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Abstract

Type IIA brane configurations are used to construct $N = 2$ supersymmetric gauge theories in two dimensions. Using localization of chiral multiplets in ten-dimensional spacetime, supersymmetric non-linear sigma models with target space such as CP^{n-1} and the Grassmann manifolds are studied in detail. The quantum properties of these models are realized in M theory by taking the strong Type IIA coupling limit. The brane picture implies an equivalence between the parameter space of $N = 2$ supersymmetric theories in two dimensions and the moduli space of vacua of $N = 2$ supersymmetric gauge theories in four dimensions. Effects like level-rank duality are interpreted in the brane picture as continuation past infinite coupling. The BPS solitons of the CP^{n-1} model are identified as topological excitations of a membrane and their masses are computed. This provides the brane realization of higher rank tensor representations of the flavor group.

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1 Introduction

The realization of supersymmetric gauge theories using various branes in string theory, with the aid of some string theory dualities, enables us to make various predictions on the dynamical effects in the strong coupling regime, as was first exhibited in [1]. Phenomena in theories with eight supercharges were studied subsequently in [2, 3, 4, 5, 6, 7, 8]. Theories with four supercharges were constructed in [9] and studied further in [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Supersymmetric gauge theories can also be studied from string theory by realizing the gauge symmetry as a singular geometry of the string compactification [26]. This method has also been developed extensively in various directions.¹

One important aspect of the discussion in theories with four supercharges is the realization of chiral symmetry and chiral gauge theory in terms of branes. A proposal of chiral symmetry realization was given in [15] in which it was also proposed how the different chiral multiplets arise from the brane construction. The proposal was examined in [4] by calculating superpotentials for various brane configurations, and there was an agreement with field theory expectation in all the cases. In [21] it was explained how chiral multiplets can be localized in ten-dimensional space-time, by making use of the fact that the theories in question actually live in five dimensions with one direction being in a finite interval. The chiral matter is localized on one boundary of the interval, injecting an anomaly which flows through the interval to be absorbed by the chiral matter of opposite chirality which is localized at the other boundary. The four dimensional theory is then anomaly free. However, this is not a chiral theory in the ordinary sense. For this, we need to realize chiral multiplet in a more general representation of the gauge group rather than just the (anti-)fundamental or the adjoint.

One of the aims of this paper is to examine these ideas by studying $N = 2$ (i.e. $(2, 2)$) supersymmetric theories in two dimensions that can be considered as the dimensional reduction of chiral theories in four dimensions. Chiral matter in four dimensions does not lead to gauge anomalies upon dimensional reduction to two dimensions and hence theories which would be anomalous in four dimensions are consistent theories in two dimensions. We construct such gauge theories using branes of the Type IIA superstring following the proposals of [15], and compare with what we know about these theories. We will find considerable agreement between

¹In particular, in [27] it was noted that solutions of some $N = 2$ theories are realized as the configurations of Type IIA fivebranes in flat space-time.

them, providing further support for the proposal of chiral symmetry realization.

There are many interesting features in $N = 2$ theories in two dimensions, and many exact results have been obtained. Moreover, some of these theories are even completely integrable. In section 2, the general background for $N = 2$ theories in two dimensions is summarized. One motivation of the study of such well-understood systems using branes is to translate interesting field theory phenomena to the language of branes. In such a way more phenomena can be captured using the branes in cases where the field theory tools are not as powerful as in two dimensions. In particular, we obtain a hint for realizing non-fundamental representation of the gauge group.

However the interplay between gauge theories and branes goes both ways. Another aim of this paper is to use brane configurations to deepen the understanding of $N = 2$ supersymmetric theories in two dimensions. Using the brane construction we get new interpretation for phenomena which are not clear from current methods in field theory. In some cases we obtain also some predictions, qualitative and quantitative, which were not known before.

In section 3, we construct brane configurations in Type IIA string theory. We examine the space of vacuum configurations in this set-up, and compare with the space of classical vacua of the field theory.

In section 4 we solve the proposed models of section 3 by taking the strong Type IIA coupling, going over to M theory using the methods of [3]. We see that it correctly captures important quantum effects, such as running of the Fayet-Iliopoulos coupling and the anomaly of an R-symmetry group. We also show that it correctly reproduces the number of quantum vacua together with the discrete chiral symmetry breaking, in the cases corresponding to the $\mathbb{C}P^{n-1}$ and Grassmannian sigma models. In particular, we show that the relation of the quantum cohomology ring of the $\mathbb{C}P^{n-1}$ model is realized in the M theory picture.

In section 5, we discuss continuation past infinite coupling which is realized by an interchange of two NS branes [1, 9] and interpret it as a transition between two gauge theories which leads to level rank duality [28, 29] of WZW models. Other brane motions lead to new transitions in two dimensions which are discussed in section 5.1.

One more important viewpoint emerges from the brane construction. The brane picture demonstrates a relation between p -dimensional theories with 4 supercharges and $p + 2$ -dimensional theories with 8 supercharges ($p \leq 4$). The relation is that the parameter space of

the p -dimensional theory is the moduli space of vacua of the $p + 2$ -dimensional theory. This viewpoint was first introduced and emphasized in [4] for the $p = 3$ case; The Coulomb branch of the 5d theory is then the space of real masses of the 3d theory, while the Higgs branch of the 5d theory is the space of complex mass parameters for the 3d theory. In section 5.2, we will touch this correspondence for the $p = 2$ case.

The simplest model in two dimensions with $N = 2$ supersymmetry and with chiral matter is the $\mathbb{C}P^{n-1}$ model. This model was studied intensively in the past mainly because of various features which serve as a toy model for QCD. This model is asymptotically free and has a theta angle with instantons. All these features were attractive for modeling similar phenomena in QCD. From the construction of the $\mathbb{C}P^{n-1}$ model in terms of branes this is not surprising. Actually the brane construction provides an explanation of why the $\mathbb{C}P^{n-1}$ model is a good toy model for QCD. One way of interpretation of the brane system is as a D2 brane propagation on the world volume of a configuration of branes which realizes some limit of $N = 1$ supersymmetric Yang-Mills theory with gauge group $SU(n)$. The world volume theory which is realized on the finite D2 brane is the $\mathbb{C}P^{n-1}$ model. Thus the D2 brane probes some of the features of $N = 1$ supersymmetric Yang-Mills theory.

In section 6 we discuss the realization of solitons in the $\mathbb{C}P^{n-1}$ model as the topological excitations of a membrane in which new boundary circles are created. Each boundary circle is interpreted as an open string end point which carries a quantum number of the fundamental representation of the flavor group $SU(n)$. Note that a string has only two ends and thus can realize only up to second rank tensor representation, while a membrane can have many boundaries and so higher rank tensor representations can be realized. Indeed, it turns out that the fundamental solitons interpolating adjacent vacua have one boundary and are in the fundamental representation of the flavor group $SU(n)$, but solitons interpolating ℓ -separated vacua have ℓ boundaries and are in the ℓ -th anti-symmetric representation of $SU(n)$. These properties in fact agrees with the field theory knowledge. For example, the fundamental solitons of the $\mathbb{C}P^{n-1}$ model are known to be the elementary chiral multiplets in the fundamental of $SU(n)$ which corresponds in the brane picture to the elementary open Type IIA strings. The mass spectrum of $\mathbb{C}P^{n-1}$ solitons is also computed and it also agrees with the field theory results. The analysis is generalized to the case with deformation by mass term and we determine the mass spectrum of the solitons, which could not be achieved by field theory argument.

2 $N = 2$ Theories in Two Dimensions

In this section, we describe general properties of $N = 2$ supersymmetric field theories in two dimensions. We also describe a gauged linear sigma model realization of supersymmetric non-linear sigma models [30, 31, 32]. In particular, we discuss the soliton spectrum of the CP^{n-1} model and its deformation by mass term.

2.1 General Background on $N = 2$ SUSY Field Theories

$N = 2$ supersymmetry in two dimensions can be obtained by dimensional reduction from $N = 1$ supersymmetry in four dimensions.

$N = 1$ supersymmetry algebra in four dimensions contains four supercharges which transform as Majorana spinors under $d = 4$ Lorentz group (one left handed and one right handed spinors which are conjugate to each other). The SUSY algebra contains one $U(1)$ R-symmetry under which the left handed supercharges have charge -1 and the right handed ones have charge $+1$.

By dimensional reduction to two dimensions (i.e. eliminating the dependence of fields on two coordinates $x^{2,3}$), the four-dimensional Lorentz group is broken to the two-dimensional Lorentz group and an internal symmetry group associated with the rotations in the $x^{2,3}$ directions which we call $U(1)_A$. A left (right) handed spinor in four dimensions becomes one Dirac spinor in two dimensions — one left and one right handed spinors with opposite $U(1)_A$ charge ∓ 1 (± 1). The four supercharges are thus two Dirac spinors $Q_{L,R}$ and $\bar{Q}_{L,R}$ (L, R denotes the two-dimensional chirality and absence/presence of *bar* indicates the four-dimensional chirality) which carry $U(1)_A$ charge $-1, +1$ and $+1, -1$ respectively. These are related to each other under conjugation by $(Q_L)^\dagger = \bar{Q}_L$ and $(Q_R)^\dagger = \bar{Q}_R$. They obey the commutation relation

$$\{Q_L, \bar{Q}_L\} = 2(H + P), \quad (2.1)$$

$$\{Q_R, \bar{Q}_R\} = 2(H - P), \quad (2.2)$$

$$\text{and } Q_L^2 = Q_R^2 = \bar{Q}_L^2 = \bar{Q}_R^2 = 0 \quad (2.3)$$

where H and P are Hamiltonian and momentum operators. In the absence of the central charge which we will describe shortly, the other commutators vanish.

The $U(1)$ R-symmetry in four-dimensions can reside in two dimensions as another internal

symmetry — which we call $U(1)_V$ — under which the supercharges $Q_{L,R}$ and $\bar{Q}_{L,R}$ carry charge $-1, -1$ and $+1, +1$ respectively. Thus, there are two $U(1)$ R-symmetry groups, $U(1)_V$ and $U(1)_A$. The action on the supercharges is exhibited as

$$\begin{array}{cc} Q_R & \bar{Q}_L \\ Q_L & \bar{Q}_R \end{array} \quad (2.4)$$

where the upper (lower) row is assigned a $U(1)_A$ charge $+1$ (-1) while the right (left) column is assigned a $U(1)_V$ charge $+1$ (-1). Of course these R-symmetries can be broken explicitly by a tree level superpotential or, in the quantum theory, by an anomaly. A basic example for such effects is provided at the end of this subsection.

Representations

The two basic representations of the $N = 1$ SUSY algebra in four-dimensions, (anti-)chiral and vector multiplets, go down to the corresponding representations of the two-dimensional $N = 2$ SUSY algebra.

A chiral multiplet consists of one complex scalar field ϕ and a Dirac fermion $\psi_{L,R}$. The action of the two R-symmetries is exhibited (together with its conjugate anti-chiral multiplet consisting of $\phi^\dagger, \bar{\psi}_{L,R}$) as

$$\begin{array}{cc} \psi_R & \bar{\psi}_L \\ \phi & \phi^\dagger \\ \psi_L & \bar{\psi}_R \end{array} \quad (2.5)$$

where the $U(1)_A$ charge of the scalar component is zero, while the $U(1)_V$ charge is not specified since it can be shifted by a constant. The chiral multiplet is represented in the $N = 2$ superspace as a chiral superfield Φ obeying ²

$$\bar{D}_L \Phi = \bar{D}_R \Phi = 0, \quad (2.6)$$

which can be expanded as $\Phi = \phi + \sqrt{2}\theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha F$ where F is a complex auxiliary field.

² We follow the convention of [33, 32] in which

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} - i\bar{\theta}^\alpha \left(\frac{\partial}{\partial x^0} + \epsilon_\alpha \frac{\partial}{\partial x^1} \right), & \bar{D}_\alpha &= -\frac{\partial}{\partial \bar{\theta}^\alpha} + i\theta^\alpha \left(\frac{\partial}{\partial x^0} + \epsilon_\alpha \frac{\partial}{\partial x^1} \right), \\ Q_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i\bar{\theta}^\alpha \left(\frac{\partial}{\partial x^0} + \epsilon_\alpha \frac{\partial}{\partial x^1} \right), & \bar{Q}_\alpha &= -\frac{\partial}{\partial \bar{\theta}^\alpha} - i\theta^\alpha \left(\frac{\partial}{\partial x^0} + \epsilon_\alpha \frac{\partial}{\partial x^1} \right), \end{aligned}$$

where $\alpha = L, R$ and $\epsilon_L = 1, \epsilon_R = -1$. Indices are lowered and raised by $\epsilon_{\alpha\beta}$ with $\epsilon_{LR} = 1$ and its inverse $\epsilon^{\alpha\beta}$.

A vector multiplet consists of a vector field A_μ , Dirac fermions $\lambda_{L,R}$ and $\bar{\lambda}_{L,R}$ which are conjugate to each other, and a complex scalar σ which comes from the $x^{2,3}$ components of the four-dimensional vector field. These are all in the adjoint representation of the gauge group. The action of the two R-symmetry groups is given by

$$\begin{array}{ccc} & \sigma & \\ \bar{\lambda}_L & & \lambda_R \\ & A_\mu & \\ \bar{\lambda}_R & & \lambda_L \\ & \sigma^\dagger & \end{array} \quad (2.7)$$

where the vector field A_μ is neutral under both. The vector multiplet is represented in the $N = 2$ superspace as a vector superfield V satisfying $V^\dagger = V$ which can be expanded in the Wess-Zumino gauge as

$$\begin{aligned} V = & \theta^L \bar{\theta}^L (A_0 + A_1) + \theta^R \bar{\theta}^R (A_0 - A_1) - \theta^R \bar{\theta}^L \sigma - \theta^L \bar{\theta}^R \sigma^\dagger \\ & - i \theta^\alpha \theta_\alpha \bar{\theta}^\beta \bar{\lambda}_\beta + i \bar{\theta}^\alpha \bar{\theta}_\alpha \theta^\beta \lambda_\beta - \frac{1}{2} \theta^\alpha \theta_\alpha \bar{\theta}^\beta \bar{\theta}_\beta D \end{aligned} \quad (2.8)$$

where D is a real auxiliary field. The super field strength is defined by $\Sigma = \{\bar{\mathcal{D}}_L, \mathcal{D}_R\}/2$ where $\mathcal{D}_\alpha = e^{-V} D_\alpha e^V$ and $\bar{\mathcal{D}}_\alpha = e^V \bar{D}_\alpha e^{-V}$. This is a twisted chiral superfield:

$$\bar{\mathcal{D}}_L \Sigma = \mathcal{D}_R \Sigma = 0. \quad (2.9)$$

The lowest component of Σ is the complex scalar field σ .

D-term, F-term and twisted F-term

There are three kinds of supersymmetric couplings.

One is the D-term which can be expressed as

$$\int d^4\theta K \quad (2.10)$$

where $\int d^4\theta$ is the integration over all the Grassmannian coordinates $\theta^L, \theta^R, \bar{\theta}^L, \bar{\theta}^R$ and K is some real combination of superfields. The D-term appears in the following as the gauge or matter kinetic term. This term is invariant under both $U(1)_V$ and $U(1)_A$.

Second one is the F-term

$$\int d^2\theta W = \int d\theta^L d\theta^R W|_{\bar{\theta}^L, \bar{\theta}^R=0} \quad (2.11)$$

plus its Hermitian conjugate. Here W is a holomorphic combination of chiral superfields and is called superpotential as usual. The F-term is always invariant under $U(1)_A$, but is invariant under $U(1)_V$ only when it is possible to assign $U(1)_V$ charge to chiral superfields so that the superpotential carries charge 2. The latter condition is equivalent to saying that W is quasi-homogeneous of degree 2 with respect to $U(1)_V$. Note that even if W is not quasi-homogeneous, a discrete subgroup of $U(1)_V$ can be unbroken.

Third one is the twisted F-term

$$\int d^2\tilde{\theta} \tilde{W} = \int d\theta^L d\bar{\theta}^R \tilde{W}|_{\theta^R=\bar{\theta}^L=0} \quad (2.12)$$

plus its Hermitian conjugate. Here \tilde{W} is a holomorphic combination of twisted chiral superfields and is called twisted superpotential. This preserves $U(1)_V$ but breaks $U(1)_A$ unless \tilde{W} is quasi-homogeneous of degree 2 with respect to $U(1)_A$. In a gauge system, the Fayet-Iliopoulos D-term $-r \int d^4\theta \text{Tr}V$ and the theta term $i\theta \text{Tr}F_A/2\pi$ can be described by a single twisted F-term with

$$\tilde{W}_{\text{FI},\theta} = \frac{i\tau}{4} \text{Tr}\Sigma \quad (2.13)$$

where $\tau = ir + \theta/2\pi$. Since (2.13) is homogeneous of degree 2, this does not break the R-symmetry $U(1)_A$. However, in a gauge system $U(1)_A$ is often broken by an axial anomaly as in non-linear sigma model based on non-Calabi-Yau manifolds. Again, even if $U(1)_A$ is broken by an anomaly, a discrete subgroup can remain unbroken.

Central Extension and BPS Bound

Consider a massive $N = 2$ SUSY field theory with a discrete set of vacua. If we put such a system on the flat Minkowski space $\mathbf{R}^{1,1}$, there can be solitonic states in which the boundary condition of fields at the left spatial infinity $x^1 = -\infty$ (specified by one vacuum) is different from the one at the right infinity $x^1 = +\infty$ (specified by another vacuum). As is well known [34], in such a theory the $N = 2$ SUSY algebra admits a central extension associated with the topological charge of the soliton sectors. Note that the central extension is impossible in a theory with unbroken $U(1)_V$ and $U(1)_A$ R-symmetry, since the central term should commute also with R-symmetry generators [35].

Let us consider a massive theory in which $U(1)_V$ is broken by a superpotential. For example, $N = 2$ Landau-Ginzburg (LG) models with non quasi-homogeneous superpotentials. In

addition to (2.1), (2.2) and (2.3), the algebra reads as

$$\{Q_L, Q_R\} = 2Z, \quad \{\bar{Q}_L, \bar{Q}_R\} = 2Z^*, \quad (2.14)$$

$$\{Q_L, \bar{Q}_R\} = 0, \quad \{\bar{Q}_L, Q_R\} = 0, \quad (2.15)$$

$$[F_A, Q_L] = -Q_L, \quad [F_A, Q_R] = Q_R, \quad [F_A, \bar{Q}_L] = \bar{Q}_L, \quad [F_A, \bar{Q}_R] = -\bar{Q}_R \quad (2.16)$$

where F_A is the generator of $U(1)_A$ R-symmetry. One of the most important consequence of this algebra is that the mass of the particle in a sector with central charge Z is bounded from below by [34]

$$M \geq |Z|. \quad (2.17)$$

This follows from the positive semi-definite-ness of the anti-commutator of $(H - P)Q_L - Z\bar{Q}_R$ and its hermitian conjugate $(H - P)\bar{Q}_L - Z^*Q_R$. This bound is saturated for states on which the condition $(H - P)Q_L = Z\bar{Q}_R$ (called *BPS condition*) is satisfied. For energy-momentum eigenstates satisfying the BPS condition, Q_L and \bar{Q}_L are proportional to \bar{Q}_R and Q_R respectively, and thus the SUSY multiplet consists of two states rather than four. This is called a *BPS multiplet*.

In a LG model with chiral superfields $X = (X^1, \dots, X^d)$ and Lagrangian

$$S = \int d^2x d^4\theta K(X, X^\dagger) + \left(\int d^2x d^2\theta W(X) + \text{h.c.} \right), \quad (2.18)$$

with a non quasi-homogeneous superpotential $W(X)$, the vacua are the critical points of the superpotential, $\partial W = 0$. For a solitonic state in such a system, the LG field $X(x^1)$ satisfies the boundary condition such that $X(x^1 = -\infty)$ is one critical point, say a , and $X(x^1 = +\infty)$ is another one, say b . Then, the central charge Z_{ab} in such a solitonic sector is [34, 36]

$$Z_{ab} = 2(W(b) - W(a)). \quad (2.19)$$

Indeed we can see the BPS bound $M \geq 2|W(b) - W(a)|$ from a classical argument [37]. Let $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ be the Kähler metric. Then the energy of a static configuration is

$$E = \int dx^1 \left\{ g_{i\bar{j}} \partial_1 X^i \partial_1 \bar{X}^{\bar{j}} + g^{i\bar{j}} \frac{\partial W}{\partial X^i} \frac{\partial \bar{W}}{\partial \bar{X}^{\bar{j}}} \right\} \quad (2.20)$$

$$= \int dx^1 \left| \partial_1 X^i - \alpha g^{i\bar{j}} \frac{\partial W}{\partial \bar{X}^{\bar{j}}} \right|^2 + 2\text{Re}(\alpha^*(W(b) - W(a))), \quad (2.21)$$

for any phase α , $|\alpha| = 1$. The second term of the RHS is maximum if we choose α to be the phase of $W(b) - W(a)$, and thus, we obtained the bound $E \geq 2|W(b) - W(a)|$. Note that this bound is independent of the Kähler metric. In LG theories, the Kähler potential gets quantum corrections, but the superpotential is not. Thus, this bound is exact quantum mechanically. It is important to note that for a BPS saturated configuration $\partial_1 X^i = \alpha g^{i\bar{j}} \partial_{\bar{j}} \bar{W}$, the trajectory along the spatial direction x^1 of the superpotential W is a straight line

$$\partial_1 W = \frac{W(b) - W(a)}{|W(b) - W(a)|} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W}, \quad (2.22)$$

connecting the two critical values $W(a)$, $W(b)$ [37].

For a theory in which $U(1)_A$ is broken (say, by an anomaly), the same thing can be said. In addition to (2.1)-(2.3), the $N = 2$ SUSY algebra reads

$$\{Q_L, Q_R\} = 0, \quad \{\bar{Q}_L, \bar{Q}_R\} = 0, \quad (2.23)$$

$$\{Q_L, \bar{Q}_R\} = 2\tilde{Z}, \quad \{\bar{Q}_L, Q_R\} = 2\tilde{Z}^*, \quad (2.24)$$

$$[F_V, Q_L] = -Q_L, \quad [F_V, Q_R] = -Q_R, \quad [F_V, \bar{Q}_L] = \bar{Q}_L, \quad [F_V, \bar{Q}_R] = \bar{Q}_R \quad (2.25)$$

where F_V is the generator of $U(1)_V$. As in the previous case, there is a BPS bound

$$M \geq |\tilde{Z}|, \quad (2.26)$$

with the BPS condition being $(H - P)Q_L = \tilde{Z}Q_R$. In a LG model for twisted chiral LG fields with a non quasi-homogeneous twisted superpotential \tilde{W} , the central charge in a solitonic sector is again given by the difference of the critical values of $4\tilde{W}$ at the two spatial infinities. (The extra factor 2 is due to the convention.) In [38, 37], it has been argued that for any massive $N = 2$ theory, one can define some kind of superpotential (“holomorphic function” on the discrete set of vacua) such that the central charge in a solitonic sector is the difference of the values of the superpotential. As we will see however, in a theory with continuous Abelian symmetries in addition to the R-symmetry, there can be a contribution to the central charge from charges of such Abelian groups:

$$Z = 2\Delta W + \sum_i m_i S_i \quad \text{or} \quad \tilde{Z} = 4\Delta\tilde{W} + \sum_i \tilde{m}_i S_i \quad (2.27)$$

where S_i are the Abelian charges and m_i or \tilde{m}_i are parameters such that the Abelian symmetries are enhanced to some non-Abelian symmetry as $m_i \rightarrow 0$ or $\tilde{m}_i \rightarrow 0$ (typically mass parameters).

Note that the central charge in such a case is not determined just by the asymptotic condition at the spatial infinities.

Mirror Symmetry

As noted in [39], there is an interesting automorphism of the $N = 2$ supersymmetry algebra given by

$$\begin{aligned} F_A &\longleftrightarrow F_V \\ Q_R &\longleftrightarrow \bar{Q}_R \end{aligned} \tag{2.28}$$

with other generators kept intact.

Two $N = 2$ theories are said to be mirror to each other when there is an identification under which the $N = 2$ SUSY generators are mapped according to the above automorphism. Such mirror pairs were first found and used effectively in the study of $N = 2$ super *conformal* field theory associated with the sigma models with Calabi-Yau target space (see [40] and references therein).³

Although it has not extensively been noted in the past, a notion of mirror symmetry exists also for massive $N = 2$ theories. In such a case, the automorphism involves also the central elements. Look at the two types of massive theories considered right above, where one type has unbroken $U(1)_A$ and the other has unbroken $U(1)_V$. Then, (2.28) together with

$$Z \longleftrightarrow \tilde{Z} \tag{2.29}$$

defines an isomorphism of the $N = 2$ algebras with central extension: (2.1)-(2.3), (2.14)-(2.16) is mapped to (2.1)-(2.3), (2.23)-(2.25). The basic example of mirror symmetry in massive theories is the pair of $N = 2$ Sine-Gordon theory and supersymmetric CP^1 sigma model, or more generally, the pair of $N = 2$ A_{n-1} affine Toda field theory and SUSY CP^{n-1} sigma models. In the affine Toda field theory $U(1)_V$ is broken by superpotential to \mathbf{Z}_{2n} which is further broken spontaneously to \mathbf{Z}_2 , while in the CP^{n-1} model $U(1)_A$ is anomalously broken to \mathbf{Z}_{2n} which is again further broken spontaneously to \mathbf{Z}_2 . Under the identification, the spontaneously broken discrete \mathbf{Z}_n symmetries are also mapped to each other. Equivalence of soliton spectrum and S-matrices were observed in [41] where we need to take a certain limit of the coupling in the affine Toda side. The mirror pair is recently generalized to pairs of other kind of target space

³ In fact, the automorphism (2.28) extends to an automorphism of the infinite $N = 2$ superconformal algebra.

of positive first Chern class and affine Toda-type field theory in the study of twisted $N = 2$ theories coupled to gravity [42] (see also [43]).

2.2 Gauged Linear Sigma Models

Let us consider $N = 2$ supersymmetric $U(k)$ gauge theory in two dimensions with n_1 chiral multiplets Q^i in the fundamental representation \mathbf{k} and n_2 chiral multiplets $\tilde{Q}_{\tilde{j}}$ in the anti-fundamental representation $\bar{\mathbf{k}}$ ($i = 1, \dots, n_1; \tilde{j} = 1, \dots, n_2$).

The kinetic term of the Lagrangian of the theory is given by

$$\mathcal{L}_{kin} = \frac{1}{4} \int d^4\theta \left(Q^\dagger e^{2V} Q + \tilde{Q} e^{-2V} \tilde{Q}^\dagger - \frac{1}{2e^2} \text{Tr}(\Sigma^\dagger \Sigma) \right), \quad (2.30)$$

where Σ is the twisted chiral superfield representing the field strength of the $U(k)$ vector superfield V and e is the gauge coupling constant which has dimension of mass.

In addition, we consider the Fayet-Iliopoulos (FI) and the theta terms

$$\mathcal{L}_{\text{FI},\theta} = \frac{i\tau}{4} \int d^2\tilde{\theta} \text{Tr}\Sigma + h.c. \quad (2.31)$$

where the FI parameter r and the theta parameter θ are combined in the form

$$\tau = ir + \theta/2\pi. \quad (2.32)$$

Also, we can consider the mass term

$$\mathcal{L}_{mass} = \sum_{i,\tilde{j}} \int d^2\theta m_i^{\tilde{j}} \tilde{Q}_{\tilde{j}} Q^i + h.c., \quad (2.33)$$

where $\tilde{Q}_{\tilde{j}} Q^i$ is the natural gauge invariant combination. $m_i^{\tilde{j}}$ are complex parameters which we call the *complex masses*. Note that this term can be considered as coming from the mass term which already exists in four dimensions.

Actually, there is another kind of mass term which cannot be considered as coming from any coupling in four-dimensional theories. This can be obtained by first gauging the flavor symmetry $U(n_1) \times U(n_2)$ and giving a background value to the scalar component of the vector superfield, and then setting the fields to be vanishing. This can be written as

$$\mathcal{L}_{\widetilde{mass}} = \int d^4\theta \left(Q^\dagger e^{2V_1} Q + \tilde{Q} e^{-2V_2} \tilde{Q}^\dagger \right) \quad (2.34)$$

where V_1 and V_2 are given by

$$V_1 = \theta^R \bar{\theta}^L \tilde{m} + h.c., \quad V_2 = \theta^R \bar{\theta}^L \widehat{m} + h.c.. \quad (2.35)$$

This preserves $N = 2$ supersymmetry if and only if \tilde{m} and \widehat{m} are (independently) diagonalizable:

$$\tilde{m} = \begin{pmatrix} \tilde{m}_1 & & \\ & \ddots & \\ & & \tilde{m}_{n_1} \end{pmatrix}, \quad \widehat{m} = \begin{pmatrix} \widehat{m}_1 & & \\ & \ddots & \\ & & \widehat{m}_{n_2} \end{pmatrix}. \quad (2.36)$$

We call these the *twisted masses*. This is the two-dimensional version of the “real mass term” which were considered in [10, 44, 45]. Note that the shift of \tilde{m} and \widehat{m} by matrices $c\mathbf{1}_{n_1}$ and $c\mathbf{1}_{n_2}$ proportional to identity matrices can be absorbed by a redefinition of the σ field, and thus is irrelevant.

The Space of Classical Vacua

After integrating out the auxiliary fields, the potential energy of this system is

$$\begin{aligned} U = & \frac{e^2}{2} \text{Tr} (QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} - r)^2 + \frac{1}{8e^2} \text{Tr} [\sigma, \sigma^\dagger]^2 \\ & + \frac{1}{2} \|\sigma Q - Q\tilde{m}\|^2 + \frac{1}{2} \|\sigma^\dagger Q - Q\tilde{m}^\dagger\|^2 + \|Qm\|^2 \\ & + \frac{1}{2} \|\tilde{Q}\sigma - \widehat{m}\tilde{Q}\|^2 + \frac{1}{2} \|\tilde{Q}\sigma^\dagger - \widehat{m}^\dagger\tilde{Q}\|^2 + \|m\tilde{Q}\|^2. \end{aligned} \quad (2.37)$$

We describe the space of classical vacua which is the space of zeros of U modulo gauge transformations. First of all, for the second term to be vanishing σ must be diagonalizable:

$$\sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}. \quad (2.38)$$

The structure of the whole space depends of the parameters r , m , \tilde{m} and \widehat{m} .

- (i) When all these parameters are turned off, the space of classical vacua is a singular space in which there are roughly two branches: In one branch (“Coulomb branch”), σ is a non-zero diagonal matrix and $Q = \tilde{Q} = 0$, while in the other branch (“Higgs branch”), $\sigma = 0$ but Q and \tilde{Q} can take non-zero values obeying $QQ^\dagger = \tilde{Q}^\dagger \tilde{Q}$. Of course, there are “mixed branches” in which first l rows of Q and first l columns of \tilde{Q} are non-vanishing and only the last $k - l$ of σ_a ’s are non-vanishing. Note that when $n_2 = 0$ (or $n_1 = 0$), the “Higgs branch” is trivial $Q = 0$ ($\tilde{Q} = 0$).

- (ii) When $r = 0$ and a generic complex mass term is turned on, the space of classical vacua consists only of “Coulomb branch” in which $Q = \tilde{Q} = 0$ and σ is an unconstrained diagonal matrix. Upon specialization to rank $m < \min\{n_1, n_2\}$, “Higgs branch” becomes possible.
- (iii) When $r = 0$ and a generic twisted mass is turned on, it is again only of “Coulomb branch”, but when some of the twisted masses for Q and some for \tilde{Q} coincide, there emanate “Higgs branches” at which some of σ_a ’s are tuned at the values of such twisted masses.
- (iv) When $r > 0$ and all other parameters are turned off, for the first term to be vanishing, Q is non-zero and actually must have rank k . This is possible only when $n_1 \geq k$. In this case, we must have $\sigma = 0$ for the third term to be vanishing. Thus, the space consists only of “Higgs branch”. Note that when $n_2 = 0$, this space is compact, that is, there is no infinite “flat direction”.
- (v) When $r < 0$ and all other parameters are turned off, \tilde{Q} must have rank k for the first term to be vanishing. This is possible only when $n_2 \geq k$. In this case, $\sigma = 0$ for the sixth term to be vanishing. The space thus consists again only of “Higgs branch”. The same remark as (iv) applies to the case $n_1 = 0$.

At the tree level, fluctuation around each vacuum consists of massless and massive modes which are tangent and transverse to the space of classical vacua respectively. The mass of the transverse modes depends on the choice of vacuum and, in general, some massive modes become massless at some special points such as the intersection of “Higgs” and “Coulomb” branches. However, for the cases such as (iv) and (v), the mass² of the transverse modes are bounded from below by a constant of order $e^2|r|$. Thus, for this parameter region, if we take the limit $e^2 \rightarrow \infty$ (or equivalently the long distance limit), the massive modes decouple and the system approaches to a supersymmetric non-linear sigma model whose target space is the corresponding space of classical vacua. Namely the space of solutions of

$$QQ^\dagger - \tilde{Q}^\dagger\tilde{Q} = r \tag{2.39}$$

modulo $U(k)$ gauge transformations. In what follows, we mainly study such non-linear sigma models realized by this gauged linear sigma models.

Global Symmetry

The group of global symmetry of the system is at the tree level $SU(n_1) \times SU(n_2) \times U(1)_a \times U(1)_A \times U(1)_V$, where $U(1)_A$ and $U(1)_V$ are the two R-symmetry groups and $SU(n_1) \times SU(n_2)$ is the semi-simple part of the flavor symmetry group $U(n_1) \times U(n_2)$ which acts on Q and \tilde{Q} as $k \times (\mathbf{n}_1, \mathbf{1})$ and $k \times (\mathbf{1}, \bar{\mathbf{n}}_2)$. The vector combination of the center of $U(n_1) \times U(n_2)$ is the same as the action of the center $U(1)$ of the gauge group $U(k)$ and is not considered as a global symmetry. The rest of the center is called $U(1)_a$ here. If we turn on mass terms, part of these symmetries are explicitly broken: A generic twisted mass preserves the $U(1)$ symmetry groups but breaks $SU(n_1) \times SU(n_2)$ to its maximal torus. A generic complex mass breaks $SU(n_1) \times SU(n_2)$ completely but preserves $U(1)_A$ and a combination of $U(1)_V$ and $U(1)_a$. These are restored by transforming the mass parameters in a suitable way. A choice of classical vacuum in the ‘‘Higgs branch’’ spontaneously breaks (part of) the flavor group $SU(n_1) \times SU(n_2) \times U(1)_a$.

The above is a brief description of the symmetry of the classical system. In the quantum theory, however, there are two major corrections to what have been said.

One is the anomaly of $U(1)_A$. It acts oppositely on the left and right handed fermions in each representation of $U(k)$ and hence is generically anomalous. Under the action of $e^{i\alpha} \in U(1)_A$, the fermion determinant in a fixed gauge field A changes by a phase shift

$$(2n_1 - 2n_2) \frac{i\alpha}{2\pi} \int \text{Tr}(F_A) \quad (2.40)$$

where F_A is the curvature of A . This shows that the $U(1)_A$ R-symmetry is broken to its discrete subgroup consisting of $2n_1 - 2n_2$ roots of unity:

$$U(1)_A \longrightarrow \mathbf{Z}_{2(n_1 - n_2)}. \quad (2.41)$$

In the case $n_1 = n_2$, the whole $U(1)_A$ is unbroken. Note that in the general case $U(1)_A$ can be restored by making it act on the theta parameter by the shift $2(n_1 - n_2)\alpha$ which absorbs (2.40).

In two-dimensional quantum field theory, a continuous symmetry cannot be spontaneously broken. Therefore, even in the case when some flavor symmetry appears to be spontaneously broken at the tree level, it must be restored in the full quantum theory unless it is broken by an anomaly. This is the second correction to the classical statement about the global symmetry.

Renormalization

The theory is super-renormalizable with respect to the gauge coupling constant e , as its mass dimension shows. However, the FI parameter r is dimensionless and this must be renormalized due to a one loop ultra-violet divergence which is present in the case $n_1 \neq n_2$.

In order to see this, we look at the term in the Lagrangian which depends linearly on the auxiliary field D in the vector superfield:

$$- \text{Tr} \left\{ D \left(QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} - r \right) \right\} . \quad (2.42)$$

The effective Lagrangian contains a term of this kind in which $QQ^\dagger - \tilde{Q}^\dagger \tilde{Q}$ is replaced by its expectation value. At the one loop level, the expectation value is given by

$$\sum_{i=1}^{n_1} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \dots} - \sum_{j=1}^{n_2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \dots} \quad (2.43)$$

where $+\dots$ are terms depending on the complex/twisted masses and background value of σ . This integral is logarithmically divergent in the limit $k \rightarrow \infty$ which is regularized by introducing a UV cut off Λ_{UV} . There is no higher loop divergence, and the effective Lagrangian can be made finite by renormalizing the bare FI parameter r_0 as

$$r_0 = \frac{n_1 - n_2}{2\pi} \log \left(\frac{\Lambda_{UV}}{\Lambda} \right) , \quad (2.44)$$

to cancel the divergence of (2.43) as $\Lambda_{UV} \rightarrow \infty$. Note that we are forced to introduced a dimensionful constant Λ which is an analog of the dynamical scale of four-dimensional gauge theories. The effective FI parameter at an energy scale μ is then given by $r(\mu) = \frac{n_1 - n_2}{2\pi} \log(\mu/\Lambda)$. In other words, the theory for $n_1 > n_2$ is asymptotically free with respect to the coupling g given by $r = 1/g^2$, while for $n_2 > n_1$ it is so with respect to g defined by $r = -1/g^2$.

This has an important implication. Consider a theory with $n_1 > n_2$. It is always possible to find a scale μ at which $r(\mu)$ is positive: take μ to be much larger than Λ . The space of zeroes of the renormalized potential U at that scale is of the type (iv) in the above discussion, and the theory is, in the limit $e^2 \rightarrow \infty$, interpreted as the non-linear sigma model with the target space given by (2.39). The FI parameter $r = r(\mu)$ is interpreted as the size of the target space, or equivalently its Kähler class. Indeed, the way r runs is exactly the same as the running of the Kähler class of the sigma model, since the one-loop beta function of the latter is given by

the first Chern class which, being equal to the anomaly of $U(1)_A$, is proportional to $n_1 - n_2$. Likewise, a theory with $n_2 > n_1$ is always of the type (v) in which $r < 0$ and is interpreted in the limit $e^2 \rightarrow \infty$ also as the non-linear sigma model with the target space given by (2.39). A theory with $n_1 = n_2$ is quite different and has separate “phases” corresponding to $r > 0$ and $r < 0$. (See [32] for discussion of this type of theories.)

2.2.1 The $\mathbb{C}P^{n-1}$ Model

Here we consider the case $k = 1$, $n_1 = n$, $n_2 = 0$ in some detail. We first consider the theory with $\widetilde{m} = 0$ in which the global symmetry is $SU(n) \times \mathbf{Z}_{2n} \times U(1)_V$. The space of solutions of the D-term equation (2.39) modulo $U(1)$ gauge transformation is the space of vectors in \mathbb{C}^n of length² = r modulo phase rotation, and hence is the same as $n - 1$ dimensional complex projective space $\mathbb{C}P^{n-1}$. Thus, the theory describes in the $e^2 \rightarrow \infty$ limit the supersymmetric sigma model with target space $\mathbb{C}P^{n-1}$ which has been studied extensively from various view points [30, 31, 46, 47, 48, 49, 41, 37, 50, 42].

Low Energy Effective Action

We consider integrating out the chiral superfield Q and obtain the effective Lagrangian as a functional of the vector superfield. Due to the gauge invariance, the result should be expressed in terms of the twisted chiral superfield Σ representing the field strength. The terms with at most two derivatives and not more than four fermions can be written as

$$\frac{1}{4} \int d^4\theta K(\Sigma, \Sigma^\dagger) + \left(\int d^2\tilde{\theta} \widetilde{W}_{eff}(\Sigma) + h.c. \right). \quad (2.45)$$

One can *exactly* determine the effective superpotential $\widetilde{W}_{eff}(\Sigma)$ [46, 37]. It is given by

$$\widetilde{W}_{eff}(\Sigma) = \frac{1}{4} \left[i\tau\Sigma - \frac{n}{2\pi} \Sigma \left(\log(\Sigma/\mu) - 1 \right) \right], \quad (2.46)$$

where τ is the complex combination $\tau = ir(\mu) + \theta/2\pi$ of the effective FI parameter at the scale μ and the theta parameter.

The part of the effective Lagrangian (2.45) which depends on the auxiliary field D and the field strength $F_A = F_{01} dx^0 \wedge dx^1$ is

$$K_{\sigma\bar{\sigma}} \left(D^2 + F_{01}^2 \right) + 2\widetilde{W}'_{eff}(\sigma) (D - iF_{01}) + \overline{2\widetilde{W}'_{eff}(\sigma)} (D + iF_{01})$$

$$= \frac{1}{2e_{\text{eff}}^2} (D^2 + F_{01}^2) - r_{\text{eff}} D + \frac{\theta_{\text{eff}}}{2\pi} F_{01} \quad (2.47)$$

where $1/e_{\text{eff}}^2 = 2K_{\sigma\bar{\sigma}}$, and r_{eff} and $\theta_{\text{eff}}/2\pi$ are the imaginary and the real parts of

$$\begin{aligned} \tau_{\text{eff}} &= i r_{\text{eff}} + \frac{\theta_{\text{eff}}}{2\pi} \\ &:= -4i \frac{\partial \tilde{W}_{\text{eff}}(\sigma)}{\partial \sigma} = \tau - \frac{n}{2\pi i} \log(\sigma/\mu). \end{aligned} \quad (2.48)$$

Integrating out the auxiliary field D , we get the energy density $e_{\text{eff}}^2 r_{\text{eff}}^2/2$. Actually, there is a contribution of the gauge field F_{01} to the energy density. This follows from the fact [51] that the theta parameter θ_{eff} creates a constant electric field proportional to the minimum absolute value $|\tilde{\theta}_{\text{eff}}|/2\pi$ of $\theta_{\text{eff}}/2\pi + \mathbf{Z}$. The contribution is then $e_{\text{eff}}^2 (\tilde{\theta}_{\text{eff}}/2\pi)^2/2$. Thus, the potential energy of the effective theory is

$$U_{\text{eff}} = \frac{e_{\text{eff}}^2}{2} \left[r_{\text{eff}}^2 + \left(\frac{\tilde{\theta}_{\text{eff}}}{2\pi} \right)^2 \right] \quad (2.49)$$

$$= \frac{e_{\text{eff}}^2}{2} \left| \tilde{\tau} - \frac{n}{2\pi i} \log(\sigma/\mu) \right|^2. \quad (2.50)$$

A supersymmetric vacuum is the zero of this potential energy. There are n such zeros. One of them is given by

$$\sigma = \mu \exp\left(\frac{2\pi i \tau}{n}\right) = \Lambda e^{i\theta/n}, \quad (2.51)$$

and the others are obtained by the action of the discrete \mathbf{Z}_{2n} subgroup of $U(1)_A$ R-symmetry. Namely, this discrete chiral symmetry is spontaneously broken to \mathbf{Z}_2 .

Solitons in The CP^{n-1} Model

Since we have a discrete set of vacua, we expect that there exist solitons which interpolate different vacua at the two spatial infinities $x^1 \rightarrow \pm\infty$. If we only look at the effective Lagrangian (2.45), solitonic configurations are forbidden by the Gauss law which is given by the variation of (2.45) or equivalently (2.47) with respect to A_0 :

$$\frac{\partial}{\partial x^1} \left(\frac{1}{e_{\text{eff}}^2} F_{01} \right) + \frac{\partial}{\partial x^1} \left(\frac{\theta_{\text{eff}}}{2\pi} \right) = 0. \quad (2.52)$$

Recall from (2.48) that

$$\theta_{\text{eff}} = \theta - n \arg(\sigma). \quad (2.53)$$

Then, integrating (2.52) over the space coordinate x^1 and using the fact that $F_{01} = 0$ for the vacua at the spatial infinities $x^1 \rightarrow \pm\infty$, we see that $\arg \sigma(+\infty) - \arg \sigma(-\infty) = 0$, which means that a configuration cannot interpolate different vacua.

However, we can see that solitons *do* exist if we take into account the effect of the n massive chiral multiplets (Q^i, ψ^i) minimally coupled to the $U(1)$ gauge field A_μ as $-D_\mu Q^\dagger D^\mu Q + i\bar{\psi}\gamma^\mu D_\mu \psi$ [30]. The Gauss law is then modified as

$$\frac{\partial}{\partial x^1} \left(\frac{1}{e_{\text{eff}}^2} F_{01} \right) + \frac{\partial}{\partial x^1} \left(\frac{\theta_{\text{eff}}}{2\pi} \right) + j^0 = 0, \quad (2.54)$$

where j^0 is the time component of the electric current $j^\mu = iQ^\dagger D^\mu Q - iD^\mu Q^\dagger Q - \bar{\psi}\gamma^\mu \psi$ of the fields Q^i, ψ^i . Then, integrating over the spatial coordinate, we have

$$\arg \sigma(+\infty) - \arg \sigma(-\infty) = \frac{2\pi}{n} \int dx^1 j^0. \quad (2.55)$$

As a consequence of this identity, the *fundamental solitons* which interpolate the neighboring vacua $\sigma(-\infty) = \Lambda e^{\frac{i\theta}{n}} \rightarrow \sigma(+\infty) = \Lambda e^{\frac{i\theta}{n} + \frac{2\pi i}{n}}$ carry the electric charge $= +1$. Indeed, these are the elementary fields Q^i, ψ^i which constitute BPS doublets in the fundamental representation of the flavor group $SU(n)$ [30]. Likewise, solitons interpolating vacua by ℓ steps, $\sigma(-\infty) = \Lambda e^{\frac{i\theta}{n}} \rightarrow \sigma(+\infty) = \Lambda e^{\frac{i\theta}{n} + \frac{2\pi i \ell}{n}}$, carry electric charge ℓ . It is known that the corresponding solitons consist of BPS saturated bound states of ℓ elementary fields which transform as the ℓ -th anti-symmetric representation of $SU(n)$ [47, 48]. The mass of such solitons is known to be [47, 49, 37, 50]

$$M_\ell = \frac{n}{2\pi} \Lambda |e^{2\pi i \ell / n} - 1|. \quad (2.56)$$

This coincides with what we naively expect from the twisted superpotential (2.46): Indeed, in spite of the presence of the logarithm we can unambiguously determine the value of $\widetilde{W}_{\text{eff}}$ at the n vacua, by replacing τ in (2.46) by $\tilde{\tau}$. The value of $4\widetilde{W}_{\text{eff}}$ at the j -th vacuum is $(n/2\pi)\mu e^{\frac{2\pi i(\tau+j)}{n}}$. Then, (2.56) is just the difference $4|\Delta\widetilde{W}_{\text{eff}}|$ of the values at $j = \ell$ and $j = 0$.

Inclusion of Twisted Mass

Let us consider the theory with general twisted mass in which the flavor symmetry $SU(n)$ is broken to $U(1)^{n-1}$. Upon integrating out the chiral superfields Q , we obtain the effective Lagrangian (2.45) with the twisted superpotential

$$\widetilde{W}_{\text{eff}}(\Sigma) = \frac{1}{4} \left[i\tau\Sigma - \frac{1}{2\pi} \sum_{i=1}^n (\Sigma - \widetilde{m}_i) \left(\log \left(\frac{\Sigma - \widetilde{m}_i}{\mu} \right) - 1 \right) \right]. \quad (2.57)$$

Subsequent analysis is the same as in the case $\widetilde{m} = 0$ and we have n vacua corresponding to the n roots of

$$\prod_{i=1}^n (\sigma - \widetilde{m}_i) = \mu^n e^{2\pi i t}. \quad (2.58)$$

The soliton spectrum is also similar. In the presence of a soliton with electric charge ℓ , the phase $\sum_{i=1}^n \arg(\sigma - m_i)$ changes by $2\pi\ell$. As for the degeneracy, there would be no much difference from the case $\widetilde{m} = 0$ at least for small \widetilde{m} , and we expect $\binom{n}{\ell}$ solitons to exist in the ℓ -th sector.

What are the masses of these solitons? The theory with twisted mass has not been studied in the past. A naive guess is the difference of the twisted superpotential (2.57). However, in a theory with non-coincident twisted mass, we cannot unambiguously determine the values of (2.57) at the vacua nor the difference of the values at two vacua, as we now see. Consider, for example, two nearby vacua as depicted in Figure 1. The difference of the values of $4\widetilde{W}_{eff}$ could

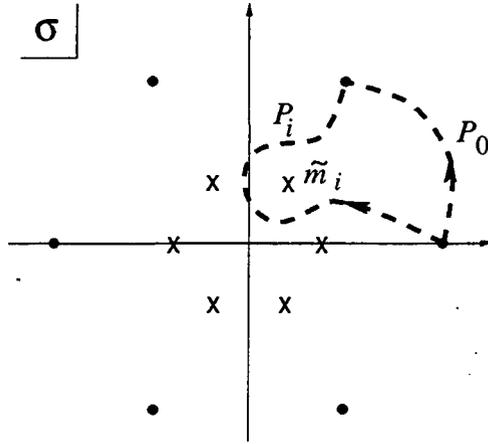


Figure 1: The Two Paths

be obtained by tracing the values of \widetilde{W}_{eff} along some path, say P_0 , connecting them. However, if we choose another path, say P_i , then the difference changes by

$$4\widetilde{W}_{eff}|_{P_i} = 4\widetilde{W}_{eff}|_{P_0} - i\widetilde{m}_i. \quad (2.59)$$

So, there is an ambiguity in defining the superpotential, and that is proportional to the twisted mass \widetilde{m} and vanishes in the CP^{n-1} model $\widetilde{m} = 0$. Actually, this ambiguity is related to the fact that there is a continuous Abelian symmetry $U(1)^{n-1}$ for generic values of \widetilde{m} .

We will see in section 6 using branes, that the central charge is determined unambiguously as the linear combination (as (2.27)) of the difference of the superpotential values and the twisted masses times the $U(1)$ charges. Indeed, the central charge is not determined just by the boundary condition at the two spatial infinities. The ambiguity of defining the superpotential cancels with the ambiguity of the choice of $U(1)$ charges. This is reminiscent of the formula for the central charge in the $N = 2$ theories in four dimensions [52] in which the ambiguity is related to the $SL(2, \mathbf{Z})$ duality of the low energy maxwell theory. The central charge formula is thus modified, in the presence of twisted mass terms by the amount

$$\sum_i S_i \widetilde{m}_i. \tag{2.60}$$

A similar phenomenon appears also with real central charges in $N = 2$ supersymmetry in three dimensions. See a discussion in [44].

Relation to the $SU(n)$ $N = 1$ super YM Theory in Four Dimensions

The supersymmetric CP^{n-1} sigma-model in two dimensions has many properties in common with the $N = 1$ supersymmetric Yang-Mills theory in four dimensions with gauge group $SU(n)$.

Both have n vacua with mass gap and have discrete \mathbf{Z}_{2n} chiral symmetry which is broken spontaneously to \mathbf{Z}_2 . The CP^{n-1} model is asymptotically free with respect to g defined by $r = 1/g^2$ as the $N = 1$ Yang-Mills theory is with respect to the gauge coupling constant g_4 where the one-loop beta functions are both proportional to n . The twisted superpotential of the CP^{n-1} model is holomorphic with respect to the complex combination $\tau = ir + \theta/2\pi$, while the superpotential of the $N = 1$ Yang-Mills theory is so with respect to the combination $\tau_4 = 4\pi i/g_4^2 + \theta_4/2\pi$ where θ_4 is the theta parameter in four dimensions. Moreover, the effective twisted superpotential (2.46) is the same as the effective superpotential of the $N = 1$ super YM theory of [53] under the replacement $\Sigma \rightarrow S = W_\alpha^2$, $\tau \rightarrow \tau_4$ and $\mu \rightarrow \mu^3$.

As a consequence, there is some close resemblance between the solitons in the CP^{n-1} model and domain walls in the $N = 1$ YM theory. In particular, both are BPS saturated. Recently, there is some interest in the study of domain walls in the $N = 1$ YM theory [54, 55] and the exact tension has been computed. In particular, in [23] domain walls in super YM theory are studied in the brane framework and are claimed to be a D-brane for the QCD string. It would

be interesting to see the relation with the $\mathbb{C}P^{n-1}$ solitons in the brane framework. The brane description of solitons in the $\mathbb{C}P^{n-1}$ model is given in section 6.

There is another similarity of the two systems. As noted in the previous subsection, the $\mathbb{C}P^{n-1}$ model is dual under mirror symmetry to $N = 2$ supersymmetric affine A_{n-1} Toda field theory. On the other hand, the $N = 1$ $SU(n)$ super YM theory is, when formulated on $\mathbf{R}^3 \times S^1$, described by a theory with chiral superfields with the superpotential being the A_{n-1} affine Toda potential [56, 57, 10, 44].

Relation of the two-dimensional systems to four-dimensional gauge theories will be further discussed in section 5 in terms of branes.

2.2.2 Other $k = 1$ Theories

Let us consider a $U(1)$ gauge theory with general n_1, n_2 . It is easy to obtain the effective twisted superpotential for generic values of σ : It is such that the effective FI and theta parameters are given by

$$\begin{aligned} \tau_{eff} &= -4i \frac{\partial \widetilde{W}_{eff}(\sigma)}{\partial \sigma} \\ &= \tau - \frac{1}{2\pi i} \sum_{i=1}^{n_1} \log \left(\frac{\sigma - \widetilde{m}_i}{\mu} \right) + \frac{1}{2\pi i} \sum_{\tilde{j}=1}^{n_2} \log \left(\frac{\sigma - \widetilde{m}_{\tilde{j}}}{\mu} \right) \end{aligned} \quad (2.61)$$

If the twisted masses are generic, there are $\max\{n_1, n_2\}$ vacua corresponding to the zeroes of (2.61). Something special happens when some of \widetilde{m}_i and some of $\widetilde{m}_{\tilde{j}}$ coincide. Consider, for example, the case with $\widetilde{m}_1 = \widetilde{m}_1$. Then, there is a cancellation of two terms in (2.61). However, this does not mean that the vacua are only the zeroes of (2.61) (as many as $\max\{n_1 - 1, n_2 - 1\}$), because the fields Q^1 and \tilde{Q}_1 are massless at $\sigma = \widetilde{m}_1$ and the above description purely in terms of σ breaks down. These massless fields span a one dimensional non-compact complex manifold with a metric $dzd\bar{z}/\sqrt{r + |z|^2}$ which is asymptotically \mathbb{C}/\mathbb{Z}_2 , where r is the r_{eff} at $\sigma = \widetilde{m}_1$. Thus, the theory has $\max\{n_1 - 1, n_2 - 1\}$ vacua and the vacua of the sigma model with such a target space.

Further discussion using branes on this is given in section 5.1.

2.2.3 The Grassmannian Model

Finally, we consider the case $k \geq 2, n_1 = n$, and $n_2 = 0$ with vanishing twisted mass $\widetilde{m} = 0$. The D-term equation (2.39) for $Q = (Q^{ai})$ can be considered as requiring the k vectors v_1, \dots, v_k in \mathbf{C}^n given by $v_a = (Q^{ai})_{i=1, \dots, n}$ to be orthogonal to each other and have length² = r : $v_a^\dagger v_b = r\delta_{a,b}$. The $U(k)$ gauge transformation can be considered as the unitary rotation of such orthogonal k -frames in \mathbf{C}^n and does not change the k -plane in \mathbf{C}^n which they span. Namely, the space of classical vacua is the space of k -planes in \mathbf{C}^n , which is the complex Grassmann manifold $G(k, n)$. Thus, the theory describes in the $e^2 \rightarrow \infty$ limit the supersymmetric non-linear sigma model with target space $G(k, n)$.

Low Energy Effective Action

Like in the Abelian case, one can exactly perform the integration over the chiral superfield Q . Although a manifestly supersymmetric form of the effective Lagrangian with respect to the full field strength Σ is not known, it is easy to determine the effective twisted superpotential for the case in which Σ is diagonal, $\Sigma = \text{diag}(\Sigma_1, \dots, \Sigma_k)$. It is given by

$$\widetilde{W}_{\text{eff}}(\Sigma) = \frac{1}{4} \sum_{a=1}^k \left[i\tau \Sigma_a - \frac{n}{2\pi} \Sigma_a \left(\log(\Sigma_a/\mu) - 1 \right) \right]. \quad (2.62)$$

If the diagonal entries are well-separated $|\sigma_a - \sigma_b| \gtrsim \Lambda$, the off diagonal components of Σ are heavy and it is appropriate to integrate them out as well. It does not give a contribution to $\widetilde{W}_{\text{eff}}$ since the off-diagonal components are in a complex and its conjugate representations of the diagonal $U(1)$ gauge groups. Thus, we may take (2.62) as the effective superpotential in the region in which σ_a 's are well-separated.

Thus, a supersymmetric vacuum is at $\sigma = \text{diag}(\sigma_1, \dots, \sigma_k)$ where each entry σ_a is one of the n -roots of

$$\sigma^n = \mu^n e^{2\pi i t}. \quad (2.63)$$

Since the approximation is valid only when σ_a 's are well-separated, distinct entries must be at distinct roots of (2.63). The number of vacua is thus the number of possible selections of k -elements among n -roots, namely,

$$\binom{n}{k}. \quad (2.64)$$

We have not considered the region in which some of the diagonal entries are near one another, but it seems likely from the following reasons that there is no supersymmetric vacuum in such a region. First, if the distinct entries of σ were allowed to coincide, there would be supersymmetric vacua even in the case $k > n$ where there is no supersymmetric vacuum at the tree level. Second, for the case $k \leq n$, the number (2.64) already agrees with the Euler number of the Grassmannian $G(k, n)$ which is the Witten index of the Grassmannian sigma model [58]. We will see in section 4 that this is consistent with the observation in [1] that s -configurations of the branes are not supersymmetric.

Given that these are the only supersymmetric vacua, we see that the discrete chiral symmetry \mathbf{Z}_{2n} is spontaneously broken to some subgroups. A generic vacuum breaks it to \mathbf{Z}_2 , but some special vacua keep larger subgroups unbroken.

Relation to $U(k)/U(k)$ Gauged WZW Models

It was shown in [59] that the low energy limit, or the dynamics of vacuum, of the present model is described by the topological field theory realized as the $U(k)$ Wess-Zumino-Witten (WZW) model with the whole adjoint $U(k)$ group being gauged. The level of the WZW model is $n - k$ for the $SU(k)$ part and n for the $U(1)$ part of the group $U(k)$. By Abelianization of [60], this gauged WZW model can be considered, when formulated on the flat space-time, as the theory of $U(1)^k$ sigma model (represented by k free bosons ϕ_a , $\phi_a \equiv \phi_a + 2\pi$) coupled to $U(1)^k$ gauge fields A_a by

$$\frac{n}{2\pi} \sum_{a=1}^k \int \phi_a dA_a. \quad (2.65)$$

The identification of the two systems is essentially based on the observation that this is the same under $\phi_a = \arg(\sigma_a)$ as the terms in $\int d^2\tilde{\theta} \tilde{W}_{eff} + c.c.$ depending on $\arg(\sigma_a)$'s.

3 A Type IIA Configuration

In this section, we construct brane configurations such that the world-volume dynamics at long distances describes $N = 2$ supersymmetric gauge theories in two dimensions. In particular, we propose configurations which lead to the $N = 2$ theories obtained by dimensional reduction of four-dimensional $N = 1$ *chiral* gauge theories, a typical example of which is the gauged linear

sigma model realizing the CP^{n-1} or Grassmannian sigma models. In this and later sections, we provide evidence for the construction and study such a gauge theory using the branes.

Following [1], the authors in [9] constructed a brane configuration in Type IIA string theory whose world-volume dynamics at long distances describes $N = 1$ supersymmetric QCD in four dimensions. The configuration is in a flat space-time with time coordinate x^0 and space coordinates x^1, \dots, x^9 and involves two NS 5-branes spanning world-volumes in the 012345 and 012389 directions, k parallel D4-branes stretched between them spanning world-volumes in the 01236 directions, and n parallel D6 branes located between the two NS 5-branes and spanning world-volumes in the 0123789 directions. This yields $N = 1$ supersymmetric $SU(k)$ QCD with n flavors in four dimensions. One of the important features of such a theory is the chiral flavor symmetry $SU(n)_L \times SU(n)_R$. In a generic position of the D6 branes, we can only see the diagonal subgroup $SU(n)$ in the brane configuration, because the configuration of D4 and D6 branes is locally of theories with 8 supercharges. In such a configuration, strings ending on the D4 and D6 branes create the chiral multiplet Q in the representation $(\mathbf{k}, \mathbf{n}, \mathbf{1})$ and the chiral multiplet \tilde{Q} in the representation $(\bar{\mathbf{k}}, \mathbf{1}, \bar{\mathbf{n}})$ at the same point, and therefore it is also difficult to obtain chiral matter in such a configuration.

In the presence of the NS 5-brane spanning world-volume in the 012389 directions — which we call the NS' 5-brane — the configuration is locally of theories with 4 supercharges. (In the absence of this brane the number of supercharges is 8.) In particular, if the D6-branes and the NS' 5-brane have the same x^6 value, the D6-branes break into two pieces at the intersection with the NS' 5-brane [15]. The pieces in $x^7 > x^7(\text{NS}'5)$ will be called *upper-half* D6-branes, while the other pieces will be called *lower-half* D6-branes. In [15], it is proposed that the strings ending on the D4 and the upper-half D6-branes create the fundamental chiral multiplets Q and the strings ending on the D4 and the lower-half D6-branes create the anti-fundamental chiral multiplets \tilde{Q} . Taking this for granted, if we could take away the lower-half D6-branes from the configuration, we would expect only the fundamental chiral multiplet Q , leading to a chiral gauge theory. However, in the absence of any other branes, such a configuration breaks the charge conservation relation, and cannot be a stable one, reflecting the gauge anomaly of the corresponding four-dimensional theory with only left-handed fermions in a complex representation of the gauge group. Indeed in [21] it was argued how, in the presence of D8 branes, a semi-infinite D6 brane can end on a NS fivebrane. This configuration leads to a

localization of a four dimensional chiral matter in ten dimensional space time. The presence of the chiral matter still induces an anomaly in the four dimensional theory. However, the theory being really five dimensional on an interval, gives rise to a flow of the anomalous current along the five dimensional interval. This current is absorbed by a chiral matter with an opposite chirality which is localized at the other end of the interval.

Upon application of T-duality on the 23 directions, we obtain a configuration of NS and NS' 5-branes, k D2-branes with world-volume in the 016 directions, and n D4-branes with world-volume in the 01789 directions. In this situation, when the D4-branes have the same x^6 value as the NS' 5-brane and breaks at the NS' into upper-half $x^7 > x^7(\text{NS}')$ and lower-half $x^7 < x^7(\text{NS}')$ pieces, we can take away the lower-half pieces because D4-branes can end on a NS 5-brane. This situation is different from the T-dual D6 - NS configuration since now the D4 brane can move along the NS' brane in the 23 coordinates while the D6 brane can not move in any directions, its boundary being the whole NS' brane. Moreover, two dimensional chiral multiplets do not give rise to anomalies as their four dimensional analogs do. In this case the presence of a T-dual to D8 brane, as in [21] - a D6 brane, is not necessary for localizing the chiral matter in ten dimensional space time. This is reflected by the fact that a D4 brane can end on a NS brane without having a D6 brane in the background. In this paper, we will study such a configuration. Namely, the configuration involves

- A NS 5-brane with world-volume 012345 located at a point in the 6789 directions,
- A NS' 5-brane with world-volume 012389 located at a point in the 4567 directions,
- k D2-branes with world-volumes 016 stretched between the NS and NS' 5-branes,
- n_1 upper-half D4-branes with world-volumes 01789 ending on the NS' 5-brane from above in the x^7 direction, and
- n_2 lower-half D4-branes with world-volumes 01789 ending on the NS' 5-brane from below in the x^7 direction.

The configuration preserves 4 supercharges among the 32 of Type IIA string theory. This can be seen from direct calculation of the broken supersymmetries. Another way to see this is to note that this configuration is obtained from the configuration of [9] by T-duality (and removing a part of the branes).

We note that this configuration is invariant under the rotations in the 01, 23, 45 and 89

planes. These will be inherited as the Lorentz invariance and as global symmetries $U(1)_{2,3}$, $U(1)_{4,5}$ and $U(1)_{8,9}$ in the field theory on the D2-brane world-volume which is now explained.

3.1 Field Theory on the D2-brane World-volume

First note that the D2-branes are finite in the x^6 direction, and any of the other branes has more than one (semi-)infinite directions in addition to the 01. Thus, as in [1], we study the dynamics of the D2-branes and consider the positions of the other branes to be fixed parameters. Since the brane configuration is invariant under 4 supercharges — two left and two right — the world-volume dynamics at long distances describes an $N = (2, 2)$ (or simply $N = 2$) supersymmetric field theory in two dimensions. By the rotational invariance of the configuration, at least classically, the theory possesses the Lorentz invariance and global symmetry groups $U(1)_{2,3}$, $U(1)_{4,5}$ and $U(1)_{8,9}$.

Light fields on the D2-brane world-volumes are given as follows.

An open string ending on the D2-branes creates $N = 8$ $U(k)$ vector multiplet, but 6 out of 8 scalars are killed by the boundary condition at the NS and NS' ends, and only an $N = 2$ $U(k)$ vector multiplet, or equivalently a $U(k)$ twisted chiral multiplet Σ remains. The value $x^2 + ix^3$ of the D2-branes correspond to the eigenvalues of the scalar component σ of the twisted chiral multiplet. On dimensional grounds, these must be related by $x^2 + ix^3 = \ell_{st}^2 \sigma$, where ℓ_{st} is the string length. These scalar components transform in the vector representation (i.e. charge 2) under $U(1)_{2,3}$.

As in [15], we propose that strings ending on the D2-branes and the upper-half D4-branes create fundamental chiral multiplets $Q^{i=1, \dots, n_1}$, while strings ending on the D2-branes and the lower-half D4-branes create anti-fundamental chiral multiplets $\tilde{Q}_{j=1, \dots, n_2}$. In addition to the arguments given in [21], we will collect more evidence for this proposal in the following sections by observing that it has the right consequences expected from the field theory analysis. The scalar components of these chiral multiplets are singlets under $U(1)_{4,5}$ but transform in the spinor representation (i.e. charge 1) under $U(1)_{8,9}$.

The positions of the NS and NS' and D4-branes give parameters of the theory. Note that the position of the NS brane is specified by its $x^{6,7,8,9}$ -value, but we may put these to be zero

$x^{6,7,8,9}(\text{NS}) = 0$ by a choice of the center of the coordinate. Also, the position of the NS' brane is specified by its $x^{4,5,6,7}$ -value but we may put $x^{4,5}(\text{NS}') = 0$ by a choice of the origin of the 45 plane because the NS brane has world-volume in these directions. The remaining parameters are

- The x^6 -value of the NS' brane determines the bare gauge coupling constant of the $U(k)$ gauge theory

$$\frac{x^6(\text{NS}')\ell_{st}}{g_{st}} = 1/e^2, \quad (3.1)$$

where g_{st} is the Type IIA string coupling constant.

- The x^7 -value of the NS' brane is the Fayet-Iliopoulos parameter of the $U(1)$ factor of the $U(k)$ gauge group

$$\frac{x^7(\text{NS}')}{g_{st}\ell_{st}} = -r. \quad (3.2)$$

- The $x^2 + ix^3$ value of the upper-half or lower-half D4-branes are the twisted masses for the chiral multiplets Q^i or $\tilde{Q}_{\tilde{j}}$:

$$\ell_{st}^{-2}(x^2 + ix^3)|_{\text{D4}_i} = \tilde{m}_i, \quad \ell_{st}^{-2}(x^2 + ix^3)|_{\text{D4}_{\tilde{j}}} = \tilde{m}_{\tilde{j}}. \quad (3.3)$$

- In the original configuration, all the D4 branes end on the NS' brane. Thus, their $x^4 + ix^5$ value should be zero. However, if one of the upper-half D4 branes and one of the lower-half branes rejoin, they can be separated from the NS' branes and thus, in particular can have a non-zero $x^4 + ix^5$ value. If i -th upper and j -th lower D4-branes rejoin, the $x^4 + ix^5$ value corresponds to the complex mass $m_i^{\tilde{j}}$ which enters into the tree level superpotential as

$$m_i^{\tilde{j}}\tilde{Q}_{\tilde{j}}Q^i. \quad (3.4)$$

Note, however, that we cannot have the general superpotential $\sum_{i,\tilde{j}} m_i^{\tilde{j}}\tilde{Q}_{\tilde{j}}Q^i$ in this brane set-up. This is because the brane configuration chooses some diagonal embedding of the matrix $m_i^{\tilde{j}}$. The most general mass matrix is given by the Higgs branch of a four dimensional theory as explained below.

In this set-up, it is easy to identify the chiral flavor symmetry $U(n_1) \times U(n_2)$.⁴ The n_1 upper-half D4-branes are responsible for the $U(n_1)$ factor and the n_2 lower-halves are responsible for

⁴Note that the axial and the vector $U(1)$ subgroups of this are the same as the axial part of the group $U(1)_{4,5} \times U(1)_{8,9}$ and the $U(1)$ subgroup of $U(k)$ gauge group respectively.

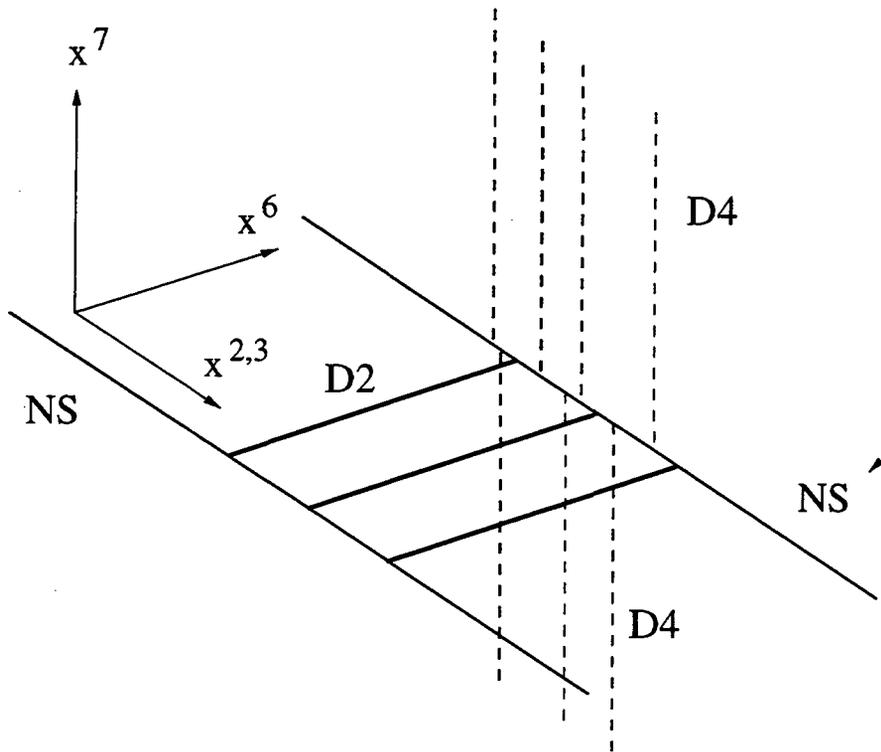


Figure 2: The IIA configuration for general twisted mass

the $U(n_2)$ factor. These are broken if the D4-branes are separated in the 23 directions, that is, if the twisted masses \widetilde{m}_i or \widehat{m}_j are turned on. In fact, the twisted masses can be interpreted as the scalar component of the $N = 2$ $d = 4$ $U(n_1) \times U(n_2)$ vector multiplet of a gauge system associated with the system of these D4 branes and the NS' 5-brane where, the branes being semi-infinite in the x^7 direction, the gauge dynamics is frozen.

Let us examine the situation from the point of view of a four dimensional observer which lives in the common directions, 0189, of the D4 and NS' branes. From the point of view of such an observer, the theory in question is a weakly coupled $SU(n_1) \times SU(n_2)$ gauge theory with $N = 2$ supersymmetry in four dimensions. In addition there are hypermultiplets which transform in the bi-fundamental representation. This consists of two chiral multiplets in the (n_1, \bar{n}_2) and (\bar{n}_1, n_2) representations. Such a theory is known to have a moduli space of vacua which contains two branches. The Coulomb branch of this theory is $n_1 + n_2 - 2$ complex dimensional while in the Higgs branch, we can turn on a matrix of rank which is at most

$\min\{n_1, n_2\}$. We want to identify the moduli of this theory with the parameters of the two dimensional theory. This identification is possible due to the hierarchy in scales that exists in the problem since the moduli of the four dimensional theory are associated with motion of the heavy branes which are slowly varying compared to the D2 branes.

The Coulomb branch of the four dimensional theory is identified with the motion of the D4 branes along the 23 directions and thus gives the twisted mass parameters \widetilde{m}_i and \widehat{m}_j . The transition from the Coulomb branch to the Higgs branch of the four dimensional theory is done by reconnecting D4 branes from both sides of the NS' brane and moving them in the 456 directions. Thus the Higgs branch of the four dimensional theory parametrizes the complex mass parameters, m_i^j , of the two dimensional theory. Note that since the Higgs branch contains four real scalars for a hypermultiplet and the complex mass parameters are only two real, there are two additional parameters in the four dimensional theory, per one hypermultiplet. These parameters correspond to the x^6 position of the D4 branes and to the reduction of the D4 gauge field in the x^7 direction, A_7 . There are some examples in which these parameters affect the low energy dynamics of the two dimensional theory. See [4] for examples of such effects, the discussion there being on a three dimensional analog of the system discussed here. It would be interesting to study further such effects.

In summary, we list the fields and parameters of the theory, together with the transformation properties under the gauge group $U(k)$ and the global symmetry groups. Note that the symmetries $U(1)_{2,3}$, $U(1)_{4,5}$, and $U(1)_{8,9}$ can be considered as $U(1)_A$, $U(1)_V$, and another $U(1)_V$ R-symmetry groups respectively. The charges in the list denote the charges of the scalar components of the superfields:

	$U(k)$	$SU(n_1)$	$SU(n_2)$	$U(1)_{2,3}$	$U(1)_{4,5}$	$U(1)_{8,9}$	
Σ	adj	1	1	2	0	0	
Q_i	k	n₁	1	0	0	1	
\widetilde{Q}_j	$\overline{\mathbf{k}}$	1	$\overline{\mathbf{n}}_2$	0	0	1	
r	1	1	1	0	0	0	
\widetilde{m}_i	1	adj \oplus 1	1	2	0	0	
\widehat{m}_j	1	1	adj \oplus 1	2	0	0	
m_i^j	1	$\overline{\mathbf{n}}_1$	n₂	0	2	0	(3.5)

3.2 Classical Space of Vacua

In this subsection we describe the brane realization of the classical space of vacua as described in section 2.2. There are five cases discussed in that section depending on the values of the parameters r, m, \bar{m} and \widehat{m} . In terms of branes these are the positions in 7, 45 and 23, respectively, of the various branes other than D2. In the following, we analyze the vacuum configuration of the D2-branes under various positions of other branes. The numbering here is correlated with the one in section 2.2.

1. When all parameters are turned off – all the heavy branes are at the origin of the ten dimensional space (the NS' brane is at the origin of 45 and 7 and the semi-infinite D4 branes are at the origin of 23), there are two branches. The Coulomb branch is parametrized by the 23 positions of the D2 branes between the two NS branes. The Higgs branch is parameterized by the positions of segments of D2 branes which break along the D4 branes and move in the 789 directions. The two branches meet at the origin of moduli space. There are mixed branches in which l D2 branes break with positions in 789 (Higgs) and $k - l$ D2 branes do not break and have arbitrary positions in 23 (Coulomb). When there are no lower (upper) D4 branes the D2 branes can not break and there is no Higgs branch.
2. When the NS' is at $x^7 = 0$ and pairs of lower and upper D4 branes form into infinite D4 branes and move away in the 45 directions there is a Coulomb branch where the D2 branes are free to move in between the two NS branes. If only $m < \min\{n_1, n_2\}$ infinite branes leave the 45 origin a Higgs branch is possible.

There is a different branch which is not visible in the field theory. This is the motion of the infinite D4 branes in the x^6 direction. Let z_i denote the x^6 positions of the infinite D4 branes and $t_1(t_2)$ denote the x^6 position of the NS (NS') brane. When $z_i < t_2$ for some i , the analysis is the same and the parameter z_i is irrelevant. When $z_i < t_1$ there is a D2 brane created between the D4 and the NS brane [1] which still makes the z_i position irrelevant. When $z_i > t_2$ there is a phase transition and the states coming from the D4 brane decouple from the system. The number of massless multiplets is changed.

3. When the NS' is at $x^7 = 0$ and the semi-infinite branes have arbitrary 23 positions there is again a Coulomb branch given by arbitrary positions of D2 branes in the 23 directions.

However when the 23 positions of a lower and an upper D4 branes coincide they can combine and form an infinite D4. Then, a Higgs branch can emanate from that point by breaking the D4 branes and moving them along the 789 directions. The 23 values of the D4 branes must therefore coincide with the 23 values of the D4 branes for such a breaking to happen.

4. When the x^7 position of the NS' brane is negative (which corresponds to r positive due to the minus sign in (3.2)), the D2 branes are stretched between the NS brane and the upper-half D4-branes. To avoid s-configurations [1] the right ends of distinct D2 branes are in distinct upper D4-branes. For this we need $n_1 \geq k$. The D2 branes are fixed to the origin of 23 and hence there is only a Higgs branch. When $n_2 \neq 0$ the D2-branes can break at the infinite D4-branes and the resulting segment can move in the 789 directions, while when $n_2 = 0$ there is no direction of the D2-brane motion.
5. For negative position of the NS' brane the analysis is the same by exchanging the lower and upper D4 branes.

It is evident that the space of classical field theory vacua almost agrees with the space of vacuum configurations of the D2-branes in this Type IIA set-up. A non-compact flat direction corresponds to a direction of the D2-brane motion, while a compact direction corresponds to the absence of such direction.

4 Quantum Properties Via M Theory

The description in the previous section of Type IIA brane configurations is missing basic properties of the quantum theory — running of the FI parameter r , the anomaly of the $U(1)_A$ R-symmetry, and the spontaneous breaking of its non-anomalous discrete subgroup, which are present for theories with $n_1 \neq n_2$. For instance, the Type IIA configuration is invariant under $U(1)_{2,3}$ which is identified with $U(1)_A$ and no sign of anomaly was observed.

In this section, we see that these properties are correctly reproduced by considering the corresponding configuration in M theory where membranes and fivebranes are involved. We will also see that the number of vacua agrees with the one known in field theory for the case $n_2 = 0$ ($\mathbb{C}P^{n-1}$ or Grassmannian model).

4.1 M Theory Description of the Configuration

A system of branes ending on other branes cannot be described by any conformal field theory of weakly coupled superstrings. In [3] it was shown that the system of D4-branes ending on NS 5-branes with two transverse directions can be described as a smooth configuration of a single fivebrane in M theory on $\mathbf{R}^{10} \times S^1$ in the eleven-dimensional supergravity limit, and that, for some purposes, it gives us nice or even exact results of the gauge theory on the brane in the long distance limit. A D4-brane is a fivebrane wrapped on the S^1 and NS 5-brane is a fivebrane at a point in the S^1 . The configuration of a D4-brane ending on a NS 5-brane is interpreted as a wrapped fivebrane merging smoothly with an unwrapped fivebrane. In the brane configuration given in the previous section, the NS brane is described as a flat fivebrane in M theory, but the system of upper- and lower-half D4-branes ending on the NS' brane is described by a curved fivebrane. The D2-branes are described as membranes stretched between the two fivebranes. The dynamics of such open membranes ending on fivebranes has not been fully understood. However, as we will see in the next subsection, there are features of quantum theory that can be captured correctly without the detailed knowledge of the membrane dynamics, e.g., renormalization of the FI parameter r and anomalous breaking of an R-symmetry. In some cases, the description by M theory determines the vacuum structure of the theory and, moreover, enables us to obtain some exact information, such as soliton spectrum and masses. Conversely, it would be interesting to investigate the dynamics of open membranes by making use of knowledge of two-dimensional quantum field theory.

We introduce the coordinate x^{10} of the circle S^1 in the eleventh direction where x^{10} is identified with $x^{10} + 2\pi$. The flat eleven-dimensional space-time $\mathbf{R}^{10} \times S^1$ is given by the metric

$$ds^2 = -(dx^0)^2 + \sum_{\mu=1}^9 (dx^\mu)^2 + R^2(dx^{10})^2 \quad (4.1)$$

where R is the radius of the circle S^1 . Recall that the string length ℓ_{st} and the eleven-dimensional Planck length ℓ_{11} are related by $\ell_{11} = g_{st}^{1/3} \ell_{st}$ where the string coupling constant g_{st} is given by $g_{st}^{2/3} = R/\ell_{11}$.

Below, we give an M theory description of the Type IIA configuration in section 3 in some detail.

The D4-Branes Ending on the NS' Brane

In M theory, the system of upper and lower D4-branes ending on the NS' brane can be described as a single fivebrane of the form $\mathbf{R}^4 \times C$ where \mathbf{R}^4 spans the coordinate $x^{0,1,8,9}$ and C is a real two-dimensional surface embedded in the four-dimensional space spanning the coordinates $x^{2,3,7,10}$ which is at a fixed position in the $x^{4,5,6}$ directions. Since it preserves eight supercharges, it must be holomorphically embedded with respect to some complex structure. By introducing the complex coordinates ⁵

$$\sigma = \ell_{st}^{-2}(x^2 + ix^3), \quad s = R^{-1}x^7 + ix^{10}, \quad (4.2)$$

the $x^{2,3,7,10}$ part of the space-time can be considered as a hyper-Kähler surface with a flat Kähler metric

$$g = \ell_{st}^4 |d\sigma|^2 + R^2 |ds|^2, \quad (4.3)$$

and a holomorphic two-form

$$\Omega = R\ell_{st}^2 d\sigma \wedge ds = \ell_{11}^3 d\sigma \wedge ds. \quad (4.4)$$

The curve C corresponding to a straight NS' brane is at a point in the s -cylinder and is coordinatized by σ , while the curve for a straight D4-brane is at a point in the σ -plane and is coordinatized by s modulo a shift by $2\pi i$. When an upper-half D4-brane ends on an NS' brane at a point $\sigma = \widetilde{m}$, then, the D4-brane bends the NS' brane in such a way that s -value of the NS' brane varies as a function of σ as if the D4-brane plays the role of a source of the Laplace equation for $s = s(\sigma)$: $\partial_\sigma \partial_{\bar{\sigma}} s = -\delta(\sigma - \widetilde{m})$. Since the fivebrane becomes a D4-brane extending to $x^7 = +\infty$ as we approach $\sigma = \widetilde{m}$, the real part of $s(\sigma)$ must diverge at $\sigma = \widetilde{m}$ but it should not diverge nowhere else because there is no other D4-branes. Also, the imaginary part of $s(\sigma)$ admits a 2π shift as we go around $\sigma = \widetilde{m}$, but no other kind of multi-valued-ness is allowed. A unique holomorphic function satisfying these properties is

$$s = -\log(\sigma - \widetilde{m}) + \text{constant}. \quad (4.5)$$

Likewise, if we consider n_1 upper-half D4-branes at $\sigma = \widetilde{m}_1, \dots, \widetilde{m}_{n_1}$ and n_2 lower-half D4-branes at $\sigma = \widehat{m}_1, \dots, \widehat{m}_{n_2}$, the curve C is described by

$$s = -\sum_{i=1}^{n_1} \log(\sigma - \widetilde{m}_i) + \sum_{j=1}^{n_2} \log(\sigma - \widehat{m}_j) + \text{constant}. \quad (4.6)$$

⁵We use for the complex combination of $x^{2,3}$ the same symbol σ as the scalar component of the twisted chiral superfield, as these are identified in section 3.1. The factor ℓ_{st}^{-2} is the tension of the fundamental string.

We recall that we have chosen the coordinates so that the D4 and NS' branes are at $x^4 = x^5 = 0$. Let L be the x^6 -value of them. Then, by introducing a single valued coordinate of the s -cylinder as $t = \exp(s)$, the fivebrane is described by

$$t \prod_{i=1}^{n_1} (\sigma - \widetilde{m}_i) = q \prod_{\widetilde{j}=1}^{n_2} (\sigma - \widehat{m}_{\widetilde{j}}), \quad (4.7)$$

$$x^4 = x^5 = x^6 - L = 0, \quad (4.8)$$

where q is a non-zero constant.

The NS Brane

The NS brane is described in M theory as just the flat fivebrane spanning the coordinates $x^{0,1,2,3,4,5}$ which is located at a point in the remaining directions. In section 3, we have chosen the origin of the coordinates so that it is located at $x^{6,7,8,9} = 0$. Likewise, we can also choose the origin so that the NS brane is at $x^{10} = 0$. Namely, the NS brane is the fivebrane at $s = 0$, or equivalently,

$$t = 1 \quad \text{and} \quad (4.9)$$

$$x^6 = x^8 = x^9 = 0. \quad (4.10)$$

The D2 Branes

The D2 branes are described as membranes stretched between the two fivebranes. For a fixed time x^0 , it is basically an infinite strip where the two boundaries are at the two fivebranes, namely, one is constrained by (4.9)-(4.10) and the other by (4.7)-(4.8). There can be a “topological excitation” in which the membrane is not just a strip but can have additional boundaries or genus. In such a case, we require any boundary to be in one of the two fivebranes. Indeed, in section 6 we will see such excitations as solitonic states of some models.

4.2 Quantum Properties of the Theory

The Fayet-Iliopoulos and Theta Parameters

We recall that in the Type IIA description the difference $\Delta x^7 = x^7(\text{NS}') - x^7(\text{NS})$ of the x^7 -values of the NS and NS' branes is interpreted as the FI parameter r of the $U(1)$ part of the gauge group. More precisely,

$$r = -\frac{\Delta x^7}{g_{st}\ell_{st}}. \quad (4.11)$$

However, in the present situation, the NS' brane does not have a definite x^7 value and varies as a function of σ . Correspondingly, r varies as a function of σ and we interpret this as the effective FI coupling at the mass scale $|\sigma|$. Since $g_{st}\ell_{st} = R$, it is given by

$$r(\sigma) = -\frac{\Delta x^7}{R} = -\text{Re}(s) \quad (4.12)$$

$$= \sum_{i=1}^{n_1} \log |\sigma - \widetilde{m}_i| - \sum_{j=1}^{n_2} \log |\sigma - \widehat{m}_j| - \log |q|. \quad (4.13)$$

Indeed, at large $|\sigma|$ it behaves as $r \sim (n_1 - n_2) \log |\sigma|$, and this agrees with what we expect from the renormalization (2.44) of the FI parameter, up to a factor of 2π for which we have not been careful. Moreover, for a suitable choice of $\log q$ this is exactly the same (modulo the 2π factor) as the effective FI coupling r_{eff} as a function of the scalar component of the twisted chiral superfield Σ (see (2.48) and (2.61)).

In addition to this, we also interpret the separation of NS and NS' branes in the x^{10} direction as the theta parameter of the $U(1)$ part of the gauge group. This can be understood in the following way. Consider a (generically non-supersymmetric) configuration of a membrane stretched between the fivebranes where the boundary on the right has a fixed x^{10} value as well as the one on the left does. Since the gauge field on the D2-brane is dual to the scalar field representing the position of the membrane in the eleventh direction, we have $F_{01} = g_{st}\ell_{st}^{-1} \partial_6 x^{10}$.

⁶ Namely,

$$F_{01} \sim g_{st}\ell_{st}^{-1} \frac{\Delta x^{10}}{\Delta x^6} \sim e^2 \Delta x^{10}. \quad (4.14)$$

On the other hand, we know from field theory [51] that the theta parameter θ creates a constant electric field $F_{01} \sim e^2 \theta$. Thus, we can identify the separation Δx^{10} as the theta parameter.

⁶ Actually, it is $F_{01} - B_{01} = g_{st}\ell_{st}^{-1} \partial_6 X^{10}$ [61]. In this situation, however, $B_{01} = 0$ as the directions 01 being parallel to the fivebranes.

Altogether, the complex combination $\Delta s = \Delta x^7/R + i\Delta x^{10}$ as a function of σ is interpreted as the effective coupling constant $2\pi i\tau_{\text{eff}} = -2\pi r_{\text{eff}} + i\theta_{\text{eff}}$. By (4.7) and (4.9), it is given by

$$\Delta s = -\sum_{i=1}^{n_1} \log(\sigma - \widetilde{m}_i) + \sum_{j=1}^{n_2} \log(\sigma - \widehat{m}_j) + \log q. \quad (4.15)$$

This agrees with the field theory knowledge (2.48),(2.61) if we identify the constant q as

$$q = \mu^{n_1-n_2} e^{2\pi i\tau} = \Lambda^{n_1-n_2} e^{i\theta}. \quad (4.16)$$

The Axial Anomaly

The Type IIA configuration has an invariance under the rotations in the 23 plane. Indeed, if we set all the twisted mass to be zero, the configuration is invariant under the action of $e^{i\alpha} \in U(1)_{2,3}$, $\sigma \rightarrow \sigma e^{2i\alpha}$, up to the positions of the D2-branes which depends on the choice of vacua. In the M theory configuration, however, the symmetry is reduced due to the modification of the fivebrane on the right. If we set $\widetilde{m}_i = \widehat{m}_i = 0$, the fivebrane on the right is described by the equation $t\sigma^{n_1} = q\sigma^{n_2}$, and this is invariant under $e^{i\alpha} \in U(1)_{2,3}$ only if $e^{2i(n_1-n_2)\alpha} = 1$. Namely, $U(1)_{2,3}$ invariance is broken to its discrete subgroup $\mathbf{Z}_{2(n_1-n_2)}$. This corresponds to the anomalous breaking of the $U(1)_A$ R-symmetry (2.41). This discrete symmetry might be further broken by the configuration of membranes, which corresponds to the spontaneous breaking by a choice of vacuum.

Validity of the Approximation

The theory on the branes is in general different from the conventional quantum field theory because the former interacts with gravity and string excitations in the bulk and involves the modes associated with Kaluza-Klein reduction on the interval. If we are to draw some information on the two-dimensional field theory from branes, we must at least be able to find a limit in which all these extra modes decouple from the system.

In the present context, there are essentially three parameters that characterize the brane configurations (for this part of the section, we turn off the mass parameters): the separation $\Delta x^6 (= L)$ of the two fivebranes in the x^6 direction, their separation Δx^7 in the x^7 direction or

the parameter q that characterize the fivebrane on the right, and the radius R of the circle in the eleventh direction. On the other hand, there are only two parameters that characterize the field theory: the bare gauge coupling constant e^2 and the FI parameter r or the scale parameter Λ that organizes the running of r . They are related by ⁷

$$1/e^2 = \frac{|\Delta x^6| \ell_{11}^3}{R^2}, \quad r = \frac{|\Delta x^7|}{R}. \quad (4.17)$$

Since the theory is asymptotically free with respect to the coupling constants e and $1/r$, if $1/r$ is very small at the scales of gravity, string and Kaluza-Klein excitations, and e is small compared to these scales, we can neglect the effects of these extra modes at enough lower energies. This condition is satisfied if the radius R is much smaller than ℓ_{11} and the other parameters $|\Delta x^6|$, $|\Delta x^7|$.

Note that this is the weak coupling Type IIA limit of the M theory and is not a parameter region in which the low energy supergravity approximation of M theory is valid. However, as far as the qualitative features as well as quantities that depends only on the combinations (4.17) are concerned, it seems that we can make some prediction and perform a computation by going to a region in which R is large where we can use the eleven-dimensional supergravity approximation.

There is, however, one thing which one must be careful. The condition that $1/r$ is very small at the Planck scale ℓ_{11}^{-1} , the string scale $\ell_{st}^{-1} = R^{1/2} \ell_{11}^{-3/2}$ and the scale $1/|\Delta x^6|$ of Kaluza-Klein modes is equivalent with the condition that Λ is much smaller than these energy scales. The identification (4.16) yields $\Lambda = |q^{1/(n_1-n_2)}|$ and this is proportional to the characteristic length $|\Delta x^{2,3}| = \ell_{st}^2 |q^{1/(n_1-n_2)}|$ of the fivebrane on the right. In short,

$$\Lambda = R |\Delta x^{2,3}| / \ell_{11}^3. \quad (4.18)$$

Thus, if we are to increase the radius R beyond ℓ_{11} and keep Λ to be small compared to the scales ℓ_{11}^{-1} and $R^{1/2} \ell_{11}^{-3/2}$, then, the length $|\Delta x^{2,3}|$ becomes very small (compared to ℓ_{11}). Since this length is the distance of the different branches of the part of the fivebrane on the right described by $\sigma^{n_1-n_2} t = q$, if it is small, we are probing the system of parallel and nearly coincident fivebranes. Namely, the (2,0) superconformal field theory in six dimensions of [62, 63]. Use of supergravity approximation is therefore valid only if we increase Λ . However, if the result of

⁷We thank correspondence with Andreas Karch on related issues.

some computation in this approximation depends only on Λ (and other quantities that appears in the field theory) and is totally independent of ℓ_{11} , we can still expect the result to be a good prediction. One such example in the past is the BPS mass formula in the $N = 2$ theories in four-dimensions [3] in which the distance of the fivebranes corresponding to the D4-branes is of order Λ and must be small in the field theory limit, but should be large for the supergravity approximation to be valid. In the present paper, we compute the mass of the BPS states of the CP^{n-1} model and its deformation (see section 6).

4.3 The Structure of Vacua

A supersymmetric vacuum of the two-dimensional field theory is realized as a configuration of membranes which preserves four of the supercharges of M theory. The condition of four unbroken supercharges is simply that each of the k membranes are located at a point in the directions transverse to 016. Namely, the worldvolume of each membrane is a straight strip $\mathbf{R}^2 \times I$, where \mathbf{R}^2 is the 01 part of the space-time and I is a segment in $0 \leq x^6 \leq L$ located at a definite position in the 2345789 and 10 directions. Since it is stretched between the two fivebranes, the two ends of I must satisfy the conditions (4.9)-(4.10) and (4.7)-(4.8).

Below, we determine the vacua of the CP^{n-1} and Grassmannian models in this M theory frame work. These cases can be treated without detailed knowledge of the dynamics of membranes, except that we need in the Grassmannian case to make use of the rule [1] that s -configurations are not supersymmetric. We discuss other cases in the next section.

The CP^{n-1} Model

We first consider configurations corresponding to the CP^{n-1} model in which $k = 1$, $n_1 = n$, $n_2 = 0$ and all the twisted masses are turned off. In this case, the fivebrane on the right is described by (4.8) and $\sigma^n t = q$. Therefore, a configuration corresponding to a supersymmetric vacuum is $\mathbf{R}^2 \times I$ where I is a segment in $0 \leq x^6 \leq L$ at $x^4 = x^5 = x^8 = x^9 = 0$, $t = 1$ and at one of the roots of

$$\sigma^n = q. \tag{4.19}$$

Since (4.19) has n roots, there are n configurations preserving four supersymmetries, in agreement with the field theory result. Also, it is evident that each choice breaks the discrete

subgroup \mathbf{Z}_{2n} of $U(1)_{2,3}$ to \mathbf{Z}_2 , which corresponds to the spontaneous breaking of the discrete R-symmetry group $\mathbf{Z}_{2n} \subset U(1)_A$.

It is interesting to note that the relation (4.19) is nothing but the quantum cohomology relation of the \mathbf{CP}^{n-1} model [64], which represents the instanton correction to the chiral ring.

It is easy to extend this result to the case where the twisted masses are turned on. In this case, the equation (4.19) is modified as

$$\prod_{i=1}^n (\sigma - \widetilde{m}_i) = q. \quad (4.20)$$

The number of vacua is still n since the number of roots is, in agreement with the field theory result. It would be interesting to determine the chiral ring of the model with twisted mass in the field theory frame work. A natural guess is that it is described by (4.20).

The Grassmannian Model

The Grassmannian sigma model with target space $G(k, n)$ is described by k membranes stretched between the fivebranes in which $n_1 = n$, $n_2 = 0$, and all $\widetilde{m}_i = 0$. A configuration preserving four supersymmetries is a set of k membranes $\mathbf{R}^2 \times I_{(a)}$, $a = 1, \dots, k$, where each $I_{(a)}$ is a segment in $0 \leq x^6 \leq L$ at $x^4 = x^5 = x^8 = x^9 = 0$, $t = 1$ and at one of the n roots of (4.19).

The structure of vacua depends on whether two or more membranes can be on top of each other. To answer this from first principles, we need a detailed knowledge of the dynamics of open membranes stretched between the fivebranes, which we do not have presently. However, the configuration of coincident membranes in this set-up is locally the same as the (T-dual of) s -configuration of [1].⁸ Indeed, since the fivebrane on the right is obtained from the D4-branes ending on NS' brane, we may view a point on it as a point of one of the D4-branes. Then, the configuration of coincident membranes can be viewed as a configuration of two (or more) D2-branes stretched between a single NS 5-brane and a single D4-brane, a T-dual of the s -configuration of [1]. Then, provided the rule that s -configurations are not supersymmetric is correct, a configuration of two or more membranes on top of each other is not supersymmetric.

⁸To remind the reader, s -configurations are configurations in which k D3 branes are stretched between a NS fivebrane and a D5 brane in type IIB. The statement is that for $k > 1$ the configuration breaks supersymmetry even though apparently it need not.

It is plausible to admit this since the rule has passed through various different checks [1, 11, 22]. It would be interesting to prove it directly from the study of membrane dynamics. We leave it as an open problem.

Thus, a configuration corresponding to a supersymmetric vacuum is given by a choice of k *distinct* roots among the n roots of (4.19). Thus, the total number of such configuration is $\binom{n}{k}$, in agreement with the field theory result. A choice of such configuration breaks the discrete symmetry group Z_{2n} . The pattern of breaking Z_{2n} also agrees with what we expect from the field theory.

It is interesting here also to note that the quantum cohomology ring of the Grassmannian $G(k, n)$ is the ring of symmetric polynomials of $\sigma_1, \dots, \sigma_k$ with each σ_i obeying the constraint (4.19) (see [65, 37, 59]).

We will discuss more about the Grassmannian model in the next section.

5 Continuation Past Infinite Coupling

Let us now turn to study non-trivial dynamics of the two dimensional theories at hand by moving branes in space-time. There is a very simple trick which can be applied to any brane configuration. We can reorder positions of branes in the x^6 direction. Let us review first what are the consequences of this operation in various dimensions and supersymmetries.

In [1] a configuration of N_c threebranes stretched between two NS fivebranes together with N_f D fivebranes was used to construct $N = 4$ supersymmetric $U(N_c)$ gauge theory with N_f flavors in three dimensions. It was shown there that when the two NS fivebranes exchange their position the theory changes its matter content. There is a phase transition to a $U(N_f - N_c)$ gauge theory with N_f flavors. The gauge coupling of a given theory is proportional to the inverse distance between the two NS branes. Thus, the transition goes through infinite coupling for both gauge theories. In this sense we call one theory continuation past infinite coupling of the other. The transition is performed in the Higgs branch of both theories and thus allows us to study the equivalence between the Higgs branches of the two theories.

The authors in [9] applied the same trick as in [1] to study the dynamics of $N = 1$ supersymmetric gauge theories in four dimensions. For this case the exchange of the two NS branes leads to Seiberg Duality [66]. The two gauge theories which are involved are similar to

the theories in the three dimensional analog and are $SU(N_c)$ with N_f flavors while the dual is $SU(N_f - N_c)$ with N_f flavors and some mesons with a superpotential. See a detailed discussion in [17]. Other applications of this effect can be found in [13, 15, 16, 18, 25].

Let us try to apply the trick of [1] to the system at hand. In this case the number of matter fields is roughly half the number of fields that we have in the two previous examples, however since the technique is the same, we will get qualitatively equivalent results for this theory as well. There are two ways to look at the problem. We have the type IIA picture which describes the semiclassical limit of the theory and the M theory picture which describes the quantum behavior of the system. We will first describe the exchange of the NS branes in the type IIA picture. Then we will describe the same transition in the M theory setup and see that the two pictures are really different. This serves as a good example for the quantum correction of the transition and shows that it is a truly quantum effect.

Type IIA Description of the Transition

Consider the system of branes as in section 3 with $n_1 = n, n_2 = 0$ and $k < n$. We sketch this configuration in figure 3. This configuration of branes describes $U(k)$ gauge theory with n

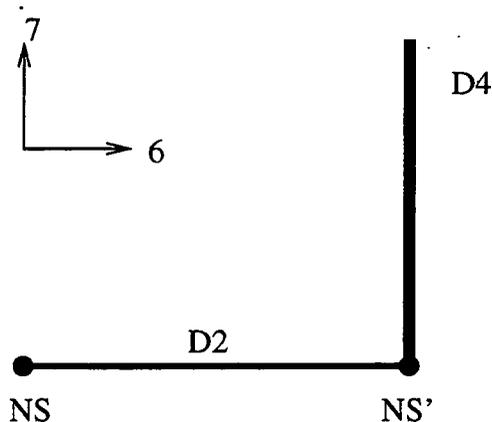


Figure 3: $U(k)$ gauge theory coupled to n chiral multiplets. There are k D2 branes stretched between NS and NS' fivebranes. n semi-infinite (upper) D4 branes end on the NS' fivebrane.

chiral multiplets in the fundamental representation. As reviewed in section 2.2.3, in the infrared

this theory was shown to describe a sigma model on the space of Grassmannians $G(k, n)$.⁹

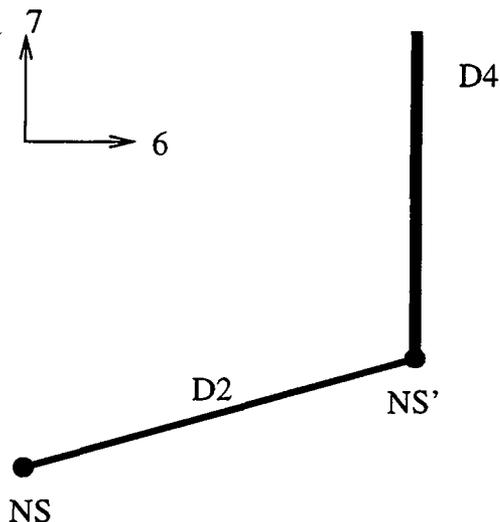


Figure 4: Turning on a FI term when the D2 branes are in between the two NS branes. The D2 branes must end on the NS branes to avoid charge violation. As a result their orientation in space-time is changed and thus supersymmetry is broken.

We want to exchange the x^6 position of the NS fivebrane and the NS' fivebrane. We need to avoid possible charge flow between the two fivebranes when they pass. Therefore we want the fivebranes to avoid each other in space time. For this we need to change the relative distance in the x^7 direction. This corresponds to turning on a FI parameter for the $U(1)$ part in the $U(k)$ gauge group. However in a generic situation this has the effect of breaking supersymmetry. Such a breaking is visible in the brane construction by changing the orientation of the D2 branes in space. Indeed if we move the NS fivebrane in the x^7 direction, when some D2 branes are stretched in between the two fivebranes, the orientation of the D2 branes in space gets some angle in the 67 plane and thus breaks supersymmetry. This is sketched in figure 4. We can avoid such a breaking by letting the D2 branes end on other branes. The simplest way to do it is to introduce D4 branes on which the D2 branes can end. The only D4 branes present are the semi infinite branes which end on the NS' brane. As such they can not move independently in the x^6 direction because the NS' brane is point like in this direction. An attempt to move the D4 branes away from the NS' brane will result in RR charge violation. However we can

⁹See, for example, [65, 59] and references therein.

use lower semi-infinite branes which come from infinity in the 23 directions and reconnect with the upper D4 branes to form infinite D4 branes. Once they are infinite they can leave the NS' brane as there is no charge violation in this case. So we move m lower D4 branes from infinity to reconnect with m upper D4 branes to form m infinite D4 branes. This can be done only if $n \geq m$. Such a process corresponds to turning on m chiral multiplets in the anti fundamental representation. The 23 position of the lower D4 branes being the twisted mass \hat{m}_j . When the lower and upper branes reconnect and the resulting infinite D4 brane moves in the x^6 direction, The chiral symmetry $SU(n) \times SU(m)$ is broken explicitly.

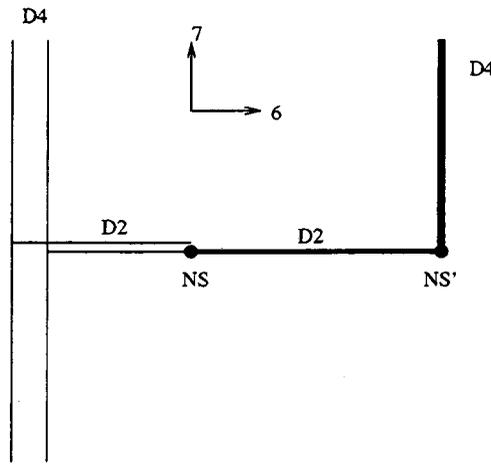


Figure 5: A Type IIA description of the exchange of two NS branes. This is a classical transition which is corrected quantum mechanically. In the figure there are D2 branes stretched between infinite D4 branes and a NS brane. The D2 branes are created when the D4 and the NS branes pass each other in the x^6 direction.

Next the D4 branes should cross the NS brane in the x^6 direction. They share only one transverse direction and therefore can not avoid each other in space. When they cross a D2 brane is created [1] which is stretched in between them. If m D4 branes cross the NS brane there are m D2 branes which are stretched in between the NS brane and each of the D4 branes. These D2 branes are not free to move. They are stuck on one side to the NS brane and on the other side to the D4 branes. We can still reconnect such branes with the k D2 branes that are stretched between the two NS branes. This is done by moving the latter branes to touch the former branes which are stack. Such a process corresponds to changing the expectation values of the adjoint matrix σ in such a way that we go to a point where the quark fields

become massless. Such a process can not be possibly correct quantum mechanically because in two dimensions there is no moduli space of vacua, at most a discrete set of vacua are allowed. Nevertheless let us proceed with the semiclassical description. In such a way we reduce the number of branes attached to the NS brane. If we do this for all k D2 branes between the two NS branes there are no branes attached to the NS brane left. At this stage we can move the NS brane in the x^7 direction without breaking supersymmetry. Note that the number of D4 branes, m , must be greater or equal to k for supersymmetry not to be broken. From the above considerations we see that the minimal number of lower branes needed for these processes to preserve supersymmetry is $m = k \leq n$. So we choose this value and proceed. The resulting configuration is sketched in figure 5.

At this point we can move the NS brane past the NS' brane by first moving in the x^7 direction and then in the x^6 direction. We encounter an apparent puzzle. There are two ways to turn on the FI parameter, r . One is to turn $r > 0$ and the other is to turn $r < 0$. In the first case the NS brane will meet the upper D4 branes and when crossing will create a D2 for each D4 brane. In contrast, in the second case, the NS brane will not meet any D4 branes and thus there will be no creation of D2 branes at all. This is in contradiction with the expectation that the transition will be independent of the path chosen!

We clearly see that the type IIA picture has some ambiguity. We will see how this is cured in the M theory description of this transition. For now let us recall, as reviewed in section 2.2, that the FI parameter gets renormalized in the quantum theory (which is equivalent to going to M theory limit). Indeed the $r > 0$ region which seems to be special at the classical theory is smoothed and can be continued to all values of r . In terms of branes what happens is that the D4 branes bend the five brane in such a way that the $r > 0$ region is extended. So let us assume that the region of $r > 0$ is valid for every value of r .

With this assumption, when the NS brane crosses the $n - k$ D4 branes there are $n - k$ D2 branes created. The NS brane can then go back to the origin of the x^7 direction and now the resulting theory is $U(n - k)$ with n fundamental fields. Classically there seems to be some additional matter and couplings however the picture modifies in the quantum theory. This completes the description of this transition in the type IIA picture which clearly has many loopholes and some apparent inconsistencies.

M Theory Description of the Transition

Actually the description of this transition in the M theory picture is much simpler and avoids all the problems mentioned. The M theory picture gives us one straight fivebrane at $t = 1$ spanning worldvolume in the 012345 directions and another fivebrane with world volume $\mathbf{R}^4 \times \Sigma$ where \mathbf{R}^4 is the world volume spanned by 0189 and Σ is a degenerate Riemann surface in the 237 and 10 directions which is described by the equation $\sigma^n t = q$. The two fivebranes are connected by k membranes. In a vacuum configuration, the membranes are located at the roots of $\sigma^n = q$. Recall that k of the n roots are occupied by the membranes and the remaining $n - k$ roots are not, since one root cannot be occupied by two or more membranes because s -configurations are not supersymmetric [1].

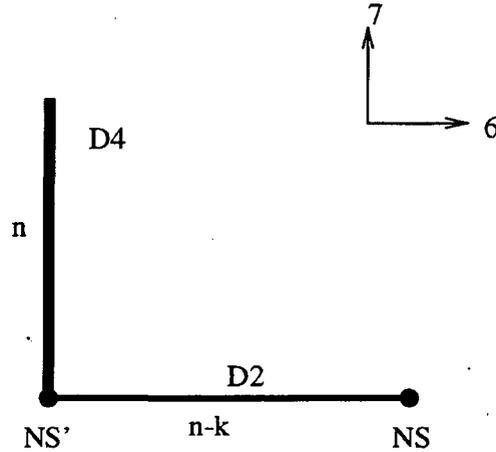


Figure 6: Continuation past infinite Coupling for the Grassmannian Model. There are n upper D4 branes which end on NS' brane and $n - k$ D2 branes stretched between the NS and NS' branes. This is the type IIA description of the transition which is done in the M theory limit.

In contrast to the situation in the type IIA picture, the two fivebranes cannot avoid each other in space. Thus the process of getting lower D4 branes becomes unnecessary. Instead the two fivebrane cross each other transversely in the space perpendicular to the 01 directions. In particular, they intersect at the n roots of $\sigma^n = q$ at $t = 1$, $x^{4,5,8,9} = 0$. Now the use of the transition of [1] implies¹⁰ that, whenever there was a membrane stretched before the transition, there will be no membrane after that, and vice-versa. That is, after the exchange in the x^6

¹⁰See also a discussion in [67, 68].

direction we are left with $n - k$ occupied positions of membranes and k positions which are not occupied by membranes. The resulting configuration is sketched in figure 6 and the theory at hand thus describes $U(n - k)$ gauge theory with n fundamental chiral multiplets.

The transition we have found implies that there are two microscopically inequivalent theories which are equivalent in the IR. This is, in some sense, “Seiberg’s duality in two dimensions.” What the brane picture demonstrates is how the transition past infinite coupling implies this equivalence.

Brane Proof of Level-Rank Duality

Let us recall that the number of vacua for the $U(k)$ theory with n fundamental fields is given by the Witten index which is nothing but the Euler characteristic of the Grassmannian $G(k, n)$. It is given by $\binom{n}{k}$. What we have just found is that the number of vacua is consistent with the transitions as the formula is invariant under the exchange $k \leftrightarrow n - k$. This is what is expected since the Grassmannian $G(n - k, n)$ is “dual” to $G(k, n)$ and these are essentially the same.

There is one interesting point of view. The transition we have found is nothing but the level-rank duality of the WZW model, as described in [28] (see also [29] and references therein). Recall that the dynamics of vacua of the Grassmannian sigma model with target space $G(k, n)$ is described by the $U(k)/U(k)$ gauged WZW model with the level of $U(k) \sim SU(k) \times U(1)$ being $(n - k, n)$. The level-rank duality says that the space of conformal blocks of $SU(k)$ WZW model with level $n - k$ is dual to the one of $SU(n - k)$ WZW model with level k . The $U(k)/U(k)$ gauged WZW model, being a topological field theory, has as its correlation functions the dimension of the space of conformal blocks. If we use this, we see that our system is equivalent with the $U(n - k)/U(n - k)$ gauged WZW model with the level of $U(n - k) \sim SU(n - k) \times U(1)$ being (k, n) , which describes the sigma model with target space $G(n - k, n)$ which in turn is given by the $U(n - k)$ gauge theory with n fundamental chiral multiplets. This is exactly what we have seen in the above discussion of brane motions. In other words, we have shown the level-rank duality in the brane framework.

It would be interesting to study the interplay between this transition and the methods used in the literature to study this duality. The brane picture demonstrates that Seiberg Duality

in four dimensions and level rank duality in two dimensions follow from the same qualitative level.

5.1 More Brane Motion

We have performed all possible motions of branes in space for the $G(k, n)$ models. We can now study transitions in other models by moving the branes around. Let us add more branes into the picture. Suppose that a lower D4 brane comes from infinity in 23 to the origin. We get a massless anti-fundamental field of $U(k)$. As in the last subsection, the lower D4 brane can join an upper D4 brane and move away from the NS' brane in the 456 directions. Let us assume that the motion is only in the x^6 direction. As discussed in section 3.2, if the position of the D4 brane is to the right of the NS' brane, the corresponding matter field decouples from the low energy theory. So we will discuss the case where the D4 brane is to the left of the NS' brane. Then, there seems to be a flat direction for which a D2 brane can break and move in between the infinite D4 brane and the NS' - upper D4 branes system. This is in contradiction with Coleman's theorem which states that there are no flat directions in two dimensions, as mentioned in the last section. To understand how this is possible let us look at the M theory solution for this problem.

The equation which describes the addition of the lower D4 brane in type IIA is

$$\sigma^n t = q\sigma. \quad (5.1)$$

This demonstrates that indeed an infinite D4 brane in type IIA can decouple. In the M theory picture two fivebranes are formed. One at $\sigma = 0$ and the other at

$$\sigma^{n-1} t = q. \quad (5.2)$$

Next we want to break a membrane which will be stretched in between the two fivebranes. However we encounter a problem since the two equations have no common solution in the 23 direction. That is unless q is equal to zero which is infinitely far away. Such absence of solution may lead us to conclude that the supersymmetric vacuum is broken in contrast to the naive expectation that there is a space of flat directions.

To understand better let us turn on twisted masses \widetilde{m} and \widehat{m} for the two fields Q and \widetilde{Q} , involved. Then, the equation describing the fivebrane on the right is

$$\sigma^{n-1}(\sigma - \widetilde{m})t = q(\sigma - \widehat{m}). \quad (5.3)$$

First, consider the case with $\widetilde{m} = \widehat{m}$. This gives a position \widetilde{m} to the infinite D4 brane which in the M theory picture has the equation $\sigma - \widetilde{m} = 0$. Together with equation (5.2) we see that the x^7 position of the membrane is determined to be given by the equation $t = \frac{q}{\widetilde{m}^{n-1}}$. Thus, for a given \widetilde{m} , the x^7 and x^{10} positions are fixed. We also see that when the twisted mass is zero, the membrane is running away to infinity in this direction. There is still an option to turn on some arbitrary values in the 89 directions which will have the interpretation of expectation value for meson fields. This new flat direction corresponds to the sigma model based on a complex one dimensional non-compact space which is discussed in section 2.2.2. The mechanism for freezing motion in this direction is not clear. We will assume that this motion is frozen. The total set of vacua is thus $n - 1$ massive vacua corresponding to the solution of (5.2) and the vacua of this non-compact sigma model.

Second, we consider the generic values of \widetilde{m} and \widehat{m} . As the equation (5.3) shows, the number of vacua is n . Recall that a supersymmetric vacuum here is interpreted as a stable brane configuration which is in accord with the supersymmetry. This condition is indeed satisfied. For this case the Witten index, $\text{Tr}(-1)^F$, does not depend on the value of the twisted mass parameter.

There are two generalizations to the models considered so far. One type of generalization is to replace the degenerate Riemann surface of the NS' brane and D4-branes system by a general Riemann surface. The theory then will be of a D2 brane propagating between a NS brane and the Riemann surface. One possible system is to take a Riemann surface which describes a particular four dimensional theory like those studied in [3]. An example is discussed in the next section.

Another generalization is to replace also the NS brane with an arbitrary Riemann surface. In some sense the NS brane is a very degenerate Riemann surface. Then our aim would be to study the dynamics of the D2 brane when it propagates between two Riemann surfaces. More generally, we can take a series of Riemann surfaces localized at points in x^6 . There can be arbitrary numbers of D2 branes propagating between each two adjacent Riemann surfaces.

5.2 D2 brane Propagation on $N = 2$ Supersymmetric QCD.

We have studied in detail the propagation of a D2 brane on a four dimensional theory which is somewhat degenerate. As a four dimensional theory it is weakly coupled and frozen,

in the brane language. The closest realistic theory that the $\mathbb{C}P^{n-1}$ model captures in this brane realization is $N = 1$ supersymmetric QCD. The genus zero curve which describes the chiral ring of the $\mathbb{C}P^{n-1}$ model coincides with part of the genus zero curve which describes $N = 1$ supersymmetric QCD. The correspondence between the $\mathbb{C}P^{n-1}$ model and this theory was known for a long time. This is reviewed in section 2.2.1. The $\mathbb{C}P^{n-1}$ was used as a two dimensional toy model for studying confinement and other four dimensional phenomena. Both models share domain walls (solitons in the two dimensional theory). From the brane point of view this is not a surprise. The brane picture really provides a string theory explanation for this correspondence. It tells us that studying the two dimensional model, probes some qualitative features of the four dimensional theory. At some cases, as for the ratios between domain wall tensions versus soliton mass ratios, the correspondence is even quantitative! This may be just the beginning of an interesting interplay.

The aim of this section is to continue this approach and study the propagation of the D2 brane on $N = 2$ supersymmetric QCD in four dimensions. This theory is described by some D4 branes stretched between two NS branes. We will choose, as in the previous models, a NS brane at one end of the D2 brane and in the other end we will take the 4d theory in question. So let us start with a single D2 brane stretched between a NS brane to the left and 2 NS' branes to the right. According to a conjecture by [9], this configuration describes a $U(1)$ theory with adjoint field x subject to a superpotential $W = x^3$. The x field is associated with the 45 position of the D2 brane. This superpotential can be perturbed by moving the NS' branes in the 45 directions resulting in a superpotential which satisfies

$$\frac{\partial W}{\partial x} = (x - a)(x - b). \quad (5.4)$$

a and b are now the positions of the two NS' branes in the 45 directions.

Let us assume that this description is correct and add more branes to the picture. We put n D4 branes in the interval between the three NS branes (the two NS' branes are on the same x^6 position). This process can be thought of as putting the branes at far infinity in 45 directions and then slowly moving them to the origin of 45. The resulting theory describes a $U(1)$ theory coupled to n chiral fields of charge $+1$ and their complex conjugates, with the adjoint field subject to the superpotential as above.

Next, let us move the NS' branes far apart in the x^7 direction. Such a process does not change

the superpotential. The x^7 distance, being a real parameter, can not enter the superpotential if we assume, by supersymmetry, holomorphic dependence. We can now move the D4 branes in the x^6 direction to touch the two NS' branes. Once they touch, they can break and form a system of n finite D4 branes between two NS' branes together with n semi-infinite lower (upper) D4 branes. To a four dimensional observer, living in the 0189 coordinates, this configuration is nothing but finite $N = 2$ supersymmetric QCD with gauge group $SU(n)$ coupled to $2n$ flavors represented by the semi-infinite branes!

What is the two dimensional theory? We can repeat our analysis from the previous models and find that there are n Q_1 and \tilde{Q}_1 fields coming from localization near the first NS' brane and n Q_2 and \tilde{Q}_2 fields coming from localization near the second NS' brane. The adjoint field is still there subject to the superpotential.

Going back to the four dimensional theory, varying the positions in the 456 directions is now interpreted as the Higgs branch of this theory. In addition, once the D4 branes break, each part can move independently and we can get all the models which are derived from the finite theory by turning on expectation values or mass terms. In particular pure $N = 2$ $SU(n)$ YM, and other models. The Coulomb branch of this theory is identified with the motion of the finite D4 branes along the NS' branes in the 23 directions and is described by the genus $n - 1$ Riemann surface

$$R_n(\sigma)t^2 + tP_n(\sigma) + Q_n(\sigma) = 0, \quad (5.5)$$

where R, P and Q are polynomials of order n in σ . For the two dimensional theory the 4d Higgs branch is the space of complex mass parameters together with x^6 and A_7 motions. The 4d Coulomb branch is the space of twisted mass parameters for the 2d theory. We can continue with the identification further. There is a complex modulus for the Riemann surface which describes the coupling constant and theta angle of the four dimensional theory. This is identified with the FI coupling and theta angle of the two dimensional theory. This identification requires some explanation which will be given below. However it is sufficient to see that the FI coupling of the two dimensional theory receives non-perturbative corrections which come from contribution of 2d instantons which are stretched between the two NS' branes. This is the first model in which such an effect happens for a two dimensional FI parameter! We find that two dimensional FI parameters behave just like four dimensional gauge couplings.

Let us go back to describe more about the FI parameter. Consider a D2 brane which propagates between the Riemann surface of the 4d theory and the NS brane. Let us assume that we have m finite D4 branes and $n_1(n_2)$ upper (lower) D4 branes. This describes $SU(m)$ gauge theory with $n_1 + n_2$ flavors. Near the first NS' brane we have a $U(1)$ theory coupled to m charge +1 fields Q and n_1 charge -1 fields \tilde{Q} . There is a FI term given by the distance between the NS' and the NS in the x^7 direction. Near the second NS' brane we have a similar theory but it is a *different* $U(1)$ coupled to n_2 charge +1 fields Q and m charge -1 fields \tilde{Q} . The FI term is given by the x^7 distance between the second NS' brane and the NS brane. We see that there are two $U(1)$ theories with two FI couplings. In the intermediate region between the two NS' branes both $U(1)$ theories are broken and moving along the four branes is a transition between the two $U(1)$ theories.

There are two combinations for the FI couplings – a sum and a difference – for which only the difference is relevant for the physical system. It is this quantity which is identified with the τ parameter of the Riemann surface for the 4d theory.

This situation is generalized easily to k D2 branes and also for including more than two NS' branes. It would be interesting to study this system and the 2d – 4d correspondence in more detail.

One more important point is in order. At some special points in the Coulomb branch of the four dimensional theory there are additional massless multiplets – monopoles or dyons. If there are more than one such additional massless multiplets then there can be Higgs branches which emanate from this singularity in the Coulomb branch. Such branches are not visible from a semi-classical point of view since the singularities appear at strong coupling. In terms of branes what happens is that a Riemann surface becomes degenerate at the singularity points and leads to disconnected Riemann surfaces. This is interpreted as two fivebranes which locally come together and form a massless hypermultiplet. When the two branes move in a different direction away from the singularity, if this is possible, then a Higgs branch emanates from the singular point.

The correspondence between the moduli space of vacua of the four dimensional theory and the space of parameters of the two dimensional theory now has a new prediction. Since the new Higgs branch which emanates is interpreted as complex mass parameters for the two dimensional theory, there are new couplings which are not visible from the semi-classical Lagrangian

description. Clearly the new couplings are visible from the brane point of view as the new direction for which the two fivebranes can move.

6 Solitons Via M Theory

In this section, we give a description of the BPS saturated solitons in $N = 2$ field theory in two dimensions, which were discussed in section 2, in terms of branes in M theory.

We recall from section 2.2 that the fundamental BPS soliton in the $\mathbb{C}P^{n-1}$ model is the elementary field Q in the fundamental representation of the flavor group $SU(n)$ [30]. The corresponding statement in the Type IIA description would be that the fundamental soliton is the Type IIA string stretched between the D2 brane and the D4 branes on the right. In the M theory description the string and the D2 brane become a single membrane which winds around the eleventh dimension in the region near the fivebrane on the right, and is stretched between the two fivebranes. We will show that this must be the case on topological grounds. Recall also that the ℓ -th soliton of the $\mathbb{C}P^{n-1}$ model (which interpolates the two vacua separated by ℓ -steps) is in the ℓ -th anti-symmetric representation of $SU(n)$ (ℓ -th exterior product of the fundamental). In the Type IIA description, this would be a bound state of ℓ elementary strings stretched between the D2-brane and the D4 branes. We will also see this in M theory including the fact that they form the ℓ -th anti-symmetric representation of $SU(n)$, as a consequence of a constraint on the topology of the membrane which preserves two of the four supersymmetries. In addition, we compute the BPS mass by defining the superpotential. These results are generalized to the case with twisted masses in which the soliton mass has not been computed from field theory due to an ambiguity in defining the values of the twisted superpotential. We will see that the soliton masses can be determined unambiguously in the brane framework.

Solitons in the supersymmetric $\mathbb{C}P^{n-1}$ model in two dimensions are closely related to the domain walls in the $N = 1$ $SU(n)$ super Yang-Mills theory in four-dimensions, as noted in section 2.2. Recently, the domain wall separating the adjacent vacua of super Yang-Mills theory was studied in [23] in the M theory framework and claimed to be the D-branes for QCD strings. It would be interesting to see the implication of the present general description of the $\mathbb{C}P^{n-1}$ solitons to the study of the domain walls.

6.1 The \mathbb{CP}^{n-1} Solitons

Brane Description of BPS Solitons

As explained in section 2, a soliton in two dimensional field theory is a configuration of fields which are at one vacuum in one spatial infinity $x^1 \rightarrow -\infty$ and are at another vacuum in the other spatial infinity $x^1 \rightarrow +\infty$. Likewise, the soliton is described in M theory as a configuration of a membrane that depends on the spatial coordinate x^1 so that it interpolates two different vacua in this direction.

We first describe the solitons in the \mathbb{CP}^{n-1} model. Thus, we consider the configuration of section 4 with $k = 1$, $n_1 = n$ and $n_2 = 0$, where there are two fivebranes and one membrane stretched between them. The two fivebranes are a flat fivebrane at $x^8 = x^9 = x^6 = 0, t = 1$ and a curved fivebrane defined by $\sigma^n t = q$ which is at $x^4 = x^5 = x^6 - L = 0$. For fixed t and q there are n solutions for σ which implies that there are n different vacua. A vacuum configuration is given by a membrane, with the time direction being omitted, with world-volume a strip $\mathbf{R} \times I_j$ stretched between the two fivebranes ($j = 0, 1, \dots, n-1$), where \mathbf{R} is the one-dimensional space with coordinate x^1 and I_j is a segment $0 \leq x^6 \leq L$ which is located at $x^4 = x^5 = x^8 = x^9 = 0$, $t = 1$ and $\sigma = e^{\frac{2\pi i j}{n}} q^{1/n}$. Therefore, a solitonic configuration will be given by a membrane over a real two-dimensional surface Σ in the ten-dimensional space $\mathbf{R}^9 \times S^1$ which is stretched between the fivebranes and interpolates two different segments, say I_0 and I_ℓ , in the x^1 direction. Namely, Σ is a surface with boundaries and two ends which are constrained by the following: All the boundaries are in the two fivebranes, and at one end $x^1 \rightarrow -\infty$, Σ looks like $\mathbf{R} \times I_0$, while at the other end $x^1 \rightarrow +\infty$, it looks like $\mathbf{R} \times I_\ell$.

Note here that what was the string and D2 branes in the Type IIA limit combine into a single membrane which can only have boundaries on the fivebranes. A string and a D2 brane become a single membrane on a Riemann surface just like D4 branes and NS fivebranes become a single fivebrane in M theory. This is indeed a supersymmetric configuration which breaks locally one quarter of the supersymmetry charges. Such a solitonic configuration is a BPS state when Σ is a supersymmetric cycle in the sense that it preserves two of the four supersymmetries. Here, we don't specify the precise condition for a cycle to be supersymmetric in this situation, although it should follow from an argument as in [69]. It seems plausible, however, to require that a supersymmetric cycle is a minimal surface. Also, there must be two fermionic zero modes

coming from the $2 = 4 - \frac{4}{2}$ broken supersymmetry. Since some of the boundaries must be at $x^4 = x^5 = 0$ and others at $x^8 = x^9 = 0$, and since there is no other condition involving $x^{4,5,8,9}$, the minimal surface condition implies that Σ is totally at $x^4 = x^5 = x^8 = x^9 = 0$. Thus, we can consider Σ to be a surface in the four-manifold with coordinates t and σ .

In this paper, we do not touch the issue of existence and uniqueness of the supersymmetric cycle. Rather, we assume that there exists a unique BPS configuration for each topological type, unless there is an obstruction from fermionic zero modes. The verification for this is an interesting open problem.

In what follows, we find a restriction on the topology of such a cycle coming from the boundary conditions. It turns out that this restriction is very strong and has a surprising consequence. In particular, Σ cannot be just a strip but has a topology of a disc with holes, where each hole is in the fivebrane on the right and winds once around the eleventh dimension. This means that the configuration represents a bound state of Type IIA strings each of which carries a quantum number of the fundamental representation of $SU(n)$.

Fundamental Soliton = Type IIA String

We first consider a solitonic configuration interpolating adjacent vacua, i.e., $\sigma = 1$, $t = 1$ and $\sigma = e^{\frac{2\pi i}{n}}$, $t = 1$ (for this and the next part of the section, we put $q = 1$ for simplicity). Since Σ is stretched between the two fivebrane, one boundary J_l of Σ is restricted to be in the fivebrane on the left, i.e., at $t = 1$ but σ is free, while another boundary J_r is restricted to be in the fivebrane on the right, i.e., in the surface $\sigma^n = t^{-1}$. Due the condition at the two ends, the boundary J_l is a line in the σ -plane at $t = 1$ which connects the two points $\sigma = 1$ and $\sigma = e^{\frac{2\pi i}{n}}$, while the boundary J_r is a line in the surface $\sigma^n = t^{-1}$ which connects the two points $\sigma = 1$, $t = 1$ and $\sigma = e^{\frac{2\pi i}{n}}$, $t = 1$. If we consider the projection to the t^{-1} -plane, we see that J_l is mapped to one point $t^{-1} = 1$, but J_r is mapped to a circle starting and ending at the same point which winds at least once around $t^{-1} = 0$. Here we choose a shortest path connecting $\sigma = 1$ and $\sigma = e^{\frac{2\pi i}{n}}$ so that the image of J_r in the t^{-1} -plane winds exactly once counter-clockwise around $t^{-1} = 0$.¹¹ Thus, the surface has a circle boundary (of infinite length) consisting of J_l , J_r and two segments I_0 and I_1 at $x^1 = \mp\infty$, which is mapped to a circle in

¹¹ There are choices such that it winds $1 \pm n, 1 \pm 2n, \dots$ times, but we will see shortly that these cases are actually equivalent to the case with winding number one.

the t^{-1} -plane that winds once around $t^{-1} = 0$. Since $t^{-1} = 0$ nor $t = 0$ are not points in the space-time, Σ cannot be just a strip with disc topology because, if it were a disc, some point of Σ would be mapped either to $t^{-1} = 0$ or to $t = 0$.

To avoid the points $t^{-1} = 0$ and $t = 0$, Σ must have another circle boundary C that is mapped to a circle in the t^{-1} -plane which winds once clockwise around $t^{-1} = 0$. The minimal choice of such a surface is an annulus. By the condition that all the boundaries of Σ be in one of the two fivebranes, the boundary C , not being at $t = 1$, must be in the fivebrane on the right. In particular, C winds once around $t^{-1} = 0$ in the t^{-1} -plane while satisfying the equation $\sigma^n = t^{-1}$. Then, the image of C cannot be a circle which is totally away from $t^{-1} = 0$ since such a “circle” is mapped in the σ -plane to a line starting and ending at two distinct points related by the $e^{\frac{2\pi i}{n}}$ rotation, which is not a circle. This means that C must start and end at $\sigma = t^{-1} = 0$. Thus, the image of the surface Σ in the σ and t^{-1} planes look like the one depicted in Figure 7.

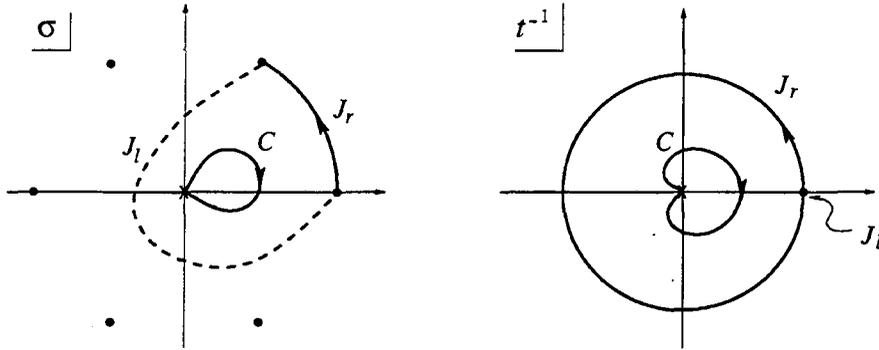


Figure 7: The Image of Σ in the σ - and the t^{-1} -Plane

In the above description, however, it is not easy to classify the topology of circles starting and ending at $\sigma = t^{-1} = 0$ since $\sigma = 0$ is a degenerate zero. In order to “regularize” this, we consider turning on small distinct twisted masses for the fields Q_i . For convenience, we choose the masses to be $\widetilde{m}, \widetilde{m}e^{\frac{2\pi i}{n}}, \widetilde{m}e^{\frac{4\pi i}{n}}, \dots, \widetilde{m}e^{\frac{2\pi(n-1)i}{n}}$, so that the fivebrane on the right is described by

$$\sigma^n - \widetilde{m}^n = t^{-1}. \quad (6.1)$$

In this situation, the circle C can wind around $t^{-1} = 0$ without approaching $t^{-1} = 0$, since the image in the σ -plane can wind once around one of the roots of $\sigma^n - \widetilde{m}^n = 0$ because any

root is a simple zero of this equation. One such configuration looks like the one depicted in Figure 8. Since there are n roots of $\sigma^n = \widetilde{m}^n$, there are n -kinds of topological types of such configurations.

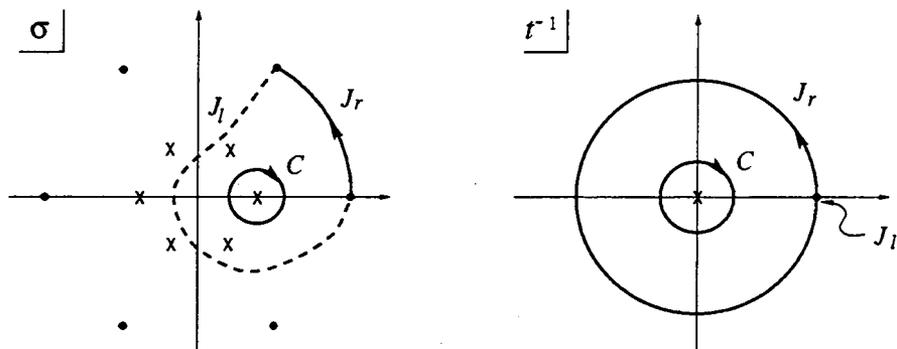


Figure 8: One of the n Possible Configurations

This can be interpreted as follows. Recall that the roots $\widetilde{m}e^{\frac{2\pi ij}{n}}$ ($j = 0, 1, \dots, n-1$) at $x^6 = L$ are interpreted as the asymptotic position in σ of the n upper-half D4-branes which are the parts of fivebrane wrapped once around the circle S^1 in the eleventh dimension. Recall also that the membrane wrapped once around the eleventh dimension is interpreted as the Type IIA string. Thus, the membrane configuration in which the circle C winds around the j -th root can be considered in the Type IIA string theory as a configuration in which an open string is stretched between the D2-brane and j -th upper-half D4-brane. As explained in section 3, such an open string generates the chiral multiplet Q_j . In total, Q_j , $j = 0, \dots, n-1$ constitute the fundamental representation of $SU(n)$.

In this way we have seen in M theory description that the solitons interpolating adjacent vacua are interpreted as the elementary Type IIA strings which in turn give rise to the elementary chiral multiplet Q in the fundamental representation of the flavor group $SU(n)$, in accord with the field theory knowledge [30] (see section 2.2).

The Exclusion Principle of Type IIA Strings

Let us next consider more general solitons, interpolating I_0 and I_ℓ for $\ell = 2, 3, \dots, n-1$, where we work in the “regularized” configuration (6.1) in which I_0 is at $\sigma = (1 + \widetilde{m}^n)^{1/n}$, $t = 1$

while I_ℓ is at $\sigma = (1 + \widetilde{m}^n)^{1/n} e^{\frac{2\pi i \ell}{n}}$. In this case, the boundary J_l at the fivebrane on the left is a line in the σ -plane at $t = 1$ connecting I_0 and I_ℓ while the boundary J_r at the fivebrane on the right is a line in the surface $\sigma^n - \widetilde{m}^n = t^{-1}$ connecting I_0 and I_ℓ . In particular, the image of J_l in the t^{-1} -plane is at a point $t^{-1} = 1$ while the image of J_r starts and ends at that point winding ℓ -times around $t^{-1} = 0$. Thus, the surface Σ has a large boundary circle (which is actually of infinite length) consisting of J_l , J_r and the two ends, which winds ℓ -times counter-clockwise around 0 in the t^{-1} -plane.

To avoid the points $t^{-1} = 0, \infty$ which are not in the space-time, Σ must have some other boundary circles. Since the large boundary winds ℓ -times around 0 in the t^{-1} -plane, there must be other ℓ boundary circles C_1, \dots, C_ℓ (which can be connected and rejoined), each of which winds once clockwise around 0 in the t^{-1} -plane. Since these boundaries must be in the fivebranes and since these are not identically at $t^{-1} = 1$, they all must be in the fivebrane on the right, namely, at $x^6 = L$, $\sigma^n - \widetilde{m}^n = t^{-1}$. The fact that C_i are circles means that each of them winds once around one of the n roots of $\sigma^n - \widetilde{m}^n = 0$ in the σ -plane.

How these boundary circles C_i choose the roots? We show that two or more distinct circles cannot choose one common root. In other words, one boundary cannot wind twice or more times around one root. We show this in the case $\ell = 2$ which captures the essence.

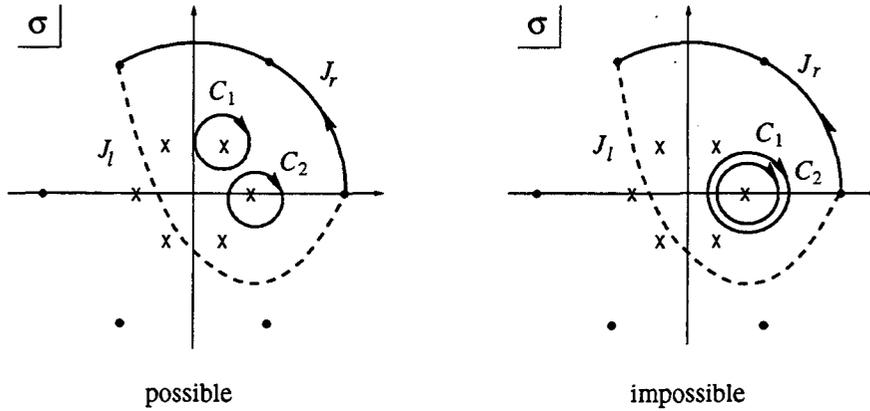


Figure 9: Possible and Impossible Configurations with $\ell = 2$

Suppose that the two circles C_1 and C_2 winds a common root, say $\sigma = \widetilde{m}$, as in the RHS of Figure 9. We first compactify the surface Σ by capping the three boundary circles by three discs. The three boundaries are C_1 , C_2 and the large boundary circle (of infinite length) consisting of

J_l and J_r joined by I_0 and I_ℓ . We denote the compactified surface by $\bar{\Sigma}$. In the case where C_1 and C_2 are rejoined, we consider the corresponding discs to be rejoined as well. The projection of Σ to the σ -plane can be considered as a complex valued function. Then, one can deform it so that it defines a meromorphic function of the compact Riemann surface $\bar{\Sigma}$ with respect to a suitably chosen complex structure. By the boundary condition, $\sigma - \tilde{m}$ is a function which has one simple pole and two simple zeros (or one double zero). The Riemann-Roch theorem implies that there is no Riemann surface having a meromorphic function with this property. This completes the proof that the RHS of Figure 9 is impossible. Note that the LHS is possible because it implies that $\sigma - \tilde{m}$ has one simple zero and one simple pole, which is possible for $\bar{\Sigma} = \mathbb{CP}^1$.

If the boundary on the left, J_l , winds once around $\sigma = \tilde{m}$ as in Figure 10, two distinct small boundaries can wind around $\sigma = \tilde{m}$ on topological grounds. However, this configuration has

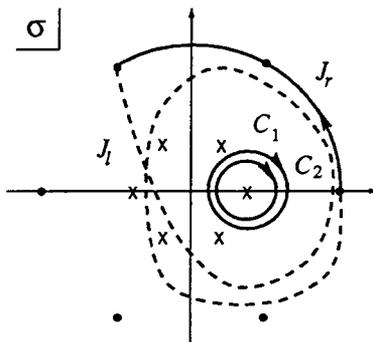


Figure 10: Possible but Non-BPS Configuration

twice as many fermionic zero modes as the one in Figure 8 or in the LHS of Figure 9. Since the latter, being a BPS configuration, preserves half of the four supersymmetries, it carries two fermionic zero modes. Therefore, the configuration in Figure 10 has four zero modes and it is likely that these can be interpreted as the Goldstino associated with the breaking of all of the supersymmetry. Namely, we claim that it is not a BPS configuration.

Let us consider the case $\ell = n$, where the large boundary winds $t^{-1} = 0$ n -times. By the “exclusion principle” of the boundaries which we have just proved, there are n -circle boundaries C_1, \dots, C_n where each C_j winds once around each $\tilde{m}e^{\frac{2\pi ij}{n}}$ of the roots of $\sigma^n = \tilde{m}^n$. This configuration is actually unstable as it can be shrunk to a point, as the Figure 11 shows. As

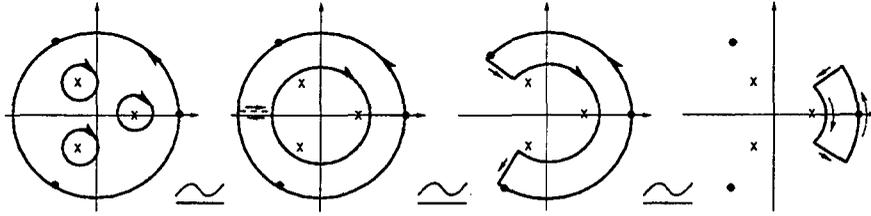


Figure 11: The Process of Shrinking in the Case $\ell = n$ (The σ -Planes)

a consequence of this, we see that the solitonic configuration interpolating I_0 and $I_{\ell+n}$ in the counter-clockwise direction in the σ -plane is equivalent to a configuration interpolating I_0 and I_ℓ in the same direction, where m is any positive integer.

By a similar reasoning, one can show that the solitonic configuration interpolating I_0 and I_{n-1} in the counter-clockwise direction is equivalent to the solitonic configuration interpolating I_0 and I_{-1} in the clockwise direction. (See Figure 12) Note that the orientation of the small boundary circle on the RHS of the figure is inverse to the one we have been considering.

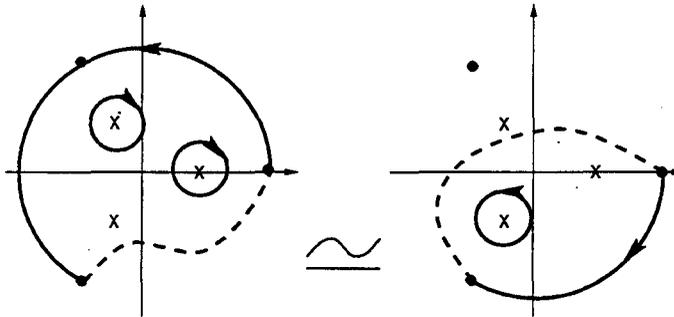


Figure 12: $\ell = n - 1$ is equivalent to $\ell = -1$

All these have quite natural interpretations. First of all, a configuration of membrane with several small boundaries which are attached to the fivebrane on the right and wind once around the circle S^1 in the eleventh direction is interpreted in Type IIA string theory as some bound state of elementary open strings and a D2-brane, where the string end points carry quantum numbers of $SU(n)$ fundamental representation. For a solitonic configuration interpolating I_0 and I_ℓ , there are ℓ such small boundaries, meaning that there are ℓ string end points, and thus, the corresponding state is in some ℓ -fold tensor product of the fundamental representation of

$SU(n)$. The “exclusion principle” of the small boundaries of a BPS configuration implies that this tensor product representation for the BPS states is actually the anti-symmetric tensor representation, i.e., ℓ -th exterior power

$$\bigwedge^{\ell} \mathbf{C}^n \quad (6.2)$$

of the fundamental representation \mathbf{C}^n . This representation has dimension

$$\binom{n}{\ell} \quad (6.3)$$

which coincides with the number of possible topological types. The fact that configuration interpolating I_0 and I_{n-1} is equivalent with the one interpolating I_0 and I_{-1} , with the orientation of the small boundary circles being flipped, corresponds to the equivalence

$$\bigwedge^{n-1} \mathbf{C}^n \cong \overline{\mathbf{C}^n} \quad (6.4)$$

as a representation of $SU(n)$. That there is no stable configuration interpolating I_0 and I_n means that there are no BPS states in such a sector.

These reproduce what we know from field theory argument [30, 47, 48, 49] as essential properties of the \mathbf{CP}^{n-1} solitons in a very interesting way.

6.2 The Twisted Superpotential and The BPS Mass

Finally, we compute the mass of these solitons. As explained in section 2.2, the mass of a BPS state is given by the difference of the values of superpotential at the two spatial infinities. Thus, we start by defining superpotential in the M theory framework, by essentially following the path made in [23]. Note that what we call “superpotential” here is actually twisted superpotential since we are considering a theory in which $U(1)_V$ is unbroken.

Configuration of a membrane at a fixed two-dimensional space-time point is a segment I in the nine-dimensional space with coordinates x^2, \dots, x^{10} which is stretched between the two fivebranes. A superpotential is a function of the space of such configurations I satisfying the two basic properties: It is a holomorphic function, and its critical points are vacuum configurations in which the supersymmetry is totally unbroken. As we have seen in section 4, a vacuum configuration is a straight segment in the x^6 direction, namely, one of the n segments

I_0, I_1, \dots, I_{n-1} parametrized by $0 \leq x^6 \leq L$. In order for the “holomorphic function” to make sense, we need to introduce the complex structure of the space of configurations. Since we are interested in the configurations which are totally at $x^4 = x^5 = x^8 = x^9 = 0$ and in $0 \leq x^6 \leq L$, we may consider a configuration I as given by a pair of functions $\sigma(x^6), t(x^6)$ of the segment $0 \leq x^6 \leq L$ which are constrained by the condition $t(0) = 1$ and $\sigma(L)^n - \widetilde{m}^n = t(L)^{-1}$ for ending on the fivebranes.¹² Then, the superpotential must be a holomorphic functional on the space of such pair of functions.

As in [23], at least locally the superpotential can be defined in the following way up to additive constant. Let Ω be the holomorphic two form

$$\Omega = d\sigma \wedge \frac{dt}{t}. \tag{6.5}$$

Given two configurations, say I and I' , the difference of the values of superpotential is defined by

$$\widetilde{W}(I) - \widetilde{W}(I') = \int_{\Sigma} \Omega \tag{6.6}$$

where Σ is a one-parameter family of configurations interpolating I and I' . Then, it is easy to see that this satisfies the basic requirements. Consider a variation of a configuration $I \rightarrow I + \delta I$. Then,

$$\delta \widetilde{W} = \int_I \left(\delta \sigma \frac{dt}{t} - \frac{\delta t}{t} d\sigma \right). \tag{6.7}$$

The fact that this is independent on $\delta \bar{\sigma}$ and $\delta \bar{t}$ means that \widetilde{W} indeed depends holomorphically on I . Also, a critical configuration is given by the one satisfying $dt = d\sigma = 0$, i.e., a straight segment in which $t(x^6)$ and $\sigma(x^6)$ are constant functions.

One may wonder how to fix the normalization of the superpotential. Note that the holomorphic two form Ω is related to the one $\mathbf{\Omega}$ given in (4.4) associated with the eleven-dimensional space-time metric (4.1) by $\mathbf{\Omega} = \ell_{11}^3 \Omega$ where ℓ_{11} is the eleven-dimensional Planck length. Later we will see that the above normalization of \widetilde{W} is correct up to a numerical factor.

Now we need to check that a superpotential is actually defined globally by (6.6) up to additive constant. First of all, as we have observed in the above discussion, two configurations cannot always be interpolated by a one parameter family of segments. Thus, we must relax the condition on Σ by allowing it to have some boundary circles which are in the fivebranes.

¹²Of course, the restriction to $x^4 = x^5 = x^8 = x^9 = 0, 0 \leq x^6 \leq L$ is not essential and we could develop the following argument for general configuration, but it makes no difference in the final result.

Namely, Σ is a real two-dimensional surface in the four-manifold with complex coordinate t and σ which has boundaries like

$$\partial\Sigma = I - I' + J_R - J_L + C_1 + \cdots + C_s \quad (6.8)$$

where $J_{L,R}$ are lines in the left and right fivebranes connecting the end points of I and I' , and C_1, \dots, C_s are circles in the fivebrane on the right. What we must show is that the difference $\widetilde{W}(I) - \widetilde{W}(I')$ given by (6.6) is independent on the choice of such a surface Σ . Let us take another surface Σ' with the boundary

$$\partial\Sigma' = I - I' + J'_R - J'_L + C'_1 + \cdots + C'_{s'}. \quad (6.9)$$

Then, the difference of the superpotential changes by

$$\int_{\Sigma-\Sigma'} \Omega = \int_{\partial\Sigma-\partial\Sigma'} \sigma \frac{dt}{t} \quad (6.10)$$

$$= \left(\oint_{J_R-J'_R} - \oint_{J_L-J'_L} + \sum_{j=1}^s \oint_{C_j} - \sum_{j'=1}^{s'} \oint_{C'_{j'}} \right) \sigma \frac{dt}{t} \quad (6.11)$$

where we have used $\Omega = d(\sigma dt/t)$. Note first that the integration of $\sigma dt/t$ over $J_L - J'_L$ vanishes, since t is constant $t \equiv 1$ along J_L and J'_L which are in the fivebrane on the left. Other boundaries are all closed circles in the fivebrane on the right in which

$$\sigma \frac{dt}{t} = -n \frac{\sigma^n d\sigma}{\sigma^n - \widetilde{m}^n} = -n d\sigma - \widetilde{m}^n \frac{d\sigma}{\sigma^n - \widetilde{m}^n}. \quad (6.12)$$

Thus, the difference $\widetilde{W}(I) - \widetilde{W}(I')$ changes by the sum of residues of the differential of the second term on the right hand side, which are proportional to \widetilde{m} . This vanishes in the limit $\widetilde{m} \rightarrow 0$. Therefore, in this $\widetilde{m} = 0$ case, the superpotential is indeed globally defined by (6.6) up to additive constant.

The actual computation of the superpotential is straightforward. Since we are interested in the mass of the BPS solitons, we compute the difference of the values at the vacuum configurations I_0 and I_ℓ . Then, we can take as Σ the solitonic membrane configuration which we discussed in the previous part of this section. Since $\Omega = d(\sigma dt/t)$, we have

$$\widetilde{W}(I_\ell) - \widetilde{W}(I_0) = \int_{\partial\Sigma} \sigma \frac{dt}{t} \quad (6.13)$$

$$= \left(\int_{I_\ell} - \int_{I_0} + \int_{J_r} - \int_{J_l} + \sum_{j=1}^{\ell} \oint_{C_j} \right) \sigma \frac{dt}{t}. \quad (6.14)$$

Since I_ℓ, I_0, J_l are at $t = 1$, the corresponding integrals vanish. Also, the integration over the small boundaries C_j in the fivebrane on the right vanish in the limit $\tilde{m} \rightarrow 0$ as we have seen using (6.12). Thus, the only non-vanishing term is the integration over the path J_r in the fivebrane on the right in which

$$\sigma \frac{dt}{t} = -n d\sigma$$

in the case $\tilde{m} = 0$ by (6.12). Since J_r is a path connecting $\sigma = q^{1/n}$ and $\sigma = q^{1/n} e^{\frac{2\pi i \ell}{n}}$, the difference of the superpotential values is

$$\begin{aligned} \tilde{W}(I_\ell) - \tilde{W}(I_0) &= - \int_{J_r} n d\sigma \\ &= n q^{1/n} \left(1 - e^{\frac{2\pi i \ell}{n}} \right). \end{aligned} \quad (6.15)$$

The mass of the BPS soliton is basically the absolute value of this difference (6.15). Here we comment on the reason for this, which shows also that the normalization of the definition of \tilde{W} is correct up to a numerical factor (i.e. a factor which is independent of the parameters of the system). The membrane action contains the volume of its worldvolume, and hence, the energy is the area of its spatial part. Even though the area is generally infinite, we may regularize it by considering a difference of the area of an excited configuration and the one of the ground configuration. It is a natural guess, although we do not presently have a proof of it, that such an area is bounded from below by the absolute value of the integration of the holomorphic two form Ω of the $x^{2,3,7,10}$ part of the space-time (4.4), and that the BPS configuration with two unbroken supersymmetry saturates this. For a finite time interval Δx^0 , the action is given by

$$E \Delta x^0 = \frac{1}{\ell_{11}^3} \int dx^0 \left| \int_{\Sigma} \Omega \right| = \frac{1}{\ell_{11}^3} \Delta x^0 \left| \int_{\Sigma} \ell_{11}^3 \Omega \right| = \Delta x^0 \left| \int_{\Sigma} \Omega \right|. \quad (6.16)$$

This is the reason why the absolute value of (6.15) is the mass of the BPS soliton (up to a numerical factor which we have not been careful enough to fix). In view of the identification of the parameters (4.16), the mass of the soliton is given by

$$M_\ell = n \Lambda \left| 1 - e^{2\pi i \ell / n} \right|, \quad (6.17)$$

which coincides with what we know from field theory (2.56) [47, 48, 49, 37], up to a numerical factor.

General Twisted Mass

It is easy to extend the above analysis to the case where general twisted masses \tilde{m}_i are turned on. The one form $\sigma dt/t$ is expressed in the fivebrane on the right as

$$\sigma \frac{dt}{t} = -\sigma \sum_{i=1}^n \frac{d\sigma}{\sigma - \tilde{m}_i} = -n d\sigma - \sum_{i=1}^n \frac{\tilde{m}_i d\sigma}{\sigma - \tilde{m}_i}. \quad (6.18)$$

Thus, there is an ambiguity in defining the superpotential due to the residue of this form, as we have seen right above in a special case. Namely, the ambiguity is proportional to $2\pi i \tilde{m}_i$. This is exactly the ambiguity we have observed in section 2.2 in the field theory discussion (up to the usual factor of 2π). However, we can nevertheless define unambiguously the central charge, or the masses of the BPS solitons. For a BPS configuration given by a surface Σ , the central charge is simply defined as the integration of Ω over Σ :

$$\tilde{Z} = \int_{\Sigma} \Omega. \quad (6.19)$$

Consider, for example, a configuration interpolating neighboring vacua with a single small boundary circle C which winds around $\sigma = \tilde{m}_i$, as depicted in Figure 13. Then, the integration

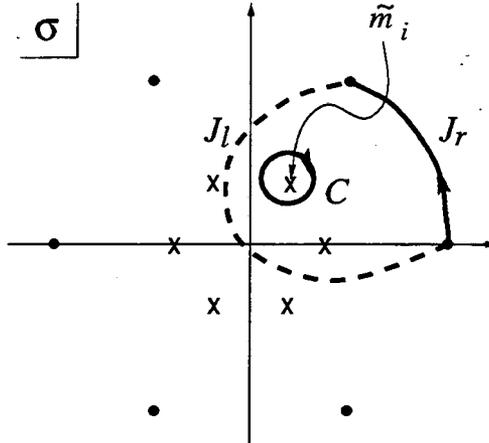


Figure 13: The Contours

(6.19) is reduced to the contour integration of the one form (6.18) along the solid lines C and J_r . This is actually the same as the integration over the path P_i considered in section 2.2, Figure 1, since $J_r + C - P_i$ is a boundary of a surface on which the one form (6.18) has no

residue. Namely, the same central charge can be expressed in two ways

$$\tilde{Z} = \int_{P_i} \sigma \frac{dt}{t} = \int_{P_0} \sigma \frac{dt}{t} + 2\pi i \tilde{m}_i, \quad (6.20)$$

where P_0 is the notation for J_r used in section 2.2 (compare Figure 13 and Figure 1). This corresponds to the equation (2.59). The first expression is interpreted as the difference of the values of the superpotential associated with a choice of the path (P_i) and the second is interpreted as the sum of the one associated with another choice (P_0) and the twisted mass times the $U(1)$ charge carried by the soliton. In general, the central charge is expressed as

$$\tilde{Z} = \Delta \tilde{W} + 2\pi i \sum_{i=1}^n \tilde{m}_i S_i, \quad (6.21)$$

where S_i is the charge of the i -th $U(1)$ of the group $U(1)^n$, the subgroup of the flavor group $U(n)$ (modulo the gauge group $U(1)$) which remains unbroken by the twisted masses. Note that a change of the path defining the superpotential changes the $U(1)$ charges S_i by an amount related to the topological charge.

Finally, let us see what happens if we send one of the mass, say \tilde{m}_n , to infinity. If we do this by keeping fixed Λ_L and $e^{i\theta_L}$ defined by

$$\Lambda^n e^{i\theta} = \tilde{m}_n \Lambda_L^{n-1} e^{i\theta_L}, \quad (6.22)$$

then, one of the n roots of $\prod_{i=1}^n (\sigma - \tilde{m}_i) = q = \Lambda^n e^{i\theta}$ is of order \tilde{m}_n and goes to infinity, while the rest becomes the $n - 1$ roots of $\prod_{i=1}^{n-1} (\sigma - \tilde{m}_i) = \Lambda_L^{n-1} e^{i\theta_L}$ and are finite. Namely, one of the n vacua runs away to infinity and only $n - 1$ of them remain. The mass of the BPS soliton interpolating two vacua will stay finite if the small boundary circles do not wind around $\sigma = \tilde{m}_n$ and the boundary J_r stays finite. However even a BPS soliton may disappear by acquiring an infinite mass, if the boundary circle winds around $\sigma = \tilde{m}_n$ or the boundary J_r is infinitely elongated. If we consider the process

$$\tilde{m} = 0 \longrightarrow \tilde{m} = \text{diag}(0, 0, \dots, 0, \tilde{m}_n = \infty), \quad (6.23)$$

then, the soliton spectrum changes as

$$\bigwedge_{\ell}^{\ell} \mathbf{C}^n \longrightarrow \bigwedge_{\ell}^{\ell} \mathbf{C}^{n-1}, \quad \ell = 1, \dots, n-2 \quad (6.24)$$

$$\bigwedge_{n-1}^{\mathbf{C}^n} \longrightarrow \emptyset. \quad (6.25)$$

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