SMALL ANGULAR SCALE SIMULATIONS OF THE MICROWAVE SKY

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Abstract

Simulations of the microwave sky good to small angular scales are described. Three physical effects are considered: initial temperature fluctuations corresponding to photon density fluctuations located on the last scattering surface, Doppler temperature fluctuations produced by the peculiar velocity field on the same surface, and the gravitational Sachs-Wolfe effect. A gauge-invariant formalism is used in order to study both sub-horizon and super-horizon scales. The simulations are smoothed with a Gaussian beam function with beam width $\sigma = 17'$ ($\theta_{FWHM} \sim 0.67^{\circ}$). The background is a flat Universe with reduced Hubble constant $h = 0.5$. It contains cold dark matter plus a small quantity of baryons ($\Omega_B = 0.03$). A scale-invariant initial spectrum of density fluctuations is assumed. The initial, Doppler, and Sachs-Wolfe contributions are compared. The total effect is also analyzed. The simulations are tested and their predictions are compared with the observational data from the MAX experiment.

Subject headings: cosmic microwave background—cosmology: theory—large-scale structure of the universe

1 Introduction

Large scale simulations of the microwave sky are based on the expansion of the temperature contrast in spherical harmonics,

$$\frac{\delta T}{T}(\theta,\phi) = \sum_{l=1}^{l_{max}} \sum_{m=-l}^{m=+l} a_{lm} Y_{lm}(\theta,\phi). \tag{1}$$

Typically $l_{max} \leq 40$, since the number of coefficients, a_{lm}, to be calculated is $(l_{max}+1)^2$ and there are questions of numerical accuracy in high order calculations. In order to make high resolution maps — resolutions $\sim 0.13^o$— l_{max} must be of order 10^3 so that the number of coefficients to be calculated is of order 10^6 and the 10^6 spherical harmonics must be evaluated accurately in roughly 10^6 locations. Thus, for high angular resolution, the spherical harmonic expansion seems unsuitable even with modern computing power. In order to make simulations on small angular scales efficiently, a method based on the Fast Fourier Transform (FFT) is used.

The temperature contrast is expanded as follows:

$$\frac{\delta T}{T}(\vec{x}) = \frac{1}{L^3} \sum_{l,m,n=-N}^{N} e^{-i\vec{k}\cdot\vec{x}} \frac{\delta T}{T}(\vec{k}) \tag{2}$$

where the components of the wavenumber $\vec{k}$ are $(2\pi l/L, 2\pi m/L, 2\pi n/L)$, L being the size of a large enough cube (hereafter referred to as the elemental cube) located in the 3-dimensional space at decoupling time. One of the faces of this cube is located on the Last Scattering Surface (LSS). In this paper, comoving coordinates are used; thus the $\vec{k}$ and $\vec{x}$ components are adimensional. Our method can be divided into three main steps: (i) the quantities $\frac{\delta T}{T}(\vec{k})$ are numerically generated as described below, (ii) Eq. (2) is used to compute the required temperature contrast $\frac{\delta T}{T}(\vec{x})$, $\vec{x}$ being the coordinates of a point of the LSS, and (iii) the angular coordinates of the points located on the LSS are computed in order to obtain the angular distribution of temperatures.

In this paper the universe is assumed to be flat. The temperature contrast $\frac{\delta T}{T} = \frac{T-T_B}{T_B}$ is defined with respect to the background temperature T_B. The value of the scale factor, a, at decoupling time is assumed to be $a_D = 1$. $H_0 = 100h\ km\ s^{-1} Mpc^{-1}$ is the Hubble constant, and h the dimensionless reduced Hubble constant. Ω_B is the density parameter of the baryonic component. We take $\Omega_B = 0.03$ and $h = 1/2$. Units are such that the speed of light is 1. The physical wavenumber q and the comoving wavenumber k are related by $q = k/a$. Likewise, the cosmological time t and the conformal time τ are related by $d\tau = dt/a$. An overdot stands for a derivative with respect to τ. The signature $(-,+,+,+)$ is used. Greek indices refer to space-time coordinates (0,1,2,3) and Latin indices to spatial ones (1,2,3).

In a flat universe with $\Omega_B = 0.03$, the decoupling takes place at redshift $Z_D = 1170$ and the angle subtended by the effective horizon at decoupling is $\sim 0.84^o$ (Kolb & Turner 1994); therefore, if we consider any experiment with $\theta_{FWHM} \sim 0.5^o$ and a chop angle of the same order—as in the MAX experiment and the COBRAS/SAMBA project—sub-horizon and super-horizon scales contribute to the measured anisotropy. The fact that super-horizon scales are important suggests a relativistic gauge-invariant study of the problem. From a theoretical point of view, a Newtonian study is only admissible for sub-horizon scales. *Before decoupling*, the Universe can be considered as a system formed by two decoupled gravitating fluids: the dark matter fluid and a second fluid formed by radiation and baryonic matter. This system will be studied by using the multifluid extension (Abbott & Wise 1984, Kodama & Sasaki 1984) of Bardeen's formalism (Bardeen, 1980); baryonic matter and radiation evolve in thermal equilibrium, while dark matter evolves decoupled. This last component only influences the rest of the universe through the gravitational interaction. *After decoupling*, dark matter, radiation, and baryonic matter are considered as three decoupled gravitating fluids.

The so-called secondary anisotropies are not studied in this paper. The Zel'dovich Sunyaev effect, a possible reionization of the Universe, some topological defects and some nonlinear gravitating objects could produce these anisotropies.

We are interested in the so-called primary anisotropy. Only scales larger than $\sim 0.13°$—namely, scales larger than the angular scale corresponding to the thickness of the LSS ($\sim 7h^{-1}$ Mpc, Kaiser & Silk 1986)—contribute to this anisotropy. The primary anisotropy is the superposition of three main effects: (1) the initial anisotropy $(\frac{\delta T}{T})_{in}$ produced by photon density fluctuations on the LSS, (2) the Doppler anisotropy $(\frac{\delta T}{T})_{d}$ produced by the radial component, v_r, of the velocity field on the LSS, and (3) the Sachs-Wolfe gravitational anisotropy $(\frac{\delta T}{T})_{sw}$ appearing as a result of the motion of the CMB photons in the gravitational field of the energy density fluctuations. Scales larger than 4.5° do not produce significant contributions to the effects (1) and (2) and, consequently, these effects are simulated by using the FFT and angular scales between 0.13° and 4.5°. In the case (3), the effect produced by scales between 0.13° and 4.5° is calculated in the same way as in cases (1) and (2); this effect appears to be subdominant. The contribution of scales greater than 4.5° is separately computed by using a standard code based on spherical harmonics. This splitting of scales becomes possible because all the scales evolve independently in the linear regime (Bardeen, 1980).

It is not easy to give a general formula for each of the three above effects applying to both sub-horizon and super-horizon scales. Let us make some comments about each of these scales:

a) *In the case of sub-horizon scales*, spurious effects produced by the gauge are not important and, consequently, a Newtonian study suffices (Kolb & Turner, 1994).

The Doppler effect produced by sub-horizon scales is:

$$(\frac{\delta T}{T})_d = -\vec{v} \cdot \vec{n} \ , \tag{3}$$

where $\vec{v}$ is the peculiar velocity of the baryonic matter and $\vec{n}$ is the unit vector along the line of sight of the observer. This velocity $\vec{v}$ is created by the gravitational effect of the total energy density contrast, plus the effect of pressure. In the sub-horizon case, one obtains the following formula for the components of the peculiar velocity field $\vec{v}$ (Kolb & Turner, 1994):

$$v^j(\vec{x}) = \frac{1}{L^3} \sum_{l,m,n=-N}^{N} e^{-i\vec{k}\cdot\vec{x}} v^j_{\vec{k}} \tag{4}$$

with

$$v^j_{\vec{k}} = -i\frac{\dot{a}}{a}\frac{k^j}{k^2}\delta_{\vec{k}} \tag{5}$$

(Peebles, 1980), where $\delta_{\vec{k}}$ is the Fourier Transform (FT) of the total density contrast producing the peculiar accelerations and velocities. This formula is only valid in the case of a flat background.

Formulae (4) and (5) include pressure effects if these effects are taken into account in the calculation of $\delta_{\vec{k}}$.

The initial fluctuations on the LSS are

$$(\frac{\delta T}{T})_{in} = \frac{1}{3}(\frac{\delta\rho}{\rho})_b = \frac{1}{4}(\frac{\delta\rho}{\rho})_\gamma \ , \tag{6}$$

where $(\frac{\delta\rho}{\rho})_b$ and $(\frac{\delta\rho}{\rho})_\gamma$ are the baryonic and radiation density contrasts on the LSS, respectively.

Finally, in the case of sub-horizon scales (Kolb & Turner, 1994), the FT of the Sachs-Wolfe anisotropy is given by:

$$(\frac{\delta T}{T})_{sw}(\vec{k}) = -\frac{1}{2}H_0^2(1+Z_D)^2k^{-2}\delta_{\vec{k}} \ . \tag{7}$$

This last formula follows from the well known Sachs-Wolfe relation

$$(\frac{\delta T}{T})_{sw} = \frac{1}{3}\phi_g \tag{8}$$

and the equation

$$\nabla^2\phi_g = 4\pi G\rho \tag{9}$$

which holds in the case of Newtonian sub-horizon scales.

For sub-horizon scales, the three above effects are to be linearly superposed after computations based on Eqs. (3)–(9).

b) *In the case of super-horizon scales*, the peculiar velocities, the total density contrast and the density contrasts of baryons and radiation strongly depend on the gauge. Spurious gauge-dependent effects are very important. The above Newtonian analysis does not apply and a gauge-invariant study is necessary. This study will be based on the general gauge-invariant analysis of the CMB due to Abbot and Schaefer (1986), which is based on the gauge-invariant study of uncoupled gravitating fluids due to Abbot and Wise (1984).

Our treatment of the CMB anisotropy is valid for both super-horizon and sub-horizon scales. It is rigorous and general. Its main limitation appears as a result of the assumption that recombination and decoupling are simultaneous and instantaneous effects ($Z_D = 1170$) separating a fully opaque universe from a completely transparent one. Future improvements should consider recombination and decoupling as coupled processes lasting a finite nonvanishing time; in such a case, it is well known that the LSS has a certain thickness. Here the anisotropies produced by scales smaller than this thickness are neglected. This will not introduce significant error as we are smoothing with a 17' gaussian beam.

The plan of this paper is as follows: the multifluid extension of the Bardeen's formalism is briefly described in Sec. 2. The application of this formalism to the study of the CMB anisotropy is considered in Sec. 3. Initial conditions at equivalence and decoupling are discussed in Sec. 4. Simulations of the anisotropy effects are described in Sec. 5. Results and comparisons with observations are presented in Sec. 6, and Sec. 7 contains a general discussion.

2 The multifluid extension of Bardeen's formalism

In this paper we are only interested in scalar fluctuations, which couple to density fluctuations (Bardeen 1980). Vector and tensor perturbations are not considered.

Abbot and Wise (1984) extended Bardeen's formalism to the case of several uncoupled gravitating fluids. Let us give a brief summary of the approach. For scalar fluctuations, the quantities describing the spacetime and the fluids are expanded in terms of the solutions of the scalar Helmholtz equation: the so-called scalar harmonics $Q(\vec{x})$. In the case of a flat background, plane waves satisfy the scalar Helmholtz equation and, consequently, Fourier's expansions appear.

By using the quantities $Q_i = -\frac{1}{k}\partial_i Q$, and $Q_{ij} = \frac{1}{k^2}\partial_i\partial_j Q + \frac{1}{3}\delta_{ij} Q$, the metric tensor $g_{\alpha\beta}$ is expanded as follows

$$g_{00} = -a^2[1 + 2AQ] \ , \tag{10}$$

$$g_{0i} = -a^2 B Q_i \ , \tag{11}$$

$$g_{ij} = a^2\{[1 + 2H_L Q]\delta_{ij} + 2H_T Q_{ij}\} \ , \tag{12}$$

and the components of the energy-momentum tensor $T^{a\alpha}_{\beta}$ are:

$$T^{a0}_{0} = -\rho_a(1 + \delta_a Q) \ , \tag{13}$$

$$T_0^{ai} = -(\rho_a + p_a)v_a Q^i \ , \tag{14}$$

$$T_i^{a0} = (\rho_a + p_a)(v_a - B)Q_i \ , \tag{15}$$

$$T_j^{ai} = p_a[(1 + \pi_L^a Q)\delta_j^i + \pi_T^a Q_j^i] \ , \tag{16}$$

where the index a labels the fluids, ρ_a and p_a are the unperturbed energy density and pressure, and the functions $A, B, H_L, H_T, \delta_a, v_a, \pi_L^a$ and π_T^a define the perturbations. These functions depend on the time and the wavenumber k; from them, the following gauge-invariant quantities can be defined:

$$\Phi_A = A + \frac{1}{k}\dot{B} + \frac{1}{k}\frac{\dot{a}}{a}B - \frac{1}{k^2}(\ddot{H}_T + \frac{\dot{a}}{a}\dot{H}_T) \ , \tag{17}$$

$$\Phi_H = H_L + \frac{1}{3}H_T + \frac{1}{k}\frac{\dot{a}}{a}B - \frac{1}{k^2}\frac{\dot{a}}{a}\dot{H}_T \ , \tag{18}$$

$$\epsilon_a = \delta_a + 3\frac{(\rho_a + p_a)}{\rho_a}\frac{1}{k}\frac{\dot{a}}{a}(v_a - B) \ , \tag{19}$$

$$v_s^a = v_a - \frac{1}{k}\dot{H}_T \ . \tag{20}$$

Inside the horizon, quantity $\frac{1}{k}\frac{\dot{a}}{a}$ is smaller than 1 and, consequently, $\epsilon_a \sim \delta_a$.

Abbot and Wise gave the equation governing the evolution of the quantities $E_a = \rho_a \epsilon_a a^3$. In the case of a dark matter fluid (E_d) plus a radiation fluid (E_γ), we easily obtain the following evolution equations for E_d and E_γ:

$$\begin{aligned} \frac{d^2E_d}{dt^2} \ &+ \ 2H\frac{dE_d}{dt} + \frac{12\pi GH}{q^2 + 12\pi G(\rho_d + \frac{4}{3}\rho_\gamma)}\{\frac{8}{3}\rho_\gamma\frac{dE_d}{dt} - 2\rho_d\frac{dE_\gamma}{dt}\} \\ &-8\pi G\rho_d(E_\gamma + \frac{E_d}{2}) = 0 \ , \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{d^2E_\gamma}{dt^2} \ &+ \ 3H\frac{dE_\gamma}{dt} + \frac{12\pi GH}{q^2 + 12\pi G(\rho_d + \frac{4}{3}\rho_\gamma)}\{\rho_d\frac{dE_\gamma}{dt} - \frac{4}{3}\rho_\gamma\frac{dE_d}{dt}\} \\ &+\frac{k^2E_\gamma}{3} - \frac{4\pi G}{3}(4\rho_\gamma E_\gamma + 4\rho_\gamma E_d - 3\rho_d E_\gamma) = 0 \ , \end{aligned} \tag{22}$$

where $q = k/a$. Since the subdominant baryonic component is neglected, Eqs. (21) and (22) lead to an evolution equation for the total density contrast $\epsilon \sim \epsilon_\gamma + \epsilon_d$. These equations will be important in order to evolve the quantities ϵ, ϵ_γ, and ϵ_d from equivalence to decoupling. ¿From recombination to decoupling, Silk damping erases initial temperature fluctuations on small angular scales (photon diffusion); this phenomenon is not described by Eqs. (21) and (22). Our simulations model the Silk damping by introducing a cutoff at the angular scale subtended by the thickness of the LSS.

3 The FT of the total anisotropy

The FT, $\frac{\delta T}{T}(\vec{k})$, of the total anisotropy $\frac{\delta T}{T}(\vec{x})$ can be easily obtained from previous computations due to Abbott and Schaefer (1986). These authors obtained $\frac{\delta T}{T}(\vec{x})$ as an expansion in terms of the scalar harmonics $Q(\vec{x}, \vec{k})$. In the flat case considered in this paper, this reduces to Fourier's expansion. After eliminating any monopolar contribution and the dipole produced by the motion of the observer, Eq. (26) of Abbott and Schaefer (1986) can be rewritten as follows:

$$\frac{\delta T}{T}(\vec{x}) = \{\frac{1}{4}\epsilon_\gamma + \Phi_{_A} - \frac{1}{k}\frac{\dot{a}}{a}v^s_\gamma - \frac{i}{k}v^s_\gamma(\vec{n}\cdot\vec{k})\}e^{-i\vec{k}\cdot\vec{x}} + \int_{\tau_e}^{\tau_0}(\dot{\Phi}_{_A} - \dot{\Phi}_{_H})e^{-i\vec{k}\cdot\vec{x}}d\tau \ , \qquad (23)$$

where the sums over the components of $\vec{k}$—or over m, n, l—have been omitted for the sake of briefness. All the terms involved in the last equation have been written in terms of gauge-invariant variables. It is worthwhile to analyze these terms in some detail.

The last term is negligible because it is well known that potentials $\Phi_A = -\Phi_H$ are quasi-constant in the case of a flat background. This term was derived and studied by Martínez-González, Sanz, & Silk (1990). It is only important in the presence of nonlinear gravitating objects.

In order to interpret the remaining terms, they must be rewritten in an appropriate form. In the case of super-horizon scales, the use of the synchronous gauge leads to easy interpretations. In this gauge, the following relation holds:

$$\epsilon_a \sim \delta_a \tag{24}$$

and, for super-horizon scales, one gets:

$$\dot{\delta}_k = \frac{\dot{a}}{a}\delta_k \ . \tag{25}$$

On account of Eqs. (24) and (25)—which are justified in Sec. 4—the following interpretations become straightforward:

(1) By using Eq. (24), the term $\frac{1}{4}\epsilon_\gamma$ can be rewritten in terms of δ_γ; the resulting term is formally identical to the right hand side (r.h.s) of Eq. (6). This fact suggests that the term $\frac{1}{4}\epsilon_\gamma$ accounts for the initial temperature fluctuations. The same conclusion is achieved in the case of sub-horizon scales, in which Eq. (24) is also valid.

(2) Taking into account Eqs. (24) and (25) plus the relations (Abbott and Schaefer 1986):

$$\Phi_A = -\frac{3}{2}\frac{\epsilon}{k^2}H^2 \ , \tag{26}$$

$$v^s_\gamma \sim v_\gamma \sim v_b \ , \tag{27}$$

and

$$v_\gamma^s = -\frac{\dot{\epsilon}_\gamma}{k} , \tag{28}$$

one easily finds:

$$\Phi_A - \frac{1}{k}\frac{\dot{a}}{a} v_\gamma^s \sim -\frac{1}{2}\frac{H^2}{k^2} \mid \delta_{\vec{k}} \mid ; \tag{29}$$

hence, the term $\Phi_A - \frac{1}{k}\frac{\dot{a}}{a} v_\gamma^s$ becomes formally identical to the r.h.s. of Eq. (7) and, consequently, this term allows us to estimate the Sachs-Wolfe effect.

(3) From Eqs. (24), (25) and (28), the term $-\frac{i}{k} v_\gamma^s (n^j.k^j)$ can be rewritten as $-\vec{n} \cdot (-i\frac{\dot{a}}{a}\frac{\vec{k}}{k^2}\delta_{\vec{k}})$; hence, according to Eqs. (3) and (5), this term accounts for the Doppler effect appearing as a result of the peculiar motions of baryonic matter on the LSS.

Taking into account Eqs. (23) and (28) plus the above discussion, the formula

$$\frac{\delta T}{T}(\vec{k}) = \frac{1}{4}\epsilon_\gamma - \frac{3}{2}\frac{\epsilon}{k^2}(\frac{\dot{a}}{a})^2 + \frac{1}{k^2}\frac{\dot{a}}{a}\dot{\epsilon}_\gamma + \frac{i}{k^2}\dot{\epsilon}_\gamma(\vec{n}\cdot\vec{k}) \tag{30}$$

gives the FT of the total anisotropy.

The physical meaning of each term has been elucidated by considering particular cases. The above discussion about the terms involved in Eq. (23) leads to the following remarkable conclusion: If the super-horizon scales are studied in the synchronous gauge, Eqs. (3), (6) and (7) apply. Note that these equations were initially obtained for sub-horizon scales.

4 Conditions at equivalence and decoupling

For fluctuations due to cold dark matter on scales greater than $\sim 7h^{-1}$ Mpc, the power spectrum at equivalence, $\mid \delta(k) \mid^2$, is (Peebles 1983, Bond & Efstathiou 1984, Davis et al. 1985, Kolb & Turner 1994):

$$\mid \delta(k) \mid^2 = \frac{AL^3 k^{1+6\alpha}}{(1 + \beta q + w q^{1.5} + \gamma q^2)^2} \tag{31}$$

where: $\beta = 1.7(\Omega_0 h^2)^{-1}$ Mpc, $w = 9.0(\Omega_0 h^2)^{-1.5}$ $Mpc^{1.5}$, $\gamma = 1.0(\Omega_0 h^2)^{-2}$ Mpc^2, and $q = k(1 + Z_D)$ is the present value of the physical wavenumber in units of Mpc^{-1} In the case of sub-horizon scales, this spectrum is valid whatever the gauge may be; however, for super-horizon scales, the above form corresponds to the synchronous gauge ($A = B = 0$, Kolb & Turner 1994, Bond & Efstathiou, 1984).

The quantities $\epsilon(k)$, $\epsilon_\gamma(k)$—at decoupling time—are necessary in order to compute the total anisotropy by using Eq. (30). The computation of these quantities can be divided into two main steps: (1) At equivalence, the quantities $\epsilon(k)$ and $\epsilon_\gamma(k)$ are calculated—as detailed in Sec. 4.1—from the power spectrum $\mid \delta(k) \mid^2$ given by Eq. (31), and (2) from equivalence to decoupling, these quantities are evolved by using Eqs. (21) and (22). The resulting values are shown in Sec. 4.2.

4.1 Initial conditions at equivalence

In the first step, the spectra $\mid \epsilon(k) \mid^2$ and $\mid \epsilon_\gamma(k) \mid^2$ are computed at equivalence.

Since we are interested in adiabatic fluctuations, the equalities

$$\delta_d = \delta_b = \frac{3}{4}\delta_\gamma = \frac{6}{7}\delta \tag{32}$$

apply at equivalence, δ being the density contrast of the total energy density. This equation leads to an evident relation between the initial spectra $\mid \delta(k) \mid^2$ and $\mid \delta_\gamma(k) \mid^2$ at equivalence; hence, our attention is hereafter focused on $\mid \delta(k) \mid^2$.

Since ϵ is gauge-invariant, the initial spectra $\mid \epsilon(k) \mid^2$ can be calculated in any gauge from Eq. (19). This equation involves the quantities δ and v. For super-horizon scales—in the synchronous gauge—the peculiar velocity v only has a decaying mode (Kolb & Turner 1994); hence, this velocity can be neglected at equivalence and the equality $\epsilon \sim \delta$ holds in the synchronous gauge (see Sec. 3, where this conclusion was used). Since the same equality applies in the case of sub-horizon scales (in any gauge), we conclude that the relation $\mid \epsilon(k) \mid \sim \mid \delta(k) \mid$ is valid for any scale and, consequently, Eq. (31) defines the initial spectrum $\mid \epsilon(k) \mid^2$ at decoupling. The same argument applies for $\mid \epsilon_\gamma(k) \mid^2$ and the spectrum $\mid \delta_\gamma(k) \mid^2$ defined by Eqs. (31) and (32).

¿From the above discussion, it follows that the initial spectra $\mid \epsilon(k) \mid^2$ and $\mid \epsilon_\gamma^2(k) \mid$ can be calculated from Eqs. (31) and (32), but $\epsilon(k)$ and $\epsilon_\gamma(k)$ are complex numbers whose phases must be also defined. The assignation of phases is studied in the next Section.

4.2 From equivalence to decoupling

Let us now focus our attention on the phases of $\epsilon(k)$ and $\epsilon_\gamma(k)$. These numbers must satisfy two coupled equations, which can be easily obtained from Eqs. (21) and (22) and the relation $\epsilon_d \sim \epsilon - \epsilon_\gamma$. In order to find a complex solution of these equations, it is useful to put $\epsilon = g e^{i\beta}$ and $\epsilon_\gamma = g_\gamma e^{i\beta_\gamma}$, where g and g_γ are real numbers and β and

β_γ are phases. It is evident that the equations are satisfied if: (a) the real numbers g and g_γ satisfy the same formal equations as ϵ and ϵ_γ, (b) the phases β and β_γ do not depend on the time but only on $\vec{k}$, and (c) the phases of ϵ and ϵ_γ are identical for any $\vec{k}$; namely $\beta = \beta_\gamma$ for any $\vec{k}$. The phases are fixed in the next section according to these considerations.

It is worthwhile to point out that, *a priori*, the real numbers g and g_γ may be negative at any time. If the initial conditions at equivalence $g =\mid \epsilon \mid$ and $g_\gamma =\mid \epsilon_\gamma \mid$ are assumed, then at decoupling, the integration of Eqs. (21) and (22) give the functions $g(k)$, $g_\gamma(k)$ and $\dot{g}_\gamma(k)$ displayed in Figs. 1, 2, and 3. ¿From these Figures, it follows that g is positive for any k, while $g_\gamma(k)$ and $\dot{g}_\gamma(k)$ appear to be negative for some k values. This is a result of the fact that Eqs. (21) and (22) are not equations for the evolution of the spectra.

It follows from the above considerations that the initial phases assigned to $g =\mid \epsilon \mid$ (see Sec. 5) must also be assigned to g, $g_\gamma(k)$ and $\dot{g}_\gamma(k)$ at all times, in particular, at decoupling time. In this way, a proper superposition of the terms involved in Eq. (30) is achieved.

4.3 Normalization

For $\alpha = 0$ and small k values (large spatial scales), the spectrum (31) reduces to the scale-invariant Harrison-Zel'dovich spectrum $\mid \delta(k) \mid^2 = AL^3k$, where the normalization parameter A evolves as a^2. This is the key for our normalization of the spectrum (31). Since the COBE quadrupole ($Q_{rms-PS} = 17 \pm 5\ \mu K$, Smoot et al. 1992) is produced by very large features (Harrison-Zel'dovich part of the spectrum), the value of the parameter A at equivalence appears to be:

$$A_E = \frac{96\pi^2}{5H_0^4}(1+Z_E)^{-2}Q^2_{COBE} \tag{33}$$

where Z_E denotes the value of Z at equivalence. Eq. (33) can be easily obtained from Eq. (9.144) in Kolb & Turner, 1994.

For super-horizon scales—small k values—and arbitrary α values, the parameter A evolves as a^2 and the spectrum has the form $\mid \delta(k) \mid^2 = AL^3k^{1+6\alpha}$. On account of these facts, Eq. (25) can be easily derived.

For light neutrinos (hot dark matter), the power spectrum at equivalence—in the synchronous gauge—is (Bond & Szalay, 1983):

$$\mid \delta(k) \mid^2 = AL^3k^{1+6\alpha}e^{-4.61(k/k_\nu)^{1.5}} \tag{34}$$

where $k_\nu = 0.16(m_\nu/30eV)\ Mpc^{-1}$, m_ν being the neutrino mass. This equation can be used in the same way as Eq. (31). In other cases, such as a mixing of cold and hot dark matter with adiabatic fluctuations or cold dark matter with isocurvature perturbations, formulae playing the same role as (31) and (34) are available in the literature (Kolb & Turner 1994, van Dalen & Schaefer 1992).

5 Simulations

In order to simulate $\frac{\delta T}{T}(\vec{x})$ by using Eq. (30), the complex numbers $\epsilon(\vec{k})$ and $\epsilon_\gamma(\vec{k})$ are necessary. Let us begin with the number $\epsilon(\vec{k})$. This number is the three-dimensional

(3D) FT of the total density contrast. It is usually assumed (Peebles, 1980) that, if the function $\epsilon(\vec{x})$ is expanded in the form:

$$\epsilon(\vec{x}) = \frac{1}{L^3} \sum_{l,m,n=-N}^{N} e^{-i\vec{k}\cdot\vec{x}} \epsilon(\vec{k}) \tag{35}$$

the complex numbers $\epsilon(\vec{k})$ have the following features: (1) they satisfy periodic boundary conditions, (2) $\mid \epsilon(\vec{k}) \mid$ has the same values for any realization of $\epsilon(\vec{x})$, (3) $\epsilon^*(\vec{k}) = \epsilon(-\vec{k})$ for any $\vec{k}$, and (4) the phases of any pair of $\epsilon(\vec{k})$ complex numbers either are related according to condition (3) or they are statistically independent random numbers. The distribution of these independent phases is uniform.

Condition (1) can be assumed because the size of the elemental cube is larger than the significant scales of the considered effects (no correlations at scales comparable with the cube size). Since the two point correlation function is the FT transform of the power spectrum, the condition (2) means that every realization of $\epsilon(\vec{x})$ has the same two-point correlation function. Condition (3) ensures that the imaginary part of $\epsilon(\vec{x})$ vanishes and, finally, the central limit theorem plus condition (4) imply that the distribution of $\epsilon(\vec{x})$ is approximately Gaussian; in fact, according to Eq. (35), $\epsilon(\vec{x})$ is the superposition of a large number of statistically independent variables $\epsilon(\vec{k})$.

Our code is based on the simulation of a set of $\epsilon(\vec{k})$ numbers satisfying conditions (1)-(4). The moduli of these numbers are fixed from the chosen spectrum computed at decoupling time (see Sec. 4.1). For adiabatic fluctuations, the form of $\mid \epsilon(\vec{k}) \mid^2$ depends on the nature of dark matter, the ratio between baryonic and dark matter abundances, and the form of the primordial spectrum. In this paper, only the spectrum (31) is used. Other spectra will be considered elsewhere. Simulations of $\epsilon(\vec{k})$ are lengthy but straightforward.

According to the discussion of Sec. 4, after the phases of the $\epsilon(\vec{k})$ are fixed, the phases of the remaining complex numbers involved in Eq. (30) also become fixed.

5.1 Geometry of the simulation

As stated before, $\frac{\delta T}{T}(\vec{x})$ is computed on the FFT elemental cube. Since a face of this cube is assumed to be located on the LSS, it is evident that we are identifying a face of the cube with a part of the LSS; hence, we are neglecting the spatial curvature of this part. This restricts our simulations to small parts of the microwave sky. Regions of $\sim 20^{\circ} \times 20^{\circ}$ (or $\sim 1\%$ of the total sky) will be mapped; in one of these regions, the assumption of flatness would produce a deformation of the true maps, but this deformation is not expected to hide their main features. Hereafter, the spatial coordinates are chosen in such a way that the equation of the face located on the LSS is $z = 0$.

All the FT considered until now are 3D, but we are going to see that, in order to compute a physical quantity $\xi(x, y, z)$ on the face $z = 0$, the original 3D FT can be reduced to a well defined 2D FT; in fact, by using the 3D FT one easily obtains:

$$\xi(x,y)_{LSS} = \xi(x,y,0) = \frac{1}{L^3} \sum_{l,m,n=-N}^{N} e^{-\frac{2\pi i}{L}(lx+my)} \xi(k_x, k_y, k_z) = \frac{1}{L^2} \sum_{l,m=-N}^{N} e^{-\frac{2\pi i}{L}(lx+my)} \xi^{(2)}(k_x, k_y) \qquad (36)$$

where

$$\xi^{(2)}(k_x, k_y) = \frac{1}{L} \sum_{n=-N}^{N} \xi(k_x, k_y, k_z) \tag{37}$$

This means that, on the LSS ($z = 0$), the quantity $\xi(x,y)_{LSS}$ can be considered as the 2D inverse FT of the function $\xi^{(2)}(k_x, k_y)$, which can be easily calculated from the corresponding 3D function ξ by using Eq. (37). On account of this fact, we can compute $\frac{\delta T}{T}$ by using Eq. (30) and the 2D FFT.

Since the FT involves periodic boundary conditions, our simulations are not physically significant in all the points of the elemental square. Only the simulation of the central region is physically admissible; on account of this fact, Fourier transforms are performed in a big $\sim 40^\circ \times 40^\circ$ square and the resulting simulation is only considered valid in the central $\sim 20^\circ \times 20^\circ$ region. The curvature could be important in order to get a physically significant mapping of the big square; however, such a square is only an auxiliary element without any physical significance. The number of points on each edge of the big square is taken to be 2N=512. Similar simulations have been obtained by using a $\sim 80^\circ \times 80^\circ$ square and 1024 points per edge; hence, the use of a number of points greater than 512 is not necessary.

In order to obtain the angular distribution of the temperature contrast on the LSS, one must take into account that, in a flat background, the relation between a small angle $\Delta\theta$ and the distance on the LSS, D_θ, subtended by this angle is

$$D_\theta = -\frac{2}{H_0} \frac{1}{1+Z_D} [(1+Z_D)^{-1/2} - 1] \Delta\theta \tag{38}$$

This relation allows us to assign angular coordinates to the points of the mapped region of the LSS. In the absence of lens effects, only small deformations should be produced by the neglecting of the curvature of small ($\sim 20^\circ \times 20^\circ$) regions.

Simulations of larger regions are being studied. Such a region could be considered as a curved surface located inside the Fourier elemental cube; but this assumption would compel us to use the 3D FT and interpolations inside the elemental cube for particular points on the surface; hence, the numerical cost would increase. A mosaic of well matched $\sim 20^\circ \times 20^\circ$ regions would be preferable from the point of view of the numerical cost. However, the matching of neighboring regions seems to be problematic. Even if the values of the complex numbers $\frac{\delta T}{T}(\vec{k})$ are taken to be identical on the common edges (phase space), large discontinuities are expected on the borders (physical space) and, consequently, some kind of smoothing would be necessary. In this paper, only realizations of a $\sim 20^\circ \times 20^\circ$ region are considered. These realizations suffice for comparisons with the data from experiments—such as MAX—which measure small regions of the sky.

5.2 Smoothing

After the FFT is used and angular coordinates are assigned to the points of the LSS, the resulting distribution of the temperature contrast is smoothed by using a Gaussian beam having beam width σ. The resulting temperature contrast is given by

$$\left(\frac{\delta T}{T}\right)_\sigma (\vec{n}) = \int \frac{dS\,'}{r^2} \frac{1}{2\pi\sigma^2} e^{[-\theta^2/2\sigma^2]} \frac{\delta T}{T}(\vec{n}\,') \tag{39}$$

where θ is the angle formed by the directions $\vec{n}$ and $\vec{n}\,'$.

The beam width used in this paper is $\sigma = 17'$; on account of the relation $\sigma = 0.425\theta_{FWHM}$, the chosen σ value corresponds to $\theta_{FWHM} = 0.67^{\circ}$. This choice is suitable in order to compare our predictions with the latest results obtained in the MAX experiment, in which the values $\theta_{FWHM} \sim 0.55^{\circ}$ and $\theta_{FWHM} \sim 0.75^{\circ}$ are used (Clapp et al. 1994). Since $(\frac{\delta T}{T})_{\sigma}$ cannot be calculated on the edges—or too near the edges—of the $\sim 20^{\circ} \times 20^{\circ}$ regions, the simulations of $(\frac{\delta T}{T})_{\sigma}$ are extended over slightly smaller parts of the microwave sky ($\sim 17^{\circ} \times 17^{\circ}$).

6 Results and comparisons with observations

Plots 4–8 correspond to $\sim 17^{\circ} \times 17^{\circ}$ regions of the sky. The true distribution of $\frac{\delta T}{T}(\vec{n})$ has been smoothed with a Gaussian beam having $\sigma = 17'$. Plots 4, 5, and 6 only involve the contributions produced by structures having angular scales between 0.13° and 4.5°. Fig. 4 is a simulation of the initial anisotropy. Fig. 5 corresponds to a subdominant part of the Sachs-Wolfe effect (small scales). Fig. 6 is a simulation of the Doppler anisotropy produced on the LSS. Fig. 7 is a simulation of the Sachs-Wolfe effect for scales larger than 4.5°. This contribution has been computed by using a code based on Eq. (1). Fig. 8 is a simulation of the total primary anisotropy, which is the superposition of the effects described in Figs. 4–7.

In Figs. 4–8, the maxima (minima) of the positive (negative) $(\frac{\delta T}{T})_{\sigma}$ fluctuations are shown; these values only appear in some scarce points of the maps and they change from simulation to simulation. These changes are particularly relevant when we are dealing with large scales (Fig. 7) producing large spots which could have a size comparable to that of the simulated region. For comparisons with the observational data, the most appropriate quantities are the temperature differences corresponding to a separation angle ζ between two observation directions. These differences are denoted $\frac{\Delta T}{T}$; they can be calculated from the $(\frac{\delta T}{T})_\sigma$ values obtained in the simulations.

For the five simulations displayed in Figs. 4–8, 13500 temperature differences, $\frac{\Delta T}{T}$, between pairs of points separated by the angle $\zeta = 1.4^\circ$ have been computed; these pairs have been randomly placed in the simulated region. In the case of Fig. 8 (total primary anisotropy), the rms value of the computed differences is $\sim 3.67 \times 10^{-5}$. The frequencies of the differences have a nearly gaussian distribution in each case; for the sake of briefness, only the frequencies corresponding to the simulation of the total anisotropy (Fig. 8) are given in Fig (9); however, the rms values and the maximum and minimum $\frac{\Delta T}{T}$ differences are given for all the cases corresponding to Figs. 4–8; this information is contained in Table 1. As expected, the contributions of the partial effects of Figs. 4–7 to the total anisotropy couple in a complicated way (strong cancellation) and the rms value of the total anisotropy is much smaller than the sum of the rms values of the partial contributions. The phases involved in Eq. (30) determine this coupling (see Secs. 4 and 5). As it can be seen in Table 1, the rms values of the initial and Doppler anisotropies are similar, while the rms values corresponding to the Sachs-Wolfe effect on scales smaller and larger than 4.5° are subdominant. Since the angle $\theta_{FWHM} \sim 0.67^\circ$ and the separation angle $\zeta = 1.4^\circ$

are near the angle corresponding to the so-called Doppler peak, the rms value of the Doppler contribution appears to be significant. Although these angles have been chosen with the essential aim of maximizing the Doppler effect, the contribution of the initial anisotropy is also important.

7 Conclusions and discussion

Small scale simulations of the microwave background are expected to be very useful in order to analyze observational data from both current and future experiments. The mathematical and physical foundations of the simulations presented in this paper have been rigorously studied. A gauge-invariant formalism is used. The theoretical limitations of the discrete Fourier transform are taken into account in order to define the region where our computations are significant. The resulting simulations have been tested by computing the temperature differences $\frac{\Delta T}{T}$ for $\zeta = 1.4°$; the order of the rms calculated from these differences is 10^{-5} and the distribution is very similar to a Gaussian one.

The main assumptions of this paper are: adiabatic Gaussian energy density fluctuations with a spectrum of the form (31) normalized according to COBE data; a flat background having $h = 0.5$ and no cosmological constant; a LSS corresponding to $Z_D = 1170$; vanishing contributions of angular scales smaller than $0.13°$; vanishing contributions of scales greater than $4.5°$ except in the case of the Sachs-Wolfe effect; a beam width $\sigma = 17'$ and a chopping angle $\zeta = 1.4°$ for the computation of temperature differences.

Since the dominant dark matter component does not undergo any Silk damping, the estimation of the Sachs Wolfe effect does not require any cutoff at $0.13°$. Since the electron velocities are altered by the photon diffusion, the Silk damping influences the Doppler effect; however, only scales larger than $0.13°$ would produce significant Doppler contributions and, consequently, the cutoff at $0.13°$ is not expected to be very effective. The initial anisotropy is erased at scales smaller than $\sim 0.13°$ as a result of the photon diffusion; hence, the estimate of this anisotropy requires either a cutoff at $\sim 0.13°$ or a more detailed study of the recombination-decoupling period. This study is in progress.

According to the above comments, only the initial anisotropy could undergo some modifications when an accurate code based on a rigorous description of the Silk damping is used; these modifications could be particularly significant for $\theta_{FWHM} \sim 0.13°$; however, in the MAX case ($\theta_{FWHM} \sim 0.67°$), the initial anisotropies on scales smaller than $\sim 0.13°$ are averaged and a detailed description of the Silk damping should not produce important improvements on our results.

Under the above assumptions, relative temperature differences corresponding to $\zeta \sim 1.4°$ have a rms value 3.67×10^{-5}. This value is very similar to the rms values obtained from the latest measurements of the MAX experiment (Clapp. et al. 1994).

Several simulations similar to those of Figs. 4–8 have been obtained. The rms value and the Gaussian character of the temperature differences are very stable; they do not undergo significant changes from simulation to simulation. In a IBM 30-9021 VF, the CPU time of each simulation is $\sim$ 50 minutes.

If the normalization is not performed according to the COBE quadrupole, but according to the usual prediction $\langle | a_{2m} |^2 \rangle^{1/2} = 2 \times 10^{-5}$, (Kolb & Turner 1994), the above rms value is magnified by a factor $\sim$ 2. In such a case, detection would be easier, but the resulting anisotropies would be too large to be compatible with the observational values reported by Clapp. et al. 1994. In other scenarios corresponding, for example, to other spectra, the situation could be different. Variations of the spectra and the free parameters involved in our simulations are being studied.

References

Abbot, L.F., & Schaefer, R.K., 1986, ApJ, 308, 546

Abbot, L.F., & Wise, M.B., 1984, Nucl. Phys., 237B, 226

Bardeen, J.M., 1980, Phys. Rev., 22D, 1882

Bond, J.R. & Szalay, A.S., 1983, ApJ, 276, 443

Bond, J.R. & Efstathiou, G., 1984, ApJ, 285, L45

Clapp, A.C. et al., 1994, ApJ, 433, L57

Davies, M. et al., 1985, ApJ, 292, 371

Kaiser, N. & Silk, J., 1986, Nature, 324, 529

Kolb, E.W. & Turner, M.S., 1994, The Early Universe, Addison-Wesley Publishing company

Martínez-González, E., Sanz, J.L., & Silk, J., 1990, ApJ, 355, L5

Peebles, P.J.E., 1980, The Large Scale Structure of the Universe, Princeton Univ. Press, Princeton, NJ

Peebles, P.J.E., 1983, ApJ, 263, L1

Sachs, R.K., & Wolfe, A.M., 1967, ApJ, 147,73

Smoot et al., 1992, ApJ, 396, L1

van Dalen, A. & Schaefer, R.A., 1992, 398,33

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Figure Captions

FIG. 1.–Plot of the quantity $g \times 10^5$, at decoupling, as a function of the wavenumber k. The FT of the total density contrast is $\epsilon = ge^{i\alpha}$.

FIG. 2.–The same plot as in Fig. 1 for the quantity $g_\gamma \times 10^5$. The FT of the radiation density contrast is $\epsilon_\gamma = g_\gamma e^{i\alpha}$.

FIG. 3.–The same plot as in Fig. 1 for the quantity $\dot{g}_\gamma \times 10^4$.

FIG. 4.–Simulation of the initial $(\frac{\delta T}{T})_\sigma$ fluctuations in a $17^\circ \times 17^\circ$ region of the LSS. Only the first term of the r.h.s. of Eq. (30) is taken into account.

FIG. 5.–Simulation of the Sachs-Wolfe $(\frac{\delta T}{T})_\sigma$ fluctuations produced by scales between 0.13° and 4.5° in a $17^\circ \times 17^\circ$ region. Only the second and the third terms of the r.h.s. of Eq. (30) are considered.

FIG. 6.–Simulation of the Doppler $(\frac{\delta T}{T})_\sigma$ fluctuations in a $17^\circ \times 17^\circ$ region. Computations are based on the fourth term of the r.h.s. of Eq. (30).

FIG. 7.–Simulation of the same effect as in Fig. 5 for scales larger than 4.5°. Spherical harmonics are used.

FIG. 8.–Simulation of the total $(\frac{\delta T}{T})_\sigma$ fluctuations in a $17^\circ \times 17^\circ$ region. All the terms of the r.h.s of Eq. (30) and all the significant scales are considered.

FIG. 9.–This plot shows the frequency f of the $\frac{\Delta T}{T}$ differences in two cases. The dotted line corresponds to ~ 13500 temperature differences numerically obtained from the simulation of Fig. 8. The solid line is a Gaussian distribution with the same rms as the set of numerical differences. The chopping angle is $\zeta = 1.4^\circ$

TABLE 1

TEMPERATURE DIFFERENCES $\frac{\Delta T}{T}$ FOR $\zeta = 1.4^{\circ}$

simulation	rms value	minimum difference	maximum difference
Fig. 4	2.54×10^{-5}	-9.00×10^{-5}	8.68×10^{-5}
Fig. 5	1.47×10^{-5}	-5.38×10^{-5}	5.78×10^{-5}
Fig. 6	2.25×10^{-5}	-7.86×10^{-5}	8.16×10^{-5}
Fig. 7	3.68×10^{-6}	-1.18×10^{-5}	1.17×10^{-5}
Fig. 8	3.65×10^{-5}	-1.25×10^{-4}	1.28×10^{-4}