ\rangle_0\langle QQQ\rangle_0 + (\langle QQ\rangle_0)^2 = 0. \quad (20)$$

By using the expansion of the free energy (14) we can convert (20) into a relation for the instanton numbers. After some algebra we find

$$N_d^{(0)} = (3d-4)!\sum\frac{N_\ell^{(0)}N_k^{(0)}}{(3\ell-1)!(3k-1)!}k^2\ell[3k\ell + \ell - 2k] \quad (21)$$

which is the well-known result of Kontsevich and Manin [2].

The above procedure may be made more systematic as follows: it is possible to show that in the small phase space and at genus=0, the $L_1 Z = 0$ condition may be rewritten as

$$A^\alpha A^\beta\Big[\langle\mathcal{O}_\alpha\mathcal{O}_\mu\mathcal{O}_\gamma\rangle_0\,\langle\mathcal{O}^\gamma\mathcal{O}_\beta\mathcal{O}_\nu\rangle_0 - \langle\mathcal{O}_\alpha\mathcal{O}_\beta\mathcal{O}_\gamma\rangle_0\,\langle\mathcal{O}^\gamma\mathcal{O}_\mu\mathcal{O}_\nu\rangle_0\Big] = 0, \quad (22)$$

where

$$A^\alpha \equiv (q_\alpha - 1)t^\alpha - (C)_0{}^\alpha. \quad (23)$$

$\mathcal{C}$ is the matrix representation of the first Chern class

$$(\mathcal{C})_{\alpha\beta} \equiv \int_M c_1 \wedge \omega_\alpha \wedge \omega_\beta. \tag{24}$$

Repeated indices $(\alpha, \beta, \gamma)$ are summed in (22). In this form the $L_1$ equation has the structure of the genus-0 associativity equation and it reproduces the results based on the associativity of quantum cohomology ring [2, 3, 4, 5].

We may also consider the genus-1 instanton numbers and check against the recent results of ref. [13, 17]. Genus-1 free energy of $CP^2$ has an expansion

$$F_1 = \frac{-t^Q}{8} + \sum_{d=1} N_d^{(1)} \frac{(t^R)^{3d}}{(3d)!} \exp(d\, t^Q). \tag{25}$$

We again consider the equation $L_1 Z = 0$ for simplicity. This time we use the TRR at genus-1 [16]

$$\langle \sigma_n(\mathcal{O}_\alpha) \rangle_1 = \frac{n}{24} \langle \sigma_{n-1}(\mathcal{O}_\alpha) \mathcal{O}_\beta \mathcal{O}^\beta \rangle_0 + n \langle \sigma_{n-1}(\mathcal{O}_\alpha) \mathcal{O}_\beta \rangle_0 \langle \mathcal{O}^\beta \rangle_1 \tag{26}$$

and the flow equation eq.(18). After some algebra we find

$$\begin{aligned} &\langle Q \rangle_0 \langle Q \rangle_1 + 3t^R \langle QR \rangle_0 \langle Q \rangle_1 + \frac{1}{8} t^R \langle QQR \rangle_0 - \frac{1}{4} \langle QQQ \rangle_0 \\ &-6 \langle QQ \rangle_0 \langle Q \rangle_1 + \frac{1}{8} \langle QQ \rangle_0 - 9 \langle R \rangle_1 = 0 \end{aligned} \tag{27}$$

where $\langle \cdots \rangle_1$ denotes genus-1 correlators. Using the free-energy expansion (25) we find

$$N_n^{(1)} = N_n^{(0)} \frac{1}{72} n(n-1)(n-2) + \sum_{k+\ell=n} (3n-1)! \frac{N_k^{(0)}}{(3k-1)!} \frac{N_\ell^{(1)}}{(3\ell)!} \frac{\ell}{9} (3k^2 - 2k). \tag{28}$$

The above equation has a form somewhat different from the one of [13], however, predicts the same instanton numbers. We can also check that the genus-1 instanton numbers of $CP^3$ are correctly reproduced by Virasoro conditions.

Let us next describe the derivation of the Virasoro conditions eqs.(1)–(5). $L_{-1}$ is the well-known string equation which has a universal form for all manifolds $M$ [18]. $L_0$ operators has been obtained in [19] by using the intersection theory on the moduli space of Riemann surfaces. Its general form is given by

$$\begin{aligned} L_0 \;=\; & \sum_{m=0}^{\infty} (m + b_\alpha) t_m^\alpha \frac{\partial}{\partial t_m^\alpha} + \sum_{m=1}^{\infty} m (\mathcal{C})_\alpha{}^\beta t_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta} + \frac{1}{2\lambda^2} (\mathcal{C})_{\alpha\beta} t^\alpha t^\beta \\ & + \frac{1}{24} \left( \frac{3 - \dim M}{2} \chi(M) - \int_M c_1(M) c_{\dim M - 1}(M) \right) \end{aligned} \tag{29}$$

where $\chi(M)$ denotes the Euler characteristic of $M$ and the $\mathcal{C}$ is the matrix of the first Chern class eq.(24). In the case of $CP^N$, $(\mathcal{C})_{\alpha\beta} = \delta_{\alpha+\beta+1,N}$ and (29) reduces to (4).

We also recall the recursion relation for descendant two-point functions obtained in our previous work [11]

$$(b_\alpha+b_\beta+n)\langle\sigma_n(\mathcal{O}_\alpha)\mathcal{O}_\beta\rangle_0 = n\Big(M_\beta{}^\gamma\langle\sigma_{n-1}(\mathcal{O}_\alpha)\mathcal{O}_\gamma\rangle_0-\langle\sigma_{n-1}(c_1(M)\wedge\mathcal{O}_\alpha)\mathcal{O}_\beta\rangle_0\Big) \tag{30}$$

where

$$M_{\alpha\beta} = (b_\alpha + b_\beta)\langle\mathcal{O}_\alpha\mathcal{O}_\beta\rangle_0 + (\mathcal{C})_{\alpha\beta}. \tag{31}$$

Let us now derive $L_1$. We first take the $t^\gamma$-derivative of the equation $L_0 Z = 0$ and keep the genus-0 terms

$$\begin{aligned}&\sum_{\alpha=0}^{N}\sum_{m=0}^{\infty}(m+b_\alpha)t_m^\alpha\langle\sigma_m(\mathcal{O}_\alpha)\mathcal{O}_\gamma\rangle_0 + b_\gamma\langle\mathcal{O}_\gamma\rangle_0\\ &+(N+1)\sum_{\alpha=0}^{N}\sum_{m=1}^{\infty}mt_m^\alpha\langle\sigma_{m-1}(\mathcal{O}_{\alpha+1})\mathcal{O}_\gamma\rangle_0 + (N+1)t_{\gamma+1} = 0.\end{aligned} \tag{32}$$

Then multiply (32) by $M_\beta{}^\gamma$ and use the recursion relation eq.(30). We obtain

$$\begin{aligned}&\partial_\beta\Bigg[\sum_{\alpha=0}^{N}\sum_{m=0}^{\infty}\frac{(m+b_\alpha)(m+b_\alpha+b_\beta+1)}{m+1}t_m^\alpha\langle\sigma_{m+1}(\mathcal{O}_\alpha)\rangle_0\\ &+(N+1)\sum_{\alpha=0}^{N}\sum_{m=0}^{\infty}(2m+2b_\alpha+b_\beta+1)t_m^\alpha\langle\sigma_m(\mathcal{O}_{\alpha+1})\rangle_0\\ &+(N+1)^2\sum_{\alpha=0}^{N}\sum_{m=1}^{\infty}mt_m^\alpha\langle\sigma_{m-1}(\mathcal{O}_{\alpha+2})\rangle_0 + \frac{1}{2}\sum_{\alpha=0}^{N}b^\alpha b_\alpha\langle\mathcal{O}^\alpha\rangle_0\langle\mathcal{O}_\alpha\rangle_0\\ &+\frac{1}{2}(N+1)^2\sum_{\alpha=0}^{N}t^\alpha t_{\alpha+2}\Bigg] - b_\beta(2b_\beta+1)\langle\sigma_1(\mathcal{O}_\beta)\rangle_0\\ &-2(N+1)b_\beta\langle\mathcal{O}_{\beta+1}\rangle_0 + \sum_{\alpha=0}^{N}b^\alpha b_\beta\langle\mathcal{O}_\alpha\mathcal{O}_\beta\rangle_0\langle\mathcal{O}^\alpha\rangle_0 = 0.\end{aligned} \tag{33}$$

Note that $\beta$ is not summed in (33). We introduce an auxiliary equation

$$\tilde{L}_0 \;:\; \sum_{\alpha=0}^{N}\sum_{m=0}^{\infty}t_m^\alpha\langle\sigma_m(\mathcal{O}_\alpha)\rangle_0 = 2F. \tag{34}$$

The above equation follows from the dilaton equation

$$\langle\sigma_1(P)\sigma_{n_1}(\mathcal{O}_1)\cdots\sigma_{n_s}(\mathcal{O}_s)\rangle_g = (2g-2+s)\langle\sigma_{n_1}(\mathcal{O}_1)\cdots\sigma_{n_s}(\mathcal{O}_s)\rangle_g. \tag{35}$$

In fact by taking derivatives of $\tilde{L}_0$ and putting all the variables $t_m^\alpha = 0$ except $t_1^P = -1$ we find

$$-\langle\sigma_1(P)\sigma_{n_1}(\mathcal{O}_1)\cdots\sigma_{n_s}(\mathcal{O}_s)\rangle_0+s\langle\sigma_{n_1}(\mathcal{O}_1)\cdots\sigma_{n_s}(\mathcal{O}_s)\rangle_0 = 2\langle\sigma_{n_1}(\mathcal{O}_1)\cdots\sigma_{n_s}(\mathcal{O}_s)\rangle_0. \tag{36}$$

(36) agrees with (35) at $g = 0$. We now take the $t^\gamma$-derivative of $\tilde{L}_0$ and multiply $M_\beta{}^\gamma$. We obtain

$$\partial_\beta \left[ \sum_{\alpha=0}^{N} \sum_{m=0}^{\infty} \frac{(m + b_\alpha + b_\beta + 1)}{m+1} t_m^\alpha \langle \sigma_{m+1}(\mathcal{O}_\alpha) \rangle_0 + (N+1) \sum_{\alpha=0}^{N} \sum_{m=0}^{\infty} t_m^\alpha \langle \sigma_m(\mathcal{O}_{\alpha+1}) \rangle_0 \right]$$

$$-2(N+1)\langle \mathcal{O}_{\beta+1} \rangle_0 - (2b_\beta + 1)\langle \sigma_1(\mathcal{O}_\beta) \rangle_0 - \sum_{\alpha=0}^{N} (b_\alpha + b_\beta) \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle_0 \langle \mathcal{O}^\alpha \rangle_0 = 0. \quad (37)$$

Next consider the linear combination (33) $- b_\beta \times$ (37)

$$\partial_\beta \left[ \sum_{\alpha=0}^{N} \sum_{m=0}^{\infty} \frac{(m + b_\alpha - b_\beta)(m + b_\alpha + b_\beta + 1)}{m+1} t_m^\alpha \langle \sigma_{m+1}(\mathcal{O}_\alpha) \rangle_0 \right.$$

$$+(N+1) \sum_{\alpha=0}^{N-1} \sum_{m=0}^{\infty} (2m + 2b_\alpha + 1) t_m^\alpha \langle \sigma_m(\mathcal{O}_{\alpha+1}) \rangle_0$$

$$+(N+1)^2 \sum_{\alpha=0}^{N-2} \sum_{m=1}^{\infty} m t_m^\alpha \langle \sigma_{m-1}(\mathcal{O}_{\alpha+2}) \rangle_0 + \frac{1}{2} \sum_{\alpha=0}^{N} b^\alpha b_\alpha \langle \mathcal{O}_\alpha \rangle_0 \langle \mathcal{O}^\alpha \rangle_0$$

$$\left. + \frac{1}{2}(N+1)^2 \sum_{\alpha=0}^{N-2} t^\alpha t_{\alpha+2} + \frac{1}{2} b_\beta (1 + b_\beta) \sum_{\alpha=0}^{N} \langle \mathcal{O}_\alpha \rangle_0 \langle \mathcal{O}^\alpha \rangle_0 \right] = 0. \quad (38)$$

Thus the equation becomes a total derivative and we can integrate it in $t^\beta$ (we ignore integration constants). Since the choice of $\beta$ is arbitrary in eq.(38), we obtain two equations: one of them comes from the $\beta$-independent terms and the other one from the terms linear in $b_\beta$ (terms quadratic in $b_\beta$ give the same equation as the latter),

$$L_1 : \sum_{\alpha=0}^{N} \sum_{m=0}^{\infty} \frac{(m + b_\alpha)(m + b_\alpha + 1)}{m+1} t_m^\alpha \langle \sigma_{m+1}(\mathcal{O}_\alpha) \rangle_0$$

$$+(N+1) \sum_{\alpha=0}^{N-1} \sum_{m=0}^{\infty} (2m + 2b_\alpha + 1) t_m^\alpha \langle \sigma_m(\mathcal{O}_{\alpha+1}) \rangle_0 \quad (39)$$

$$+(N+1)^2 \sum_{\alpha=0}^{N-2} \sum_{m=1}^{\infty} m t_m^\alpha \langle \sigma_{m-1}(\mathcal{O}_{\alpha+2}) \rangle_0 + \frac{1}{2} \sum_{\alpha=0}^{N} b^\alpha b_\alpha \langle \mathcal{O}_\alpha \rangle_0 \langle \mathcal{O}^\alpha \rangle_0$$

$$\frac{1}{2}(N+1)^2 \sum_{\alpha=0}^{N-2} t^\alpha t_{\alpha+2} = 0,$$

$$\tilde{L}_1 : -\sum_{\alpha=0}^{N} \sum_{m=0}^{\infty} \frac{1}{m+1} t_m^\alpha \langle \sigma_{m+1}(\mathcal{O}_\alpha \rangle_0 + \frac{1}{2} \sum_{\alpha=0}^{N} \langle \mathcal{O}_\alpha \rangle_0 \langle \mathcal{O}^\alpha \rangle_0 = 0. \quad (40)$$

Derivation of $L_2$ is similar. We take the derivative of $L_1$ and multiply the $M$-matrix and then integrate it with the help of equations derived from $\tilde{L}_0, \tilde{L}_1$. We find $L_2$ and an additional equation $\tilde{L}_2$

$$L_2 : \sum_{\alpha=0}^{N} \sum_{m=0}^{\infty} \frac{(m + b_\alpha)(m + b_\alpha + 1)(m + b_\alpha + 2)}{(m+1)(m+2)} t_m^\alpha \langle \sigma_{m+2}(\mathcal{O}_\alpha) \rangle_0$$

$$+(N+1)\sum_{\alpha=0}^{N-1}\sum_{m=0}^{\infty}\frac{(m+b_\alpha)(m+b_\alpha+1)(m+b_\alpha+2)}{m+1}(\sum_{\ell=0}^{2}\frac{1}{m+b_\alpha+\ell})t_m^\alpha\langle\sigma_{m+1}(\mathcal{O}_{\alpha+1})\rangle_0$$

$$+(N+1)^2\sum_{\alpha=0}^{N-2}\sum_{m=0}^{\infty}(\sum_{\ell=0}^{2}(m+b_\alpha+\ell))t_m^\alpha\langle\sigma_m(\mathcal{O}_{\alpha+2})\rangle_0+(N+1)^3\sum_{\alpha=0}^{N-3}\sum_{m=1}^{\infty}mt_m^\alpha\langle\sigma_{m-1}(\mathcal{O}_{\alpha+3})\rangle_0$$

$$+\sum_{\alpha=0}^{N}b^\alpha b_\alpha(b_\alpha+1)\langle\sigma_1(\mathcal{O}_\alpha)\rangle_0\langle\mathcal{O}^\alpha\rangle_0+\frac{N+1}{2}\sum_{\alpha=0}^{N-1}b^\alpha b_\alpha(b_\alpha+1)(\sum_{\ell=-1}^{1}\frac{1}{b_\alpha+\ell})\langle\mathcal{O}_{\alpha+1}\rangle_0\langle\mathcal{O}^\alpha\rangle_0$$

$$+\frac{1}{2}(N+1)^3\sum_{\alpha=0}^{N-3}t^\alpha t_{\alpha+3}=0, \tag{41}$$

$\tilde{L}_2$ :

$$\sum_{\alpha=0}^{N}\sum_{m=0}^{\infty}\frac{b_\alpha+m+1}{(m+1)(m+2)}t_m^\alpha\langle\sigma_{m+2}(\mathcal{O}_\alpha)\rangle_0+(N+1)\sum_{\alpha=0}^{N-1}\sum_{m=0}^{\infty}\frac{1}{(m+1)}t_m^\alpha\langle\sigma_{m+1}(\mathcal{O}_{\alpha+1})\rangle_0$$

$$-\sum_{\alpha=0}^{N}b_\alpha\langle\mathcal{O}^\alpha\rangle_0\langle\sigma_1(\mathcal{O}_\alpha)\rangle_0-\frac{N+1}{2}\sum_{\alpha=0}^{N-1}\langle\mathcal{O}^\alpha\rangle_0\langle\mathcal{O}_{\alpha+1}\rangle_0=0. \tag{42}$$

Higher equations $L_n$, $n\geq 2$ and their associated ones $\tilde{L}_n$ will be derived in a similar manner. We postpone our discussions on $\tilde{L}_n$ and concentrate on $L_n$ equations. The last step is to convert them into differential operators and we find exactly the form of Virasoro operators given in eq.(5). We postulate the validity of the Virasoro conditions at all genera. Note that the quadratic terms of the correlators of the form $\langle\cdots\rangle\langle\cdots\rangle$ are promoted to second derivative terms in the Virasoro operator which relate amplitudes of different genera.

It is interesting to see if we can construct the negative branch $L_{-n}$, $n\geq 2$ of Virasoro operators and compute the central charge of the algebra. It turns out that in the case of $CP^N$ with $N$ =even it is possible to construct $\{L_{-n}\}$. They are given by

$$L_{-n}=\sum_{m=0}^{\infty}\sum_{\alpha,j}(N+1)^jA_\alpha^{(j)}(m,n)t_{m+n+j}^\alpha\partial_{m,\alpha+j}$$
$$+\frac{\lambda^2}{2}\sum_{\alpha,j}\sum_{m=0}^{n+j-1}(N+1)^jB_\alpha^{(j)}(m,n)t_{n-m+j-1}^\alpha t_{m,\alpha+j},\quad n\geq 1 \tag{43}$$

The coefficients are defined by

$$A_\alpha^{(j)}(m,n)\equiv(-1)^j\frac{(m+1)(m+2)\cdots(m+n+j)}{(b_\alpha+m+j+1)(b_\alpha+m+j+2)\cdots}$$
$$\frac{}{\cdots(b_\alpha+m+j+n-1)}\times\sum_{1\leq\ell_1\leq\ell_2\leq\cdots\leq\ell_j\leq n-1}\left(\prod_{i=1}^{j}\frac{1}{b_\alpha+m+j+\ell_i}\right), \tag{44}$$

and

$$B_\alpha^{(j)}(m,n)\equiv(-1)^j\frac{m!(n-m+j-1)!}{(b^\alpha-j)(b^\alpha+1-j)\cdots(b^\alpha+m-1-j)}$$

$$\times\frac{}{(b_\alpha+j)(b_\alpha+1+j)\cdots(b_\alpha+n-m-2+j)}\sum_{0\le\ell_1\le\ell_2\le\cdots\le\ell_j\le n-2}\Big(\prod_{i=1}^{j}\frac{1}{b_\alpha+j-m+\ell_i}\Big). \tag{45}$$

We can check the entire algebra and find that the central charge is given by

$$c=N+1. \tag{46}$$

This suggests that there exists a realization of our algebra by means of $N+1$ free scalar fields. In fact it is possible to express Virasoro operators in terms of $N+1$ free scalars and the system resembles that of a logarithmic conformal field theory. Details will be discussed elsewhere. In the case of $N$ =odd some factors in the denominators of $A_\alpha^{(j)}(m,n), B_\alpha^{(j)}(m,n)$ vanish and the above expressions become singular. We do not understand the origin of this disparity between even and odd values of $N$.

Let us now consider the possible form of Virasoro operators for a general Fano manifold $M$. A natural conjecture is

$$L_n=\sum_{m=0}^{\infty}\sum_{\alpha,\beta}\sum_{j}C_\alpha^{(j)}(m,n)(\mathcal{C}^{\,j})_\alpha{}^\beta t_m^\alpha\partial_{m+n-j,\beta}\qquad n\ge 1 \tag{47}$$

$$+\frac{\lambda^2}{2}\sum_{\alpha,\beta}\sum_{j}\sum_{m=0}D_\alpha^{(j)}(m,n)(\mathcal{C}^{\,j})_\alpha{}^\beta\partial_m^\alpha\partial_{n-m-j-1,\beta}+\frac{1}{2\lambda^2}\sum_{\alpha\beta}(\mathcal{C}^{\,n+1})_\alpha{}^\beta t^\alpha t_\beta,$$

$$L_{-n}=\sum_{m=0}^{\infty}\sum_{\alpha,\beta}\sum_{j}A_\alpha^{(j)}(m,n)(\mathcal{C}^{\,j})_\alpha{}^\beta t_{m+n+j}^\alpha\partial_{m,\beta},\qquad n\ge 1 \tag{48}$$

$$+\frac{\lambda^2}{2}\sum_{\alpha,\beta}\sum_{j}\sum_{m=0}B_\alpha^{(j)}(m,n)(\mathcal{C}^{\,j})_\alpha{}^\beta t_{n-m+j-1}^\alpha t_{m,\beta},$$

$$b_\alpha=q_\alpha-\frac{(\dim \mathrm{M}-1)}{2}$$

where $\mathcal{C}^{\,j}$ is the $j$-th power of the matrix $\mathcal{C}$. (47), (48) reduces to (5), (43) in the case of $CP^N$. The operators (47), (48) (together with $L_0$ (eq.(29))) form a Virasoro algebra with a central charge

$$c=\sum_\alpha 1=\chi(M), \tag{49}$$

if the following condition is satisfied

$$\frac{1}{4}\sum_\alpha b^\alpha b_\alpha=\frac{1}{24}\Big(\frac{3-\dim \mathrm{M}}{2}\chi(M)-\int_M c_1(M)\wedge c_{dimM-1}(M)\Big). \tag{50}$$

(50) follows from $[L_1,L_{-1}]=2L_0$. (Note that we are considering the case where there are no odd-dimensional cohomologies, $dimH^{odd}(M)=0$ and hence $\sum dimH^{even}=\chi(M)$.)

Eq.(50) is a curious formula depending only on the geometrical data of $M$. It in fact holds in the case of projective spaces,

$$LHS = \frac{1}{4}\sum_{\alpha=0}^{N}\Big(\alpha + \frac{1-N}{2}\Big)\Big(N - \alpha + \frac{1-N}{2}\Big) = -\frac{(N^2-1)(N+3)}{48},$$
$$RHS = \frac{1}{24}\Big(\frac{3-N}{2}(N+1) - (N+1)^2\frac{N}{2}\Big) = -\frac{(N^2-1)(N+3)}{48}. \quad (51)$$

However, it is possible to show that (50) also holds in other classes of Fano varieties, i.e. Grassmannians, rational surfaces (point blow-ups of $CP^2$ and $P^1 \times P^1$), etc. Thus for these classes of Fano manifolds our Virasoro conditions may also correctly determine their quantum cohomology.

We have tested the operators (47) in the case of Grassmannian manifold $Gr(2,4)$ for which genus-0 instanton data exist [5]. We found that Virasoro conditions in fact reproduce correct instanton numbers of $Gr(2,4)$. Thus we conjecture that our Virasoro conditions are also valid for the Grassmannian manifolds. It is a very interesting problem to find exactly the class of manifolds for which our construction works.

In the above we have concentrated on the discussions of the Virasoro conditions. How about the additional constraints $\tilde{L}_n = 0$? It is easy to check that they are in fact satisfied by the genus-0 correlation functions of the $CP^N$ model. It seems that these are the analogues of the W-constraints in the theory of two-dimensional gravity coupled to minimal models. There are, however, higher genus corrections to these equations which we do not know how to control at present. We would like to have a better understanding of these equations in the near future.

It is quite encouraging for us that the problem of world-sheet instantons seem to possess a simple organizing principle and the theory has a structure which is a natural generalization of the 2-dimensional gravity. We note that in the case of general target manifold $M$ the central charge of the Virasoro algebra is equal to its Euler number (49) which is the number of supersymmetric vacua of the non-linear $\sigma$-model. Thus the theory appears to be a free field theory of $\chi(M)$ scalar fields each of which describes the fluctuation around a supersymmetric vacuum. The presence of the mass gap in the system may explain the decoupling of different vacua and the free field behavior of the theory (private communication by Witten). It will be very interesting to see if the free field or Virasoro structure persists in the case of Fano varieties which have no mass gap.

It will also be quite interesting to construct solutions to the Virasoro conditions possibly by some matrix integrals. In the simplest case of $CP^1$ we already have a matrix model with a logarithmic action which reproduces the quantum cohomologies at all genera [20, 15]. Similar construction for more general Fano varieties will be extremely valuable.

We thank discussions with M.Jinzenji and E.Witten. Research of T.E. is supported in part by Grant-in-Aid for Scientific Research on Priority Area 213 "Infinite Analysis", Japan Ministry of Education. Research of K.H. is supported by NSF grant PHY-951497 and DOE grant DE-AC03-76SF00098.

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Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098