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Renormalization Group Invariance of Exact Results in Supersymmetric Gauge Theories*

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Abstract

We clarify the notion of Wilsonian renormalization group (RG) invariance in supersymmetric gauge theories, which states that the low-energy physics can be kept fixed when one changes the ultraviolet cutoff, provided appropriate changes are made to the bare coupling constants in the Lagrangian. We first pose a puzzle on how a quantum modified constraint (such as $\mathrm{Pf}(Q^iQ^j) = \Lambda^{2(N+1)}$ in $\mathrm{SP}(N)$ theories with $N+1$ flavors) can be RG invariant, since the bare fields Q^i receive wave function renormalization when one changes the ultraviolet cutoff, while we naively regard the scale Λ as RG invariant. The resolution is that Λ is *not* RG invariant *if* one sticks to canonical normalization for the bare fields as is conventionally done in field theory. We derive a formula for how Λ must be changed when one changes the ultraviolet cutoff. We then compare our formula to known exact results and show that their consistency requires the change in Λ we have found. Finally, we apply our result to models of supersymmetry breaking due to quantum modified constraints. The RG invariance helps us to determine the effective potential along the classical flat directions found in these theories. In particular, the inverted hierarchy mechanism does not occur in the original version of these models.

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1 Introduction

The last two years have seen remarkable progress in understanding the dynamics of supersymmetric gauge theories (for a review, see [1]). It is now worthwhile to consider model building implications of strong supersymmetric gauge dynamics, especially in the areas of composite models or dynamical supersymmetry breaking. Quantitative results are often required in many phenomenological applications. For instance, the exact vacuum structure and mass spectrum are needed for realistic models of dynamical supersymmetry breaking. Similarly, the Yukawa couplings must be determined in a realistic composite model. It is, therefore, useful to have a closer look at the quantitative results which follow from exactly solved supersymmetric gauge theories.

Actually, a detailed look at these exact results leads to some possible confusions. For instance, the quantum modified constraint in SP(N) theories with $N+1$ flavors:

$$\mathrm{Pf}(Q^iQ^j) = \Lambda^{2(N+1)} \tag{1.1}$$

appears inconsistent at the first sight. The left-hand side involves quantum fields which acquire wave function renormalization, while the right-hand side appears renormalization group (RG) invariant.

It is the purpose of this paper to clarify possible confusions associated with the RG invariance of exact results. RG analysis always contains two steps. The first step is naive dimensional analysis which changes all dimensionful parameters by the same factor e^t; it in particular changes the cutoff scale M where the theory is defined to e^tM. The second is the readjustment of the bare parameters in the Lagrangian as the cutoff scale is changed from e^tM back to M, keeping the low energy physics fixed. The first part is of course trivial. The second part requires care. We will show that the scale Λ actually changes when one changes the cutoff back to M, and hence is *not* RG invariant. We will discuss in detail why this is true and how an improved understanding helps to avoid possible further confusions. Although most of the essential ingredients in this paper are already contained in the seminal work of Shifman and Vainshtein [2], we hope that our paper will help in clarifying this subtle issue and in applying RG invariance to practical problems. See also other analyses in Refs. [3, 4].

The main result of the paper is quite simple. When one changes the

ultraviolet cutoff M to $M' < M$ by integrating out a momentum slice, and keeps the same form for the Lagrangian, *i.e.* canonical kinetic terms for the bare chiral superfields,* one needs to replace the holomorphic gauge coupling constant† as

$$\frac{8\pi^2}{g_h^2} \to \frac{8\pi^2}{g_h'^2} = \frac{8\pi^2}{g_h^2} + b_0 \ln \frac{M}{M'} - \sum_i T_F^i \ln Z_i(M', M) \tag{1.2}$$

where $b_0 = -3C_A + \sum_i T_F^i$ is the one-loop β-function, T_F^i is the β-function coefficient for a chiral multiplet $T_F^i \delta^{ab} = \mathrm{Tr}_i T^a T^b$, and $Z_i(M', M)$ is the coefficient of the kinetic term for the chiral multiplet i when the modes between M' and M are integrated out. Employing this formula, it is straightforward to check the consistency of various results.

The above formula implies that the dynamical scale which appears in exact results:

$$\Lambda(M, 1/g_h^2) \equiv M \exp(8\pi^2/g_h^2 b_0) \tag{1.3}$$

is not RG invariant in theories with matter multiplets. Under the change of the cutoff and bare parameters, it changes to

$$\Lambda(M', 1/g_h'^2) = \Lambda(M, 1/g_h^2) \prod_i Z_i(M, M')^{-T_F^i/b_0}. \tag{1.4}$$

This observation can solve possible confusions about the RG invariance of exact results and effective potentials. We point out that a naive argument (regarding Λ RG invariant) gives a qualitatively incorrect conclusion for the vacuum structure of a theory breaking supersymmetry dynamically. The result is interesting for model building: the naive understanding allows the inverted hierarchy mechanism to be realized in these models, whereas the correct understanding shows that this is impossible.

*We stick to canonical normalization for chiral superfields simply because the first part of the RG analysis (naive scaling) preserves the normalization of the fields and therefore this choice makes the application of the RG analysis simpler. When one would like to keep the holomorphy of the gauge coupling constant manifest, one needs to keep track of the wave function renormalization in a different manner. The same results are obtained for physical quantities either way: see Section 4.7 for an example.

†A holomorphic gauge coupling is defined by the coefficient of the WW operator in the Lagrangian with $W_\alpha = \bar{D}^2 e^{-2V_h} D_\alpha e^{2V_h}$. We will explain how the holomorphic gauge coupling constant is related to the one in canonical normalization $W_\alpha = \bar{D}^2 e^{-2g_c V_c} D_\alpha e^{2g_c V_c}$ used in the perturbation theory in Section three.

The paper is organized as follows. In the next section, we formulate the Wilsonian renormalization program in the context of supersymmetric gauge theories. We derive the formula Eq. (1.2) in this section. The same result is derived by perturbative calculations in section three. Section four describes various examples where our formula guarantees the consistency of known results. In section five, we apply our improved understanding to a particular model where one might naively expect the inverted hierarchy mechanism to work. A careful application of our formalism demonstrates that this is not the case. We conclude in section six.

2 Wilsonian Renormalization Group

In this section, we review the notion of Wilsonian Renormalization Group and apply it to supersymmetric gauge theories. Based on the Shifman–Vainshtein [2] result that the renormalization of the gauge kinetic term is exhausted at one-loop, and the anomalous Jacobian of the path integral under the rescaling of the quantum fields [5], we determine the correct readjustment of the bare couplings to derive Eq. (1.2).

A field theory is normally defined by specifying the bare parameters λ_i^0 and some cutoff scale M. All Green's functions can then be calculated as functions of λ_i^0 and M. To work out Green's functions at energy scales much below the cutoff M, it is convenient to "integrate out" physics between μ and M in a path integral and write down a new Lagrangian with a cutoff μ which is close to the energy scale of the interest. It is a non-trivial fact (related to the renormalizability of the theory) that one does not need to specify infinite number of bare couplings for all possible operators: those for relevant (*i.e.* dimension ≤ 4) operators are enough to define the theory. Therefore, one can define the RG flow of the finite number of bare parameters as one changes the cutoff gradually by "integrating out" modes.

Practically, we determine the bare couplings from experiments. By measuring the amplitudes corresponding to the relevant operators at low energies E (which we refer to loosely as $\lambda_i(E)$), we can work backwards and determine what values of λ_i^0 are needed to reproduce the measured $\lambda_i(E)$. If we work with a different cutoff M', but wish to reproduce the same observed values of the $\lambda_i(E)$, a different set of bare parameters $\lambda_i^{0\prime}$ must be chosen. However, once this choice is made, the predictions for all other low energy

amplitudes are identical * whether we work with the theory based on (λ_i^0, M) or $(\lambda_i^{0\prime}, M')$.

We should emphasize that none of our discussions depend on the precise way in which the theory is cutoff at the scale M, nor the precise way of integrating out modes. The point is simply that it is possible to change the bare couplings λ_i^0 with the cutoff M while keeping the low-energy physics fixed. This is the formal definition of the "integrating out modes" procedure. The way in which the λ_i^0 must change with M, while keeping the low energy physics fixed, is encoded in Renormalization Group Equations (RGE's) for the λ_i^0:

$$M\frac{d}{dM}\lambda_i^0 = \beta_i^0(\lambda^0). \tag{2.1}$$

All of the usual results of RG analysis follow from the above considerations. The procedure is always the same: for any quantity of interest, first, one rescales all parameters (including the cutoff) by naive dimensional analysis, then one changes the cutoff back to the original one while simultaneously changing the bare couplings in accordance with Eq. (2.1). As an example, consider the 1PI 4-point function $\Gamma^4(p_i; \lambda^0, M)$ for a $\lambda^0\phi^4/4!$ theory with cutoff M, and with all the (Euclidean) momenta $|p_i| \sim \mu \ll M$. If we just compute Γ^4 in perturbation theory, we find

$$\Gamma^4(p_i; \lambda^0, M) = \lambda^0 - \frac{3}{16\pi^2}\lambda^{0\,2}\ln\frac{M}{\mu} + \cdots \tag{2.2}$$

where $\cdots$ stands for higher order terms in perturbation theory and non-logarithmic corrections which depend on p_i and μ. For $\mu \ll M$, the logarithm in the above becomes large and the 1-loop term becomes comparable to the tree-level piece, making perturbation theory unreliable. Let us now apply the procedure outlined above for applying the Wilsonian RGE. First, we rescale everything by dimensional analysis:

$$\Gamma^4(p_i; \lambda^0, M) = \Gamma^4(e^t p_i; \lambda^0, e^t M). \tag{2.3}$$

* Actually, for the amplitudes for two theories with cutoffs M, M' to be *exactly* the same, an infinite number of higher dimension operators will have to be included in the definition of the theory with cutoff M'. However, it is always possible to absorb the effects of higher dimension operators into the relevant operators, up to calculable finite corrections suppressed by powers in E/M, E/M'.

Next, we use the Wilsonian RGE to bring the cutoff on the RHS of the above back from $e^t M$ to M while changing λ^0 appropriately:

$$\Gamma^4(e^t p_i; \lambda^0, e^t M) = \Gamma^4(e^t p_i; \lambda(\lambda^0; t), M) \tag{2.4}$$

where $\lambda(\lambda^0; t)$ is the solution of $(d/dt)\lambda = -\beta^0(\lambda)$ with $\lambda(\lambda^0; 0) = \lambda^0$. We then have

$$\begin{aligned} \Gamma^4(p_i; \lambda^0, M) &= \Gamma^4(e^t p_i; \lambda(\lambda^0; t), M) \\ &= \lambda(\lambda^0; t) - \frac{3}{16\pi^2}\lambda(\lambda^0; t)^2 \ln\frac{M}{e^t \mu} + \cdots \end{aligned} \tag{2.5}$$

and if we choose t so that $e^t \mu \sim M$, the logarithms on the second line of the above are small and the perturbation expansion is reliable. In particular we have the standard result

$$\Gamma^4(p_i; \lambda^0, M) = \lambda(\lambda^0, t \sim \ln M/\mu) + \text{small calculable corrections.} \tag{2.6}$$

Let us now consider the Wilsonian RGE for supersymmetric gauge theories with matter. With some cutoff M, the theory is specified by the bare Lagrangian

$$\mathcal{L}(M) = \frac{1}{4}\int d^2\theta \frac{1}{g_h^2} W^a W^a + h.c. + \int d^4\theta \sum_i \phi_i^\dagger e^{2V_i} \phi_i, \tag{2.7}$$

where $V_i = V^a T_i^a$, and T_i^a are generators in the representation of the chiral superfield ϕ_i. We are working with the holomorphic normalization for the gauge coupling

$$\frac{1}{g_h^2} = \frac{1}{g^2} + i\frac{\theta}{8\pi^2}. \tag{2.8}$$

Actually, there is a hidden parameter in the above Lagrangian: the coefficient of the matter field kinetic term $Z_i(M)$. However, we have chosen to work with canonical normalization for the bare matter field kinetic terms and we have set $Z_i(M) = 1$. There are two reasons for taking canonical normalization: (1) this is the conventional choice in field theory, (2) it is easy to compare Lagrangians with different cutoffs with fixed normalization of the bare fields, since the naive dimensional analysis part of the RG analysis preserves the normalization of the kinetic term. Now with this choice of the normalization,

when we change the cutoff from M to M', how should the bare parameters be changed to keep the low energy physics fixed? Shifman and Vainshtein argued that, to all orders of perturbation theory, the couplings should be changed so that the Lagrangian with cutoff M' becomes

$$\begin{aligned}\mathcal{L}(M') &= \frac{1}{4}\int d^2\theta \left(\frac{1}{g_h^2} + \frac{b_0}{8\pi^2}\ln\frac{M}{M'}\right) W^a W^a + h.c. \\ &\quad + \int d^4\theta \sum_i Z_i(M,M')\phi_i^\dagger e^{2V_i}\phi_i. \end{aligned} \tag{2.9}$$

That is, the holomorphic coupling receives only 1-loop contributions. However, the matter field kinetic terms do not remain canonical in going from M to M'.

One can easily understand that the change of $1/g_h^2$ is exhausted at 1-loop in perturbation theory as long as the change is holomorphic. This is because holomorphy and periodicity in θ demand that one can expand the dependence in Fourier series of $\exp(-8\pi^2/g_h^2)$,

$$\frac{1}{g_h^2} + \sum_{n\geq 0} a_n \left(\frac{M}{M'}\right) \exp\left(-n\frac{8\pi^2}{g_h^2(M)}\right). \tag{2.10}$$

The sum is limited to the positive frequencies $n \geq 0$ to ensure that the theory has a well-defined weak coupling limit $g_h^2 \to 0$. The terms with $n > 0$ can never arise in perturbation theory, and we drop them. The function $a_0(M/M')$ must satisfy the consistency condition $a_0(M/M') + a_0(M'/M'') = a_0(M/M'')$, and hence it must be a logarithm. This proves the one-loop law of the change in holomorphic gauge coupling constant.

The point is, however, that the change in $1/g_h^2$ is holomorphic only when the normalization for the matter field kinetic terms (which is manifestly non-holomorphic, being only a function of g) is allowed to change from 1 to $Z(M,M')$.

In order to go back to canonical normalization for the matter fields, one simply redefines $\phi = Z(M,M')^{-1/2}\phi'$. However, the path integral measure $D\phi$ is not invariant under this change, $D(Z(M,M')^{-1/2}\phi') \neq D\phi'$; there is an anomalous Jacobian [5]. In our case, $Z(M,M')$ is positive and real, but it is sensible to look at $D(Z^{-1/2}\phi')$ for a general complex number Z since ϕ' is a chiral superfield. When $Z = e^{i\alpha}$ is a pure phase, the field redefinition is a chiral rotation on the fermionic component of ϕ' and the Jacobian is the

one associated with the chiral anomaly. This Jacobian is exactly known [5] and is cutoff independent:

$$\begin{aligned} & D(e^{-i\alpha/2}\phi')D(e^{+i\alpha/2}\phi'^{\dagger}) \\ = \; & D\phi' D\phi'^{\dagger} \exp\left(\frac{1}{4}\int d^4y \int d^2\theta \frac{T_F(\phi)}{8\pi^2} \ln(e^{i\alpha}) W^a W^a + h.c.\right). \end{aligned} \tag{2.11}$$

In the case where Z is a general complex number, the Jacobian will in general have F terms and D terms (such as $Re(\ln Z)W^*W^*WW$). However, since R symmetry is at least good in perturbation theory, the F terms can only contain W^aW^a, and its coefficient is the same as in Eq. (2.11) with $\ln e^{i\alpha}$ replaced by $\ln Z$. The D terms are all higher dimensional operators suppressed by powers of the cutoff and can be neglected.†

Therefore, if we wish to keep canonical normalization for the matter fields in changing the cutoff from M to M', the Lagrangian at cutoff M' must be given by

$$\mathcal{L}'(M') = \frac{1}{4}\int d^2\theta \frac{1}{g_h'^2} W^a W^a + h.c. + \int d^4\theta \sum_i \phi_i^{\dagger} e^{2V_i} \phi_i \tag{2.12}$$

where

$$\frac{1}{g_h'^2} = \frac{1}{g_h^2} + \frac{b_0}{8\pi^2} \ln \frac{M}{M'} - \sum_i \frac{T_F(\phi^i)}{8\pi^2} \ln Z_i(M, M'). \tag{2.13}$$

We can rephrase the above results in terms of the scale $\Lambda(M, 1/g_h^2)$ (see Eq. (1.3)). If we change the cutoff from M to M', and always work with canonical normalization for the matter fields, we have

$$\Lambda(M, 1/g_h^2) \to \Lambda(M', 1/g_h'^2) = \Lambda(M, 1/g_h^2) \prod_i Z_i(M, M')^{-T_F^i/b_0}. \tag{2.14}$$

So far we have considered the case with zero superpotential, but the extension to the general case is obvious. For instance, suppose we add a

†In a general non-supersymmetric theory, it is not possible to simply throw away higher dimension operators suppressed by the cutoff, since loops with these operators may contain power divergences which negate the cutoff suppression; what *can* be done is to set the operators to zero with an appropriate modification of the renormalizable couplings. However, in supersymmetric theories, the non-renormalization theorem makes it impossible for the higher dimensional D terms to ever contribute to the coefficient of W^aW^a which is an F term, and so the higher dimensional D terms really can be dropped [6]

superpotential term of the form $\int d^2\theta W = \int d^2\theta \lambda^{ijk}\phi_i\phi_j\phi_k$. Then by the non-renormalization theorem, λ^{ijk} stays the same if we allow non-canonical kinetic terms We, however, insist on working with canonical kinetic terms, and we must have $\lambda'^{ijk} = Z_i(M,M')^{-1/2}Z_j(M,M')^{-1/2}Z_k(M,M')^{-1/2}\lambda^{ijk}$.

3 Perturbative Derivation

In this section, we rederive the result obtained in the previous section by perturbative calculations. We first review how one can relate perturbative results to the exact results, and then discuss how we change the bare parameters as we change the ultraviolet cutoff. The final result is the same as Eq. (1.2).*

Comparison of the perturbative results to the exact results is a somewhat confusing issue. The so-called anomaly puzzle is one famous example of such a confusion. In supersymmetric theories, the $\mathrm{U}(1)_R$ current belongs to the same supermultiplet as the trace of the energy-momentum tensor, and hence the chiral anomaly and the trace anomaly are related. On the other hand, the chiral anomaly is exhausted at one-loop (Adler–Bardeen theorem) while the trace anomaly is not in $N = 1$ theories. Shifman and Vainshtein made a breakthrough on this question by discriminating two definitions of coupling constants: "canonical" and "holomorphic".† The holomorphic gauge coupling g_h runs only at one-loop, while the canonical gauge coupling g_c has higher order β-functions. There is a simple relation between them, the Shifman–Vainshtein formula,

$$\frac{8\pi^2}{g_h^2} = \frac{8\pi^2}{g_c^2} + C_A \ln g_c^2, \tag{3.1}$$

*Note that the analysis in this section is not independent from the one in the previous section; it is simply a reanalysis in a different language. The one-loop exhaustion of the renormalization of W^aW^a used in the previous section and NSVZ β-function used in this section are closely related [6].

†We find the terminology by Shifman and Vainshtein rather confusing. In our understanding, what they call "1PI" coupling constant is not what appears in 1PI effective actions; they are still coupling constants in Wilsonian effective action. The only difference between them is that one employs canonical normalization for gauge field kinetic term in "1PI" couplings while holomorphic normalization in "Wilsonian" couplings. We will rather refer to them as "canonical" and "holomorphic" gauge coupling constants in this paper. We will discuss more on this issue in our forthcoming paper [6].

where $f^{acd}f^{bcd} = C_A\delta^{ab}$. The difference $C_A \ln g^2$ appears due to an anomalous Jacobian in the path integral when one rescales the vector multiplet V_h which appears in the field strength $W_\alpha = \bar{D}^2 e^{-2V_h} D_\alpha e^{2V_h}$ to the one in canonical normalization $V_h = g_c V_c$. The Lagrangian written in terms of V_h does not need the gauge coupling constant in the exponent, and hence does not need to separate the θ angle from the gauge coupling constant. This normalization of the vector multiplet therefore keeps holomorphicity of the gauge coupling constant manifest (holomorphic normalization) while the canonical one requires an explicit dependence on the gauge coupling constant in the exponent.

There still remains the question how the canonical gauge coupling constant g_c in the Wilsonian action is related to the perturbative definitions of the running coupling constant in popular schemes such as $\overline{\mathrm{DR}}$. We are not aware of a complete answer to this question,[‡] even though one can work out the relation between the two coupling constants at each order in perturbation theory [7].

There is a known "exact" β-function in supersymmetric gauge theories by Novikov–Shifman–Vainshtein–Zakharov (NSVZ) [8]. Our understanding is that this exact β-function applies to the canonical gauge coupling constant in a Wilsonian action and is hence appropriate for our analysis [6]. Therefore we employ the NSVZ β-function for our perturbative analysis to determine the necessary change of the bare parameters to keep the low-energy physics fixed as we change the ultraviolet cutoff. The exact NSVZ β-function is given by

$$\mu\frac{dg^2}{d\mu} = \beta = -\frac{g^4}{8\pi^2}\frac{3C_A - \sum_i T_F^i(1-\gamma_i)}{1 - C_A g^2/8\pi^2}, \tag{3.2}$$

with $\gamma_i = (\mu d/d\mu)\ln Z_i(\mu, M)$. Of course the β-function for the gauge coupling constant is the same up to two-loop order in any schemes. From the strictly perturbative point of view, one can regard our analysis as a two-loop analysis in, say, $\overline{\mathrm{DR}}$. The RGE can then be integrated with the NSVZ β-function, and one finds that

$$\left(\frac{8\pi^2}{g_c^2(\mu)} + C_A \ln g_c^2(\mu) + \sum_i T_F^i \ln Z_i(\mu, M)\right)$$

[‡]At least in some models, one can define a regularized Wilsonian action of the theory and compare the canonical gauge coupling constant in the Wilsonian action to the perturbative definition [6].

$$= \left(\frac{8\pi^2}{g_c^2(M)} + C_A \ln g_c^2(M) \right) + b_0 \ln \frac{M}{\mu}. \tag{3.3}$$

The combination in the bracket runs only at one-loop, and the wavefunction renormalization factors are by definition unity at the cutoff scale, $Z_i(M, M) = 1$. The β-function coefficient is given by $b_0 = -3C_A + \sum_i T_F^i$.

Now the strategy is to change the bare parameters M and $g_c^2(M)$ while keeping the low-energy physics ($g_c^2(\mu)$) fixed. Naively, the change required appears to come from $b_0 \ln(M/\mu)$ in the right-hand side, and the change

$$\left(\frac{8\pi^2}{g_c^2(M')} + C_A \ln g_c^2(M') \right) = \left(\frac{8\pi^2}{g_c^2(M)} + C_A \ln g_c^2(M) \right) + b_0 \ln \frac{M}{M'} \tag{3.4}$$

might appear to be enough. However, this is not correct, because the wave function renormalization factor $Z_i(\mu, M)$ in the left-hand side also depends on M implicitly due to the boundary condition $Z_i(M, M) = 1$.

The trick is that the wave function renormalization is multiplicative:

$$Z_i(\mu, M) = Z_i(\mu, M') Z_i(M', M). \tag{3.5}$$

Then Eq. (3.3) can be rewritten as

$$\begin{aligned} &\left(\frac{8\pi^2}{g_c^2(\mu)} + C_A \ln g_c^2(\mu) + \sum_i T_F^i \ln Z_i(\mu, M') \right) \\ &= \left(\frac{8\pi^2}{g_c^2(M)} + C_A \ln g_c^2(M) \right) + b_0 \left(\ln \frac{M'}{\mu} + \ln \frac{M}{M'} \right) - \sum_i T_F^i \ln Z_i(M', M). \end{aligned} \tag{3.6}$$

It is now clear that the correct change of the bare parameters is

$$\begin{aligned} &\left(\frac{8\pi^2}{g_c^2(M')} + C_A \ln g_c^2(M') \right) \\ &\quad = \left(\frac{8\pi^2}{g_c^2(M)} + C_A \ln g_c^2(M) \right) + b_0 \ln \frac{M}{M'} - \sum_i T_F^i \ln Z_i(M', M), \end{aligned} \tag{3.7}$$

which keeps the low-energy physics ($g_c^2(\mu)$) fixed.

The final step is to rewrite the above relation in terms of the holomorphic gauge coupling g_h using the Shifman–Vainshtein formula Eq. (3.1),

$$\frac{8\pi^2}{g_h^2}(M') = \frac{8\pi^2}{g_h^2}(M) + b_0 \ln \frac{M}{M'} - \sum_i T_F^i \ln Z_i(M', M) \tag{3.8}$$

This is indeed the same relation as obtained in the previous section.

If a chiral superfield has a coupling λ^{ijk} in the superpotential, it is renormalized only due to wave function renormalization because of the non-renormalization theorem. The low-energy coupling is given by

$$\lambda^{ijk}(\mu) = \lambda^{ijk} Z_i^{-1/2}(\mu, M) Z_j^{-1/2}(\mu, M) Z_k^{-1/2}(\mu, M). \tag{3.9}$$

By using the multiplicative property of the wave function renormalization again, the change of the bare parameter is

$$\lambda'^{ijk} = \lambda^{ijk} Z_i^{-1/2}(M', M) Z_j^{-1/2}(M', M) Z_k^{-1/2}(M', M) \tag{3.10}$$

to keep $\lambda^{ijk}(\mu)$ fixed when one changes the cutoff.

4 Examples

In this section, we apply our result Eqs. (1.2,1.4) to many examples. The RG invariance is checked usually with two steps, (1) naive dimensional analysis, and (2) the change of cutoff parameters. For simplicity of the presentation, we do not discuss the first part since it is rather trivial. The non-trivial part of the analysis is the correct application of the change of bare parameters as derived in previous sections.

4.1 Quantum Modified Moduli Space (I)

In SP(N) theories with $N+1$ flavors, Intriligator and Pouliot found the quantum modified constraint [9]

$$\mathrm{Pf}(Q^i Q^j) = \Lambda^{2(N+1)}. \tag{4.1}$$

Dine and Shirman [3] correctly emphasized that the fields in the left-hand side are bare fields in a Wilsonian action with an ultraviolet cutoff M. The Lagrangian of the model is simply

$$\mathcal{L} = \int d^2\theta \, \frac{1}{4g_h^2} W^a W^a + h.c. + \int d^4\theta \, Q^{i\dagger} e^{2V} Q^i \tag{4.2}$$

in terms of bare fields.

As explained in Section 2, a Wilsonian RG allows the change of ultraviolet cutoff while keeping the low-energy physics fixed by appropriately changing the bare coupling constants in the theory. With the same Lagrangian given at a different cutoff M', a coupling constant g'_h, and bare fields Q'^i, we must find

$$\mathrm{Pf}(Q'^i Q'^j) = \Lambda^{2(N+1)}. \tag{4.3}$$

if Λ were a RG invariant quantity. Note that we need to keep the form of the Lagrangian the same no matter how we change the cutoff; therefore the fields Q' must have canonical kinetic terms as Q do.

The relation between the bare fields in two different Lagrangians, Q and Q' can be calculated. When one integrates out modes between M' and M, the original bare fields Q acquire corrections to the kinetic terms by a factor $Z_Q(M', M)$. The bare fields Q' have canonical normalization in the Lagrangian with the cutoff M', and hence they are related by

$$Q' = Z_Q^{1/2}(M', M)Q. \tag{4.4}$$

Therefore, the left-hand sides of the constraint equations are related by

$$\mathrm{Pf}(Q'^i Q'^j) = Z_Q^{N+1}(M', M)\mathrm{Pf}(Q^i Q^j), \tag{4.5}$$

and hence the right-handed sides must also differ by Z_Q^{N+1}. Then Eq. (4.3) is inconsistent.

Our result (1.4) says that the dynamical scale of the theory with cutoff M' is related to the original one by

$$\Lambda' = \Lambda \left[Z_Q^{-T_F/b_0}(M', M)\right]^{2N_f} = \Lambda Z_Q^{1/2}(M', M) \tag{4.6}$$

with $T_F = 1/2$ and $b_0 = -2(N+1)$. Now it is easy to see that the quantum modified constraint holds between the primed fields and the primed dynamical scale:

$$\mathrm{Pf}(Q'^i Q'^j) = \Lambda'^{2(N+1)}. \tag{4.7}$$

This is a consistency check that the quantum modified constraint is RG invariant.

4.2 Quantum Modified Moduli Space (II)

It is amusing to see how the quantum modified constraints are RG invariant in more complicated cases. Let us look at SU($2k+1$) models with one anti-symmetric tensor A, three fundamentals Q^a ($a = 1,2,3$) and $2k$ anti-fundamentals $\tilde{Q}_i$ ($i = 1,\ldots,2k$) [10]. The moduli space can be described by the gauge invariant polynomials

$$\begin{aligned} M_i^a &= \tilde{Q}_i^\alpha \, Q_\alpha^a \\ X_{ij} &= A_{\alpha\beta} \, \tilde{Q}_i^\alpha \, \tilde{Q}_j^\beta \\ Y^a &= Q^a_{\alpha_{2k+1}} \, \epsilon^{\alpha_1 \ldots \alpha_{2k+1}} \, A_{\alpha_1\alpha_2} \ldots A_{\alpha_{2k-1}\alpha_{2k}} \\ Z &= \epsilon^{\alpha_1 \ldots \alpha_{2k+1}} \, A_{\alpha_1\alpha_2} \ldots A_{\alpha_{2k-3}\alpha_{2k-2}} \, Q^a_{\alpha_{2k-1}} \, Q^b_{\alpha_{2k}} \, Q^c_{\alpha_{2k+1}} \, \epsilon_{abc} \, , \end{aligned} \tag{4.8}$$

and the quantum modified constraint

$$Y \cdot M^2 \cdot X^{k-1} - \frac{k}{3} \, Z \, \mathrm{Pf} X = \Lambda^{4k+2} \, . \tag{4.9}$$

By following Eq. (1.4), we find

$$\Lambda'^{-b_0} = \Lambda^{-b_0} Z_A^{k-1/2} Z_Q^{3/2} Z_{\tilde{Q}}^k \tag{4.10}$$

with $b_0 = -(2k+1)$. The quantum modified constraint is indeed RG invariant as we change the cutoff from M to M', replacing all fields by primed fields (with canonical kinetic terms) *and* the dynamical scale Λ by Λ'.

4.3 Matching Equations (I)

When there is a massive chiral superfield, the gauge coupling constants in a theory with a massive field (high-energy theory) and the other theory where the massive field is integrated out (low-energy theory) are related by matching equations. For SU(N) gauge group with a single massive vector-like pair in the fundamental (Q) and anti-fundamental ($\tilde{Q}$) representations, the holomorphic coupling constants in high-energy ($g^2_{h,HE}$) and low-energy ($g^2_{h,LE}$) theories are related by

$$\frac{8\pi^2}{g^2_{h,LE}} = \frac{8\pi^2}{g^2_{h,HE}} + \ln \frac{M}{m} \tag{4.11}$$

where m is the bare mass of the field. This form can be completely fixed (up to a possible constant) by the holomorphy in $8\pi^2/g_h^2$ and m, and the anomaly under the chiral U(1) rotation of the matter fields. We drop the possible constant in the following equations, and it can be easily recovered if necessary.

Under the change of the cutoff, we rewrite the left-hand side as

$$\frac{8\pi^2}{g_{h,LE}'^2} = \frac{8\pi^2}{g_{h,LE}^2} + b_{0,LE} \ln \frac{M}{M'} - \sum_{i \neq Q, \tilde{Q}} T_F^i \ln Z_i(M', M) \tag{4.12}$$

where the sum does not include the massive field Q, $\tilde{Q}$ which are integrated out in the low-energy theory. The coupling in the high-energy theory is also rewritten as

$$\frac{8\pi^2}{g_{h,HE}'^2} = \frac{8\pi^2}{g_{h,HE}^2} + b_{0,HE} \ln \frac{M}{M'} - \sum_i T_F^i \ln Z_i(M', M), \tag{4.13}$$

but here the sum includes the massive field. The β-functions are related as $b_{0,HE} = b_{0,LE} + 1$. Now the matching equation reads as

$$\begin{aligned} \frac{8\pi^2}{g_{h,LE}'^2} &= \frac{8\pi^2}{g_{h,HE}'^2} - \ln \frac{M}{M'} + \frac{1}{2}(\ln Z_Q(M', M) + \ln Z_{\tilde{Q}}(M', M)) + \ln \frac{M}{m} \\ &= \frac{8\pi^2}{g_{h,HE}'^2} + \ln \frac{M'}{m'} \end{aligned} \tag{4.14}$$

with $m' = Z_Q^{-1/2}(M', M) Z_{\tilde{Q}}^{-1/2}(M', M) m$. Therefore, the matching equation takes the same form with the new cutoff and bare parameters.

One can also check the consistency with the perturbative calculations on matching of canonical gauge coupling constants. For instance in $\overline{\mathrm{DR}}$ scheme, the one-loop matching equation* is simply $g_{c,LE}^2(m_r) = g_{c,HE}^2(m_r)$, where m_r is the renormalized mass of the chiral multiplet. By using the Shifman–Vainshtein relation (3.1) between the canonical and holomorphic gauge couplings and NSVZ exact β-function (using the integrated form Eq. (3.3)), the

*Recall that one-loop matching is required when one employs two-loop RGE. We are not aware of $\overline{\mathrm{DR}}$ calculations of two-loop matching which can tell us whether m_r must be the on-shell mass or $\overline{\mathrm{DR}}$ mass where the latter is more likely.

matching condition between the canonical gauge couplings for high-energy and low-energy theories can be obtained as

$$\begin{aligned}\frac{8\pi^2}{g^2_{c,LE}(\mu)} &= \frac{8\pi^2}{g^2_{c,HE}(\mu)} + \ln\frac{\mu}{m} + \frac{1}{2}(\ln Z_Q(\mu,M) + \ln Z_{\tilde{Q}}(\mu,M)) \\ &= \frac{8\pi^2}{g^2_{c,HE}(\mu)} + \ln\frac{\mu}{m(\mu)}\,. \end{aligned} \tag{4.15}$$

Therefore the gauge coupling constants can be matched at the renormalized mass of the heavy field $\mu = m(\mu) = mZ_Q^{-1/2}(\mu,M)Z_{\tilde{Q}}^{-1/2}(\mu,M)$ as expected.

4.4 Matching Equations (II)

When a chiral superfield acquires an expectation value and the Higgs mechanisms occurs, the gauge coupling constants in a theory with the full gauge group (high-energy theory) and the other theory only with unbroken gauge group (low-energy theory) are related by matching equations. For SU(N) gauge group with an expectation value of a single vector-like pair in the fundamental and anti-fundamental representations Q and $\tilde{Q}$, they can acquire an expectation value along the D-flat direction $Q = \tilde{Q}$ and the gauge group breaks down to SU($N-1$). The holomorphic coupling constants in high-energy ($g^2_{h,HE}$) and low-energy ($g^2_{h,LE}$) theories are related by

$$\frac{8\pi^2}{g^2_{h,LE}} = \frac{8\pi^2}{g^2_{h,HE}} - \ln\frac{M^2}{\tilde{Q}Q} \tag{4.16}$$

where $\tilde{Q}$, Q are the bare fields. This form can be completely fixed (up to a possible constant) by the holomorphy in $8\pi^2/g_h^2$ and $\tilde{Q}$, Q, non-anomalous vector U(1) symmetry, and and the anomaly under the chiral U(1) rotation of the matter fields. We drop the possible constant in the following equations, and it can be easily recovered if necessary.

Under the change of the cutoff, we rewrite the left-hand side as

$$\frac{8\pi^2}{g'^2_{h,LE}} = \frac{8\pi^2}{g^2_{h,LE}} + b_{0,LE}\ln\frac{M}{M'} - \sum_{i\neq Q,\tilde{Q}} T_F^i \ln Z_i(M',M) \tag{4.17}$$

where the sum does not include the massive field Q, $\tilde{Q}$ which are integrated out in the low-energy theory. The coupling in the high-energy theory is also

rewritten as

$$\frac{8\pi^2}{g'^2_{h,HE}} = \frac{8\pi^2}{g^2_{h,HE}} + b_{0,HE} \ln \frac{M}{M'} - \sum_i T_F^i \ln Z_i(M', M), \tag{4.18}$$

but here the sum includes the massive field. The β-functions are related as $b_{0,HE} = b_{0,LE} - 2$. Now the matching equation reads as

$$\begin{aligned} \frac{8\pi^2}{g'^2_{h,LE}} &= \frac{8\pi^2}{g'^2_{h,HE}} + 2\ln\frac{M}{M'} + \frac{1}{2}(\ln Z_Q(M', M) + \ln Z_{\tilde{Q}}(M', M)) - \ln\frac{M^2}{\tilde{Q}Q} \\ &= \frac{8\pi^2}{g'^2_{h,HE}} - \ln\frac{M'^2}{\tilde{Q}'Q'} \end{aligned} \tag{4.19}$$

where the primed fields are defined by

$$\tilde{Q}' = Z_{\tilde{Q}}^{1/2}(M', M)\tilde{Q}, \qquad Q' = Z_Q^{1/2}(M', M)Q. \tag{4.20}$$

Therefore, the matching equation takes the same form with the new cutoff and bare fields.

One can also check the consistency with the perturbative calculations on matching of canonical gauge coupling constants. For instance in $\overline{\mathrm{DR}}$ scheme, the one-loop matching equation† is simply $g^2_{c,LE}(m_V) = g^2_{c,HE}(m_V)$, where m_V is the renormalized mass of the heavy gauge multiplet. By using the Shifman–Vainshtein relation (3.1) between the canonical and holomorphic gauge couplings and NSVZ exact β-function (using the integrated form Eq. (3.3)), the matching condition between the canonical gauge couplings for high-energy and low-energy theories can be obtained as

$$\begin{aligned} &\frac{8\pi^2}{g^2_{c,LE}(\mu)} + (N-1)\ln g^2_{c,LE}(\mu) \\ &\quad = \frac{8\pi^2}{g^2_{c,HE}(\mu)} + N\ln g^2_{c,HE}(\mu) - \ln\frac{\mu^2}{\tilde{Q}Q} + \frac{1}{2}(\ln Z_Q(\mu, M) + \ln Z_{\tilde{Q}}(\mu, M)), \end{aligned} \tag{4.21}$$

and hence

$$\frac{8\pi^2}{g^2_{c,LE}(\mu)} = \frac{8\pi^2}{g^2_{c,HE}(\mu)} + \left(N - \frac{1}{2}\right)\ln\frac{g^2_{c,HE}(\mu)}{g^2_{c,LE}(\mu)} + \ln\frac{m_V^2(\mu)}{\mu^2}. \tag{4.22}$$

†Here again we are not aware of $\overline{\mathrm{DR}}$ calculations of two-loop matching which can tell us whether m_V must be the on-shell mass or $\overline{\mathrm{DR}}$ mass.

Here, the renormalized gauge boson mass m_V is defined by

$$m_V^2(\mu) \equiv g_{c,LE}(\mu) g_{c,HE}(\mu) Z_Q^{1/2}(\mu, M) Z_{\tilde{Q}}^{1/2}(\mu, M) \tilde{Q} Q. \tag{4.23}$$

The matching is particularly simple: $g_{c,LE}^2(\mu) = g_{c,HE}^2(\mu)$ at the renormalized gauge boson mass $\mu = m_V(m_V)$ as expected.

4.5 Affleck–Dine–Seiberg superpotential

In SU(N) gauge theories with $N_f < N$, a non-perturbative superpotential is generated,

$$W = \frac{\Lambda^{(3N-N_f)/(N-N_f)}}{(\det \tilde{Q}^i Q^j)^{1/(N-N_f)}}. \tag{4.24}$$

Under the change of the cutoff, we find

$$\Lambda'^{(3N-N_f)} = \Lambda^{(3N-N_f)} Z_Q^{N_f/2} Z_{\tilde{Q}}^{N_f/2}, \tag{4.25}$$

and the superpotential becomes

$$W' = \frac{\Lambda'^{(3N-N_f)/(N-N_f)}}{(\det \tilde{Q}'^i Q'^j)^{1/(N-N_f)}} = \frac{\Lambda^{(3N-N_f)/(N-N_f)} (Z_Q Z_{\tilde{Q}})^{N_f/2(N-N_f)}}{(\det \tilde{Q}^i Q^j)^{1/(N-N_f)} (Z_Q Z_{\tilde{Q}})^{N_f/2(N-N_f)}} = W. \tag{4.26}$$

The Affleck–Dine–Seiberg superpotential is RG invariant.

4.6 Gaugino Condensate

When all chiral superfields are massive, they can be integrated out from the theory and the low-energy pure Yang–Mills theory develops a gaugino condensate. After matching the gauge coupling constant at the threshold, the size of the gaugino condensate is a function of the bare mass of the chiral superfields and the bare gauge coupling constant.

If there are N_f chiral superfields with the same mass m coupled to SU(N) gauge group, the size of the gaugino condensate can be calculated as

$$\langle \lambda \lambda \rangle = m^{N_f/N} \Lambda^{3-N_f/N} \tag{4.27}$$

using holomorphy and $\mathrm{U}(1)_R$ symmetry up to an overall constant. Under the change of the cutoff and bare parameters, the scale Λ changes to

$$\Lambda'^{-b_0} = \Lambda^{-b_0} Z_Q^{N_f/2} Z_{\tilde{Q}}^{N_f/2}. \tag{4.28}$$

Here, $b_0 = -(3N - N_f)$. The corresponding change of the bare mass parameter is

$$m' = m Z_Q^{-1/2} Z_{\tilde{Q}}^{-1/2}. \tag{4.29}$$

It is easy to see that the gaugino condensate is invariant under these changes.

4.7 $N = 2$ theories

An application of our formalism to $N = 2$ theories requires care because of a difference in conventions. Take $N = 2$ supersymmetric QCD with N_f hypermultiplets in fundamental representation. In $N = 1$ language, the particle content of the theory is the vector multiplet V, a chiral multiplet in the adjoint representation ϕ, N_f chiral multiplets Q_i and $\tilde{Q}_i$ $(i = 1, \ldots, N_f)$ in fundamental and anti-fundamental representations, respectively. The Lagrangian in the conventional normalization of fields in the $N = 2$ context is

$$\begin{aligned} \mathcal{L} &= \int d^4\theta \left(Re\left(\frac{1}{g_2^2}\right) 2\mathrm{Tr}\phi_2^\dagger e^{2V}\phi_2 e^{-2V} + Q_i^\dagger e^{2V} Q_i + \tilde{Q}_i^\dagger e^{-2V^T}\tilde{Q}_i \right) \\ &\quad + \int d^2\theta \left(\frac{1}{4g_2^2} W^a W^a + \sqrt{2}\tilde{Q}_i \phi Q_i \right) + h.c. \end{aligned} \tag{4.30}$$

Note that the normalization of the ϕ kinetic term is not canonical. Here we use the notation $1/g_2^2$ to refer to the gauge coupling constant in this normalization. Correspondingly, we refer to the adjoint field in this normalization as ϕ_2. In this normalization, singularities (*e.g.*, massless monopoles/dyons, roots of the baryonic branch) occur on the Coulomb branch of the theory where a symmetric polynomial of the eigenvalues of the adjoint field ϕ takes special values proportional to the dynamical scale Λ_2 in $N = 2$ normalization:

$$\phi_2^k = c_k \Lambda_2^k \equiv c_k \left(M e^{-8\pi^2/g_2^2(2N_c - N_f)} \right)^k \tag{4.31}$$

where we used $b_0 = -(2N_c - N_f)$, and c_k are appropriate constants.

One can ask the question whether the locations of such singularities are RG invariant. They are indeed RG invariant in an obvious manner in $N = 2$ normalization. First of all, we never need to change the normalization of the adjoint field ϕ because the normalization always stays $1/g_2^2$ automatically due to the $N = 2$ supersymmetry. Therefore, the left-hand side of

Eq. (4.31) is RG invariant. Moreover, there is no Z_ϕ contribution to $1/g_2^2$ when one changes the cutoff from M to M'. Second, there is no wavefunction renormalization for the hypermultiplets [11, 12]. Therefore, there is no Z_Q, $Z_{\tilde{Q}}$ contribution to $1/g_2^2$ either. As a result, Λ is RG invariant, the right-hand side of Eq. (4.31) is also RG invariant, and hence Eq. (4.31) remains the same under the change of the cutoff and bare parameters trivially.

If one employs $N = 1$ language, the analysis is far less obvious. First of all, the holomorphic gauge coupling g_1^2 in $N = 1$ language differs from g_2^2 because one scales the adjoint field to make it canonically normalized $\phi_1 = g_2^{-1}\phi_2$, and the anomalous Jacobian [5] gives

$$\frac{8\pi^2}{g_1^2} = \frac{8\pi^2}{g_2^2} + N_c \ln g_2^2. \tag{4.32}$$

The dynamical scale $\Lambda_1 = Me^{-8\pi^2/g_1^2(2N_c-N_f)}$ in $N = 1$ normalization is then related to that in $N = 2$ normalization by

$$\Lambda_1 = \Lambda_2 (g_2^2)^{-N_c/(2N_c-N_f)}. \tag{4.33}$$

When one changes the cutoff from M to M', the adjoint field ϕ_1 receives a wave function renormalization

$$Z_\phi(M', M) = \frac{g_2^2}{g_2'^2}. \tag{4.34}$$

such that the superpotential coupling $\int d^2\theta \sqrt{2} g_2 \tilde{Q}\phi_1 Q$ is always related to the gauge coupling constant as required by $N = 2$ supersymmetry.

The locations of singularities are now written as

$$\phi_1^k = c_k g_2^{-k} \left(\Lambda_1 (g_2^2)^{N_c/(2N_c-N_f)}\right)^k. \tag{4.35}$$

Now this form is RG invariant under the same analysis as we did before. Under the change of the cutoff and bare parameters, the left-hand side is replaced by

$$\phi_1'^k = Z_\phi(M', M)^{k/2} \phi_1^k, \tag{4.36}$$

while the right-hand side by

$$\begin{aligned} & g_2'^{-k} \left(\Lambda_1' (g_2'^2)^{N_c/(2N_c-N_f)}\right)^k \\ & \quad = g_2'^{-k} \left(\Lambda_1 Z_\phi^{N_c/(2N_c-N_f)} Z_Q^{N_f/2(2N_c-N_f)} Z_{\tilde{Q}}^{N_f/2(2N_c-N_f)} (g_2'^2)^{N_c/(2N_c-N_f)}\right)^k. \end{aligned} \tag{4.37}$$

The Eq. (4.35) remains invariant with $Z_Q = Z_{\tilde{Q}} = 1$ and Eq. (4.34).

5 Inverted Hierarchy

In this section we apply our understanding of the RG in supersymmetric gauge theories to a model of dynamical supersymmetry breaking, where such an understanding is necessary to resolve puzzles about the correct vacuum structure of the theory. The theories we consider are vector-like SP(N) models with $N+1$ flavors studied by Izawa, Yanagida [13] and by Intriligator, Thomas [14]. The question we ask is whether the so-called inverted hierarchy mechanism [15] operates in these models. The inverted hierarchy refers to the situation where dynamics forces the expectation value of a scalar field to be exponentially large compared to the energy scale of the potential. This occurs in O'Rafeartaigh type models of supersymmetry breaking where the scalar field is a classical flat direction. The effective potential is modified by perturbative corrections both from the gauge coupling and Yukawa coupling. The potential minimum arises where the two corrections balance against each other.

The particle content of the SP(N) models consists of 2(N+1) SP(N) fundamentals Q^i and singlets S_{ij}. The superpotential is given by

$$W = \frac{1}{2}\lambda S_{ij} Q^i Q^j. \tag{5.1}$$

The equation of motion for S_{ij} demands that $Q^i Q^j = 0$, which is in conflict with the quantum modified constraint $\mathrm{Pf}(Q^i Q^j) = \Lambda^{2(N+1)}$, and supersymmetry is broken. For non-zero S_{ij}, the flavors become massive and can be integrated out of theory. The resulting low-energy theory is pure SP(N) with a dynamical scale depending on S_{ij}. This theory exhibits gaugino condensation and generates an effective superpotential for S_{ij}

$$W_{eff}(S_{ij}) = (\mathrm{Pf}\lambda S_{ij})^{1/2(N+1)}\Lambda^2. \tag{5.2}$$

If we expand S_{ij} around $S_{ij} = \sigma J_{ij}/\sqrt{N+1}$ where J_{ij} is the symplectic matrix, all components of S_{ij} other than σ become massive. The effective potential for σ is then

$$W_{eff}(\sigma) = \lambda\sigma\Lambda^2. \tag{5.3}$$

The σ equation of motion shows that supersymmetry is broken, while the tree-level potential for σ is

$$V_{tree}(\sigma) = |\lambda\Lambda^2|^2, \tag{5.4}$$

and the vev of σ is undetermined at this level. Since supersymmetry is broken, we expect that some nontrivial potential will be generated for σ at higher orders in perturbation theory. In [16], it was argued that the potential $V(\sigma)$ is "RG improved" as $V(\sigma) = |\lambda\Lambda^2|^2 \to |\lambda(\sigma)\Lambda^2|^2$, where $\lambda(\sigma)$ is the running value of λ, which receives contributions from both the S_{ij} and Q^i wavefunction renormalizations $\lambda(\sigma) = Z_S^{-1/2}(\sigma)Z_Q(\sigma)^{-1}\lambda$. If this conclusion is correct, it is possible to realize the inverted hierarchy mechanism in this model: σ could develop a stable vev much larger than Λ, since the Z_S factors depend on λ and tend to make the potential rise for large σ, whereas the Z_Q factors depend on asymptotically free SP(N) gauge coupling and tend to make the potential rise for small σ.

On the other hand, a host of arguments indicate that this conclusion can not be correct. For instance, the superpotential for σ is exact, and so the potential is only modified by the Kähler potential for σ, which at one loop only depends on λ. Alternately, $F_\sigma = \lambda\Lambda^2$ generates a non-supersymmetric spectrum for the Q^i but not for the gauge multiplet. If we simply look at the 1-loop effective potential, the only contribution comes from the non-supersymmetric Q^i spectrum and again depends only on λ, and the potential is monotonically increasing with σ.

In order to resolve this puzzle, we must carefully consider how the potential is "RG improved". As we will show explicitly in the remainder of the section, the solution is that not only λ but also Λ runs; the correct RG improvement of the potential is $V(\sigma) = |\lambda(\sigma)\Lambda(\sigma)^2|^2$, and all the Z_Q dependence cancels in the product $\lambda(\sigma)\Lambda^2(\sigma)$.

Let us go back to the start and carefully define the problem. With cutoff M, the Lagrangian is given by

$$\mathcal{L} = \int d^4\theta \left(Q^{i\dagger}e^{2V}Q^i + \frac{1}{2}\mathrm{Tr}S^\dagger S\right) + \int d^2\theta \left(\frac{1}{4g_h^2}W^aW^a + \frac{1}{2}\lambda S_{ij}Q^iQ^j\right) + h.c. \tag{5.5}$$

In the functional integral, we would like to integrate out the Q^i and the gauge

multiplet and be left with an effective Lagrangian for S_{ij}:

$$\exp\left(-\int d^4x\mathcal{L}_{eff}(S)\right) \equiv \int DVDQ\exp\left(-\int d^4x\mathcal{L}\right), \tag{5.6}$$

with

$$\mathcal{L}_{eff}(S) = \int d^4\theta\left(\frac{1}{2}\mathrm{Tr}S^\dagger S + \delta K(\lambda S, \lambda^\dagger S^\dagger)\right) + \int d^2\theta W_{eff}(\lambda S) + h.c., \tag{5.7}$$

where both δK and W_{eff} depend further on the gauge coupling and the cutoff, $(1/g_h^2, M)$, or equivalently on (Λ, M).

Of course, when we integrate out the Q, V multiplets in perturbation theory, we never generate any effective superpotential. However, a superpotential is generated non-perturbatively, and its form is completely determined by a non-anomalous R symmetry under which λS_{ij} has charge $+2$, an anomalous U(1) symmetry under which λS_{ij} has charge -2 and Λ has charge $+1$, and the non-anomalous SU$(2N+2)$ flavor symmetry. Together, these dictate (up to an overall constant) the exact effective superpotential Eq. (5.2). One finds an effective potential along the $S_{ij} = \sigma J_{ij}/\sqrt{N+1}$ direction:

$$V(\sigma; \lambda, \Lambda; M) = \frac{|\lambda\Lambda^2|^2}{1 + (\partial^2\delta K/\partial S\partial S^\dagger)|_{S=\sigma J}}. \tag{5.8}$$

The corrections to the Kähler potential $\delta K(\lambda S, \lambda^\dagger S^\dagger)$ are certainly generated in perturbation theory. For instance, at 1-loop there is a contribution from the loop of Q's, which yields (at the leading-log)

$$\delta K(\lambda S, \lambda^\dagger S^\dagger) = -2\frac{N}{16\pi^2}(\lambda\sigma)^*(\lambda\sigma)\ln\frac{|\lambda\sigma|^2}{M^2} \tag{5.9}$$

along the σ direction and we find:

$$V(\sigma; \lambda, \Lambda; M) = |\lambda\Lambda^2|^2\left[1 + 2\frac{N\lambda^2}{16\pi^2}\left(\ln\frac{|\lambda\sigma|^2}{M^2} + 4\right) + \mathcal{O}(\lambda^4)\right]. \tag{5.10}$$

For $\sigma \ll M$, the large logs in the above expression make perturbation theory unreliable. However, we can use the same technique as in Sec. 2 to deal with this problem. We are interested in $V(\sigma; \lambda, \Lambda; M)$. First, we rescale everything by naive dimensional analysis

$$V(\sigma; \lambda, \Lambda; M) = e^{-4t}V(e^t\sigma; \lambda, e^t\Lambda, ; e^tM). \tag{5.11}$$

Next, we bring the cutoff back from $e^t M$ to M by appropriately changing the couplings. Since the form of the tree potential is known with canonical normalization for the superfields Q, S, we would like to keep them canonical. Hence, as we have argued in Sec. 2, not only λ but also Λ must be changed:

$$V(e^t\sigma;\lambda,e^t\Lambda;e^tM) = V(e^t\sigma Z_S^{1/2}(t);\lambda Z_Q^{-1}(t)Z_S^{-1/2}(t),e^t\Lambda Z_Q(t)^{1/2};M), \tag{5.12}$$

where $Z_Q(t) \equiv Z_Q(M,e^tM)$, $Z_S(t) \equiv Z_S(M,e^tM)$. However, we can choose $e^t|\sigma| \sim M$, then the logarithms in the perturbative expansion of the RHS are small and the tree value of the RHS $V_{tree}(\sigma;\lambda,\Lambda;M) = |\lambda\Lambda^2|^2$ is an excellent approximation. Doing this and combining with Eq. (5.11), we find

$$\begin{aligned} V(\sigma;\lambda,\Lambda;M) &\simeq \left|(\lambda Z_Q^{-1}(t)Z_S^{-1/2}(t))(\Lambda Z_Q^{1/2}(t))^2\right|^2_{t\sim\ln(\lambda\sigma/M)} \\ &= Z_S^{-1}(\lambda\sigma,M)|\lambda\Lambda^2|^2. \end{aligned} \tag{5.13}$$

As promised, when the RG improvement is done consistently, the Z_Q dependence drops out and we are left with a monotonically rising potential (from the Z_S factor) which does not realize the inverted hierarchy.

One can also formally check that the effective potential $V(\sigma;\lambda,\Lambda;M)$, or equivalently the effective Lagrangian Eq. (5.7), is RG invariant, *i.e.* independent of the choice of the cutoff M as long as one changes the bare couplings appropriately. First of all, the effective superpotential Eq. (5.2) is invariant by itself, because

$$\begin{aligned} &\lambda'(\mathrm{Pf}S'_{ij})^{1/2(N+1)}\Lambda'^2 \\ &= \lambda Z_Q^{-1}(M',M)Z_S^{-1/2}(M',M)(\mathrm{Pf}S_{ij})^{1/2(N+1)}Z_S^{1/2}(M',M)(\Lambda Z_Q^{1/2}(M',M))^2 \\ &= \lambda(\mathrm{Pf}S_{ij})^{1/2(N+1)}\Lambda^2. \end{aligned} \tag{5.14}$$

The Kähler potential along the σ direction

$$\int d^4\theta Z_\sigma(\lambda\sigma,M)\sigma^*\sigma \tag{5.15}$$

is also RG invariant which can be seen as follows. First, the Z_σ factor must depend on the renormalized effective mass of Q, $m_Q \equiv \lambda\sigma Z_Q^{-1}(m_Q,M)$ because it is generated from integrating out the massive Q field. This combination can be easily seen to be RG invariant. Second, it is multiplicative, $Z_\sigma(m_Q,M) = Z_\sigma(m_Q,M')Z_\sigma(M',M)$. With the definition $\sigma' =$

$Z_\sigma^{1/2}(M', M)\sigma$, the Kähler potential is RG invariant. Since the change of the field variable σ to σ' is given by a field-independent constant $Z_\sigma(M, M')$, the auxiliary equation for the F_σ changes only by an overall factor $Z_\sigma(M, M')$. On the other hand the quadratic term of F_σ also changes the the same factor and hence we conclude that the effective potential (5.8) is RG invariant.

We have demonstrated that the inverted hierarchy mechanism does not work in these models, contrary to the naive argument of RG improvement [16]. In order to achieve the inverted hierarchy mechanism as favored from the model building point of view, one needs to make the flat direction fields S_{ij} gauge non-singlet, as was recently done in [17, 18].

6 Conclusion

In this paper, we studied the renormalization group invariance of the exact results in supersymmetric gauge theories. We first clarified the notion of Wilsonian renormalization group (RG) invariance in supersymmetric gauge theories. It is a non-trivial statement that the low-energy physics can be kept fixed when one changes the ultraviolet cutoff with appropriate changes in the bare coupling constants in the Lagrangian. We derived the formula for the changes of bare couplings using two methods: one using strictly Wilsonian actions and holomorphic gauge coupling, the other using the perturbative NSVZ β-function. We used canonical normalization for the chiral superfields because it allows the most straightforward application of the renormalization group. We find that the scale Λ is *not* RG invariant. We then compared our formula to known exact results and showed that they actually require the changes in Λ we have derived.

Finally, we applied our result to models of supersymmetry breaking due to quantum modified constraints, namely SP(N) models with $N+1$ flavors. These models have a classically flat direction, and the crucial question is in what way the flat direction is lifted. The RG invariance allowed us to determine the effective potential along the classical flat direction. A naive application of RG improvement of the potential would tell us that the potential along the flat direction is modified perturbatively both by the SP(N) gauge interaction and the superpotential interaction, and hence that the flat direction may develop an expectation value exponentially larger than the supersymmetry breaking scale (inverted hierarchy mechanism). However, a

careful application of our method demonstrates that the inverted hierarchy mechanism does not occur in these models.

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