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# Drinfel'd Twist and q-Deforming Maps for Lie Group Covariant Heisenberg Algebrae

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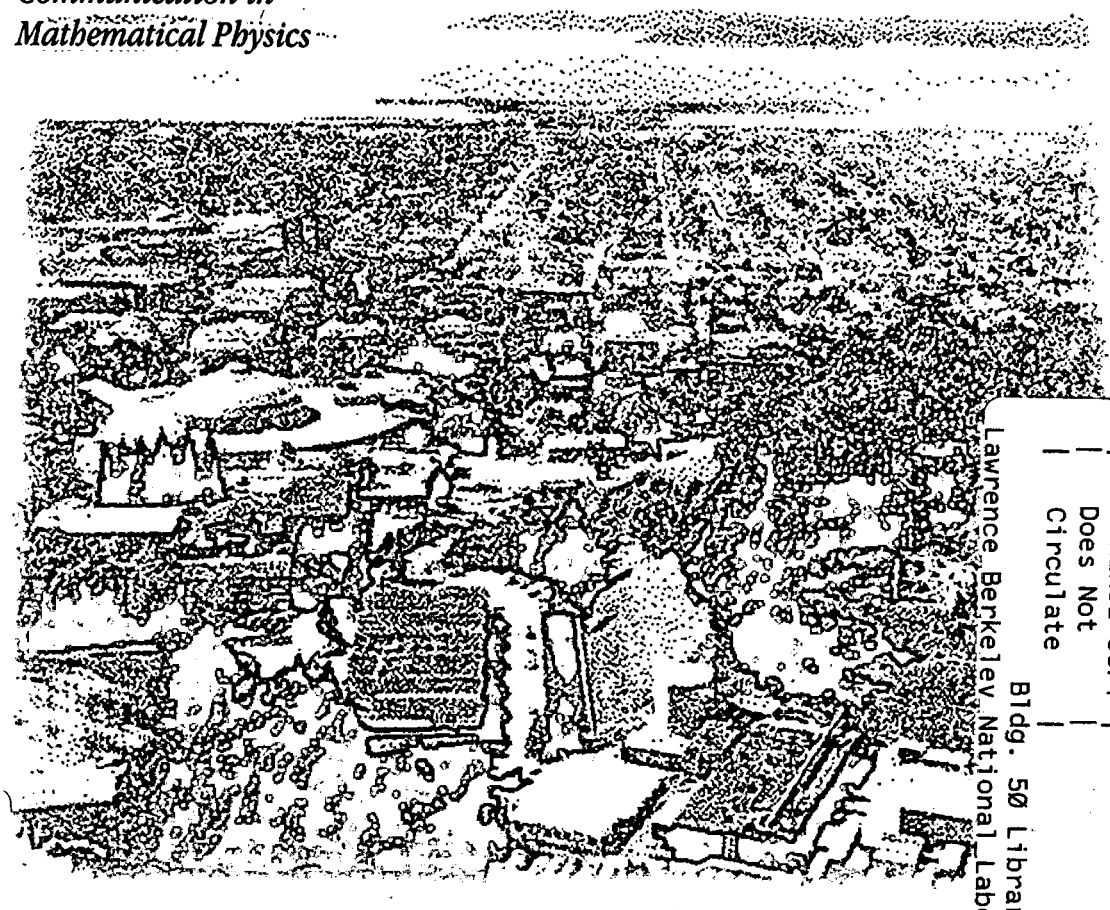

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# Drinfel'd Twist and q-Deforming Maps for Lie Group Covariant Heisenberg Algebrae

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**Abstract**

Any deformation of a Weyl or Clifford algebra can be realized through a change of generators in the undeformed algebra. $q$-Deformations of Weyl or Clifford algebrae that were covariant under the action of a simple Lie algebra $\mathbf{g}$ are characterized by their being covariant under the action of the quantum group $U_h\mathbf{g}$ , $q := e^h$. We present a systematic procedure for determining all possible corresponding changes of generators, together with the corresponding realizations of the $U_h\mathbf{g}$ -action. The intriguing relation between $\mathbf{g}$ -invariants and $U_h\mathbf{g}$ -invariants suggests that these changes of generators might be employed to simplify the dynamics of some $\mathbf{g}$ -covariant quantum physical systems.

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# 1 Introduction

Weyl and Clifford algebrae (respectively denoted by $\mathcal{A}_+, \mathcal{A}_-$ in the sequel, and collectively as "Heisenberg algebrae") are at the hearth of quantum physics. The important question whether quantum mechanics is stable under deformation of Heisenberg algebrae (within the category of associative algebrae) was addressed in the fundamental paper [1]. A general result [11] regarding the Hochschild cohomology of the universal enveloping algebra associated to a nilpotent Lie group states in particular that the first and second cohomology groups of any Weyl algebra $\mathcal{A}_+$ are trivial. This implies [17] that any deformation $\mathcal{A}_{+,h}$ ($h$ denoting the deformation parameter) of the latter is trivial, in the sense that there exists an isomorphism of topological algebrae over $\mathbf{C}[[h]]$ (a "deforming map", in the terminology of Ref. [36]), $f : \mathcal{A}_{+,h} \to \mathcal{A}_+[[h]]$, reducing to the identity when $h = 0$ (a concise and effective presentation of these results can be found in Sect.'s 1,2 of Ref. [28]). Practically this means that the generators $\tilde{A}^i, \tilde{A}^+_i$ of $\mathcal{A}_{+,h}$ can be realized as power series in $h$ with coefficients in $\mathcal{A}_+$ $A^i := f(\tilde{A}^i)$, $A^+_i := f(\tilde{A}^i)$, and the coefficients of the $h^0$ term are generators $a^i, a^+_i$ of $\mathcal{A}_+$.

Given any automorphism $g : \mathcal{A}_+[[h]] \to \mathcal{A}_+[[h]]$, $g = \mathrm{id} + O(h)$, then $g \circ f$ is a new deforming map; conversely, given two deforming maps $f, f'$, the map $f' \circ f^{-1}$ is clearly an algebra automorphism. Now, by the vanishing of the first cohomology group of $\mathcal{A}_+$, all automorphisms of $\mathcal{A}_+[[h]]$ are 'inner', *i.e.* of the form $g(a) = \alpha\, a\, \alpha^{-1}$. Hence, all deforming maps can be obtained from one through the formula

$$f_\alpha(a) := \alpha f(a) \alpha^{-1} \qquad\qquad \alpha = 1 + O(h) \in \mathcal{A}_{+,h}. \tag{1.1}$$

These results apply [28] in particular to so-called "$q$-deformations" ($q := e^h$) of Weyl algebrae which are covariant under the action of some simple Lie algebra $\mathbf{g}$; such deformations [29, 34, 4] are matched to the deformation of $U\mathbf{g}$ into the quantum group $U_h\mathbf{g}$ , in the sense that for all $q$ the deformed algebrae are in fact $U_h\mathbf{g}$ -module algebrae[1]. We shall denote by $\mathcal{A}_{+,\mathbf{g},\rho}$ the Weyl algebra with generators

[1]They should not be confused with the celebrated Biedenharn-Macfarlane $q$-oscillator (super)algebrae [2], whose generators $\alpha^i, \alpha^+_j$ fulfil ordinary (anti)commutation relations, except for the $q$-(anti)commutation relations $\alpha^i \alpha^+_i \mp q^2\, \alpha^+_i \alpha^i = 1$, and are *not* $U_h\mathbf{g}$ -covariant (in spite of the fact that they are usually used to construct a generalized Jordan-Schwinger realization of $U_h\mathbf{g}$ ). It is of interest to note that, however, the generators $\alpha^i, \alpha^+_j$ can be tipically realized as *algebraic* 'functions' of $\tilde{A}^i, \tilde{A}^+_i$ [26], whereas the generators $a^i, a^+_i$ can be tipically realized only as

$a^i, a_i^+$ belonging respectively to some representation $\rho$ of $\mathbf{g}$ and to its contragradient $\rho^\vee$, and by $\mathcal{A}^h_{+,\mathbf{g},\rho}$ its $q$-deformation. In the same way as the commutation relations among $a^i, a_i^+$ are compatible with the classical action $\triangleright : U\mathbf{g} \times \mathcal{A}_{+,\mathbf{g},\rho} \to \mathcal{A}_{+,\mathbf{g},\rho}$, so are the commutation relations among $\tilde{A}^i, \tilde{A}_i^+$ with the $q$-deformation of the latter, the 'quantum' action $\tilde{\triangleright}_h : U_h\mathbf{g} \times \mathcal{A}^h_{+,\mathbf{g},\rho} \to \mathcal{A}^h_{+,\mathbf{g},\rho}$.

At the representation-theoretic level one would be tempted to interpret deforming maps $f$ as "operator maps". Whether this is actually possible depends however not only on the explicit form of $f$, but also on the particular representation picked up. In fact, the rigidity [11, 28] of Weyl algebrae is true only in the loose sense of formal power series in $h$ [technically speaking, in the socalled $h$-adic topology], in general not in other (*e.g.* operator norm) topologies. In the case of $q$-deformations, the point $e^h = q = 1$ may yield various types of 'singularities' at the representation level: the limit $q \to 1$ of a representation may be non-smooth[2], or even ill-defined[3].

In spite of the existence of algebra isomorphisms $f : \mathcal{A}^h_{+,\mathbf{g},\rho} \to \mathcal{A}_{+,\mathbf{g},\rho}[[h]]$, and $\varphi_h : U_h\mathbf{g} \to U\mathbf{g}\,[[h]]$[10][4], the $U_h\mathbf{g}$ -module algebra structure $(U_h\mathbf{g}\,,\, \mathcal{A}^h_{\pm,\mathbf{g},\rho}, \tilde{\triangleright}_h)$ is however a *non-trivial* deformation of $(U\mathbf{g}\,, \mathcal{A}_{\pm,\mathbf{g},\rho}, \triangleright)$, *i.e.* for no $\varphi_h, f$ the equality $f \circ \tilde{\triangleright}_h = \triangleright \circ (\varphi_h \times f)$ holds. This is because $U_h\mathbf{g}$ itself as a *Hopf* algebra is a *non-trivial* deformation of $U\mathbf{g}$, in other words all $\varphi_h$'s are algebra but not coalgebra (and therefore not Hopf algebra) isomorphisms (this is related to the non-triviality of the Gerstenhaber-Schack cohomology [18]).

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*transcendental* 'functions' of $\tilde{A}^i, \tilde{A}_i^+$.

[2]E.g. spectra of hermitean operators may switch from discrete into continuos (and correspondingly their eigenvectors may become non-normalizable) [21, 24].

[3]It was shown in Ref. [29] that the set of unitary irreducible representations of $\mathcal{A}^h_{+,su(N),\rho_d,\star}$ [$\star$ denotes the natural $*$-structure of oscillators, $(\tilde{A}^i)^\star = \tilde{A}_i^+$] on separable Hilbert spaces splits into more than one (actually infinitely-many) unitarily inequivalent classes, whereas there is just one class when $q = 1$, according to Von Neumann theorem [this is to be contrasted with the set of unitary irreps of $U_h su(N)$ itself: the latter are in one-to-one correspondence with the unitary irreps of $su(N)$]. This is not in contradiction with the existence of a deforming map, since (as we verified in Ref. [12], in the concrete case of $\mathcal{A}^h_{+,su(2),\rho_d,\star}$), at $q = 1$ the inverse $f^{-1}$ of the $q$-deforming map becomes singular, as an operator map, on all but one of the Pusz-Woronowicz unitary irreducible representations [29]. $f^{-1}$ is regular (as an operator map) on this particular representation (the unique one possessing a ground state) and intertwines the latter with the (standard) Fock space representation of the corresponding undeformed Weyl algebra.

[4]The existence of the latter and its being defined up to inner automorphisms of $U\mathbf{g}\,[[h]]$ again is a consequence of the triviality of the first and second Hochschild cohomology groups of $U\mathbf{g}$.

Using $\tilde{\triangleright}_h$ and any $f$ we can draw the solid arrows in the following diagram:

$$\begin{array}{ccccc} U_h\mathbf{g} & \times & \mathcal{A}^h_{\pm,\mathbf{g},\rho} & \xrightarrow{\tilde{\triangleright}_h} & \mathcal{A}^h_{\pm,\mathbf{g},\rho} \\ & \downarrow & \mathrm{id}\times f & & \downarrow f \\ U_h\mathbf{g} & \times & \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]] & \overset{\triangleright_h}{- - \to} & \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]. \end{array} \tag{1.2}$$

In this paper we give a systematic procedure to construct all pairs $(f, \triangleright_h)$ such that the above diagram commutes (in other words $\triangleright_h$ will realize $\tilde{\triangleright}_h$ on $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$).

We start by showing (Sect. 3) that one particular $\triangleright_h$ can be naturally constructed in a ($\varphi_h$-dependent) 'adjoint-like' way. To determine the corresponding $f$ it is sufficient to identify in $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ appropriate images $A^i = f(\tilde{A}^i)$, $A^+_i = f(\tilde{A}^+_i)$: with this aim in mind, we first show [formula (3.3)] how to construct two classes of objects $A^i, A^+_i$ having the same transformation properties under $\triangleright_h$ as the generators $\tilde{A}^i, \tilde{A}^+_i$ of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ under $\tilde{\triangleright}_h$; these two classes turn out to be parametrized by some $\mathbf{g}$-invariants. The construction method is founded on the properties of the "Drinfel'd twist" [10]. Then (Sect. 5) we try to restrict our choice by requiring that the $A^i, A^+_i$ also have the same commutation rules as the $\tilde{A}^i, \tilde{A}^+_i$: this condition can be translated into a system of equations (5.0.1-5.0.3) where the twist appears only through the socalled "universal coassociator"; fortunately, the latter is known rather explicitly in terms of solutions of the socalled universal Knizhnik-Zamolodchikov [22] equation. Up to this point the whole formalism is completely $\mathbf{g}$- and $\rho$-independent. Then we solve case by case the system (5.0.1-5.0.3) for the most celebrated examples of $q$-deformed Heisenberg algebrae, *i.e.* $\mathcal{A}^h_{\pm,sl(N),\rho_d}$, $\mathcal{A}^h_{+,so(N),\rho_d}$[5], ($\rho_d$ will denote the defining representations of either $\mathbf{g}$); the solutions $A^i, A^+_i$ are determined up to an automorphism (1.1), with a $\mathbf{g}$-*invariant* $\alpha$. Coming back to the general results, in Sect. 6 we study the conditions under which $*$-structures of $\mathcal{A}_{\pm,\mathbf{g},\rho}$ realize $*$-structures of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$; imposing a $*$-structure constrains the choice of the $\mathbf{g}$-invariant $\alpha$. The subalgebrae $\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}[[h]]$, $\mathcal{A}^{h,inv}_{\pm,\mathbf{g},\rho}[[h]]$ of $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ that are invariant respectively under $\triangleright_h$ and the *classical* $U\mathbf{g}$ action $\triangleright$ coincide (Sect. 4), but we find out that invariants in the form of polynomials in $A^i, A^+_j$ are highly non-polynomial (analytic) functions in $a^i, a^+_j$, and conversely. Finally, in Sect. 7 we show how to extend our previous results to all other isomorphisms $f_\alpha; \mathcal{A}^h_{\pm,\mathbf{g},\rho} \to \mathcal{A}_{\pm,\mathbf{g},\rho}$ [formula (1.1)], while giving an outlook of the whole construction.

[5] Actually, in order to realize $\mathcal{A}^h_{+,so(N),\rho_d}$ one needs to slightly extend $\mathcal{A}_{+,so(N),\rho_d}[[h]]$ with the square root of the element representing the casimir of $so(N)$

In Ref. [12] we started the program just sketched, by sticking to the cases of arbitrary triangular deformations of the Hopf algebra $U\mathbf{g}$ (this case is easily recovered in the present setting by postulating a trivial coassociator) and of the deformation $\mathcal{A}^h_{\pm,sl(2),\rho_d}$ of $\mathcal{A}_{\pm,sl(2),\rho_d}$. We would like also to note that examples of $q$-deforming maps (in the restricted sense of the first paragraph) for Heisenberg algebrae have been explicitly determined "by hand" in past works [32, 26, 28, 24].

We are now in the conditions to give some motivations for the present work. A systematic procedure for determining $q$-deforming maps can help in understanding the relation (or contrast) between the representation theories of $\mathcal{A}_{\pm,\mathbf{g},\rho}$ and $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$. Our construction procedure is applicable in particular to Heisenberg algebrae $\mathcal{A}_{\pm,\mathbf{g},\rho}$ where $\rho$ is a direct sum of many copies of $\rho_d$'s; these are physically the most interesting cases (the different copies could correspond *e.g.* to different particles, crystal sites or space(time)-points, respectively in quantum mechanics, condensed matter physics or quantum field theory); solutions of the corresponding system (5.0.1-5.0.3) will be searched elsewhere. In the particular case of a $q$-deformation $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ of a oscillator $*$-algebra $\mathcal{A}_{\pm,\mathbf{g},\rho}$, knowledge of a ($*$-compatible) $q$-deforming map would allow to identify one of the many unitary representations of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ with the unitary Fock space representation of $\mathcal{A}_{\pm,\mathbf{g},\rho}$; correspondingly, a particle interpretation in terms of ordinary bosons and fermions would be possible [13], and $A^+_i, A^i$ could be interpreted as "composite operators" creating and destroying some sort of "dressed states". In view of the mentioned relation between $\mathbf{g}$ and $U_h\mathbf{g}$ invariants, the change of generators $a^i, a^+_i \rightarrow A^i, A^+_i$could be employed in order to simplify the dynamics of a physical system based on some complicated $\mathbf{g}$-invariant interaction Hamiltonian (similarly to what has been suggested in Ref. [33] for a 1-dim toy-model), if the functional dependence of the latter on the $q$-deformed generators $A^+_i, A^i$ were of polynomial character.

# 2 Preliminaries and notation

Some general remarks before starting. The fact that we will denote the generators of the Heisenberg algebrae by $a^i, a^+_i, A^i, A^+_i, ...$ does not necessarily mean that we have in mind creators/annihilators: only the choice of a $*$-structure may give the generators the meaning of creators/annihilators, or coordinates/derivatives, etc. (a few ones are considered in section 6). Given an algebra $B$, we will denote (with

a standard notation) by $B[[h]]$ the algebra of formal power series in $h \in \mathbf{C}$ with coefficients belonging to finite-dimensional subspaces of $B$, completed in the $h$-adic topology. Tensor products like $B[[h]] \otimes B[[h]]$ are also to be understood to be completeted in the same topology. We shall use throughout the paper the symbol $U_h\mathbf{g}$ [9], to denote the algebra on the ring $\mathbf{C}[[h]]$ (completed in the $h$-adic topology) underlying the quantum group.

## 2.1 Twisting groups into quantum groups

Let $H = (U\mathbf{g}, m, \Delta, \varepsilon, S)$ be the cocommutative Hopf algebra associated to the universal enveloping (UE) algebra $U\mathbf{g}$ of a Lie algebra $\mathbf{g}$. The symbol $m$ denotes the multiplication (in the sequel it will be dropped in the obvious way $m(a \otimes b) \equiv ab$, unless explicitly required), whereas $\Delta, \varepsilon, S$ the comultiplication, counit and antipode respectively.

Let $\mathcal{F} \in U\mathbf{g}[[h]] \otimes U\mathbf{g}[[h]]$ (we will write $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$, in a Sweedler's notation with *upper* indices; in the RHS a sum $\sum_i \mathcal{F}_i^{(1)} \otimes \mathcal{F}_i^{(2)}$ of many terms is implicitly understood) be a 'twist', *i.e.* an element satisfying the relations

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)\mathcal{F} \tag{2.1.1}$$

$$\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + O(h) \tag{2.1.2}$$

($h \in \mathbf{C}$ is the 'deformation parameter', and $\mathbf{1}$ the unit in $U\mathbf{g}$; from the second condition it follows that $\mathcal{F}$ is invertible as a power series). It is well known [8] that if $\mathcal{F}$ also satisfies the relation

$$(\mathcal{F} \otimes \mathbf{1})[(\Delta \otimes \mathrm{id})(\mathcal{F})] = (\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)(\mathcal{F}), \tag{2.1.3}$$

and $\varphi_h$ is any automorphism of $U\mathbf{g}[[h]]$ satisfying $\varphi_h = \mathrm{id} \pmod h$ (in particular, $\varphi_h = \mathrm{id}$), then one can construct a triangular non-cocommutative Hopf algebra $H_h = (U\mathbf{g}[[h]], m, \Delta_h, \varepsilon_h, S_h, \mathcal{R})$ having an isomorphic (through $\varphi_h$) algebra structure $(U\mathbf{g}[[h]], m)$, an isomorphic counit $\varepsilon_h := \varepsilon \circ \varphi_h^{-1}$, comultiplication and antipode defined by

$$\Delta_h(a) = (\varphi_h^{-1} \otimes \varphi_h^{-1})[\mathcal{F}\Delta[\varphi_h(a)]\mathcal{F}^{-1}], \qquad S_h(a) = \varphi_h^{-1}[\gamma^{-1}S[\varphi_h(a)]\gamma], \tag{2.1.4}$$

where

$$\gamma := S\mathcal{F}^{-1(1)} \cdot \mathcal{F}^{-1(2)}, \qquad \gamma^{-1} = \mathcal{F}^{(1)} \cdot S\mathcal{F}^{(2)}, \tag{2.1.5}$$

and (triangular) universal R-matrix

$$\mathcal{R} := [\varphi_h^{-1} \otimes \varphi_h^{-1}](\mathcal{F}_{21}\mathcal{F}^{-1}), \qquad \mathcal{F}_{21} := \mathcal{F}^{(2)} \otimes \mathcal{F}^{(1)}. \tag{2.1.6}$$

Condition (2.1.3) ensures that $\Delta_h$ is coassociative as $\Delta$. The inverse of $S_h$ is given by $S_h^{-1}(a) = \varphi_h^{-1}[\gamma' S[\varphi_h(a)]\gamma'^{-1}]$, where

$$\gamma' := \mathcal{F}^{(2)} \cdot S\mathcal{F}^{(1)} \qquad \gamma'^{-1} = S\mathcal{F}^{-1(2)} \cdot \mathcal{F}^{-1(1)}; \tag{2.1.7}$$

$\gamma^{-1}\gamma' \in$ Centre(U$\mathbf{g}$), and $S\gamma = \gamma'^{-1}$.

Conversely, given a $h$-deformation $H_h = (U_h, m, \Delta_h, \varepsilon_h, S_h, \mathcal{R})$ of $H$ in the form of a triangular Hopf algebra, one can find [8] and an isomorphism $\varphi_h : U_h \to U\mathbf{g}[[h]]$ an invertible $\mathcal{F}$ satisfying conditions (2.1.1), (2.1.2), (2.1.3) such that $H_h$ can be obtained from $H$ through formulae (2.1.4),(2.1.5),(2.1.7).

Examples of $\mathcal{F}$'s satisfying conditions (2.1.3), (2.1.1), (2.1.2) are provided *e.g.* by the socalled 'Reshetikhin twists' [31]

$$\mathcal{F} := e^{h\omega_{ij} h_i \otimes h_j}, \tag{2.1.8}$$

where $\{h_i\}$ is a basis of the Cartan subalgebra of $\mathbf{g}$ and $\omega_{ij} = -\omega_{ji} \in \mathbf{C}$.

A similar result to the above holds for genuine quantum groups. A well-known theorem by Drinfel'd, Proposition 3.16 in Ref. [10] (whose results are partially already implicit in preceding works by Kohno [23]), proves, for any quasitriangular deformation $H_h = (U_h\mathbf{g}, m, \Delta_h, \varepsilon_h, S_h, \mathcal{R})$ [9, 15] of $U\mathbf{g}$, with $\mathbf{g}$ a simple finite-dimensional Lie algebra, the existence of an algebra isomorphism $\varphi_h : U_h\mathbf{g} \to U\mathbf{g}[[h]]$ and an invertible $\mathcal{F}$ satisfying condition (2.1.1) such that $H_h$ can be obtained from $H$ through formulae (2.1.4),(2.1.5),(2.1.7), as well, after identifying $h = \ln q$. This $\mathcal{F}$ does not satisfy condition (2.1.16), however the (nontrivial) coassociator

$$\phi := [(\Delta \otimes \mathrm{id})(\mathcal{F}^{-1})](\mathcal{F}^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)(\mathcal{F}) \tag{2.1.9}$$

still commutes with $\Delta^{(2)}(U\mathbf{g})$,

$$[\phi, \Delta^{(2)}(U\mathbf{g})] = 0, \tag{2.1.10}$$

thus explaining why $\Delta_h$ is coassociative in this case, too. The corresponding universal (quasitriangular) R-matrix $\mathcal{R}$ is related to $\mathcal{F}$ by

$$\mathcal{R} = [\varphi_h^{-1} \otimes \varphi_h^{-1}](\mathcal{F}_{21} q^{\frac{t}{2}} \mathcal{F}^{-1}), \tag{2.1.11}$$

where $t := \Delta(\mathcal{C}) - \mathbf{1}\otimes\mathcal{C} - \mathcal{C}\otimes\mathbf{1}$ is the canonical invariant element in $U\mathbf{g}\otimes U\mathbf{g}$ ($\mathcal{C}$ is the quadratic Casimir)[6]. The twist $\mathcal{F}$ is defined (and unique) up to the transformation

$$\mathcal{F} \to \mathcal{F}T, \tag{2.1.12}$$

where $T$ is a $\mathbf{g}$-invariant [*i.e.* commuting with $\Delta(U\mathbf{g})$] element of $U\mathbf{g}[[h]]^{\otimes^2}$ such that

$$T = \mathbf{1}\otimes\mathbf{1} + O(h), \qquad (\varepsilon\otimes\mathrm{id})T = \mathbf{1} = (\mathrm{id}\otimes\varepsilon)T. \tag{2.1.13}$$

Under this transformation

$$\phi \to [(\Delta\otimes\mathrm{id})(T^{-1})](T^{-1}\otimes\mathbf{1})\phi(\mathbf{1}\otimes T)[(\mathrm{id}\otimes\Delta)(T). \tag{2.1.14}$$

A function

$$T = T\,(\mathbf{1}\otimes\mathcal{C}_i, \mathcal{C}_i\otimes\mathbf{1}, \Delta(\mathcal{C}_i)) \tag{2.1.15}$$

of the Casimirs $\mathcal{C}_i \in U\mathbf{g}$ of $U\mathbf{g}$ and of their coproducts clearly is $\mathbf{g}$-invariant. We find it plausible that any $\mathbf{g}$-invariant $T$ must be of this form; although we have found in the literature yet no proof of this conjecture, in the sequel we assume that this is true.

We will often use a 'tensor notation' for our formulae: eq. (2.1.3) will read

$$\mathcal{F}_{12}\mathcal{F}_{12,3} = \mathcal{F}_{23}\mathcal{F}_{1,23}, \tag{2.1.16}$$

and definition (2.1.9) $\phi \equiv \phi_{123} = \mathcal{F}_{12,3}^{-1}\mathcal{F}_{12}^{-1}\mathcal{F}_{23}\mathcal{F}_{1,23}$, for instance; the commas separate the tensor factors *not* stemming from the coproduct.

$\phi$ satisfies the equations

$$\begin{aligned} q^{\frac{t_{13}+t_{23}}{2}} &= \phi_{231}^{-1} q^{\frac{t_{13}}{2}} \phi_{132} q^{\frac{t_{23}}{2}} \phi_{123}^{-1} \\ q^{\frac{t_{12}+t_{13}}{2}} &= \phi_{312} q^{\frac{t_{13}}{2}} \phi_{213}^{-1} q^{\frac{t_{12}}{2}} \phi_{123} \end{aligned} \tag{2.1.17}$$

(they are equivalent resp. to $(\Delta_h\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$, $(\mathrm{id}\otimes\Delta_h)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$).

[6]To arrive at this result, Drinfel'd introduces the notion of quasitriangular quasi-Hopf algebra; the latter essentially involves the weakening of coassociativity of the coproduct into a property ("quasi-coassociativity") valid only up to a similarity transformation through an element $\phi \in U\mathbf{g}[[h]]^{\otimes^3}$ (the "coassociator"). This notion is useful because quasitriangular quasi-Hopf algebra are mapped into each other under twists [even if the latter do not satisfy condition (2.1.3)]. As an intermediate result, he shows that $U\mathbf{g}[[h]]$, beside the trivial quasitriangular quasi-Hopf structure $(U\mathbf{g}[[h]], m, \Delta, \varepsilon, S, \mathcal{R} \equiv \mathbf{1}^{\otimes^2}, \phi \equiv \mathbf{1}^{\otimes^3})$, has a non trivial one $(U\mathbf{g}[[h]], m, \Delta, \varepsilon, S, \mathcal{R} = q^{\frac{t}{2}}, \phi \neq \mathbf{1}^{\otimes^3})$.

While for the twist $\mathcal{F}$, apart from its existence, very little explicit knowledge is available, Kohno [23] and Drinfel'd [10] have proved that, up to the transformation (2.1.14),$\phi$ is given by

$$\phi_m = \hat{g}^{-1}(x)\check{g}(x), \qquad 0 < x < 1, \tag{2.1.18}$$

where $\hat{g}, \check{g}(x)$ are $U\mathbf{g}\,[[h]]^{\otimes^3}$-valued 'analytic'[7] solutions of the first order linear differential equation

$$\frac{dg}{dx} = \hbar\left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1}\right)g, \qquad 0 < x < 1 \tag{2.1.19}$$

($\hbar = \frac{h}{2\pi i}$) with the following asymptotic behaviour near the poles:

$$\hat{g}\cdot x^{-\hbar t_{12}} \xrightarrow{x\to 0} \mathbf{1}^{\otimes^3} \qquad \check{g}\cdot(1-x)^{-\hbar t_{23}} \xrightarrow{x\to 1} \mathbf{1}^{\otimes^3}. \tag{2.1.20}$$

Using eq. (2.1.19) it is straightforward to verify that the RHS of eq. (2.1.18) is indeed independent of $x$. [8] Using eq. (2.1.20) we can take the limit of eq. (2.1.18):

$$\phi_m = \lim_{x\to 0+} x^{-\hbar t_{12}}\check{g}(x). \tag{2.1.21}$$

We can formally solve the previous equations (2.1.19), (2.1.20) by a path ordered integral:

$$\hat{g}^{-1}(x) = \lim_{x_0\to 0+}\left\{x_0^{-\hbar t_{12}}\vec{P}\exp\left[-\hbar\int_{x_0}^{x} dx\left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1}\right)\right]\right\} \tag{2.1.22}$$

$$\check{g}(x) = \lim_{y_0\to 0+}\left\{\vec{P}\exp\left[-\hbar\int_{x}^{1-y_0} dx\left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1}\right)\right]y_0^{\hbar t_{23}}\right\} \tag{2.1.23}$$

($\vec{P}[A(x)B(y)] := A(x)B(y)\vartheta(y-x) + B(y)A(x)\vartheta(x-y)$), so that we can give a more explicit expression for $\phi$:

$$\phi_m = \lim_{x_0,y_0\to 0+}\left\{x_0^{-\hbar t_{12}}\vec{P}\exp\left[-\hbar\int_{x_0}^{1-y_0} dx\left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1}\right)\right]y_0^{\hbar t_{23}}\right\}. \tag{2.1.24}$$

---

[7] In the sense that the coefficients $g_n(x)$ appearing in the expansion $g(x) = \sum_{n=0}^{\infty} g_n(x)h^n$ of $g$ in $h$-powers are analytic functions of $x$ with values in a finite-dimensional subspace of $U\mathbf{g}^{\otimes^3}$.

[8] Kohno and Drinfel'd proved that $\phi$ can be obtained as the 'monodromy' of a system of three first order linear partial differential equations in three complex variables $z_i$ (the socalled universal Knizhnik-Zamolodchikov [22] equations), with an $U\mathbf{g}^{\otimes^3}$-valued unknown $f$, $\frac{\partial f}{\partial z_i} = \hbar\sum_{j\neq i}\frac{t_{ij}}{z_i - z_j}f$. The system can be reduced to the equation (2.1.19) exploiting its invariance under linear tranformations $z_i \to az_i + b$. For a review of these results see for instance Ref. [5].

Note that $\phi_m = \mathbf{1}^{\otimes^3} + O(h^2)$.

We will say that the twist $\mathcal{F}$ is *'minimal'* if the corresponding $\phi$ (2.1.9) is equal to $\phi_m$ or is trivial respectively in the case of $H_h = U_h\mathbf{g}$ or $H_h$ is a triangular deformation of $U\mathbf{g}$.

The algebra isomorphism $\varphi_h : U_h\mathbf{g} \to U\mathbf{g}[[h]]$ is defined up to an inner automorphism (a 'similarity transformation') of $U\mathbf{g}[[h]]$,

$$\varphi_{h,v}(x) := v\varphi_h(x)v^{-1}, \tag{2.1.25}$$

for any $v = \mathbf{1} + O(h) \in U\mathbf{g}[[h]]$ (we shall normalize it in such a way that $\varepsilon(v) = 1$). It is easy to check that Drinfel'd theorem [10] remains true provided one replaces $\mathcal{F}$ by $\mathcal{F}_v := (v \otimes v)\mathcal{F}\Delta(v^{-1})$ and all the objects derived from $\mathcal{F}$ correspondingly; in particular, it is easy to check that the coassociator $\phi$ remains unchanged, because it is $\mathbf{g}$-invariant

$$\phi_v = \Delta^{(2)}(v)\phi\Delta^{(2)}(v^{-1}) = \phi. \tag{2.1.26}$$

The freedom in choosing $\varphi_h$ (and $\mathcal{F}$) is usually eliminated if one requires it to satisfy additional properties, such as to lead to a specific $*$-structure for $U_h\mathbf{g}$.

The Lie algebra $g = sl(2)$ is the only $\mathbf{g}$ for which explicit $\varphi_h$'s are known[9].

[9] Let $j_0, j_+, j_- \in \mathbf{g}$,

$$[j_0, j_\pm] = \pm j_\pm \qquad\qquad [j_+, j_-] = 2j_0$$

be the classical generators, and $J_0, J^+, J^- \in \mathcal{U}_h sl(2)$

$$[J_0, J_\pm] = \pm J_\pm \qquad\qquad [J_+, J_-] = \frac{q^{2J_0} - q^{-2J_0}}{q - q^{-1}} \tag{2.1.27}$$

the quantum ones. An entire class of algebra isomorphisms $\varphi_h : U_h sl(2) \to U sl(2)[[h]]$ was given in Ref. [7]. One is [the reader can easily check the commutation rules (2.1.27)]:

$$\varphi_h(J_+) = j_+ \qquad \varphi_h(J_0) = j_0 \qquad \varphi_h(J_-) = \frac{[j - j_0]_q[j + j_0 + 1]_q}{(j - j_0)(j + j_0 + 1)} j_-.$$

The $*$-structure $(J_+)^* = J_-$, $(J_0)^* = J_0$ of $U_h su(2)$, for instance, requires changing the preceding isomorphism by the inner automorphism generated by

$$v = \sqrt{\frac{\Gamma(j - j_0 + 1)\Gamma_q(j + j_0 + 1)}{\tilde{\Gamma}_q(j - j_0 + 1)\tilde{\Gamma}(j + j_0 + 1)}}$$

[$\tilde{\Gamma}_q$ is the deformation of Euler's $\Gamma$-function defined in formula (A.4.14)], which leads to the new algebra isomorphism $\phi_v$ [6]

$$\varphi_h(J_0) = j_0 \qquad\qquad \varphi_h(J_\pm) = \sqrt{\frac{[j \pm j_0]_q[1 + j \mp j_0]_q}{(j \pm j_0)(1 + j \mp j_0)}} j_\pm.$$

For practical purposes it will be often convenient in the sequel to use the Sweedler's notation with *lower* indices $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$ for the cocommutative coproduct (in the RHS a sum $\sum_i x^i_{(1)} \otimes x^i_{(2)}$ of many terms is implicitly understood); similarly, we will use the Sweedler's notation $\Delta^{(n-1)}(x) \equiv x_{(1)} \otimes \ldots \otimes x_{(n)}$ for the $(n-1)$-fold coproduct. For the non-cocommutative coproducts $\Delta_h$, instead, we will use a Sweedler's notation with barred indices: $\Delta_h(x) \equiv x_{(\bar{1})} \otimes x_{(\bar{2})}$.

To mantain a simple notation, in the sequel we will drop the symbol $\varphi_h$ unless this may cause ambiguities.

## 2.2 Classical g-covariant Heisenberg algebrae

Let $\mathcal{A}_{\pm,\mathbf{g},\rho}$ be the unital algebra generated by $\mathbf{1}_{\mathcal{A}}$ and elements $\{a_i^+\}_{i\in I}$ and $\{a^j\}_{j\in I}$ satisfying the (anti)commutation relations

$$\begin{aligned} [a^i\,,\,a^j]_\pm &= 0 \\ [a_i^+\,,\,a_j^+]_\pm &= 0 \\ [a^i\,,\,a_j^+]_\pm &= \delta^i_j \mathbf{1}_{\mathcal{A}} \end{aligned} \tag{2.2.1}$$

(the $\pm$ sign denotes commutators and anticommutators respectively), belonging respectively to some representation $\rho$ and to its contragradient $\rho^\vee = \rho^T \circ S$ of $H$ ($^T$ is the transpose):

$$\begin{aligned} x \triangleright a_i^+ &= \rho(x)^l_i a_l^+ \\ x \triangleright a^i &= \rho(Sx)^i_l a^l \end{aligned} \qquad x \in U\mathbf{g}\,, \qquad \rho(x)^i_j \in \mathbf{C}. \tag{2.2.2}$$

Equivalently, one says that $a_i^+, a^i$ are "covariant", or "tensors", under $\triangleright$ .

$\mathcal{A}_{\pm,\mathbf{g},\rho}$ is a (left) module algebra of $(H, \triangleright)$, if the action $\triangleright$ is extended on the whole $\mathcal{A}_{\pm,\mathbf{g},\rho}$ by means of the (cocommutative) coproduct:

$$x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b). \tag{2.2.3}$$

Setting

$$\sigma(X) := \rho(X)^i_j a_i^+ a^j \tag{2.2.4}$$

for all $X \in \mathbf{g}$, one finds that $\sigma : \mathbf{g} \to \mathcal{A}_{\pm,\mathbf{g},\rho}$ is a Lie algebra homomorphism, so that $\sigma$ can be extended to all of $U\mathbf{g}$ as an algebra homomorphism $\sigma : U\mathbf{g} \to \mathcal{A}_{\pm,\mathbf{g},\rho}$;

Here $j$ is the positive root of the equation $j(j+1) - C = 0$, $C = j_- j_+ + j_0(j_0+1)$ is the Casimir, $[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$, and we have used the ordinary $\Gamma$-function, as well as its $q$-partner $\tilde{\Gamma}_q$, defined in formulae (A.4.9), (A.4.14) in the appendix.

on the unit element we set $\sigma(\mathbf{1}_{U\mathbf{g}}) := \mathbf{1}_{\mathcal{A}}$. $\sigma$ can be seen as the generalization of the Jordan-Schwinger realization of $\mathbf{g} = su(2)$ [3]

$$\sigma(j_+) = a^+_\uparrow a^\downarrow, \qquad \sigma(j_-) = a^+_\downarrow a^\uparrow, \qquad \sigma(j_0) = \frac{1}{2}(a^+_\uparrow a^\uparrow - a^+_\downarrow a^\downarrow). \quad (2.2.5)$$

Then it is easy to check the following

**Proposition 1** *The (left) action* $\triangleright : U\mathbf{g} \times \mathcal{A}_{\pm,\mathbf{g},\rho} \to \mathcal{A}_{\pm,\mathbf{g},\rho}$ *can be realized in an 'adjoint-like' way:*

$$x \triangleright a = \sigma(x_{(1)})\, a\, \sigma(Sx_{(2)}), \qquad x \in U\mathbf{g}, \qquad a \in \mathcal{A}_{\pm,\mathbf{g},\rho}. \quad (2.2.6)$$

Let us introduce the notion of $\mathbf{g}$-invariant subalgebra $\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho} \subset \mathcal{A}_{\pm,\mathbf{g},\rho}$:

$$\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho} := \{ I \in \mathcal{A}_{\pm,\mathbf{g},\rho} \ \mid \ x \triangleright I = \varepsilon(x) I \qquad \forall x \in U\mathbf{g} \} \quad (2.2.7)$$

(it is not difficult to see that the above is the natural definition of invariant subalgebrae in the Hopf algebraic language; in fact if $x \in \mathbf{g}$ then $\varepsilon(x) = 0$ and the RHS vanishes). It is easy to show that

**Proposition 2**

$$\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho} = \{ I \in \mathcal{A}_{\pm,\mathbf{g},\rho} \ \mid \ [\sigma(y), I] = 0, \quad y \in U\mathbf{g} \} \quad (2.2.8)$$

*Proof*. Given any $I \in \mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}$, $y \in U\mathbf{g}$ take $x = y_{(1)}$ in definition (2.2.7), where $y_{(1)} \otimes y_{(2)} = \Delta y$:

$$\begin{aligned} I\sigma(y) &= I\varepsilon(y_{(1)})\sigma(y_{(2)}) \overset{(4.1)}{=} (y_{(1)} \triangleright I)\sigma(y_{(2)}) \\ &\overset{(2.2.6)}{=} \sigma(y_{(1)}) I \sigma(Sy_{(2)} \cdot y_{(3)}) = \sigma(y_{(1)}) I \varepsilon(y_{(2)}) = \sigma(y) I. \ \square \end{aligned} \quad (2.2.9)$$

The simplest nontrivial invariant is the 'number of particle operator' $n := a^+_i a^i$, which satisfies

$$[n, a^+_i] = a^+_i \qquad [n, a^i] = -a^i. \quad (2.2.10)$$

¿From the previous proposition it trivially follows that, for any $I, \tilde{I} \in \mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}{}^{\otimes^2}$ the objects

$$\mathsf{a}^i := I^{(1)}\, a^i\, I^{(2)}, \qquad \mathsf{a}^+_i := \tilde{I}^{(1)} a^+_i\, \tilde{I}^{(2)} \quad (2.2.11)$$

(we are using again a Sweedler's notation) transform exactly as $a^i, a_i^+$ under $\triangleright$. If the decomposition of $\rho$ into irreducible components reads $\rho = \bigoplus_\mu \rho_\mu$, the same remains true if we define the $\mathsf{a}^i$ (resp. $\mathsf{a}_i^+$) belonging to the $\mu$-th component by plugging in the previous formula some $\mu$-dependent invariants $I_\mu, \tilde{I}_\mu, \in \mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}{}^{\otimes^2}$.

In the rest of this work we will denote by the symbols $\sigma, \triangleright, \ldots$ also the linear extensions of these operations to the corresponding algebrae of power series in $h$, $\sigma : U\mathbf{g}\,[[h]] \to \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$, $\triangleright : U\mathbf{g}\,[[h]] \times \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]] \to \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$,....

*Remark 1.* Let us note finally that other 'more exotic' algebra homomorphisms $\sigma^\alpha : U\mathbf{g} \to \mathcal{A}_{\pm,\mathbf{g},\rho}$ can be introduced by

$$\sigma^\alpha(x) := \alpha\sigma(x)\alpha^{-1}, \tag{2.2.12}$$

where $\alpha \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ is of the form $\alpha = \mathbf{1} + O(h)$ and therefore invertible. Proposition 2 remains valid after the replacement $\sigma \to \sigma^\alpha$.

## 2.3 Quantum $U_h\mathbf{g}$ -covariant Heisenberg algebrae

Examples of $U_h\mathbf{g}$ -covariant Heisenberg algebrae (denoted by $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ in the sequel) were introduced in Ref. [29, 34, 30, 4], "gluing" together a $U_h\mathbf{g}$ -covariant "quantum space function algebra" [15] (whose generators we will call here $\tilde{A}_i^+$) with its "dual" (whose generators we will call here $\tilde{A}^i$) with appropriate cross commutation relations. These $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ are deformations of corresponding $\mathcal{A}_{\pm,\mathbf{g},\rho}$ with generators $a_i^+, a^i$ belonging to certain representations $\rho, \rho^\vee$ of $\mathbf{g}$ (in the notation of the previous subsection). The cases actually considered were $\mathcal{A}^h_{+,sl(N),\rho_d}$,[29, 34] $\mathcal{A}^h_{-,sl(N),\rho_d}$,[30] $\mathcal{A}^h_{+,so(N),\rho_d}$, [4]; one could consider also $\mathcal{A}^h_{-,sp(\frac{N}{2}),\rho_d}$ [10].

The QCR ('quantum commutation relations') among $\tilde{A}^i, \tilde{A}_i^+$'s can be put in the form

$$\tilde{A}^i\tilde{A}^j = \pm P^{Fji}_{\ \ hk}\tilde{A}^k\tilde{A}^h \tag{2.3.1}$$

$$\tilde{A}_i^+\tilde{A}_j^+ = \pm P^{Fhk}_{\ \ ij}\tilde{A}_h^+\tilde{A}_k^+ \tag{2.3.2}$$

$$\tilde{A}^i\tilde{A}_j^+ = \delta^i_j\mathbf{1}_{\mathcal{A}} \pm \tilde{P}^F{}_{jk}^{\ ih}\tilde{A}_h^+\tilde{A}^k, \tag{2.3.3}$$

where: $\pm$ refers to the Weyl and Clifford case respectively; $\tilde{P}^F = q^{\pm 1}\hat{R}$ and $\hat{R}$ is

[10] Together with the generalizations in which $\rho, \rho^\vee$ are direct sum of $m \geq 1$ copies of these $\rho_d, \rho_d^\vee$ [14], these are among the few sensible cases, for the reasons we recall below.

the (numerical) 'braid matrix' [15] associated to $U_h\mathbf{g}$

$$\hat{R} := c_{\mathbf{g}}\ P\left[(\rho_{d,q})^{\otimes^2}(\mathcal{R})\right] \qquad c_{\mathbf{g}} := \begin{cases} q^{\frac{1}{N}} & \text{if } \mathbf{g} = sl(N) \\ 1 & \text{otherwise} \end{cases} \tag{2.3.4}$$

(the factor $q^{\frac{1}{N}}$ in the case $\mathbf{g} = sl(N)$ is the conventional normalization); $P$ denotes the permutation matrix; $\rho_{d,q}$ denotes the defining representation of $U_h\mathbf{g}$ ; $P^F$ is a polynomial of degree one or two in $\hat{R}$ (usually it is chosen in such a way that $(P^F)^2 = 1$). Both $P^F$ and $\tilde{P}^F$ reduce to $P$ in the limit $q \to 1$. The choice $\tilde{P}^F = q^{\mp 1}\hat{R}^{-1}$ is also possible, but will not be considered explicitly in the sequel.

$\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ is a left module algebra of $H_h = (U_h\mathbf{g}, m, \Delta_h, \varepsilon_h, S_h)$ w.r.t. to the quantum action $\tilde{\triangleright}_h$ of the latter, namely

$$(xy)\tilde{\triangleright}_h\, a = x\tilde{\triangleright}_h\,(y\tilde{\triangleright}_h\, a) \qquad\qquad x\tilde{\triangleright}_h\,(ab) = (x_{(\bar{1})}\tilde{\triangleright}_h\, a)(x_{(\bar{2})}\tilde{\triangleright}_h\, b)$$

$\forall x, y \in U_h\mathbf{g}$ , $a, b \in \mathcal{A}^h_{\pm,\mathbf{g},\rho}$; $\tilde{A}^i, \tilde{A}^+_j$ span two quantum conjugate irreducible representations $\rho_h, \rho^\vee_h = \rho^T_h \circ S_h$ of $(H, \tilde{\triangleright}_h)$:

$$\begin{aligned} x\tilde{\triangleright}_h\, \tilde{A}^+_i &= \rho_h(x)^l_i \tilde{A}^+_l \\ x\tilde{\triangleright}_h\, \tilde{A}^i &= \rho^\vee_h(x)^m_i \tilde{A}^m = \rho_h(S_h x)^i_m \tilde{A}^m. \end{aligned}$$

Let $\varphi_h$ be an algebra isomorphism $\varphi_h : U_h\mathbf{g} \to U\mathbf{g}\,[[h]]$, and $\mathcal{F}$ a corresponding twist; for any representation $\rho_h$ of $U_h\mathbf{g}$ , setting

$$\rho := \rho_h \circ \varphi_h^{-1}$$

defines a representation $\rho$ of $U\mathbf{g}\,[[h]]$[11]. ¿From formula (2.1.11) and the polynomial dependence of $P^F$ on $\hat{R}$ it follows that

$$P^F = F\,U\,F^{-1} \tag{2.3.5}$$

$$\tilde{P^F} = F\,V\,F^{-1} \tag{2.3.6}$$

where $F := \rho_d^{\otimes^2}(\mathcal{F})$, $V = c_{\mathbf{g}}\ P\, q^{\rho_d^{\otimes^2}(t/2)}$ and $U$, being a polynomial in $P\, q^{\rho_d^{\otimes^2}(t/2)}$ such that $(P^F)^2 = 1$, reduces to $U = P$.

If $\rho$ is the direct sum of $m$ copies of the ($N$-dimensional) defining representation of $\mathbf{g}$, $\rho = \bigoplus_{\mu=1}^{m} \rho_{d,\mu}$, one can consistently define [14] a $U_h\mathbf{g}$ -covariant Heisenberg algebra $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ having the same Poincaré series as its classical counterpart $\mathcal{A}_{\pm,\mathbf{g},\rho}$, following the rules of 'braiding'[12]. The generators $\tilde{A}^{\mu,i}, \tilde{A}^+_{\mu,i}$, satisfy:

[11]In particular, it is well-known that the defining representations $\rho_{d,q}, \rho_d$ of $U_h\mathbf{g}$ ,$\mathbf{g}$ coincide, in the sense that the matrix identity $\rho_{d,q}(X^i_q) = \rho_d(X^i)$, where $X^i_q$ are *e.g.* the Drinfel'd-Jimbo generators of $U_h\mathbf{g}$ and $X_i$ the corresponding Chevalley generator of $\mathbf{g}$ , holds.

[12]For an introduction to braiding see *e.g.* Ref. [25].

- QCR of the form (2.3.1), (2.3.2), (2.3.3) with the *same* $N \times N$ matrices $P^F, \tilde{P^F}$ as before, within each subalgebra ${}^{\mu}\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ generated by $\tilde{A}^{\mu,i}, \tilde{A}^+_{\mu,i}$;

- cross commutation relations which, up to a reordering of ${}^{\mu}\mathcal{A}^h_{\pm,\mathbf{g},\rho}$'s, read

$$\begin{aligned}
\tilde{A}^+_{\mu,i}\tilde{A}^+_{\nu,j} &= \pm q^{\pm 1}\hat{R}^{hk}_{ij}\tilde{A}^+_{\nu,h}\tilde{A}^+_{\mu,k} \\
\tilde{A}^{\nu,i}\tilde{A}^+_{\mu,j} &= \pm q^{\pm 1}\hat{R}^{ih}_{jk}\tilde{A}^+_{\mu,h}\tilde{A}^{\mu,k} \\
\tilde{A}^{\mu,j}\tilde{A}^{\nu,i} &= \pm q^{\pm 1}\hat{R}^{ij}_{hk}\tilde{A}^{\mu,k}\tilde{A}^{\mu,j} \\
\tilde{A}^{\mu,i}\tilde{A}^+_{\nu,j} &= \pm q^{\mp 1}(\hat{R}^{-1})^{ih}_{jk}\tilde{A}^+_{\nu,h}\tilde{A}^{\mu,k}
\end{aligned} \tag{2.3.7}$$

  when $\mu < \nu$.

If we summarize these QCR in the form (2.3.1), (2.3.2), (2.3.3) (but now with indices $i, j, \ldots$ running over the values $1, 2, \ldots, mN$), it is immediate to realize that the $mN \times mN$ matrices $P^F, \tilde{P^F}$ can be put again in the form (2.3.5), (2.3.6), where now

$$F := \rho^{\otimes^2}(\mathcal{F}), \tag{2.3.8}$$

and $U, V$ are suitable $m\,N \times m\,N$ matrices such that

$$\left[U, \rho^{\otimes^2}\left(\Delta(U\mathbf{g})\right)\right] = \left[V, \rho^{\otimes^2}\left(\Delta(U\mathbf{g})\right)\right] = 0. \tag{2.3.9}$$

*Remark 2.* It is not difficult to verify that, given an *arbitrary* (finite-dimensional) representation $\rho_h$ of $U_h\mathbf{g}$ , arbitrary $T$ satisfying the condition (2.3.9) and $V \propto P\,q^{\rho^{\otimes^2}(t/2)}$, then relations (2.3.1), (2.3.2), (2.3.3) with $P^F, \tilde{P}^F$ defined by (2.3.5), (2.3.6), (2.3.8), (2.3.9) are still compatible with the (left) $U_h\mathbf{g}$ - action $\tilde{\triangleright}_h$ . However, in general they don't generate a left *and* right ideal alone, *i.e.* without introducing additional first, third or higher degree relations, which have no classical counterpart; if, in order to define an algebra $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$, one adds the latter, then the Poincaré series of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ is smaller than that of $\mathcal{A}_{\pm,\mathbf{g},\rho}$, what makes $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ physically non-interesting. This is the reason why these $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ have not been considered in the literature. On the contrary, in the cases mentioned at the beginning of this subsection the quantum and classical Poincaré series coincide.

One can introduce the notion of $U_h\mathbf{g}$ -invariant subalgebra $\mathcal{A}^{h,inv}_{\pm,\mathbf{g},\rho} \subset \mathcal{A}^h_{\pm,\mathbf{g},\rho}$ by mimicing the classical definition (2.2.7):

$$\mathcal{A}^{h,inv}_{\pm,\mathbf{g},\rho} := \{I \in \mathcal{A}^h_{\pm,\mathbf{g},\rho} \;\mid\; x\tilde{\triangleright}_h I = \varepsilon_h(x)I \qquad \forall x \in U_h\mathbf{g}\,\}. \tag{2.3.10}$$

For later use we recall that from the projector decomposition and the properties of $\hat{R}$ [15, 34, 4] it follows that the '$q$-number operator' $\tilde{\mathcal{N}} := \tilde{A}_i^+ \tilde{A}^i$ of $\mathcal{A}^h_{\pm,sl(N)\rho_d}$ satisfies the relations

$$\tilde{\mathcal{N}} \tilde{A}_i^+ = \tilde{A}_i^+ + q^{\pm 2} \tilde{A}_i^+ \tilde{\mathcal{N}} \qquad \tilde{\mathcal{N}} \tilde{A}^i = q^{\mp 2}(-\tilde{A}^i + \tilde{A}^i \tilde{\mathcal{N}}), \tag{2.3.11}$$

and the invariant elements $\tilde{A}^+ C \tilde{A}^+ := \tilde{A}_i^+ C^{ij} \tilde{A}_j^+$, $\tilde{A} C \tilde{A} := \tilde{A}^i C_{ji} \tilde{A}^j$ of $\mathcal{A}^h_{+,so(N)\rho_d}$ satisfy the relations

$$\begin{aligned}
(\tilde{A}C\tilde{A})\,\tilde{A}^i \quad &- \quad \tilde{A}^i\,(\tilde{A}C\tilde{A}) &= 0 && (2.3.12)\\
(\tilde{A}^+C\tilde{A}^+)\,\tilde{A}_i^+ \quad &- \quad \tilde{A}_i^+(\tilde{A}^+C\tilde{A}^+) &= 0 && (2.3.13)\\
(\tilde{A}C\tilde{A})\,\tilde{A}_i^+ \quad &- \quad q^2\,\tilde{A}_i^+\,(\tilde{A}C\tilde{A}) &= (1+q^{2-N})C_{ij}\tilde{A}^j && (2.3.14)\\
\tilde{A}^i\,(\tilde{A}^+C\tilde{A}^+) \quad &- q^2\,(\tilde{A}^+C\tilde{A}^+)\,\tilde{A}^i &= (1+q^{2-N})C^{ij}\tilde{A}_j^+. && (2.3.15)
\end{aligned}$$

# 3 Realization of the quantum action and of $U_h\mathbf{g}$ - covariant generators

Having learnt from Drinfel'd theorem that a quantum group $U_h\mathbf{g}$ can be realized essentially by $U\mathbf{g}$ itself as an algebra (upon the introduction of the commuting deformation parameter $h = \log q$) and, through a similarity transformation, also as a coalgebra, it is natural to ask whether one can realize a $U_h\mathbf{g}$ -covariant Heisenberg algebra [29, 34] $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ with generators $\mathbf{1}_{\mathcal{A}}, \tilde{A}^i, \tilde{A}_j^+$ by the corresponding $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ (as characterized in the previous section).

We begin by the obvious observation that the algebra $U_h\mathbf{g}$ can be realized in $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ by the homomorphism

$$\sigma_{\varphi_h} := \sigma \circ \varphi_h : U_h\mathbf{g} \to \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]. \tag{3.1}$$

Inspired by Propostion 1, we are naturally led to

**Definition-Proposition 1** *The definition[12]*

$$x \triangleright_h a := \sigma_{\varphi_h}(x_{(\bar{1})}) a \sigma_{\varphi_h}(S_h x_{(\bar{2})}) \tag{3.2}$$

*allows to realize* $\tilde{\triangleright}_h$ *as an action on the left module* $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$, *in an 'adjoint-like' way.*

Note now that $a_i^+, a^j$ are *not* covariant w.r.t. to $\tilde{\triangleright}_h$. One may ask whether there exist some objects $A_i^+, A^j \in \mathcal{A}_{\pm,\mathbf{g},\rho}$ that *are* (and going to $a_i^+, a^j$ in the limit $h \to 0$).

The answer comes from the crucial

**Proposition 3** *[12] Let $\mathsf{a}^i, \mathsf{a}_i^+$ be defined as in formula (2.2.11), with $I, \tilde{I} = \mathbf{1}_{\mathcal{A}}^{\otimes^2} + O(h)$ (in particular, it may be $\mathsf{a}^i = a^i$ $\mathsf{a}_i^+ = a_i^+$), and let $\mathcal{F}$ be a twist associated to $\varphi_h$. The elements*

$$\begin{aligned} A_i^+ &:= \sigma(\mathcal{F}^{(1)})\mathsf{a}_i^+\sigma(S\mathcal{F}^{(2)}\gamma) \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]] \\ A^i &:= \sigma(\gamma' S\mathcal{F}^{-1(2)})\mathsf{a}^i\sigma(\mathcal{F}^{-1(1)}) \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]], \end{aligned} \tag{3.3}$$

*are "covariant" under $\triangleright_h$, more precisely belong respectively to the irreducible representation $\rho_h$ and to its quantum contragredient one $\rho_h^\vee = \rho_h^T \circ S_h$ of $(H, \triangleright_h)$, and go to $a_i^+, a^i$ in the limit $q \to 1$.*[13]

*Proof.* Due to relation (2.1.4), $\mathcal{F}$ is an intertwiner between $\Delta_h$ and $\Delta$ (we drop the symbol $\varphi_h$):

$$x_{(\bar{1}')}\mathcal{F}^{(1)} \otimes x_{(\bar{2}')}\mathcal{F}^{(2)} = \mathcal{F}^{(1)}x_{(1')} \otimes \mathcal{F}^{(2)}x_{(2')}. \tag{3.4}$$

Applying $\mathrm{id} \otimes S$ on both sides of the equation and multiplying the result by $\mathbf{1} \otimes \gamma$ from the right we find [with the help of relation (2.1.5)]

$$x_{(\bar{1}')}\mathcal{F}^{(1)} \otimes (S\mathcal{F}^{(2)})\gamma S_h x_{(\bar{2}')} = \mathcal{F}^{(1)}x_{(1')} \otimes (Sx_{(2')})(S\mathcal{F}^{(2)})\gamma.$$

Applying $\sigma \otimes \sigma$ to both sides and sandwiching $\mathsf{a}_i^+$ between the two tensor factors we find

$$\sigma(x_{(\bar{1}')})A_i^+\sigma(S_h x_{(\bar{2}')}) = \sigma(\mathcal{F}^{(1)})\sigma(x_{(1')})\mathsf{a}_i^+\sigma(Sx_{(2')})\sigma[(S\mathcal{F}^{(2)})\gamma],$$

which, together with equations (2.2.6), (2.2.2), (3.2) proves the $U_h\mathbf{g}$ -covariance of $A_i^+$.

To prove the covariance of $A^i$, let us note that relation (2.1.4) implies an analogous relation

$$\Delta_h(a)\tilde{\mathcal{F}} = \tilde{\mathcal{F}}\Delta(a), \qquad \text{with } \tilde{\mathcal{F}} := [\gamma' S\mathcal{F}^{-1\ (2)} \otimes \gamma' S\mathcal{F}^{-1\ (1)}]\Delta(S\gamma).$$

[13]The Ansatz (3.3) has some resemblance with the one in Ref. [19], prop. 3.3, which defines an intertwiner $\alpha : U\mathbf{g}[[h]] \to U_h\mathbf{g}$ of $U_h\mathbf{g}$ -modules.

This can be shown by applying in the order the following operations to both sides of eq. (2.1.4): multiplying by $\mathcal{F}'^{-1}$ from the left and from the right, applying $S\otimes S$, multiplying by $\gamma'\otimes\gamma'$ from the left and by $\Delta(S\gamma)$ from the right, replacing $a\to S_h x$, using the properties (2.1.4) and $(S_h\otimes S_h)\circ\Delta_h=\tau\circ\Delta_h\circ S_h$. Next, we observe that $A^i$ can be rewritten as

$$A^i=\sigma[\tilde{\mathcal{F}}^{(1)}S(\gamma^{-1})_{(1)}]\mathsf{a}^i\sigma[(\gamma^{-1})_{(2)}S\tilde{\mathcal{F}}^{(2)}\gamma]=\sigma(\tilde{\mathcal{F}}^{(1)})\mathsf{a}^l\sigma(S\tilde{\mathcal{F}}^{(2)}\gamma)\rho(\gamma^{-1})^i_l; \qquad (3.5)$$

whence, reasoning as for the first relation,

$$\begin{aligned}\sigma(x_{(\bar{1}')})A^i\sigma(S_h x_{(\bar{2}')}) &\overset{(3.5)}{=} \sigma(\tilde{\mathcal{F}}^{(1)})\sigma(x_{(1')})\mathsf{a}^l\sigma(Sx_{(2')})\sigma[(S\tilde{\mathcal{F}}^{(2)})\gamma]\rho(\gamma^{-1})^i_l\\ &\overset{(2.2.6)}{=} \sigma(\tilde{\mathcal{F}}^{(1)})\mathsf{a}^l\sigma[(S\tilde{\mathcal{F}}^{(2)})\gamma]\rho(\gamma^{-1}Sx)^i_l\\ &\overset{(2.1.4)}{=} \sigma(\tilde{\mathcal{F}}^{(1)})\mathsf{a}^l\sigma[(S\tilde{\mathcal{F}}^{(2)})\gamma]\rho(S_h x\cdot\gamma^{-1})^i_l\\ &\overset{(3.5)}{=} \rho_h(S_h x)^i_l A^l\end{aligned}$$

which, together with equation (2.2.6),(2.2.6), proves the second relation. □

*Remark 3.* Under the right action ${}_h\triangleleft$ ($a\,{}_h\triangleleft x:=(S_h^{-1}x)\triangleright_h a$ with $a\in\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$, $x\in U\mathbf{g}\,[[h]]$) the covariance properties of $A^i, A^+_i$ read

$$A^i\,{}_h\triangleleft x=\rho_h(x)^i_l A^l \qquad\qquad A^+_i\,{}_h\triangleleft x=\rho_h(S_h^{-1}x)^m_i A^+_m.$$

*Remark 4.* For any invertible $\mathbf{g}$-invariant elements $N_1,N_2\in U\mathbf{g}\,[[h]]\otimes U\mathbf{g}\,[[h]]$ the objects

$$\mathsf{a}'^+_i:=\sigma(N_1^{(1)})\mathsf{a}^+_i\sigma(SN_1^{(2)}) \qquad\qquad \mathsf{a}'^i:=\sigma(SN_2^{(1)})\mathsf{a}^i\sigma(N_2^{(2)}) \qquad (3.6)$$

are still of the form (2.2.11). In fact, by eq. (2.1.15) $N_1,N_2$ can depend only on $\mathcal{C}_i\otimes\mathbf{1}$, $\mathbf{1}\otimes\mathcal{C}_i$, $\mathcal{C}_{i(1)}\otimes\mathcal{C}_{i(2)}$, but $\sigma(\mathcal{C}_{i(1)})\mathsf{a}^+_i\sigma(S\mathcal{C}_{i(2)})=\mathcal{C}_i\triangleright\mathsf{a}^+_i=c\mathsf{a}^+_i$, $\sigma(S\mathcal{C}_{i(1)})\mathsf{a}^i\sigma(\mathcal{C}_{i(2)})=\mathcal{C}_i\triangleleft_h\mathsf{a}^i=c\mathsf{a}^i$, where $c\in\mathbf{C}$ is the value of the Casimir $\mathcal{C}_i$ in (the irreducible component of) the representation $\rho$ to which $a^+_i$ belongs; moreover $\sigma(\mathcal{C}_i),\sigma(S\mathcal{C}_i)\in\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}$. Thus any transformation (2.1.12) in definitions (3.3) amounts to a replacement of the type $\mathsf{a}^+_i,\mathsf{a}^i\to\mathsf{a}'^+_i,\mathsf{a}'^i$. Therefore, without loss of generality, we can assume from the starting $\mathcal{F}$ to be 'minimal' in definitions (3.3).

To conclude this section, let us give useful alternative expressions for $A^+_i, A^i$ by 'moving' to the right/left past $\mathsf{a}^+_i,\mathsf{a}^i$ the expressions $\sigma(\cdot)$ lying at their left/right in definitions (3.3). In the appendix we prove the following

**Lemma 1** *If $\mathcal{F}$ is a 'minimal', then*

$$\begin{aligned}
\mathcal{F} &= \gamma^{-1}(S\mathcal{F}^{-1(1)})\mathcal{F}^{-1(2)}_{(1)} \otimes \mathcal{F}^{-1(2)}_{(2)} && (3.7)\\
&= \mathcal{F}^{-1(1)}_{(1)} \otimes \gamma'(S\mathcal{F}^{-1(2)})\mathcal{F}^{-1(2)}_{(1)} && (3.8)\\
\mathcal{F}^{-1} &= \mathcal{F}^{(1)}_{(1)} \otimes \mathcal{F}^{(1)}_{(2)}(S\mathcal{F}^{(2)})\gamma && (3.9)\\
&= \mathcal{F}^{(2)}_{(1)}(S\mathcal{F}^{(1)})\gamma'^{-1} \otimes \mathcal{F}^{(2)}_{(2)}. && (3.10)
\end{aligned}$$

**Proposition 4** *With a 'minimal' $\mathcal{F} = \mathcal{F}'$, definitions (3.3) amount to*

$$\begin{aligned}
A^+_i &= \mathsf{a}^+_l \sigma(\mathcal{F}^{-1(2)})\rho(\mathcal{F}^{-1(1)})^l_i && (3.11)\\
A^+_i &= \rho(S\mathcal{F}^{(1)}\gamma'^{-1})^l_i \sigma(\mathcal{F}^{(2)})\mathsf{a}^+_l && (3.12)\\
A^i &= \rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)})\mathsf{a}^l && (3.13)\\
A^i &= \mathsf{a}^l \sigma(\mathcal{F}^{-1(2)})\rho(\gamma^{-1}S\mathcal{F}^{-1(1)})^i_l. && (3.14)
\end{aligned}$$

*Remark 5.* In spite of its original definition (3.2), from the latter expressions we realize that only a 'semiuniversal form' of the type $(\rho \otimes \mathrm{id})\mathcal{F}^{\pm 1}$ for $\mathcal{F}$ is involved in the definition of $A^i, A^+_j$.

*Proof of Prop.* 4. Observing that

$$\begin{aligned}
\sigma(x)a &= \sigma(x_{(1)})a\sigma(Sx_{(2)} \cdot x_{(3)}) && (3.15)\\
a\sigma(x) &= \sigma(x_{(3)}Sx_{(2)})a\sigma(x_{(1)}) && (3.16)
\end{aligned}$$

for all $x \in U\mathbf{g}$, $a \in \mathcal{A}_{\pm,\mathbf{g},\rho}$, we find

$$A^+_i \overset{(2.2.6),(2.2.2)}{=} \mathsf{a}^+_l \sigma\left[\mathcal{F}^{(1)}_{(2)}(S\mathcal{F}^{(2)})\gamma\right] \rho(\mathcal{F}^{(1)}_{(1)})^l_i \overset{(3.9)}{=} \mathsf{a}^+_l \sigma(\mathcal{F}^{-1(2)}_1)\rho(\mathcal{F}^{-1(1)}_1)^l_i, \quad (3.17)$$

$$A^i \overset{(2.2.6),(2.2.2)}{=} \rho(\mathcal{F}^{-1(1)}_{(1)})^i_l \sigma\left[\gamma'(S\mathcal{F}^{-1(2)})\mathcal{F}^{-1(1)}_{(2)}\right] \mathsf{a}^l \overset{(3.8)}{=} \rho^i_l(\mathcal{F}^{(1)}_2)\sigma(\mathcal{F}^{(2)}_2)\mathsf{a}^l. \quad (3.18)$$

Similarly one proves the other relations. □

# 4 Classical versus quantum invariants

Having defined two actions $\triangleright, \triangleright_h$ on $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$, let us ask what is the relation between their respective invariant subalgebrae $\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}[[h]] \subset \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ [see def. (2.2.7)] and

$$\mathcal{A}^{h,inv}_{\pm,\mathbf{g},\rho}[[h]] := \{I \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]] \ \mid \ x \triangleright_h I = \varepsilon_h(x)I\}. \quad (4.1)$$

It is easy to show the

**Proposition 5** $\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}[[h]] = \mathcal{A}^{h,inv}_{\pm,\mathbf{g},\rho}[[h]]$.

*Proof*. We have to show that definition (4.1) is equivalent to

$$\mathcal{A}^{h,inv}_{\pm,\mathbf{g},\rho} = \{I \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]] \ \mid \ [\sigma(y), I] = 0 \quad y \in U\mathbf{g}\,[[h]]\}.$$

The proof goes exactly as for proposition 2 if we replace $\Delta$ by $\Delta_h$ and $S$ by $S_h$. □

Thus, given an element $I \in \mathcal{A}^{inv}_{\pm,\mathbf{g},\rho} \equiv \mathcal{A}^{h,inv}_{\pm,\mathbf{g},\rho}$, we can express it as a function of $a^i, a^+_j$ or $A^i, A^+_j$, $I = f(a^i, a^+_j) = f_h(A^i, A^+_j)$. What is the relation between $f_h, f$?

Since $\triangleright$ (resp. $\triangleright_h$) acts in a linear homogeneous way on the generators $a^i, a^+_j$ (resp. $A^i, A^+_j$), we can choose a basis $\{I^n\}_{n\in\mathbf{N}}$ (resp. $\{I^n_h\}_{n\in\mathbf{N}}$) of the vector space $\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}[[h]]$ consisting of normal ordered homogeneous polynomials in $a^i, a^+_j$ (resp. $A^i, A^+_j$):

$$I^n \ := \ a^+_{j_1}...a^+_{j_{k_n}} d^{j_1....j_{k_n}}_{i_1...i_{h_n}} a^{i_1}...a^{i_{h_n}} \tag{4.2}$$

$$I^n_h \ := \ A^+_{j_1}...A^+_{j_{k_n}} D^{j_1....j_{k_n}}_{i_1...i_{h_n}} A^{i_1}...A^{i_{h_n}} \tag{4.3}$$

($k_n, h_n \in \mathbf{N} \cup \{0\}$); the coefficients $d^{j_1....j_{k_n}}_{i_1...i_{h_n}}$ (resp. $D^{j_1....j_{k_n}}_{i_1...i_{h_n}}$) make up classical (resp. quantum) $\mathbf{g}$-isotropic tensors, *i.e.* satisfy

$$\left[\left(\rho^{\otimes^{k_n}} \otimes \rho^{\vee\ \otimes^{h_n}}\right)\left(\Delta^{(b_n-1)}(x)\right)\right]^{J_n I_n}_{J'_n I'_n} d^{J'_n}_{I'_n} \ = \ \varepsilon(x) d^{J_n}_{I_n} \tag{4.4}$$

$$\left[\left(\rho_h^{\otimes^{k_n}} \otimes \rho_h^{\vee\ \otimes^{h_n}}\right)\left(\Delta_h^{(b_n-1)}(y)\right)\right]^{J_n I_n}_{J'_n I'_n} D^{J'_n}_{I'_n} \ = \ \varepsilon_h(y) D^{J_n}_{I_n} \tag{4.5}$$

$\forall x \in U\mathbf{g}\,[[h]], y \in U_h\mathbf{g}$ . Here and in the rest of the section we use the collective-index notation $I_n \equiv (i_1....i_{h_n})$, $J_n \equiv (j_1....j_{k_n})$ and the short-hand notation $b_n := h_n + k_n$. Using formula (2.1.4) it is straightforward to verify that the $d$'s and $D$'s are related to each other by

$$D^{J_n}_{I_n} \propto \left[\left(\rho^{\otimes^{k_n}} \otimes \rho^{\vee\ \otimes^{h_n}}\right)\left((\mathbf{1}^{\otimes^{k_n}} \otimes (\gamma'^{-1})^{\otimes^{h_n}})\mathcal{F}_{12....b_n}\right)\right]^{J_n I_n}_{J'_n I'_n} d^{J'_n}_{I'_n}, \tag{4.6}$$

where $\mathcal{F}_{12....b} \in U\mathbf{g}\,[[h]]^{\otimes^b}$ is an intertwiner between $\Delta^{(b-1)}$ and $\Delta_h^{(b-1)}$ (symbolically, $\mathcal{F}\varphi_h(\Delta_h(a)) = \Delta(\varphi_h(a))\mathcal{F}$), *i.e.* it is given, up to multiplication from the right by a $\mathbf{g}$-invariant tensor $Q \in U\mathbf{g}\,[[h]]^{\otimes^{b_n}}$, by

$$\mathcal{F}_{12....b} = \mathcal{F}_{(b-1)b}\mathcal{F}_{b-2,(b-1)b}....\mathcal{F}_{1,2...b}. \tag{4.7}$$

The replacement $\mathcal{F} \to \mathcal{F} \cdot T$, with $T \in U\mathbf{g}\,[[h]]^{\otimes^2}$ and $\mathbf{g}$-invariant, results also in multiplication from the right by a related $Q$.

Relation (4.6) guarantees the existence of $D$'s in one-to-one correspondence with the $d$'s, but from the practical viewpoint is not of much help for finding the $D$'s (since the universal $\mathcal{F}$ is unknown and its matrix representations are known only for few representations); the latter can be found more easily from the knowledge of $\mathcal{R}$ and a direct study of $\tilde{\triangleright}_h$.

Our question can be now reformulated as follows: what is the explicit dependence of the $I_h^n$'s on $a^i, a_i^+$? and on $I^n$? We answer here the first question.

**Proposition 6**

$$I_h^n = (\mathsf{a}^+ ... \mathsf{a}^+)_{M_n} (\mathsf{a} ... \mathsf{a})^{L_n} \times$$
$$\left[\left(\rho^{\otimes k_n} \otimes \rho^{\vee \otimes h_n} \otimes \sigma\right) \left(\phi^{-1}_{(b_n-1)b_n(b_n+1)} \phi^{-1}_{(b_n-2),(b_n-1)b_n,(b_n+1)} \cdots \phi^{-1}_{1,2...b_n,(b_n+1)}\right)\right]^{M_n L^n}_{J_n I_n} d^{J_n}_{I_n}$$

*where* $\phi_{1,2...m,m+1} := (id \otimes \Delta^{(m-2)} \otimes id)\phi_{123}$ *and* $b_n := h_n + k_n$.

The Proof is given in the appendix.

*Remark 6.* Note that in these equations $\mathcal{F}$ does not explicitly appear any more; the whole effect of twisting is concentrated in the coassociator $\phi$ of $\mathbf{g}$ and in its coproducts. Consequently, use of formula (2.1.24) allows the explicit determination of the dependence of $I_h^n$'s on $a^i, a_j^+$.

If $H_h$ is *triangular* then $\phi^{-1}$ and all its copruducts are trivial, and consequently we find

$$I_h^n = I^n. \tag{4.8}$$

But if $H_h$ is a *genuine quasitriangular* Hopf algebra as $U_h\mathbf{g}$ , then

$$I_h^n \neq I^n; \tag{4.9}$$

The $I_h^n$ will be some nontrivial function of the $I^m$'s, generally speaking a highly non-polynomial function of the latter and of the $a^i, a_j^+$'s.

This can be already verified for the simplest invariants. To the $\mathbf{g}$ -isotropic tensor $d_i^j = \delta_i^j$ there corresponds $U_h\mathbf{g}$ -isotropic tensor $D_i^j = \delta_i^j$ [by formulae (4.6), (2.1.5)], whence we can construct the invariants $I := a_i^+ a^i \equiv n$ and $I := A_i^+ A^i$, which are necessarily different: we will show in next section that *e.g.* $I = (n)_{q^2}$ in the $\mathbf{g} = sl(2)$ case. In the $\mathbf{g} = so(N), \rho = \rho_d$ case another basic isotropic tensor is the classical metric matrix $c_{ij} = c_{ji}$ (with inverse $c^{ij} = c^{ji}$, to which there corresponds the quantum metric matrix [15] $C_{ij}$, and its inverse $C^{ij}$:

$$C^{ij} = F^{ij}_{hk} c^{hk} = \rho_d(\gamma^{-1})^i_h c^{hj}. \qquad C_{ij} = c_{ih} \rho_d(\gamma)^h_j; \tag{4.10}$$

the last two equalities follow from the $so(N)$ property

$$\rho_d(Sx)^i_j = \rho_d(x)^m_l c^{li} c_{mj} \qquad\qquad x \in U\mathbf{g}\,. \tag{4.11}$$

So one can build the invariants

$$\begin{array}{llll} I^{2,0} & := & a^i c_{ij} a^j \equiv a\,c\,a & \qquad I^{0,2} := a^+_i c^{ij} a^+_j \equiv a^+\,c\,a^+ \\ I^{2,0}_h & := & A^i C_{ji} A^j \equiv A\,C\,A & \qquad I^{0,2}_h := A^+_i C^{ij} A^+_j \equiv A^+\,c\,A^+; \end{array} \tag{4.12}$$

we will see in next section that $I^{2,0}_h \neq I^{2,0}$, $I^{0,2}_h \neq I^{2,0}$.

# 5 Fulfilling the QCR of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ within $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$

In section 3 we have left some freedom in the definition of $A^i, A^+_i$: the $\mathbf{g}$-invariants $I, \tilde{I}$ appearing in the definitions (2.2.11) of $\mathsf{a}^i, \mathsf{a}^+_i$ have not been not specified. Can we choose $I, \tilde{I}$ in such a way that $A^i, A^+_i$ fulfil the QCR (quantum commutation relations) of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$? This question can be studied explicitly using the following

**Proposition 7** *If we replace $\tilde{A}^i, \tilde{A}^+_j \to A^i, A^+_j$ [with $A^i, A^+_j$ defined as in formulae (3.11), (3.13), with a minimal $\mathcal{F}$], then equations (2.3.1), (2.3.2), (2.3.3) become equivalent to*

$$\mathsf{a}^i\,\mathsf{a}^j = \pm\,(M^{-1}\,U\,M)^{ji}_{lm}\mathsf{a}^m\,\mathsf{a}^l \tag{5.0.1}$$

$$\mathsf{a}^+_i\,\mathsf{a}^+_j = \pm\,\mathsf{a}^+_l\,\mathsf{a}^+_m\,(M^{-1}\,U\,M)^{lm}_{ij} \tag{5.0.2}$$

$$\mathsf{a}^i\,\mathsf{a}^+_j = \delta^i_j \mathbf{1}_{\mathcal{A}} \pm \mathsf{a}^+_l (M^{-1}\,V\,M)^{il}_{jm}\mathsf{a}^m \tag{5.0.3}$$

*where $U \equiv \|U^{ij}_{hk}\|$, $V \equiv \|V^{ij}_{hk}\|$ are the (numerical) matrices introduced in equations (2.3.9) and $M \equiv \|M^{ij}_{hk}\|$ is the $\sigma(U\mathbf{g}\,[[h]])$-valued matrix defined by*

$$M := (\rho \otimes \rho \otimes \sigma)(\phi_m). \tag{5.0.4}$$

(The proof is given in the appendix.) We recall that, if $\rho = \rho_d$, then simply $U$ is the permutation matrix $P$, and $V \propto P\,q^{\rho_d^{\otimes^2}(\frac{t}{2})} \otimes \mathbf{1}_{\mathcal{A}}$.

*Remark 7.* The above equations have to be understood as equations in the unknown $I, \tilde{I} \in (\mathcal{A}^{inv}_{\pm,\mathbf{g},\rho})^{\otimes^2}$. They can be studied explicitly because the whole dependence on $\mathcal{F}$ is concentrated again in the coassociator $\phi$ of $\mathbf{g}$.

*Remark 8.* If $H_h$ is a triangular deformation, then $U = V = P$, $\phi = \mathbf{1}^{\otimes^3}$ (and consequently $M = \mathbf{1}^{\otimes^3}$), and the eq. (5.0.1), (5.0.2) are satisfied with trivial invariants $I, \tilde{I}$, *i.e.* with $\mathsf{a}^i = a^i$, $\mathsf{a}_i^+ = a_i^+$. This was already shown in Ref. [12].

To look for solutions of eq. (5.0.1), (5.0.2), (5.0.3) for genuine quasitriangular deformations we have to treat the $\mathbf{g}$ 's belonging to different classical series separatly. We consider here $\mathcal{A}^h_{\pm,sl(N),\rho_d}$, $\mathcal{A}^h_{+,so(N),\rho_d}$.

## 5.1 The case of $\mathcal{A}^h_{\pm,sl(N),\rho_d}$

As a basis of $\mathbf{g}$ we choose $\{E_{ij}\}_{i,j=1,\dots,N}$ with $\sum_{i=1}^{N} E_{ii} \equiv 0$ (so that there exist only $N^2 - 1$ linearly independent $E_{ij}$), satisfying

$$[E_{ij}, E_{hk}] = E_{ik}\delta_{jh} - E_{jh}\delta_{ik} \tag{5.1.1}$$

The quadratic Casimir reads

$$\mathcal{C} = E_{ij}E_{ji}, \tag{5.1.2}$$

implying

$$t = 2\, E_{ij} \otimes E_{ji} \tag{5.1.3}$$

The matrix representation of $E_{ij}$ in the fundamental representation $\rho$ takes the form

$$\rho(E_{ij}) = e_{ij} - \frac{\delta_{ij}}{N}\mathbf{1}_N, \tag{5.1.4}$$

where $e_{ij}$ is the $N \times N$ matrix with all vanishing entries but a 1 in the $i$-th row and $j$-th column, and $\mathbf{1}_N = \sum_i e_{ii}$ is the $N \times N$ unit matrix; whereas the Jordan-Schwinger realization takes the form

$$\sigma(E_{ij}) = a_i^+ a^j - \frac{\delta_{ij}}{N} n. \tag{5.1.5}$$

As a consequence $\sigma(\mathcal{C}) = n(N \pm n \mp 1) - \frac{n^2}{N}$.

¿From the previous three equations one finds

$$\begin{aligned}
(\rho\otimes\rho\otimes\sigma)\left(\tfrac{t_{12}}{2}\right) &= e_{ij}\otimes e_{ji}\otimes \mathbf{1}_{\mathcal{A}} - \tfrac{1}{N}\mathbf{1}_N\otimes\mathbf{1}_N\otimes\mathbf{1}_{\mathcal{A}} =: P - \tfrac{\mathbf{1}^{\otimes^3}}{N}, \\
(\rho\otimes\rho\otimes\sigma)\left(\tfrac{t_{23}}{2}\right) &= \mathbf{1}_N\otimes e_{ij}\otimes a_j^+ a^i - \mathbf{1}_N\otimes\mathbf{1}_N\otimes\tfrac{n}{N} =: A - \mathbf{1}_N^{\otimes^2}\otimes\tfrac{n}{N} \\
(\rho\otimes\rho\otimes\sigma)\left(\tfrac{t_{13}}{2}\right) &= e_{ij}\otimes\mathbf{1}_N\otimes a_j^+ a^i - \mathbf{1}_N\otimes\mathbf{1}_N\otimes\tfrac{n}{N} =: B - \mathbf{1}_N^{\otimes^2}\otimes\tfrac{n}{N};
\end{aligned} \tag{5.1.6}$$

$P$ denotes the permutation matrix on $\mathbf{C}^N \otimes \mathbf{C}^N$, multiplied by $\mathbf{1}_{\mathcal{A}}$.

$\mathcal{A}^{inv}_{\pm,sl(N),\rho_d}$ is generated by $n := a_i^+ a^i$. Using relations (2.2.10) we can thus commute $I^{(2)}$ to the left of $a^i$ and $\tilde{I}^{(1)}$ to the right of $a_i^+$ in formula (2.2.11, and look for $\mathsf{a}^i, \mathsf{a}_i^+$ directly in the form $\mathsf{a}^i := I a^i$, $\mathsf{a}_i^+ := a_i^+ \tilde{I}$, with $I = I(n) \in \mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}$, $\tilde{I} = \tilde{I}(n) \in \mathcal{A}^{inv}_{\pm,\mathbf{g},\rho}$. ¿From eq. (3.11), (3.13) it follows $\mathcal{N} := A_i^+ A^i = \mathsf{a}_i^+ \mathsf{a}^i = n\,\hat{I}(n-1)$, where $\hat{I}(n) := I(n)\tilde{I}(n)$. In order that $\mathcal{N}, A_i^+, A^i$ satisfies the commutations relations (2.3.11), we therefore require $\hat{I}(n) = \frac{(n+1)_{q^{\pm 2}}}{n+1}$, with $(x)_a := \frac{a^x - 1}{a-1}$. Summing up, we pick

$$\begin{aligned} \mathsf{a}^i &:= I a^i, & I &\equiv I(n), \\ \mathsf{a}_i^+ &:= a_i^+ \tilde{I} & \tilde{I}(n) &:= I^{-1}(n)\,\frac{(n+1)_{q^{\pm 2}}}{(n+1)}. \end{aligned} \tag{5.1.7}$$

These ansatz can also be written in the equivalent form

$$\mathsf{a}^i = u(n) a^i u^{-1}(n), \qquad \mathsf{a}_i^+ = v(n) a_i^+ v^{-1}(n), \tag{5.1.8}$$

where $u, v$ are constrained by the relation

$$u\,v^{-1} = y = y_{sl(N)} := \frac{\Gamma(n+1)}{\Gamma_{q^2}(n+1)} \tag{5.1.9}$$

and $\Gamma, \Gamma_{q^2}$ are the $\Gamma$-functions defined in formulae (A.4.9), (A.4.12).

We have now the right ansatz to show that the QCR of $N$-dimensional $U_h sl(N)$-covariant Heisenberg algebra are fulfilled. In the appendix we prove

**Theorem 1** *When* $\mathbf{g} = sl(N)$, *the objects* $A^i, A_i^+$ $(i = 1, 2, ..., N)$ *defined in formulae (3.11), (3.13), (5.1.7) satisfy the corresponding QCR (2.3.1), (2.3.2), (2.3.3).*

In Ref. [12] the case $\mathbf{g} = sl(2)$ was worked out explicitly. Choosing $u = v^{-1} = \sqrt{y_{sl(2)}}$, we found for the $A^i, A_i^+ \in \mathcal{A}_{+,sl(2),\rho_d}$ $(i = \uparrow, \downarrow)$,

$$\begin{aligned} A_\uparrow^+ &= \sqrt{\frac{(n^\uparrow)_{q^2}}{n^\uparrow}}\, q^{n^\downarrow} a_\uparrow^+ & A_\downarrow^+ &= \sqrt{\frac{(n^\downarrow)_{q^2}}{n^\downarrow}}\, a_\downarrow^+ \\ A^\uparrow &= a^\uparrow \sqrt{\frac{(n^\uparrow)_{q^2}}{n^\uparrow}}\, q^{n^\downarrow} & A^\downarrow &= a^\downarrow \sqrt{\frac{(n^\downarrow)_{q^2}}{n^\downarrow}}, \end{aligned} \tag{5.1.10}$$

and for the $A^i, A_i^+ \in \mathcal{A}_{-,sl(2),\rho_d}$

$$\begin{aligned} A_\uparrow^+ &= q^{-n^\downarrow} a_\uparrow^+ & A_\downarrow^+ &= a_\downarrow^+ \\ A^\uparrow &= a^\uparrow q^{-n^\downarrow} & A^\downarrow &= a^\downarrow. \end{aligned} \tag{5.1.11}$$

Here $n_i := a_i^+ a^i$ (no sum over $i$).

## 5.2 The case of $\mathcal{A}^h_{+,so(N),\rho_d}$

As a basis of $\mathbf{g} = so(N)$ we choose $\{L_{ij}\}_{i,j=1,\dots,N}$ with $L_{ij} = -L_{ji}$ (so that there exist only $\frac{N(N-1)}{2}$ linearly independent $L_{ij}$), satisfying

$$[L_{ij}, L_{hk}] = L_{ik}c_{jh} + L_{kj}c_{ih} - L_{hj}c_{ik} - L_{ih}c_{jk}; \tag{5.2.1}$$

here $c_{ij}$ denotes the (classical) metric matrix on the $N$-dimensional Euclidean space ($c_{ij} = c_{ji}$), which in the special case we choose real Cartesian coordinates takes simply the form $c_{ij} = \delta_{ij}$. In the rest of this subsection classically-covariant indices will be lowered and raised by means of multiplication by $c$: $v_i = c_{ij}v^j$ $v^i = c^{ij}v_j$, etc., and $v \cdot w := v^i w^j c_{ij} = v_i w^i = v^i w_i$

The quadratic Casimir reads

$$\mathcal{C} = \frac{1}{2} L_{ij} L^{ji}, \tag{5.2.2}$$

implying

$$t = L_{ij} \otimes L^{ji} \tag{5.2.3}$$

The matrix representation of $E_{ij}$ in the fundamental representation $\rho$ takes the form

$$\rho(L_{ij}) = e_{ih}c_{hj} - e_{jh}c_{hi}, \tag{5.2.4}$$

and the Jordan-Schwinger realization becomes

$$l_{ij} := \sigma(L_{ij}) = a_i^+ a^h c_{hj} - a_j^+ a^h c_{hi} = a^h a_i^+ c_{hj} - a^h a_j^+ c_{hi}. \tag{5.2.5}$$

It is easy to work out

$$l^2 := \sigma(\mathcal{C}) + (1 - \frac{N}{2})^2 = \left(n + \frac{N}{2} - 1\right)^2 - (a^+ \cdot a^+)(a \cdot a), \tag{5.2.6}$$

and to check that, as expected,

$$[l^2, a^+ \cdot a^+] = 0 = [l^2, a \cdot a] \tag{5.2.7}$$

A direct calculation also shows that

$$\begin{aligned} [l^2, a^i] &= -a^i(2n+1+N) + 2(a \cdot a)a_j^+ c^{ji} &= -a^i(2n-3+N) + 2a_j^+ c^{ji}(a \cdot a) \\ [l^2, a_i^+] &= a_i^+(2n+3+N) - 2c_{ij}a^j(a^+ \cdot a^+) &= a_i^+(2n-1+N) - 2c_{ij}(a^+ \cdot a^+)a^j. \end{aligned}$$

We look for "eigenvectors" of $l^2$

$$l^2 \alpha^i = \alpha^i \lambda \qquad\qquad l^2 \alpha_i^+ = \alpha_i^+ \mu,$$

in the form $\alpha^i = a^i\gamma + a_j^+ c^{ji}\,(a\cdot a)\,\delta$, $\alpha_i^+ = a_i^+\alpha + a^j c_{ji}\,(a^+\cdot a^+)\,\beta$ with "eigenvalues" $\lambda, \mu$ and "coefficients" $\alpha, \beta, \gamma, \delta$ depending on $n, l^2$. We find second order equations for $\lambda, \mu$ with solutions $\lambda, \mu = (l \pm 1)^2$, where formally $l = \sqrt{l^2}$. We can therefore consistently extend $\mathcal{A}_{+,so(N),\rho_d}$ by the introduction of a new generator $l$ [whose square is constrained to give the $l^2$ defined in eq. (5.2.6)] such that

$$\begin{aligned} \alpha_\pm^i &:= a^i(n+\tfrac{N}{2}-1\pm l) - c^{ij}a_j^+\,(a\cdot a) &&= a^i(n+\tfrac{N}{2}+1\pm l) - (a\cdot a)\,c^{ij}a_j^+, \\ \alpha_{i,\pm}^+ &:= a_i^+(n+\tfrac{N}{2}-1\pm l) - (a^+\cdot a^+)\,c_{ij}a^j &&= a_i^+(n+\tfrac{N}{2}+1\pm l) - c_{ij}a^j\,(a^+\cdot a^+) \end{aligned} \tag{5.2.8}$$

satisfy

$$\begin{aligned} l\,\alpha_{i,\pm}^+ &= \alpha_{i,\pm}^+\,(l\pm 1), \\ l\,\alpha_\pm^i &= \alpha_\pm^i\,(l\mp 1). \end{aligned} \tag{5.2.9}$$

After these preliminaries, let us determine the right $\mathsf{a}^i, \mathsf{a}_i^+$'s for $A^i, A_i^+$ to satisfy the QCR. To satisfy at once eq.'s (2.3.12),(2.3.13) we make the ansatz:

$$\mathsf{a}^i = u(n,l)a^i u^{-1}(n,l), \qquad \mathsf{a}_i^+ = v(n,l)a_i^+ v^{-1}(n,l) \tag{5.2.10}$$

This implies

$$A^+\,C\,A^+ \overset{(5.2.10)}{=} \mathsf{a}_l^+\mathsf{a}_m^+ c^{lm} = v\,a^+\cdot a^+\,v^{-1} \tag{5.2.11}$$

$$A\,C\,A \overset{(5.2.10}{=} \mathsf{a}^l\mathsf{a}^m c_{lm} = u\,a\cdot a\,u^{-1} \tag{5.2.12}$$

The QCR determine only the product $y := v^{-1}u$; we are going to show now that eq.'s (2.3.14), (2.3.15) completely determine the latter. It is immediate to check that the former implies

$$y\left[2c^{ij}a_j^+(a^+\cdot a^+)a^i\right]y^{-1}(n+2,l) - q^2 y(n-2,l)(a^+\cdot a^+)a^i y^{-1} = (1+q^{N-2})c^{ij}a_j^+.$$

Expressing $a_i^+, c_{ij}(a^+\cdot a^+)a^j$ as combinations of $\alpha_{i\pm}$ we easily move $y$ past the "eigenvectors" $\alpha_{i\pm}$ of $n, l$; factoring out (from the right) $\frac{c_{ij}}{2l}$ we end up with a LHS being a combination of $\alpha_{i,+}$, $\alpha_{i,-}$. Therefore eq. (2.3.14) amounts to the condition that the corresponding coefficients vanish:

$$(1+q^{N-2}) = (n+\frac{N}{2}+1-l)\,y(n+1,l+1)y^{-1}(n+2,l) - q^2(n+\frac{N}{2}-1-l)\,y(n-1,l-1)y^{-1}(n,l)$$

$$(1+q^{N-2}) = (n+\frac{N}{2}+1+l)\,y(n+1,l+1)y^{-1}(n+2,l) - q^2(n+\frac{N}{2}-1-l)\,y(n-1,l-1)y^{-1}(n,l)$$

Similarly, from eq. (2.3.15) it follows

$$(1+q^{N-2}) = (n+\frac{N}{2}+1-l)\, y(n,l)y^{-1}(n+1,l-1) - q^2(n+\frac{N}{2}-1-l)\, y(n-2,l)y^{-1}(n-1,l-1)$$

$$(1+q^{N-2}) = (n+\frac{N}{2}+1+l)\, y(n,l)y^{-1}(n+1,l+1) - q^2(n+\frac{N}{2}-1+l)\, y(n-2,l)y^{-1}(n-1,l+1)$$

It is straightforward to check that the last four equations are solved by

$$u\,v^{-1} = y = y_{so(N)} := \left(\frac{1+q^{N-2}}{2}\right)^{-n} \frac{\Gamma\left[\frac{1}{2}\left(n+\frac{N}{2}+1-l\right)\right]\Gamma\left[\frac{1}{2}\left(n+\frac{N}{2}+1+l\right)\right]}{\Gamma_{q^2}\left[\frac{1}{2}\left(n+\frac{N}{2}+1-l\right)\right]\Gamma_{q^2}\left[\frac{1}{2}\left(n+\frac{N}{2}+1+l\right)\right]}, \tag{5.2.13}$$

where $\Gamma, \Gamma_a$ are defined in formulae (A.4.9), (A.4.12).

We have now the right ansatz for the QCR of $N$-dimensional $U_h so(N)$-covariant Weyl algebra to be fulfilled. In the appendix we sketch the proof of

**Theorem 2** *When* $\mathbf{g} = so(N)$, *the objects* $A^i, A^+_i$ $(i = 1, 2, ..., N)$ *defined in formulae (3.11), (3.13), (5.2.10), (5.2.13) satisfy the corresponding QCR (2.3.1), (2.3.2), (2.3.3).*

# 6 $*$-Structures

Given the Hopf $*$-algebra $H_h = (U_h\mathbf{g}\,, m, \Delta_h, \varepsilon_h, S_h, \mathcal{R}\,, *_h)$, we ask now whether the $*$-structures $\dagger_h$ of $\mathcal{A}^h_{\pm,\mathbf{g}\,,\rho}$compatible with the action $\triangleright_h$ of $U_h\mathbf{g}$ ,*i.e.* such that

$$(x\tilde{\triangleright}_h\, a)^{\dagger_h} = S_h^{-1}(x^{*_h})\tilde{\triangleright}_h\, a^{\dagger_h}, \tag{6.1}$$

can be naturally realized by the ones of $\mathcal{A}_{\pm,\mathbf{g}\,,\rho}$.

We stick to the case that $H_h$ is the compact real section of $U_h\mathbf{g}$ . Then $U_h\mathbf{g}$ as an algebra is isomorphic to $U\hat{\mathbf{g}}\,[[h]]$, where $\hat{\mathbf{g}}$ is the compact section of $\mathbf{g}$ and $h \in \mathbf{R}$, and the trivializing maps $\varphi_h$ intertwine between $*_h$ and $*$

$$[\varphi_h(x)]^* = \varphi_h(x^{*_h}) \tag{6.2}$$

where $*$ is the classical $*$-structure in $U\mathbf{g}$ having the elements of $\hat{g}$ as fixed points. One can easily show [20] that there is a (unique) unitary twist $\mathcal{F}$,

$$\mathcal{F}^{*\otimes *} = \mathcal{F}^{-1}. \tag{6.3}$$

In fact, applying $*_h \otimes *_h$ to eq. (2.1.4) and using eq.'s 6.2, $(*_h \otimes *_h) \circ \Delta_h = \Delta_h \circ *_h$, $(* \otimes *) \circ \Delta = \Delta \circ *$ we find, $\forall x \in U_h\mathbf{g}$

$$\mathcal{F}\Delta[\varphi_h(x^{*_h})]\mathcal{F}^{-1} = \mathcal{F}^{-1\ *\otimes *}\Delta[\varphi_h(x^{*_h})]\mathcal{F}^{\ *\otimes *} \quad \Rightarrow \quad \left[\mathcal{F}^{\ *\otimes *}\mathcal{F}, \Delta[\varphi_h(x^{*_h})]\right] = 0$$

so that $\mathcal{F}^{\ *\otimes *}\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + O(h)$ is $\mathbf{g}$-invariant. Performing the transformation (2.1.12) with $T = (\mathcal{F}^{\ *\otimes *}\mathcal{F})^{-\frac{1}{2}}$ one gets a unitary $\mathcal{F}$.

If $\mathcal{F}$ is unitary then the corresponding $\gamma, \gamma', \phi$ clearly satisfy

$$\gamma' = \gamma^* \qquad\qquad \phi^{\ *\otimes *\otimes *} = \phi^{-1}. \tag{6.4}$$

On the other hand, it is evident that the 'minimal' coassociator $\phi_m$ (2.1.18) is also unitary (because $h \in \mathbf{R}$); one could actually show that the unitary $\mathcal{F}$ is also minimal.

If $\rho_h$ is a $*$-representation of H, the $*$-structure $(\tilde{A}^i)^{\dagger_h} = \tilde{A}_i^+$ is clearly compatible with $\tilde{\triangleright}_h$ [condition (6.1)]; the classical counterpart of $\rho_h$ is also a $*$-representation $\rho$ of $H$ (*i.e.* $\rho(x^*) = \overline{\rho^T(x)}$), and formula

$$(a^i)^\dagger = a_i^+ \tag{6.5}$$

defines in $\mathcal{A}_{\pm,\mathbf{g},\rho}$ a $*$-structure ('hermitean conjugation') $^\dagger$ compatible with $\triangleright$. Correspondingly, it is immediate to check that $\sigma$, $\sigma_{\varphi_h}$ become $*, *_h$-homomorphisms respectively,

$$\sigma \circ * = \dagger \circ \sigma \qquad\qquad \sigma_{\varphi_h} \circ *_h = \dagger \circ \sigma_{\varphi_h}, \tag{6.6}$$

and $\triangleright_h$ as defined in formula (3.2) also satisfies (6.1). Under $\dagger$ the RHS of relations (3.11), (3.12) are mapped into the RHS of relations (3.13), (3.14), provided that

$$(\mathsf{a}^i)^\dagger = \mathsf{a}_i^+; \tag{6.7}$$

in this case we find, as requested

$$(A^i)^\dagger = A_i^+. \tag{6.8}$$

If $\mathbf{g} = sl(N), so(N)$ and $\rho = \rho_d$ condition (6.7) is satisfied by choosing

$$v^{-1} = u = \begin{cases} \sqrt{y_{sl(N)}} & \text{if } \mathbf{g} = \text{sl(n)} \\ \sqrt{y_{so(N)}} & \text{if } \mathbf{g} = \text{so(n)} \end{cases} \tag{6.9}$$

$\mathcal{A}_{+,so(N),\rho_d}$ admits also an alternative $*$-structure compatible with $\triangleright_h$, namely $(\tilde{A}_i^+)^\dagger = \tilde{A}_j^+ C^{ji}$ [15] together with a nonlinear equation for $(\tilde{A}^i)^{\dagger_h}$ [27] which we omit

here; in this case one usually denotes the generators by $X_i, \partial^i$ instead of $\tilde{A}_i^+, \tilde{A}^i$, because in the classical limit they become the Cartesian coordinates and partial derivatives of the $N$-dim Euclidean space respectively. The classical limit of this $\dagger_h$ is

$$(a_i^+)^\dagger = a_j^+ c_{ji} \qquad\qquad (a^i)^\dagger = -c_{ij} a^j; \tag{6.10}$$

using relations (6.10), (4.11), $tr(\rho_d) = 0$, $Sx = -x$ if $\in g$, one finds again relations (6.6). $\triangleright_h$ as defined in formula (3.2) also satisfies (6.1). Under $\dagger$ the RHS of relation (3.11) is mapped into the RHS of relations (3.12), provided that $(\mathsf{a}_i^+)^\dagger = \mathsf{a}_j^+ c_{ji}$, *i.e.*

$$v = 1 \qquad\qquad u = y_{so(N)}; \tag{6.11}$$

in this case we find, as requested

$$(A_i^+)^\dagger = A_j^+ C_{ji}, \tag{6.12}$$

and it is not difficult to show that $(A^i)^\dagger$ is the (nonlinear) function of $A^i, A_i^+$ which was found in Ref. [27].

# 7 Summary and conclusions

Given some solutions $\mathsf{a}^i, \mathsf{a}_i^+$ [in the form (2.2.11)] of equations (5.0.1-5.0.3), the $A^i, A_i^+$ defined through formulae (3.3) (where we choose a minimal $\mathcal{F}$) satisfy the quantum commutation relations of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ and are covariant under the $U_h\mathbf{g}$ action $\triangleright_h$ defined in formula (3.2). The basic algebra homomorphism $f : \mathcal{A}^h_{\pm,\mathbf{g},\rho} \to \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ is defined iteratively starting from $f(\tilde{A}^i) := A^i$, $f(\tilde{A}_i^+) := A_i^+$. Explicit solutions $\mathsf{a}^i, \mathsf{a}_i^+$ of equations (5.0.1-5.0.3) are given by

- formulae (5.1.8), (5.1.9) for $\mathcal{A}^h_{\pm,sl(N),\rho_d}$;
- formulae (5.2.10), (5.2.13) for $\mathcal{A}^h_{+,so(N),\rho_d}$.

According to relation (1.1), all other elements of $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ satisfying the QCR of $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ can be written in the form

$$A^{\alpha i} = \alpha\, A^i \alpha^{-1} \qquad\qquad A^+_{\alpha i} = \alpha\, A_i^+\, \alpha^{-1}, \tag{7.1}$$

with $\alpha = \mathbf{1}_{\mathcal{A}} + O(h) \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$. They are manifestly covariant under the $U_h\mathbf{g}$ - action $\triangleright_{h\,\alpha}$ defined by

$$x \triangleright_{h\ \alpha} a \ := \ \sigma^\alpha_{\varphi_h}(x_{(\bar{1})})\, a\, \sigma^\alpha_{\varphi_h}(x_{(\bar{2})}), \tag{7.2}$$

where $\sigma^{\alpha}_{\phi}$ is the algebra homomorphism $U_h\mathbf{g} \to \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ defined by

$$\sigma^{\alpha}_{\varphi_h}(x) := [\sigma^{\alpha} \circ \varphi_h](x) \overset{(2.2.12)}{=} \alpha\, \sigma[\varphi_h(x)]\, \alpha^{-1}. \tag{7.3}$$

In this way we have found all possible pairs $(f_\alpha, \triangleright_{h\,\alpha})$ making the diagram (1.2) in the introduction commutative.

Note that the change $\varphi_h \to \varphi_{h,v} = v\, \varphi_h(\cdot)\, v^{-1}$ [formula (2.1.25)] of the algebra isomorphism $U_h\mathbf{g} \to U\mathbf{g}\,[[h]]$ amounts to the particular transformation $(f, \triangleright_h) \to (f_\alpha, \triangleright_{h\,\alpha})$, with $\alpha = \sigma(v)$.

In Sect 4 we have shown formula (4.3)] how to construct $\mathbf{g}$-invariants $I^n_h \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ in the form of homogeneous polynomials in $A^i, A^+_i$. It is immediate to verify that under a transformation $(f, \triangleright_h) \to (f_\alpha, \triangleright_{h\,\alpha})$ these $I^n_h$ transform into $I^{\alpha\,n}_h := \alpha\, I^n_h\, \alpha^{-1}$.

In Sect. 6 we have shown (sticking to the explicit case of $\mathcal{A}^h_{\pm,sl(N),\rho_d}$ and $\mathcal{A}^h_{\pm,sl(N),\rho_d}$) that, if $\mathcal{A}^h_{\pm,\mathbf{g},\rho}$ is a module $*$-algebra [formula (6.1)] of the compact section of $U_h\mathbf{g}$ $(q > 1)$, then one can choose $(f, \triangleright_h)$ so that $f$ is a $*$-homorphism, $f(b^{\dagger_h}) = [f(b)]^\dagger$, and $\triangleright_h$ also satisfies equation (6.1). It is straighforward to verify that $(f_\alpha, \triangleright_{h\,\alpha})$ satisfy the same constraints provided that $\alpha$ is "unitary":

$$\alpha^{\dagger_h} = \alpha^{-1}. \tag{7.4}$$

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# A Appendix

## A.1 Proof of Lemma 1

We start by observing that

**Lemma 2** *[12] If $\mathcal{T} \in U\mathbf{g}\,[[h]]^{\otimes^3}$ is $\mathbf{g}$-invariant (i.e. $[\mathcal{T}, U\mathbf{g}\,[[h]]^{\otimes^3}] = 0$) then $m_{ij}S_i\mathcal{T}$, $m_{ij}S_j\mathcal{T}$ $(i,j = 1,2,3,\ i \neq j)$ are $\mathbf{g}$-invariants belonging to $U\mathbf{g}\,[[h]]^{\otimes^2}$.*

(Here $S_i$ denotes $S$ acting on the $i$-th tensor factor, and $m_{ij}$ multiplication of the $i$-th tensor factor by the $j$-th from the right.)

*Proof.* For instance,

$$\mathcal{T}^{(1)}x_{(1)} \otimes \mathcal{T}^{(2)}x_{(2)} \otimes \mathcal{T}^{(3)}x_{(3)} \otimes x_{(4)} = x_{(1)}\mathcal{T}^{(1)} \otimes x_{(2)}\mathcal{T}^{(2)} \otimes x_{(3)}\mathcal{T}^{(3)} \otimes x_{(4)} \qquad \Rightarrow$$

$$\overset{(m_{23})^2 \circ S_3}{\Longrightarrow} \qquad \mathcal{T}^{(1)}x_{(1)} \otimes \mathcal{T}^{(2)}x_{(2)}Sx_{(3)}S\mathcal{T}^{(3)}x_{(4)} = x_{(1)}\mathcal{T}^{(1)} \otimes x_{(2)}\mathcal{T}^{(2)}S\mathcal{T}^{(3)}Sx_{(3)}x_{(4)}$$

for any $x \in U\mathbf{g}\,[[h]]$, whence (because of $a_{(1)}\,Sa_{(2)} = \varepsilon(a) = Sa_{(1)}a_{(2)}$)

$$\mathcal{T}^{(1)}x_{(1)} \otimes \mathcal{T}^{(2)}S\mathcal{T}^{(3)}x_{(2)} = x_{(1)}\mathcal{T}^{(1)} \otimes x_{(2)}\mathcal{T}^{(2)}S\mathcal{T}^{(3)}, \tag{A.1.1}$$

so that $\mathcal{T}^{(1)} \otimes \mathcal{T}^{(2)}S\mathcal{T}^{(3)} \in U\mathbf{g}\,[[h]]^{\otimes^2}$ is $\mathbf{g}$-invariant. □

We may apply the previous lemma to $\mathcal{T} = \phi$, or $\mathcal{T} = \phi^{-1}$. Looking at the definition (2.1.9) one finds in particular the following $\mathbf{g}$-invariants

$$\begin{aligned} T_1 &:= m_{12}S_1\phi = (S\mathcal{F}^{(1)}\gamma \otimes \mathbf{1})\mathcal{F}(\mathcal{F}^{(2)}_{(1)} \otimes \mathcal{F}^{(2)}_{(2)}), \\ T_2 &:= m_{23}S_3\phi = (\mathcal{F}^{-1(1)}_{(1)} \otimes \mathcal{F}^{-1(1)}_{(2)})\mathcal{F}^{-1}(\mathbf{1} \otimes \gamma^{-1}S\mathcal{F}^{-1(2)}); \end{aligned} \tag{A.1.2}$$

alternative expressions for these $T_i$ can be obtained by applying the same operations to the identities

$$\phi q^{\frac{t_{12}+t_{13}}{2}} = q^{\frac{t_{12}}{2}}\mathcal{F}^{-1}_{23,1}\mathcal{F}^{-1}_{12}\mathcal{R}_{13}\mathcal{F}_{32}\mathcal{F}_{1,23}, \tag{A.1.3}$$

$$q^{-\frac{t_{13}+t_{23}}{2}}\phi = \mathcal{F}^{-1}_{12,3}\mathcal{F}^{-1}_{21}\mathcal{R}^{-1}_{13}\mathcal{F}_{23}\mathcal{F}_{3,12}q^{-\frac{t_{23}}{2}}, \tag{A.1.4}$$

which directly follow from relations (2.1.17), (2.1.9), (2.1.11) and the observation that $[\phi, q^{\frac{t_{12}+t_{13}+t_{23}}{2}}] = 0$. Applying $m_{12}S_1$ to (A.1.4), $m_{23}S_3$ to (A.1.3) we get

$$\begin{aligned} T_1 &= (S\mathcal{F}^{(2)}\gamma'^{-1} \otimes \mathbf{1})\mathcal{F}_{21}(\mathcal{F}^{(1)}_{(1)} \otimes \mathcal{F}^{(1)}_{(2)}), \\ T_2 &= (\mathcal{F}^{-1(2)}_{(1)} \otimes \mathcal{F}^{-1(2)}_{(2)})\mathcal{F}^{-1}_{21}(\mathbf{1} \otimes \gamma' S\mathcal{F}^{-1(1)}). \end{aligned} \tag{A.1.5}$$

¿From eq. (A.1.2), (A.1.5) we easily find out that the inverse of $T_i$ take the form

$$T_1^{-1} = \mathcal{F}^{-1}\left[\gamma^{-1}(S\mathcal{F}^{-1(1)})\mathcal{F}^{-1(2)}_{(1)} \otimes \mathcal{F}^{-1(2)}_{(2)}\right] \tag{A.1.6}$$

$$= \mathcal{F}^{-1}_{21}\left[\gamma'(S\mathcal{F}^{-1(2)})\mathcal{F}^{-1(1)}_{(1)} \otimes \mathcal{F}^{-1(1)}_{(2)}\right], \tag{A.1.7}$$

$$T_2^{-1} = \left[\mathcal{F}^{(1)}_{(1)} \otimes \mathcal{F}^{(1)}_{(2)}(S\mathcal{F}^{(2)})\gamma\right]\mathcal{F} \tag{A.1.8}$$

$$= \left[\mathcal{F}^{(2)}_{(1)} \otimes \mathcal{F}^{(2)}_{(2)}(S\mathcal{F}^{(1)})\gamma'^{-1}\right]\mathcal{F}_{21}, \tag{A.1.9}$$

since $[T_i, \mathcal{F}^{\pm 1(j)}_{(1)} \otimes \mathcal{F}^{\pm 1(j)}_{(2)}] = 0$, with $i, j = 1, 2$.

If, by making a transformation (2.1.12), we arrive at a $\mathcal{F}$ reducing $\phi$ to the form $\phi_m$ (2.1.18), then it is easy to verify that, according to their definitions (A.1.2), $T_i \equiv \mathbf{1}^{\otimes^3}$. In the latter case the last four relations are equivalent to relations (3.7-dritto4).

## A.2 Proof of Proposition 6

We start by expressing $(A...A)^{I_n} \equiv A^{i_1}...A^{i_{h_n}}$, $(A^+...A^+)_{J_n} \equiv A^+_{j_1}...A^+_{j_{k_n}}$ respectively in the form $(\mathsf{a}...\mathsf{a})\sigma(\cdot)$, $(\mathsf{a}^+...\mathsf{a}^+)\sigma(\cdot)$. First note that

$$\begin{aligned} A^{i_1}A^{i_2} &\overset{(3.14)}{=} \rho(\gamma^{-1}\mathcal{F}^{-1(1)})^{i_1}_{l_1}\rho(\gamma^{-1}\mathcal{F}^{-1(1')})^{i_2}_{l_2}\mathsf{a}^{l_1}\sigma(\mathcal{F}^{-1(2)})\mathsf{a}^{l_2}\sigma(\mathcal{F}^{-1(2')}) \\ &= \rho^\vee(\mathcal{F}^{-1(1)}\gamma')^{l_1}_{i_1}\rho^\vee(\mathcal{F}^{-1(1')}\gamma')^{l_2}_{i_2}\mathsf{a}^{l_1}\sigma(\mathcal{F}^{-1(2)})\mathsf{a}^{l_2}\sigma(\mathcal{F}^{-1(2')}) \\ &\overset{(2.2.6,2.2.2)}{=} \rho^\vee(\mathcal{F}^{-1(1)}\gamma')^{l_1}_{i_1}\rho^\vee(\mathcal{F}^{-1(2)}_{(1)}\mathcal{F}^{-1(1')}\gamma')^{l_2}_{i_2}\mathsf{a}^{l_1}\mathsf{a}^{l_2}\sigma(\mathcal{F}^{-1(2)}_{(2)}\mathcal{F}^{-1(2')}) \\ &= \mathsf{a}^{l_1}\mathsf{a}^{l_2}\left[\left(\rho^{\vee l_1}_{i_1} \otimes \rho^{\vee l_2}_{i_2} \otimes \sigma\right)\left(\mathcal{F}^{-1}_{1,23}\mathcal{F}^{-1}_{23}(\gamma'^{\otimes^2} \otimes \mathbf{1})\right)\right] \end{aligned}$$

whence, by repeated application, we find

$$(A...A)^{I_n} \overset{(4.7)}{=} (\mathsf{a}...\mathsf{a})^{L_n}\left[\left((\rho^{\vee\otimes^{h_n}})^{L_n}_{I_n}\otimes\sigma\right)\left(\mathcal{F}^{-1}_{12...(h_n+1)}(\gamma'^{\otimes^{h_n}}\otimes\mathbf{1})\right)\right];$$

similarly, starting from relation (3.11) we find

$$(A^+...A^+)_{J_n} = (\mathsf{a}^+...\mathsf{a}^+)_{M_n}\left[\left((\rho^{\otimes^{k_n}})^{M_n}_{J_n}\otimes\sigma\right)\left(\mathcal{F}^{-1}_{12...(k_n+1)}\right)\right].$$

Putting these results together we find

$$(A^+...A^+)_{J_n}(A...A)^{I_n} \overset{(2.2.2),(2.2.6)}{=}$$

$$(\mathsf{a}^+...\mathsf{a}^+)_{M_n}(\mathsf{a}...\mathsf{a})^{L_n}\left[\left((\rho^{\otimes^{k_n}})^{M_n}_{J_n}\otimes(\rho^{\vee\otimes^{h_n}})^{L_n}_{I_n}\otimes\sigma\right)\left(\mathcal{F}^{-1}_{12...(b_n+1)}(\mathbf{1}^{\otimes^{k_n}}\otimes\gamma'^{\otimes^{h_n}}\otimes\mathbf{1})\right)\right],$$

whence,

$$I^n_h \overset{(4.6)}{\propto} (\mathsf{a}^+...\mathsf{a}^+)_{M_n}(\mathsf{a}...\mathsf{a})^{L_n}\left[\left((\rho^{\otimes^{k_n}})^{M_n}_{J_n}\otimes(\rho^{\vee\otimes^{h_n}})^{L_n}_{I_n}\otimes\sigma\right)\mathcal{F}^{-1}_{12...(b_n+1)}\mathcal{F}_{12...b_n,b_n+1}\right]d^{L_n}_{M_n}. \tag{A.2.1}$$

We prove now that

$$\mathcal{F}^{-1}_{12...(b+1)}\mathcal{F}_{12...b} = \phi^{-1}_{(b-1)b(b+1)}\phi^{-1}_{(b-2),(b-1)b,(b+1)}\cdots\phi^{-1}_{1,2...b,(b+1)}\mathcal{F}^{-1}_{12...m,m+1}; \tag{A.2.2}$$

then the claim will follow from relation (A.2.1) and the observation that

$$\left[\left(\rho^{\otimes^{k_n}}\otimes\rho^{\vee\otimes^{h_n}}\otimes\mathrm{id}\right)(\mathcal{F}_{12...b_n,b_n+1})\right]^{M_nL_n}_{J_nI_n}d^{L_n}_{M_n} \overset{(4.4)}{=} \varepsilon(\mathcal{F}^{(1)})\mathcal{F}^{(2)}d^{I_n}_{J_n} \overset{(2.1.1)}{=} d^{I_n}_{J_n}. \tag{A.2.3}$$

To prove relation (A.2.2) we start from

$$\phi^{-1}_{123}\mathcal{F}^{-1}_{12,3} \overset{(2.1.9)}{=} \mathcal{F}^{-1}_{1,23}\mathcal{F}^{-1}_{23}\mathcal{F}_{12} \overset{(4.7)}{=} \mathcal{F}^{-1}_{123}\mathcal{F}_{12};$$

this is relation (A.2.2) for $b=2$. Applying $\mathrm{id}\otimes\Delta\otimes\mathrm{id}$ and multiplying the result from the left by $\phi^{-1}_{234}$ we find

$$\begin{aligned}\phi^{-1}_{234}\phi^{-1}_{1,23,4}\mathcal{F}^{-1}_{123,4} &= \phi^{-1}_{234}\mathcal{F}^{-1}_{1,234}\mathcal{F}^{-1}_{2,34}\mathcal{F}_{1,23}\\ &\overset{(2.1.10)}{=} \mathcal{F}^{-1}_{1,234}\phi^{-1}_{234}\mathcal{F}^{-1}_{2,34}\mathcal{F}_{1,23}\\ &\overset{(2.1.9)}{=} \mathcal{F}^{-1}_{1,234}\mathcal{F}^{-1}_{2,34}\mathcal{F}^{-1}_{34}\mathcal{F}_{23}\mathcal{F}_{1,23},\\ &\overset{(4.7)}{=} \mathcal{F}^{-1}_{1233}\mathcal{F}_{123},\end{aligned}$$

*i.e.* relation (A.2.2) for $b=3$. Applying to the latter relation $\mathrm{id}\otimes\Delta^{(2)}\otimes\mathrm{id}$ and multiplying the result from the left by $\phi^{-1}_{345}$ we find relation (A.2.2) for $b=4$, and so on □.

## A.3 Proof of Proposition 7

$$
\begin{aligned}
0 &\overset{(2.3.1)}{=} A^i A^j \mp P^{F\,ji}_{\;\;hk} A^k A^h \\
&\overset{(3.13)}{=} (\mathbf{1} \mp P^F)^{ji}_{hk} \rho(\mathcal{F}^{(1)})^h_l \rho(\mathcal{F}^{(1')})^k_m \sigma(\mathcal{F}^{(2)}) \mathsf{a}^m \sigma(\mathcal{F}^{(2')}) \mathsf{a}^l \\
&\overset{(2.2.6),(2.2.2)}{=} (\mathbf{1} \mp P^F)^{ji}_{hk} \rho(\mathcal{F}^{(1)} \mathcal{F}^{(2')}_{(2')})^k_m \rho(\mathcal{F}^{(1)})^h_l \sigma(\mathcal{F}^{(2)}_{(1)} \mathcal{F}^{(2')}_{(1')}) \mathsf{a}^m \mathsf{a}^l;
\end{aligned}
$$

multiplying both sides from the left by $(\rho \otimes \rho \otimes \sigma)(\mathcal{F}^{-1}_{1,23} \mathcal{F}^{-1}_{23})$ and noting that

$$
P^F{}_{12} \overset{(2.3.5),(2.3.9)}{=} [(\rho \otimes \rho \otimes \sigma) \mathcal{F}_{12} \mathcal{F}_{12,3}] U_{12} [(\rho \otimes \rho \otimes \sigma) \mathcal{F}^{-1}_{12,3} \mathcal{F}^{-1}_{12}],
$$

we find

$$
\{\mathbf{1}^{\otimes 3} \mp [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}^{-1}_{1,23} \mathcal{F}^{-1}_{23} \mathcal{F}_{12} \mathcal{F}_{12,3}] U_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}^{-1}_{12,3} \mathcal{F}^{-1}_{12} \mathcal{F}_{23} \mathcal{F}_{1,23})]\}^{hk}_{lm} \mathsf{a}^m \mathsf{a}^l = 0,
$$

*i.e.* relation (5.0.1), once we take definitions (5.0.4), (2.1.9) into account. Using definition (3.11) one can prove in a similar way that relations (2.3.2), (5.0.2) are equivalent. Similarly,

$$
\begin{aligned}
0 &\overset{(2.3.3)}{=} A^i \tilde{A}^+_j - \delta^i_j \mathbf{1}_{\mathcal{A}} \mp \tilde{P}^{F\;ih}_{\;\;jk} A^+_h A^k \\
&\overset{(3.13),(3.11)}{=} \rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)}) \mathsf{a}^l \mathsf{a}^+_m \sigma(\mathcal{F}^{-1(2')}) \rho(\mathcal{F}^{-1(1')})^m_j \\
&\qquad - \delta^i_j \mathbf{1}_{\mathcal{A}} \mp \mathsf{a}^+_m \sigma(\mathcal{F}^{-1(2)}) \rho(\mathcal{F}^{-1(1)})^m_h \tilde{P}^{F\;ih}_{\;\;jk} \rho(\mathcal{F}^{(1')})^k_l \sigma(\mathcal{F}^{(2)}) \mathsf{a}^l;
\end{aligned}
$$

multiplying both sides by $\rho(\mathcal{F}^{-1(1)})^{i'}_i \sigma(\mathcal{F}^{-1(2)})$ from the left and by $\rho(\mathcal{F}^{(1')})^j_{j'} \sigma(\mathcal{F}^{(2')})$ from the right, and noting that

$$
\tilde{P}^F_{12} \overset{(2.3.6),(2.3.9)}{=} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{12} \mathcal{F}_{12,3}] V_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}^{-1}_{12,3} \mathcal{F}^{-1}_{12}],
$$

we get

$$
\begin{aligned}
0 &= \mathsf{a}^{i'} \mathsf{a}^+_{j'} - \delta^{i'}_{j'} \mathbf{1}_{\mathcal{A}} \mp \rho(\mathcal{F}^{-1(1)})^{i'}_i \sigma(\mathcal{F}^{-1(2)}) \mathsf{a}^+_m \Big\{ [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}^{-1}_{13} \mathcal{F}_{12}\, \mathcal{F}_{12,3}] \\
&\qquad \times V_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}^{-1}_{12,3} \mathcal{F}^{-1}_{12} \mathcal{F}_{13}] \Big\}^{im}_{jl} \mathsf{a}^l \rho(\mathcal{F}^{(1')})^j_{j'} \sigma(\mathcal{F}^{(2')}) \\
&\overset{(2.2.6),(2.2.2)}{=} \mathsf{a}^{i'} \mathsf{a}^+_{j'} - \delta^{i'}_{j'} \mathbf{1}_{\mathcal{A}} \mp \mathsf{a}^+_m \Big\{ [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}^{-1}_{1,23} \mathcal{F}^{-1}_{13} \mathcal{F}_{12} \mathcal{F}_{12,3}] \\
&\qquad \times V_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}^{-1}_{12,3} \mathcal{F}^{-1}_{12} \mathcal{F}_{13} F_{1,23}] \Big\}^{i'm}_{j'l} \mathsf{a}^l,
\end{aligned}
$$

whence the equivalence between relations (2.3.3), (5.0.3) follows, once one recalls the definition (2.1.9). □

## A.4 Some properties of special and q-special functions

We collect here some properties of the hypergeometric, $\Gamma$ and $\beta$ functions which can be found in standard textbooks. If the parameters $a, b, c \in \mathbf{C}$ are such that none of the quantities $c-1, a-b, a+b-c$ is a positive integer, the general solution of the hypergeometric differential equation in the complex $z$-plane

$$y''(1-z)z + y'[c-(a+b+1)z] - yab = 0 \tag{A.4.1}$$

can be expressed as some combinations

$$\begin{aligned} y(z) &= \alpha\, F(a,b,c;z) + \beta\, z^{1-c} F(1+a-c, 1+b-c, 2-c; z), && \text{(A.4.2)} \\ &= \gamma\, F(a,b,a+b+1-c;1-z) \\ &\quad +\delta\,(1-z)^{c-a-b} F(c-a, c-b, c+1-a-b; 1-z), && \text{(A.4.3)} \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ and $F(a,b,c;z)$ is the hypergeometric function

$$F(a,b,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \qquad (s)_k := s(s+1)...(s+k-1). \tag{A.4.4}$$

¿From this definition it follows that

$$F(a,b,c;0) = 1. \tag{A.4.5}$$

The combinations (A.4.2), (A.4.3) explicitly display the singular and non-singular part of the solution respectively around the poles $x = 0, 1$. From the above definition it immediately follows the property

$$\frac{d}{dz} F(a,b,c;z) = \frac{a\,b}{c} F(a+1, b+1, c+1; z). \tag{A.4.6}$$

An essential identity to determine the asymptotic behaviour of a solution $y$ around the pole $x = 0$ (resp. $x = 1$), known its asymptotic behaviour around the pole $x = 1$ (resp. $x = 0$), is

$$\begin{aligned} F(a,b,c;z) &= \frac{B(c, c-a-b)}{B(c-a, c-b)} F(a,b,a+b+1-c;1-z) && \text{(A.4.7)} \\ &+ \frac{B(c, a+b-c)}{B(a,b)} (1-z)^{c-a-b} F(c-a, c-b, c+1-a-b; 1-z); \end{aligned}$$

Here $B(a,b)$ is Euler's $\beta$-function, which can be expressed as a ratio of Euler's $\Gamma$-functions as follows

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{A.4.8}$$

$\Gamma$ can be defined by

$$\Gamma(a+1) := \int_0^\infty dt\, e^{-t} t^a, \tag{A.4.9}$$

whence it immediately follows that

$$\Gamma(a+1) = a\Gamma(a). \tag{A.4.10}$$

A less obvious property is

$$\Gamma(a)\Gamma(-a) = -\frac{\pi}{a \sin \pi a}. \tag{A.4.11}$$

The $q$-gamma function $\Gamma_q$ can be defined when $|q| < 1$ by [16]

$$\Gamma_q(a) := (1 - q^{1-a}) \prod_{k=0}^{\infty} \frac{(1 - q^{k+1})}{(1 - q^{a+k})} = (1 - q^{1-a}) \sum_{n=0}^{\infty} \frac{(q^{1-a}; q)_n}{(q^a; q)_n} q^{na}, \tag{A.4.12}$$

where $(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k)$; it satisfies the following modified version of the property (A.4.10):

$$\Gamma_q(a+1) = (a)_q \Gamma_q(a), \qquad (a)_q := \frac{(q^a - 1)}{(q - 1)}. \tag{A.4.13}$$

We introduce also a different version of the $q$-gamma function by

$$\tilde{\Gamma}_q(a) := \Gamma_{q^2}(a) q^{-\frac{a(a-3)}{2}} 0; \tag{A.4.14}$$

the latter satisfies

$$\tilde{\Gamma}_q(a+1) = [a]_q \tilde{\Gamma}_q(a), \qquad [a]_q := \frac{(q^a - q^{-a})}{(q - q^{-1})}. \tag{A.4.15}$$

## A.5 Proof of Theorem 1

*Proof.* We need to show that equations (5.0.1-5.0.3) are fulfilled. To make computations more expedite we get rid of indices by introducing the following vector notation:

$$\begin{aligned}
(aa)^{ij} &:= a^i a^j & (a^+ a^+)_{ij} &:= a_i^+ a_j^+ \\
(av)^{ij} &:= a^i v^j & (va)^{ij} &:= v^i a^j \\
(a^+ w)_{ij} &:= a_i^+ w_j & (wa^+)_{ij} &:= w_i a_j^+ \\
w \cdot a &:= w_i a^i & a^+ \cdot v &:= a_i^+ v^i,
\end{aligned} \tag{A.5.1}$$

where $v \equiv (v^i) \in \mathbf{C}^N$, $w \equiv (w_i) \in \mathbf{C}^N$ denote arbitrary covariant and controvariant vectors respectively. If we plug (5.1.7) into (5.0.1-5.0.3), factor out of (5.0.1) and (5.0.2) $I(n)\,I(n{+}1)$ and $\tilde{I}(n)\,\tilde{I}(n{+}1)$ respectively and multiply eq. (5.0.3) by $v^j w_i$, we find the equivalent system (in vector notation)

$$aa = \pm\,(M^{-1}PM)aa \tag{A.5.2}$$

$$a^+a^+ = \pm\,a^+a^+\,(M^{-1}PM) \tag{A.5.3}$$

$$\frac{(n{+}1)_{q^{\pm 2}}}{(n{+}1)}(w{\cdot}a)\cdot(a^+{\cdot}v) = w{\cdot}v\,\mathbf{1}_{\mathcal{A}} \pm q^{\pm 1}\frac{n_{q^{\pm 2}}}{n}\,wa^+\,(M^{-1}\,V\,M)\,va. \tag{A.5.4}$$

The action of $A, B, P$ on $av, va, aa, a^+w, wa^+, a^+a^+$ is easily found to be

$$\begin{aligned}
A\,va &= \pm\,(n{-}1)\,va & wa^+\,A &= \pm\,(n{-}1)\,wa^+\\
A\,aa &= \pm\,(n{-}2)\,aa & a^+a^+\,A &= \pm\,(n{-}2)\,a^+a^+\\
A\,av &= -va \mp av \pm aa\,(a^+{\cdot}v) & a^+w\,A &= -wa^+ \mp a^+w + (w{\cdot}a)\,a^+a^+\\
B\,av &= \pm\,(n{-}1)\,av & a^+w\,B &= \pm\,(n{-}1)\,a^+w\\
B\,aa &= \pm\,(n{-}2)\,aa & a^+a^+\,B &= \pm\,(n{-}2)\,a^+a^+\\
B\,va &= -av \mp va + aa\,(a^+{\cdot}v) & wa^+\,B &= -a^+w \mp wa^+ \pm (w{\cdot}a)\,a^+a^+\\
P\,va &= av & wa^+\,P &= a^+w\\
P\,av &= va & a^+w\,P &= wa^+\\
P\,aa &= \pm\,aa & a^+a^+\,P &= \pm\,a^+a^+.
\end{aligned} \tag{A.5.5}$$

As a consequence one finds, in particular,

$$\begin{aligned}
a^+w\,(A{+}P)^k &= (\mp 1)^k a^+w \pm \frac{(\pm n)^k - (\mp 1)^k}{n+1}\,(w{\cdot}a)\,a^+a^+ \qquad \Rightarrow\\
\Rightarrow \qquad a^+w\,q^{A+P} &= q^{\pm 1}a^+w \pm \frac{q^{\mp n} - q^{\mp 1}}{n+1}\,(w{\cdot}a)\,a^+a^+
\end{aligned} \tag{A.5.6}$$

Let us prove eq. (A.5.2), (A.5.3). The matrix $M$ (5.0.4) takes the form

$$M \overset{(2.1.24)}{=} \lim_{x_0,y_0\to 0^+}\left\{x_0^{-2\hbar P}\vec{P}\exp\left[-2\hbar\int_{x_0}^{1-y_0}dx\left(\frac{P}{x}+\frac{A}{x-1}\right)\right]y_0^{2\hbar A}\right\}; \tag{A.5.7}$$

the contributions of the central terms $-2\frac{\mathbf{1}^{\otimes^3}}{N}$, $-\frac{2}{N}\mathbf{1}^{\otimes^3}\otimes n$ to the integral are cancelled by the corresponding contributions from $x_0^{-\hbar t_{12}}$, $y_0^{\hbar t_{23}}$, in the limit $x_0, y_0 \to 0^+$. Since $aa$ is an 'eigenvector' both of $A$ and $P$, the path-order $\vec{P}$ becomes redundant and we find that $M$ acts trivially on $M$:

$$Maa = aa\lim_{x_0,y_0\to 0^+}\left\{x_0^{\mp 2\hbar}\exp\left[-2\hbar\int_{x_0}^{1-y_0}dx\left(\pm\frac{1}{x}\pm\frac{n-2}{x-1}\right)\right]y_0^{\pm 2\hbar(n-2)}\right\}$$

$$= \quad aa \lim_{x_0,y_0\to 0^+} (1-y_0)^{\mp 2\hbar}(1-x_0)^{\pm 2\hbar(n-2)} = aa. \tag{A.5.8}$$

Therefore $MPM\,aa = \pm\, aa$, Q.E.D. Similarly one proves eq. (A.5.3).

In order to prove eq. (A.5.4) it is convenient to recast $M^{-1}\,V\,M$ in a more manageable form. Permuting the second and third tensor factor in eq. $(2.1.17)_{(1)}$, we find

$$\phi_{213}^{-1} q^{\frac{t_{12}}{2}} \phi_{123} = q^{\frac{t_{12}+t_{13}}{2}} \phi_{132} q^{-\frac{t_{23}}{2}}, \tag{A.5.9}$$

whence

$$M^{-1}\,V\,M = PM_{21}^{-1} q^{\pm 1+\frac{1}{N}+\rho_d^{\otimes 2}(\frac{t}{2})} M \overset{(2.1.21),(5.1.6)}{=} Pq^{A+P} \lim_{x\to 0^+} x^{-2B} f(x) q^{-A}, \tag{A.5.10}$$

where $f$ is the $\sigma(Usl(N))[[h]]$-valued $N^2 \times N^2$ matrix satisfying the differential equation and asymptotic conditions

$$f' = 2\hbar\left(\frac{B}{x} + \frac{A}{x-1}\right) f \qquad\qquad \lim_{x\to 1} f(x)(1-x)^{-2A} = 1 \tag{A.5.11}$$

[the latter are obtained from eq. (2.1.19) by permuting the second and third tensor factor and by getting rid of the central terms involved in $(\rho\otimes\rho\otimes\sigma)(t_{ij})$ (formulae (5.1.6)) since, as in formula (A.5.7), the latter cancel with each other in the limit $x \to 0$].

It is convenient to introduce in $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ a grading $g$, by setting $g(b) = l \in \mathbf{Z}$ iff $[n, b] = lb$, $b \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$. Since $g(fva) = -1$, and $fv^i a^j$ is a doubly contravariant tensor, its most general expansion is

$$f(x)va = avf_1(x) + vaf_2(x) + aa(a^+\cdot v)f_3(x), \tag{A.5.12}$$

where $f_i$ are invariants with $g(f_i) = 0$; therefore $f_i = f_i(n)$. Thus we find

$$\begin{aligned}
& wa^+ \cdot (M^{-1}\,V\,M)\,va \\
& \overset{(A.5.5),(A.5.10)}{=} a^+ w\, q^{A+P} \lim_{x\to 0^+} x^{-2\hbar B} f(x) q^{-A}\, va \\
& \overset{(A.5.5),(A.5.6)}{=} \lim_{x\to 0^+} x^{\mp 2\hbar(n-1)} \left[q^{\mp 1} a^+ w \pm \frac{q^{\pm n} - q^{\mp 1}}{n+1}\,(w\cdot a)\, a^+ a^+\right] \cdot (f(x)\, va)\, q^{\mp(n-1)} \\
& \overset{(A.5.12)}{=} q^{\mp(n-1)} \lim_{x\to 0^+} x^{\mp 2\hbar(n-1)} \left[q^{\mp 1} a^+ w \pm \frac{q^{\pm n} - q^{\mp 1}}{n+1}(w\cdot a) a^+ a^+\right] \cdot \\
& \qquad\qquad \cdot \left[avf_1(x) + vaf_2(x) + aa(a^+\cdot v) f_3(x)\right] \\
& = q^{\mp(n-1)} \lim_{x\to 0^+} x^{\mp 2\hbar(n-1)} \left\{q^{\mp 1}(w\cdot v)(nf_1 \mp f_2) + (w\cdot a)(a^+\cdot v)\left[q^{\mp 1}(nf_3 \pm f_2)\right.\right.
\end{aligned}$$

$$+n\frac{q^{\pm n}-q^{\mp 1}}{n+1}\left(f_1 \pm f_2 + (n+1)f_3\right)\Bigg]\Bigg\}$$

$$= q^{\mp n}\left\{(w\cdot v)\, l_1 + (w\cdot a)(a^+\cdot v)\left[l_2 + n\frac{q^{\pm(n+1)}-1}{n+1}\, l_3\right]\right\}, \tag{A.5.13}$$

where we have defined

$$\begin{aligned} l_1 &:= \lim_{x\to 0^+} x^{\mp 2\hbar(n-1)}(nf_1(x) \mp f_2(x)) \\ l_2 &:= \lim_{x\to 0^+} x^{\mp 2\hbar(n-1)}(nf_3(x) \pm f_2(x)) \\ l_3 &:= \lim_{x\to 0^+} x^{\mp 2\hbar(n-1)}\left[f_1(x) \pm f_2(x) + (n+1)f_3(x)\right]. \end{aligned} \tag{A.5.14}$$

To evaluate the limits $l_i$ let us consider the linear system of first order differential equations satisfied by $f_i$. ¿From (A.5.11) we find

$$f_1' = \hbar\left[\pm\left(\frac{1}{1-x}+\frac{n-1}{x}\right)f_1 - \frac{f_2}{x}\right] \tag{A.5.15}$$

$$f_2' = \hbar\left[\frac{f_1}{1-x} \mp \left(\frac{n-1}{1-x}+\frac{1}{x}\right)f_2\right] \tag{A.5.16}$$

$$f_3' = \hbar\left[\mp\frac{f_1}{1-x} + \frac{f_2}{x} \mp \left(\frac{1}{1-x}-\frac{1}{x}\right)(n-1)f_3\right] \tag{A.5.17}$$

and the asymptotic conditions

$$\lim_{x\to 1} f_1(x) = 0 = \lim_{x\to 1} f_3(x) \qquad \lim_{x\to 1} f_2(x)(1-x)^{\mp 2\hbar(n-1)} = 1. \tag{A.5.18}$$

The first two equations can be solved separatly, since $f_3$ doesn't appear in them; then the third will yield $f_3$ in terms of $f_1, f_2$ just by an integration. Actually one of the combination we are interested in, $[f_1(x) \pm f_2(x) + (n+1)f_3(x)]$, satisfies a completely decoupled equation,

$$\frac{d}{dx}[f_1(x)\pm f_2(x)+(n+1)f_3(x)] = \pm 2\hbar(n-1)\left[\frac{1}{x}+\frac{1}{x-1}\right][f_1(x)\pm f_2(x)+(n+1)f_3(x)], \tag{A.5.19}$$

which [taking conditions (A.5.18) into account] is easily integrated to

$$f_1(x)\pm f_2(x)+(n+1)f_3(x) = \pm[x(1-x)]^{\pm 2\hbar(n-1)}. \tag{A.5.20}$$

This will yield therefore $f_3$ in terms of $f_1, f_2$.

Dividing (A.5.15) by $f_1$, (A.5.16) by $f_2$ we find

$$\frac{f_1'}{f_1} = 2\hbar\left[\pm\left(\frac{1}{1-x}+\frac{n-1}{x}\right) - \frac{1}{x}\frac{f_2}{f_1}\right] \tag{A.5.21}$$

$$\frac{f_2'}{f_2} = 2\hbar\left[\frac{1}{(1-x)}\frac{f_1}{f_2} \mp \left(\frac{n-1}{1-x}+\frac{1}{x}\right)\right] \tag{A.5.22}$$

taking the difference of the two, one finds a Riccati equation in the unknwon $u := \frac{f_1}{f_2}$:

$$\frac{u'}{u} = \frac{d}{dx}\ln(\frac{f_1}{f_2}) = \frac{f_1'}{f_1} - \frac{f_2'}{f_2} = 2\hbar\left[\pm n\left(\frac{1}{x} + \frac{1}{1-x}\right) - \frac{u^{-1}}{x} - \frac{u}{1-x}\right]; \qquad \text{(A.5.23)}$$

this should be supplemented with the condition $u \overset{x\to 1}{\to} 0$. To get rid of its nonlinearity one can transform it into a (linear) second order equation in an unknown $y(x)$ by a standard substitution, which in this case takes the form

$$u = \frac{y'}{y}\frac{(1-x)}{2\hbar}; \qquad \text{(A.5.24)}$$

the new equation will read

$$y''(1-x)x - y'(x \pm n2\hbar) + (2\hbar)^2 y = 0. \qquad \text{(A.5.25)}$$

We recognize the hypergeometric equation (see formula (A.4.1) in the Appendix) with parameters

$$a = \pm 2\hbar \qquad b = \mp 2\hbar \qquad c = \mp 2n\hbar. \qquad \text{(A.5.26)}$$

Its general solution can be expressed in the form (A.4.3), in terms of the hypergeometric function $F$. Imposing the condition $\lim_{x\to 1}\frac{y'(1-x)}{2\hbar y} = 0$ one finds that it must be $\delta = 0$, implying

$$\frac{f_1}{f_2} = u = \frac{1-x}{2\hbar}\frac{d}{dx}\ln\left[F(\pm 2\hbar, \mp 2\hbar, 1\pm 2\hbar n; 1-x)\right]. \qquad \text{(A.5.27)}$$

We can now replace this result in the RHS in eq. (A.5.22):

$$\frac{d}{dx}\ln(f_2) = \frac{d}{dx}\ln\left[F(\pm 2\hbar, \mp 2\hbar, 1\pm 2\hbar n; 1-x)\right] \mp 2\hbar\left(\frac{n-1}{1-x} + \frac{1}{x}\right); \qquad \text{(A.5.28)}$$

taking into account the condition (A.5.18), the latter is integrated to

$$f_2(x) = x^{\mp 2\hbar}(1-x)^{\pm 2\hbar(n-1)}F(\pm 2\hbar, \mp 2\hbar, 1\pm 2\hbar n; 1-x). \qquad \text{(A.5.29)}$$

Finally, we find

$$f_1(x) = u(x)f_2(x) = -\frac{1}{2\hbar}F'(\pm 2\hbar, \mp 2\hbar, 1\pm 2\hbar n; 1-x)x^{\mp 2\hbar}(1-x)^{1\pm 2\hbar(n-1)}. \qquad \text{(A.5.30)}$$

¿From properties (A.4.5),(A.4.6) we can easily read off the asymptotic behaviour of $f_2, f_1$ for $x \to 0^+$:

$$\begin{aligned} f_2(x) &\overset{(A.4.7)}{=} x^{\mp 2\hbar}(1-x)^{\pm 2\hbar(n-1)}\left[\frac{B(1\pm 2\hbar n, 1\pm 2\hbar n)}{B(1\pm 2\hbar(n+1), 1\pm 2\hbar(n-1))}F(\pm 2\hbar, \mp 2\hbar, \mp 2\hbar n; x)\right. \\ &\quad \left. + x^{1\pm 2\hbar n}\frac{B(1\pm 2\hbar n, -1\mp 2\hbar n)}{B(\pm 2\hbar, \mp 2\hbar)}F(1\pm 2(n+1)\hbar, 1\mp 2\hbar(n-1), 2\pm 2\hbar n; x)\right], \\ &\overset{(A.4.5),(A.4.8)}{\approx} x^{\mp 2\hbar}\frac{\Gamma(1\pm 2\hbar n)\Gamma(1\pm 2\hbar n)}{\Gamma(1\pm 2\hbar(n+1))\Gamma(1\pm 2\hbar(n-1))}; \end{aligned}$$

$$
\begin{aligned}
f_1(x) &\overset{(A.4.6)}{=} \tfrac{2\hbar}{1\pm 2\hbar n} F(1\pm 2\hbar, 1\mp 2\hbar, 2\pm 2\hbar n; 1-x) x^{\mp 2\hbar}(1-x)^{1\pm 2\hbar(n-1)} \\
&\overset{(A.4.7)}{=} \tfrac{2\hbar}{1\pm 2\hbar n} x^{\mp 2\hbar}(1-x)^{1\pm 2\hbar(n-1)} \Big[ \tfrac{B(2\pm 2\hbar n, \pm 2\hbar n)}{B(1\pm 2\hbar(n+1), 1\pm 2\hbar(n-1))} F(1\pm 2\hbar, 1\mp 2\hbar, 1\mp 2\hbar n; x) \\
&\quad + x^{\pm 2\hbar n} \tfrac{B(2\pm 2\hbar n, \mp 2\hbar n)}{B(1\pm 2\hbar, 1\mp 2\hbar)} F(1\pm 2(n+1)\hbar, 1\mp 2\hbar(n-1), 1\pm 2\hbar n; x) \Big], \\
&\overset{(A.4.5)}{\approx} \tfrac{2\hbar}{1\pm 2\hbar n} x^{\mp 2\hbar} \Big[ \tfrac{B(2\pm 2\hbar n, \pm 2\hbar n)}{B(1\pm 2\hbar(n+1), 1\pm 2\hbar(n-1))} + x^{\pm 2\hbar n} \tfrac{B(2\pm 2\hbar n, \mp 2\hbar n)}{B(1\pm 2\hbar, 1\mp 2\hbar)} \Big] \\
&\overset{(A.4.10),(A.4.8)}{=} x^{\mp 2\hbar} \Big[ \pm \tfrac{1}{n} \tfrac{\Gamma(1\pm 2\hbar n)\Gamma(1\pm 2\hbar n)}{\Gamma(1\pm 2\hbar(n+1))\Gamma(1\pm 2\hbar(n-1))} + 2\hbar x^{\pm 2\hbar n} \tfrac{\Gamma(1\pm 2\hbar n)\Gamma(\mp 2\hbar n)}{\Gamma(1\pm 2\hbar)\Gamma(1\mp 2\hbar)} \Big]
\end{aligned}
$$

For the combination $nf_1 \mp f_2$ we thus find

$$
\begin{aligned}
nf_1 \mp f_2 &\overset{x\to 0}{\approx} 2\hbar x^{\pm 2\hbar(n-1)} \frac{\Gamma(1\pm 2\hbar n)\Gamma(\mp 2\hbar n)}{\Gamma(1\pm 2\hbar)\Gamma(1\mp 2\hbar)} \\
&\overset{(A.4.10)}{=} \mp n x^{\pm 2\hbar(n-1)} \frac{\Gamma(\pm 2\hbar n)\Gamma(\mp 2\hbar n)}{\Gamma(\pm 2\hbar)\Gamma(\mp 2\hbar)} \\
&\overset{(A.4.11)}{=} \mp \frac{q-q^{-1}}{q^n - q^{-n}} x^{\pm 2\hbar(n-1)} \\
&= \mp \frac{1}{[n]_q} x^{\pm 2\hbar(n-1)}.
\end{aligned}
$$

The limits $l_i$ are thus given by

$$
\begin{aligned}
l_1 &= \mp \frac{1}{[n]_q} \\
l_3 &= \pm 1 \\
l_2 &= \frac{n}{n+1} l_3 - \frac{1}{n+1} l_1 = \pm \frac{n}{n+1} \left( 1 + \frac{1}{[n]_q} \right),
\end{aligned}
\tag{A.5.31}
$$

which plugged into eq. (A.5.13) give

$$
\begin{aligned}
wa^+ \cdot (M^{-1} V M) va &= q^{\mp n} \left[ -\frac{(w\cdot v)}{[n]_q} + (w\cdot a)(a^+\cdot v) \frac{n}{n+1} \left( \frac{1}{[n]_q} + q^{\pm(n+1)} \right) \right], \\
&= \mp \frac{q^{\mp 1}}{n_{q^{\pm 2}}} \pm (w\cdot a)(a^+\cdot v) q^{\mp 1} \frac{(n+1)_{q^{\pm 2}}}{(n+1)} \frac{n}{n_{q^{\pm 2}}};
\end{aligned}
\tag{A.5.32}
$$

eq. (A.5.4) is manifestly satisfied once we replace the latter result in it. □

## A.6 Proof of Theorem 2

We need to show that equations (5.0.1-5.0.3) are fulfilled. To do the proof one follows the same strategy adopted in the proof for $\mathbf{g} = sl(n)$.

To make computations more expedite we again get rid of indices by introducing an analogous vector notation. One can easily check that $(\rho_{d_h^i} \otimes \rho_{d_k^j} \otimes \sigma)[t_{12}, t_{23}]\, a^h\, a^k = 0$, $a_i^+\, a_j^+ (\rho_{d_h^i} \otimes \rho_{d_k^j} \otimes \sigma)[t_{12}, t_{23}] = 0$; this implies that the path order

$\vec{P}$ in the definition of $\phi_m$ becomes ineffective, so that $M\,aa = aa$, $a^+a^+\,M = a^+a^+$ and therefore $M\,\mathsf{aa} = \mathsf{aa}$, $\mathsf{a}^+\mathsf{a}^+\,M = \mathsf{a}^+\mathsf{a}^+$. Hence eq.'s (5.0.1), (5.0.2) are proved.

For the proof of eq. (5.0.3), which we omit, it is convenient to use the basis of generators $\alpha^+_{i,\mp}, \alpha^i_\pm$ instead of $a^i.a^+_i$. The following properties turn out to be useful.

The "eigenvectors" $\alpha^+_{i,\mp}, \alpha^i_\pm$ satisfy the 'orthogonality relations'

$$\begin{aligned} \alpha^+_{i,\mp}\alpha^i_\pm &= 0 = \alpha^i_\pm\alpha^+_{i,\mp} \\ \alpha^+_{i,\mp}\alpha^+_{j,\mp}c^{ij} &= 0 = \alpha^i_\pm\alpha^j_\pm c_{ij}; \end{aligned} \tag{A.6.1}$$

the above quantities indeed must vanish because they must have commutation relations with $l$ which at the same time are trivial (since they are invariants) and nontrivial [because of eq. (5.2.9)].

Moreover, a direct computation shows

$$\alpha^+_{i,\pm}\alpha^i_\pm = (l\mp1)(2l\pm2\mp N)(n-1+\frac{N}{2}\pm l) \tag{A.6.2}$$

$$\alpha^+_{i,\pm}\alpha^+_{j,\mp}c^{ij} = (a^+\cdot a^+)(l\mp1)(\pm N\mp2-2l) \tag{A.6.3}$$

$$\alpha^i_\pm\alpha^j_\pm c_{ij} = -(l\pm1)[2l\mp2\pm N]a\cdot a \tag{A.6.4}$$

Finally

$$l_{ij}\alpha^j_\pm = c_{ij}\alpha^j_\pm(\frac{N}{2}\mp l) \qquad l_{ij}\alpha^+_{j,\pm} = c^{ij}\alpha^+_{j,\pm}(\frac{N}{2}\pm l) \tag{A.6.5}$$

$$\begin{aligned} [a\cdot a, \alpha^i_\pm] &= 0 & [a\cdot a, \alpha^+_{i,\pm}] &= -2c_{ij}\alpha^j_\pm \\ [a^+\cdot a^+, \alpha^+_{i,\pm}] &= 0 & [a^+\cdot a^+, \alpha^i_\pm] &= -2c^{ij}\alpha^+_{j,\mp} \end{aligned} \tag{A.6.6}$$

$$\begin{aligned} [\alpha^i_\pm, \alpha^j_\pm] &= 0 \\ [\alpha^+_{i,\pm}, \alpha^+_{j,\pm}] &= 0 \\ [\alpha^j_\mp, \alpha^+_{i,\pm}] &= 0 \\ [\alpha^i_+, \alpha^j_-] &= 2(l^{ij} - c^{ij}l)a\cdot a \\ [\alpha^+_{i,+}, \alpha^+_{j,-}] &= 2(l^{ij} - c^{ij}l)a^+\cdot a^+. \end{aligned} \tag{A.6.7}$$

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