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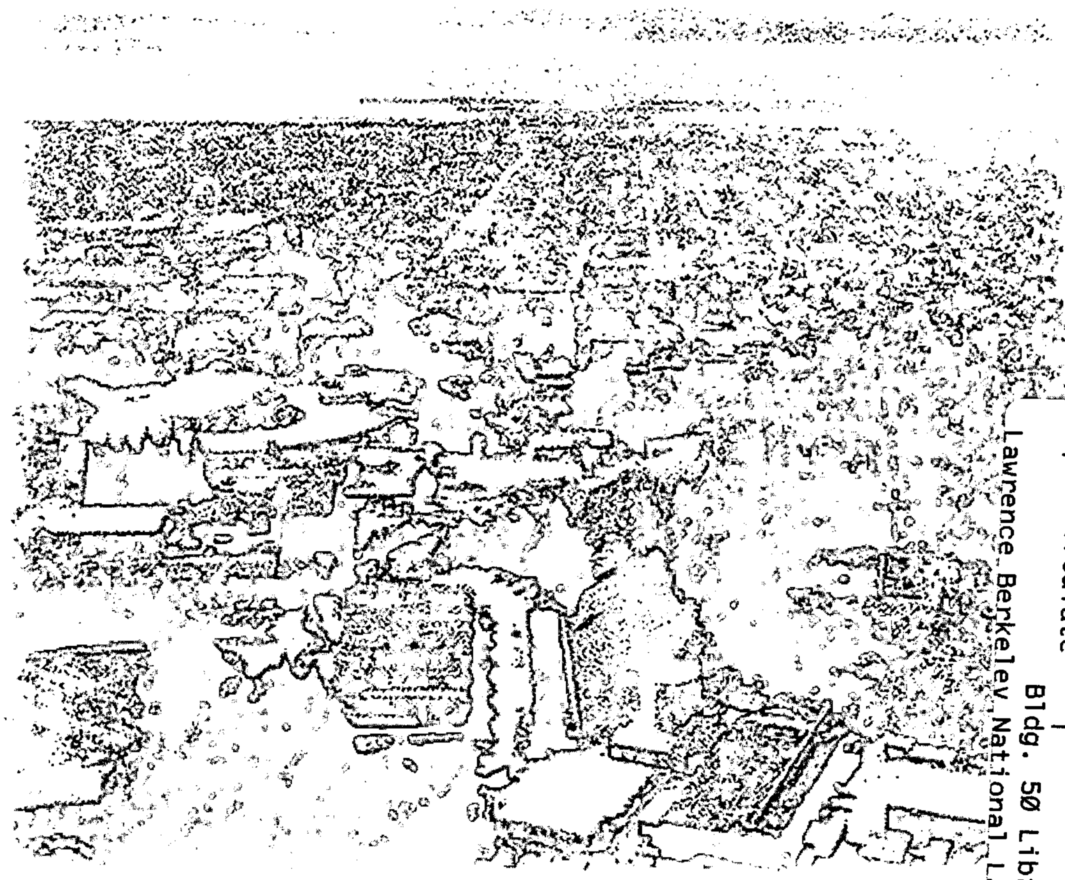

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Discrete Anomaly Matching*

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Abstract

We extend the well-known 't Hooft anomaly matching conditions for continuous global symmetries to discrete groups. We state the matching conditions for all possible anomalies which involve discrete symmetries explicitly in Table 1. There are two types of discrete anomalies. For Type I anomalies, the matching conditions have to be always satisfied regardless of the details of the massive bound state spectrum. The Type II anomalies have to be also matched except if there are fractionally charged massive bound states in the theory. We check discrete anomaly matching in recent solutions of certain $N = 1$ supersymmetric gauge theories, most of which satisfy these constraints. The excluded examples include the chirally symmetric phase of $N = 1$ pure supersymmetric Yang-Mills theories described by the Veneziano–Yankielowicz Lagrangian and certain non-supersymmetric confining theories. The conjectured self-dual theories based on exceptional gauge groups do not satisfy discrete anomaly matching nor mapping of operators, and are viable only if the discrete symmetry in the electric theory appears as an accidental symmetry in the magnetic theory and vice versa.

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1 Introduction

Understanding the dynamics of any physical system beyond perturbation theory has always been a challenging task. In the context of quantum field theories, many techniques have been developed to attack the problem with various degrees of success and applicability: computer simulation of field theories on a lattice, exact solutions using Bethe Ansatz or Yang-Baxter equation, mean field approximation, Schwinger–Dyson equation, large N, etc. If the system possesses a relatively large global symmetry, however, one of the most powerful method to study a possible low-energy spectrum of a theory is 't Hooft anomaly matching [1]. This method has been especially useful in the recent remarkable progress in supersymmetric gauge theories [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

In the method of 't Hooft anomaly matching, one compares the anomalies of the global symmetries in a model between the fundamental theory and a proposed low-energy theory. If the symmetries are not spontaneously broken, it is argued that the anomalies must match. Since the matching of anomalies often involve linear and cubic equations, the requirement that the anomalies must match usually results in a highly non-trivial consistency check of a candidate low-energy theory. Being a necessary condition, anomaly matching can not establish that the candidate theory is indeed the correct low-energy description of the given fundamental theory; it can however either be used to exclude a proposed candidate or to give a strong support for it. Because of this nature of the method, it is very important to exploit all possible available constraints. Even when only one of the constraints fails, it excludes the proposed low-energy theory.

In this paper, we will show that discrete symmetries also have anomalies which have to be matched between the fundamental and low-energy theories.* The argument can be summarized as follows. A discrete symmetry can be promoted to a continuous global symmetry by regarding certain couplings of the theory as background fields. This new continuous global symmetry has to satisfy the usual 't Hooft anomaly matching conditions. Once the background field is frozen to its actual value, the continuous symmetry is broken to a discrete one. However, the anomaly matching conditions must still be satisfied mod N for a Z_N symmetry, if one uses a normalization where all Z_N charges are integers. Furthermore, one can work out how the decoupling of massive fields can modify the anomalies and the possible modifications can be classified. The combination of the original anomaly matching and the decoupling of heavy fields give powerful constraints on the low-energy particle content.

We apply the matching of discrete anomalies to many models studied in the literature. We find that all Seiberg dualities in $N = 1$ supersymmetric gauge theories [2, 3, 4] match discrete anomalies in a highly non-trivial manner. We also find that certain conjectured dynamics of gauge theories can be excluded by this consideration. The examples include

*In this paper, we focus only on Abelian discrete symmetries while we believe that our arguments for discrete anomaly matching could be extended to non-Abelian discrete symmetries as well.

chirally symmetric vacua in $N = 1$ supersymmetric Yang–Mills theories [14] described by the Veneziano–Yankielowicz Lagrangian [15], certain non-supersymmetric chiral gauge theories [16, 17] and self-dual supersymmetric theories based on exceptional groups [18, 19, 20].

The paper is organized as follows. In Section 2, we review 't Hooft anomaly matching for continuous global symmetries. In Section 3, we discuss the possible kinds of discrete symmetries. In Section 4, we give our arguments for discrete anomaly matching and discuss the decoupling of heavy fermions. The final form of the discrete anomaly matching conditions are given in an explicit form at the end of Section 4.3. In Section 5, we apply the discrete anomaly matching conditions to the Intriligator–Seiberg solution [3] of $N = 1$ supersymmetric $SO(N)$ theories with vectors. In Section 6, other supersymmetric examples are discussed. Section 7 gives the examples excluded by the discrete anomaly matching conditions. Finally, we conclude in Section 8.

2 't Hooft Anomaly Matching

One of the most powerful tools for studying the non-perturbative low-energy dynamics of strongly interacting gauge theories are the 't Hooft anomaly matching conditions. These are highly non-trivial constraints on the massless fermion content of a confining theory, and it can also be used as a check for conjectured dualities. Since the 't Hooft anomaly matching conditions play a central role in our discussion, in this section we briefly summarize 't Hooft's original argument [1] about matching of continuous global anomalies.

Let us assume that we have a strongly interacting gauge theory, with gauge group G_{gauge}, and that there is a continuous non-anomalous global symmetry G_{global}. Since G_{global} is a global symmetry, there is generally no reason for the G^3_{global} anomaly to be vanishing. One can imagine however to include spectator fields which transform under G_{global} but not under G_{gauge}, such that the G^3_{global} anomaly is exactly canceled. Then one can weakly gauge the G_{global} group as well. Let us now consider the low-energy effective theory which contains some massless fermions, which are to be thought of as composites of the original degrees of freedom. Since G_{global} is weakly gauged, it has to be anomaly free in the low-energy effective theory as well. However, since the spectator fields do not transform under G_{gauge}, they do not participate in the strong dynamics, and hence their contribution to the G^3_{global} anomaly is identical in the high-energy and the low-energy descriptions. Therefore the G^3_{global} anomaly of the original degrees of freedom (excluding the spectators) must exactly match the contribution of the composite fields in the low-energy theory. This argument can be trivially generalized to show that, in the case the global symmetry is a product group of the form $G_{global} = G_1 \times G_2 \times \ldots \times U(1)_1 \times U(1)_2 \times \ldots$, all the G_i^3, $G_i^2 U(1)_j$, $U(1)_i U(1)_j U(1)_k$ as well as the $U(1)(\text{gravity})^2$ anomalies must match between the high-energy and the low-energy theories. Here and below, G_i refer to simple groups. To be more explicit, we list below the quantities whose values calculated in the high-energy and low-energy theories have to be precisely equal:

$$
\begin{aligned}
G_i^3 : \quad & \sum_R A_R^i \\
G_i^2 U(1)_j : \quad & \sum_R \mu_R^i q_R^j \\
U(1)_i U(1)_j U(1)_k : \quad & \sum_R q_R^i q_R^j q_R^k \\
U(1)(\text{gravity})^2 : \quad & \sum_R q_R^i
\end{aligned}
\tag{2.1}
$$

where A is the cubic anomaly coefficient defined by the relation $\mathrm{Tr}_R\{T^a, T^b\}T^c = A_R d^{abc}$ (the T's being the generators of the group G_i in a given representation R), μ_R is the Dynkin index $\mathrm{Tr}_R T^a T^b = \mu_R \delta^{ab}$, and q^i's are the $U(1)_i$ charges. The sum over R denotes the summation over all representations of fermions present in the high-energy or the low-energy descriptions.

The fact that these constraints are satisfied is the most important evidence in favor of the low-energy solutions of certain $N = 1$ supersymmetric gauge theories proposed by Seiberg and others. However, many of these theories have discrete global symmetries in addition to the continuous ones. It is a natural question to ask whether the presence of the discrete symmetries further constrains the low-energy spectrum. We will next show that this is indeed the case: discrete symmetries have to obey certain anomaly matching conditions as well. In the next section we first review the different types of discrete symmetries a theory can have and their possible origins. Then in Section 4 we show what the anomaly matching conditions for discrete symmetries are.

3 Discrete Symmetries

In this section, we review the possible origins of discrete symmetries in a quantum field theory. This is useful in order to find all non-trivial discrete symmetries of a given theory. There are two different types of discrete symmetries. One type is when the discrete symmetry commutes with the gauge group. We call these the "flavor-type" discrete symmetries. The other type is when the the discrete symmetries do not commute with the gauge transformations, and are given by outer automorphisms of the Lie algebras. We call them "color conjugation" type discrete symmetries. At the end of the section we discuss when the discrete symmetries are independent from the center of the continuous global symmetries.

3.1 Flavor-Type Discrete Symmetries

Flavor-type discrete symmetries arise when a continuous flavor symmetry of the kinetic terms of the Lagrangian is broken explicitly or spontaneously, but a discrete subgroup of the continuous symmetry is left unbroken. We review the possible mechanisms for breaking a continuous flavor symmetry to its discrete subgroup below.

3.1.1 Explicit Breaking

The simplest possibility is that a continuous global symmetry is explicitly broken by an interaction term in the Lagrangian. For example, if there is a global $U(1)$ symmetry, under which the fields ϕ_i (which could be either a bosonic or a fermionic field) have charge q_i, an interaction term

$$\mathcal{L}_{break} = \prod_i \phi_i, \qquad \sum_i q_i \neq 0, \tag{3.1}$$

breaks the global $U(1)$ to its Z_N subgroup with $N = \sum_i q_i$. The fields ϕ_i transform under this Z_N as

$$\phi_i \to e^{\frac{nq_i}{N}2\pi i}\phi_i, \qquad n = 0, 1, \ldots, N-1. \tag{3.2}$$

3.1.2 Breaking by Instantons

This happens when a global $U(1)$ is anomalous. Assume that the left-handed Weyl fermion fields ψ_i carry charges q_i of a classical $U(1)$ symmetry, and that this $U(1)$ is anomalous under the gauge group. One consequence of the anomaly is that the correlator

$$\langle \prod_i \psi_i^{\mu_i} \rangle \tag{3.3}$$

does not vanish in an instanton background [21], thus breaking the anomalous $U(1)$ symmetry. Here μ is the Dynkin index of the given fermion under the gauge group, where the index is defined as $\mathrm{Tr}T^aT^b = \mu\delta^{ab}$. The T^a's are the generators of the gauge group in the representation of the fermion ψ_i. The normalization of the Dynkin index is chosen such that it exactly corresponds to the number of fermion zero modes in a one-instanton background (*i.e.* the index of the fundamental of SU and Sp is normalized to one while the vector of SO is normalized to two). However, the correlator in Eq. (3.3) is invariant under the discrete transformations

$$\psi_i \to e^{\frac{nq_i}{N}2\pi i}\psi_i, \qquad n = 0, 1, \ldots, N-1, \tag{3.4}$$

thus a discrete Z_N subgroup with $N = \sum_i q_i\mu_i$ is left unbroken.

3.1.3 Spontaneous Breaking

It is also possible that a continuous global symmetry is spontaneously broken by an expectation value of one of the fields, but that the VEV of the field leaves a discrete rotation invariant. The general rule for the $U(1) \to Z_N$ type breaking is that if the field φ with non-vanishing expectation value $\langle\varphi\rangle \neq 0$ has charge N under a global $U(1)$ symmetry, then the Z_N subgroup of $U(1)$ under which

$$\psi_i \to e^{\frac{nq_i}{N}2\pi i}\psi_i, \quad n = 0, 1, \ldots, N-1 \tag{3.5}$$

is left unbroken. Note that this transformation has no effect on the field φ with the non-vanishing expectation value as required.

3.2 Color Conjugation Type Discrete Symmetries

The flavor-type discrete symmetries considered above all arise from breaking of the continuous flavor symmetries of the kinetic terms of the Lagrangian. However, it is possible that a theory has more symmetries than the usual global flavor symmetries. If such symmetries are present, they can not commute with the gauge group; otherwise they would be contained in the flavor symmetries of the theory. In order for such transformations to be symmetries of the theory, they must leave the Lie algebra of the gauge group invariant, and hence they must be outer automorphisms of the Lie algebra. The complete list of all possible outer automorphisms of simple gauge groups is given by (see *e.g.* [22]):

$$\begin{array}{rll} SU(N): & Z_2 & (N > 2) \\ SO(2N): & Z_2 & (N > 2) \\ E_6: & Z_2 & \\ SO(8): & S_3 & \end{array} \tag{3.6}$$

while the other simple gauge groups do not have a non-trivial outer automorphism. We call discrete symmetries based on these outer automorphisms "color conjugation type" discrete symmetries.

As the name suggests, a color conjugation is a generalization of the familiar charge conjugation in QED. Charge conjugation changes the sign of the electric charge $Q \to -Q$, and interchanges charge $+1$ fields with charge -1 fields. An immediate generalization of this for non-Abelian gauge groups is given by

$$T^a \to \mathcal{C}^{-1}T^a\mathcal{C} = -T^{a*}, \tag{3.7}$$

where the T^a's are the generators of the gauge group and $\mathcal{C}$ is the charge conjugation operator. For the case of $SU(N)$, E_6 and $SO(4k+2)$ gauge groups, this indeed defines outer automorphisms on the Lie algebras, and we call this transformation $\mathcal{C}$ charge conjugation. Charge conjugation exchanges representations with their complex conjugates. Note that the charge conjugation is trivial for real representations, and is equivalent to a gauge transformation for pseudo-real representations.

There is, however, another way to generalize the charge conjugation in QED to $SO(N)$ groups. With fields q^+ and q^- with electric charges ± 1, one can define an $SO(2)$ doublet

$(q^1, q^2) = (i(q^+ - q^-), q^+ + q^-)/\sqrt{2}$, on which the charge conjugation acts as the sign flip of the first "color", $q^1 \to -q^1$, $q^2 \to q^2$. In general, we define a "color-parity" transformation $\mathcal{P}$ on an $SO(N)$ vector by flipping the sign of one particular color (for example the first color), which defines an automorphism of the Lie algebra

$$M_{ij} \to \mathcal{P}^{-1} M_{ij} \mathcal{P} = \begin{cases} -M_{1j} & (j \neq 1) \\ M_{ij} & (i, j \neq 1) \end{cases}, \tag{3.8}$$

where the M_{ij}'s are the $SO(N)$ generators. One can view the color-parity as a non-trivial element of the $O(N)$ extension of $SO(N)$ group, *i.e.*, a parity-like transformation.* As we show in Appendix A, this definition of the color-parity transformation is equivalent to the charge conjugation of Eq. (3.7) for $SO(4k+2)$ gauge groups up to gauge transformations. On the other hand, $SO(4k)$ groups have only real or pseudo-real representations, and hence the charge conjugation Eq. (3.7) is equivalent to an $SO(4k)$ gauge transformation and thus is not an outer automorphism of the Lie algebra. The color-parity transformation is an outer automorphism for all $SO(2N)$ gauge groups, and interchange two inequivalent spinor representations (often referred to as spinor and conjugate-spinor representations).

Therefore, it is convenient to define the color conjugations by the charge conjugation (3.7) for $SU(N)$ and E_6, and by the color-parity (3.8) for $SO(2N)$ groups. The $SO(8)$ group is special and its S_3 automorphism is the triality permuting the vector, spinor and conjugate spinor representations.

Note that color conjugation type discrete symmetries are not necessarily realized within a theory, but they may map the given theory to another one. Whether this is the case depends on the matter content of the theory. Since the color conjugations usually interchange representations, only non-chiral theories have these extra discrete symmetries; in chiral theories these symmetries are broken by the matter content. Even if a color conjugation can be defined in a given theory, it is usually not very useful from the point of view of anomaly matching, since the notion of anomaly is hard to define if a symmetry interchanges representations. Thus the only interesting case is if one has a gauge group with a non-trivial outer automorphism and only self-conjugate representations. In this case, the color conjugation symmetries can mix with the usual flavor type discrete symmetries and may be important. We will see several examples of this happening in the supersymmetric $SO(N)$ examples of Sections 5.

3.3 Independence of Discrete Symmetries

We have seen above how to find the discrete symmetries of a given theory. However, one has to be careful with the identification of the non-trivial discrete symmetries. The reason

*For $SO(2N+1)$ groups, this parity-like transformation is gauge equivalent to an overall sign flip of the vector, and hence is of the flavor-type.

is that a discrete symmetry might be contained in a continuous symmetry as its discrete subgroup. As an example, let us consider QCD with F flavors. The classical theory has the continuous symmetries

$$\begin{array}{c|ccccc} & SU(N) & SU(F)_Q & SU(F)_{\bar{Q}} & U(1)_B & U(1)_A \\ \hline Q & \square & \square & 1 & 1 & 1 \\ \bar{Q} & \bar{\square} & 1 & \square & -1 & 1 \end{array} \quad , \tag{3.9}$$

where $SU(N)$ is the gauge group, the two $SU(F)$ factors are the non-Abelian global symmetries which transform either the quarks or the antiquarks[†] among each other, $U(1)_B$ is the baryon number, and $U(1)_A$ is the axial $U(1)$ under which both quarks and antiquarks transform by the same phase. This $U(1)_A$ is however anomalous, since the correlator

$$\langle Q^F \bar{Q}^F \rangle \neq 0 \tag{3.10}$$

is non-vanishing. Thus $U(1)_A$ is broken by instantons to its Z_{2F} discrete subgroup, under which

$$Q \to e^{2\pi i n/2F} Q, \quad \bar{Q} \to e^{2\pi i n/2F} \bar{Q}, \qquad n = 0, 1, \ldots, 2F - 1. \tag{3.11}$$

However this Z_{2F} symmetry is not a new symmetry of the theory. The reason is that one can choose a discrete subgroup of the continuous flavor symmetries which exactly coincides with this Z_{2F} symmetry. Take for example the center of one of the $SU(F)_Q$ flavor symmetries, which is a Z_F transformation under which only the quarks Q transform with charge one. A combination of this with discrete baryon number transformation with a phase $-\pi/F$ is exactly the Z_{2F} symmetry from the anomalous $U(1)$, thus it is part of the other continuous global symmetries. In the absence of explicit breaking terms, only those theories which do not contain matter fields in the fundamental representations (Dynkin index one) have non-trivial discrete symmetries from anomalous $U(1)$'s.

The complete list of the centers of simple groups is given in Appendix B.

4 Discrete Anomaly Matching

In this section, we will show that discrete global symmetries have to obey anomaly matching constraints as well. We will give two different arguments for this. One argument is based on considering correlators in instanton backgrounds after gauging a non-Abelian flavor symmetry and is the natural generalization of 't Hooft's original argument summarized in Section 2. In the second argument, we promote the coupling which breaks the continuous global symmetry to its discrete subgroup to a background field, thus restoring the full continuous global symmetry.

[†]Here and throughout the paper, a fermion means a left-handed two-component Weyl spinor.

We have to note that anomalies of discrete symmetries have been considered previously in Refs. [23, 24, 25]. In these papers, the authors considered the consequences of "gauging" the discrete symmetries [26] on the low-energy spectrum. The assumption was that all global symmetries are broken by quantum gravitational effects, and hence only "gauged" discrete symmetries can be realized on a realistic low-energy particle spectrum. They then considered the consequences of the anomaly cancellation for the gauged discrete symmetries. For our purpose, we do not assume that discrete symmetries have to be anomaly free. Instead, we are going to compare the anomalies of the discrete symmetries between the high-energy and the low-energy theories. Even though the spirit of our work is very different from that of Refs. [23, 24, 25], the arguments below for discrete anomaly matching will be in many aspects similar to those in [23, 24, 25]. Our first argument for discrete anomaly matching is based on instantons and resembles the spirit of Refs. [24, 25], while our second argument of restoring the continuous symmetry is closer to the attitude of Ref. [23].

We will present two classes of anomaly matching constraints. The Type I constraints (which include the $G_F^2 Z_N$ and $Z_N(\text{gravity})^2$ anomalies) have to be satisfied independently of any assumptions about the massive particle spectrum. The Type II constraints (which include the Z_N^3, $U(1)^2 Z_N$, $U(1) Z_N^2$, $Z_N^2 Z_M$ and $Z_N Z_M U(1)$ anomalies), however, may be evaded if the massive spectrum contains fractionally charged particles. We will discuss the issue of charge fractionalization in more detail in Section 4.2.3.

4.1 The Instanton Argument

Here we will show that the discrete $G_F^2 Z_N$ and $Z_N(\text{gravity})^2$ anomalies have to be matched between the high-energy and the low-energy theories (Type I constraints), where G_F is a non-Abelian flavor symmetry. Since the Z_N charges of the fields are defined only mod N, the most stringent constraint we can expect is anomaly matching mod N. We will see that for the $G_F^2 Z_N$ anomaly this is indeed the case, while for $Z_N(\text{gravity})^2$ the anomalies have to match only mod $N/2$, if N is even (and mod N if N is odd).

Let us first discuss the $G_F^2 Z_N$ anomaly. We assume that the theory we consider has a $G_F \times Z_N$ discrete symmetry which is non-anomalous under the gauge group. In general, the G_F^3 and the $G_F^2 Z_N$ anomalies do not necessarily vanish. Next we introduce spectator fields which do not transform under the gauge group such that both the G_F^3 and the $G_F^2 Z_N$ anomalies vanish (the latter mod N). Then we can weakly gauge the G_F group since the G_F^3 anomaly now vanishes. Because the $G_F^2 Z_N$ anomaly vanishes as well, this means that the Z_N symmetry is unbroken in a background of G_F instantons; it is an exact symmetry of the theory. In other words, the non-vanishing correlator in a G_F instanton background $\langle \prod_i \psi_i^{\mu_i} \rangle$ is invariant under Z_N. Thus $\sum_i \mu_i q_i = 0 \bmod N$, where μ_i is the Dynkin index of the Weyl fermions under G_F, while q_i's are the Z_N charges.

If the Z_N symmetry is not spontaneously broken, then the low-energy effective theory must have Z_N as an unbroken exact symmetry as well. This means that the non-vanishing correlator in the G_F instanton background calculated for the low-energy bound states must also be invariant under Z_N. Thus we conclude that $\sum_i \mu_i q_i = 0 \mod N$ in the low-energy theory as well. Since the spectators do not transform under the gauge group, they do not participate in forming the bound states, and their contribution to the $G_F^2 Z_N$ discrete anomaly is the same in the high-energy and in the low-energy theories. Therefore, the bound states must match the $G_F^2 Z_N$ anomaly of the original degrees of freedom mod N.

One can repeat exactly the same argument for the $Z_N(\text{gravity})^2$ anomaly (which will constrain the $\sum_i q_i$ quantity, where q_i are the Z_N charges) by considering correlators in gravitational instanton backgrounds. One has to be, however, careful with identifying the correct anomaly matching condition, because there are always even number of zero modes for a Weyl fermion in a gravitational instanton background. This is due to Rohlin's theorem [27] which states that the signature of a smooth, compact, spin four-manifold is divisible by 16. Since the $\hat{A}$-genus of a four-dimensional manifold is an eighth of the signature (see, *e.g.* [28]), there are always even numbers of zero modes for a Weyl fermion. The smallest number of zero modes is found, for instance, on a $K3$ manifold, which give two zero modes for every Weyl fermion. Even if the $Z_N(\text{gravity})^2$ anomalies differ by $N/2$ between fundamental and low-energy theories, it does not change the conclusion that the Z_N symmetry is not broken either in the high-energy or in the low-energy theory by gravitational instantons. Thus the $Z_N(\text{gravity})^2$ anomaly has to be matched only mod $N/2$, if N is even.

The origin of the possible difference of $N/2$ in the $Z_N(\text{gravity})^2$ anomaly can also be understood by considering decoupling of heavy fermions [23]. The contribution of such particles to continuous anomalies is always vanishing. This is not the case for discrete anomalies and the possible contributions of such particles must be enumerated. One can have, for example, a pair of different Weyl fermions pairing up and getting a Dirac mass. In this case, the charges of these fermions must obey $q_1 + q_2 = mN$, where q_1, q_2 and m are integers. Then these fermions contribute integer multiples of N to all anomalies, and since all the anomaly matching equations are modulo N anyway, such particles do not change these equations. However for even N, there is another possibility: a single fermion with Z_N charge $\frac{m}{2}N$ can acquire a Majorana mass. In this case, the contribution of this fermion to the $Z_N(\text{gravity})^2$ anomaly is $\frac{m}{2}N$, thus there can be a difference which is a half-integer multiple of N between the high-energy and the low-energy values of the $Z_N(\text{gravity})^2$ anomaly. The possible existence of such massive Majorana fermions leads to the weaker anomaly matching condition for the $Z_N(\text{gravity})^2$ anomaly. On the other hand, this also means that we might gain some information (even if very limited) about the massive spectrum as well. If the anomalies match only mod $N/2$, then we can conclude that there must be odd number of massive Majorana fermions with Z_N charge $N/2$ present in the theory. Such Majorana fermions however do not weaken the $G_F^2 Z_N$ anomaly matching constraint, since the Dynkin indices of real representations (as required for a Majorana fermion) are even, and therefore the contribution of heavy Majorana particles to the $G_F^2 Z_N$ anomalies is a multiple of N.

4.2 The Spurion Argument

We have shown above that the discrete $G_F^2 Z_N$ and $Z_N(\text{gravity})^2$ anomalies have to be matched mod N and mod $N/2$ between the low-energy and the high-energy theories. We used the fact that we can study correlators in the G_F or gravitational instanton background. However, this argument can obviously not be extended to the Type II anomalies, such as $U(1)^2 Z_N$, Z_N^3. Therefore we present another argument, which will show that Type II discrete anomalies have to be matched as well, not just the two discussed in the previous section, assuming there are no massive states with fractional charges. Note that the matching of the Type I anomalies is independent of the details of the massive spectrum.

We discuss only flavor-type discrete symmetries, which arise due to the breaking of a continuous global symmetry via an interaction term in the Lagrangian.* This continuous global symmetry can be restored, if we promote the coupling which breaks the continuous symmetry to a background field ("spurion"). For example in the case of explicit breaking by the interaction

$$\mathcal{L}_{break} = \lambda \prod_i \phi_i, \qquad \sum_i q_i \neq 0, \tag{4.1}$$

we can assign $U(1)$ charge $-\sum_i q_i$ to the coupling λ.

In the case of a discrete Z_N symmetry arising from an anomalous $U(1)$ symmetry, we can first add a pair of fermions ψ_0 and ψ_{-N} which exactly cancel the $U(1)G_{gauge}^2$ anomaly. For example for $SU(n)$ we add a fundamental with charge 0 and an antifundamental with charge $-N$, for $Sp(2n)$ we add a fundamental with charge 0 and another one with charge $-N$, and for $SO(n)$ we add one vector with charge $-N/2$. This latter is allowed because N is even in the case of orthogonal groups, since the smallest representation has index 2 and hence N is even.† This restores the continuous $U(1)$ symmetry, which we can explicitly break to its Z_N subgroup by adding a mass term for the extra fermions $m\psi_0\psi_{-N}$ (or $m\psi_{-N/2}\psi_{-N/2}$ for $SO(n)$). If m is taken to be sufficiently big, it will influence neither the low-energy theory nor the high-energy anomalies, but one can think of the mass parameter m as a spurion for breaking the continuous $U(1)$ symmetry to its Z_N subgroup. Alternatively, for supersymmetric theories one can promote the dynamical scale $\Lambda^{b_0} = M^{b_0} e^{-(\frac{8\pi^2}{g^2(M)}+i\theta)}$ of the theory to a background field with $U(1)$ charge $-\sum_i \mu_i q_i$, where b_0 is the coefficient of the one-loop β-function, g is the bare gauge coupling, M is the ultraviolet cutoff, μ_i are the Dynkin indices of the representations under the gauge group and q_i are the Z_N charges. This restores the anomalous $U(1)$ symmetry because the effect of an anomalous $U(1)$ symmetry is to shift the θ parameter, or in other words, a phase rotation of the scale Λ. One can undo such a

*It is straightforward to generalize the discussion to the color conjugation type discrete symmetries by enlarging the gauge group but breaking it by a spurion.

†This is true for $SO(n)$ groups with $n > 6$. Smaller SO groups with $n = 3, 4, 5, 6$ are locally isomorphic to $SU(2)$, $SU(2) \times SU(2)$, $Sp(4)$, and $SU(4)$ groups, respectively, and the other constructions based on SU or Sp groups apply.

rotation by assigning the above charge under the $U(1)$ symmetry to the scale Λ [2].

By promoting the coupling constants of the theory to background fields this way, we have restored the continuous $U(1)$ global symmetries of the theory. In this theory the 't Hooft argument of Section 2 holds, thus the anomaly matching conditions have to be satisfied for this new $U(1)$ symmetry as well, together with all other continuous global symmetries of the theory. Let us now consider what effect is generated to the anomalies involving the broken $U(1)$ symmetry by freezing the background fields to their actual value. For one, it breaks the $U(1)$ to its Z_N subgroup. However, since the background fields do not carry Z_N charge mod N, freezing of the background fields does not change any of the anomalies mod N. Thus we conclude that all the discrete anomalies involving the Z_N discrete symmetry must be matched mod N. This argument however neglects four important subtleties, which will change the final form of the discrete anomaly matching conditions slightly:

- decoupling of heavy fields
- normalization of the $U(1)$ generators
- charge fractionalization
- different units of discrete charges for mixed $Z_N - Z_M$ anomalies.

In particular, charge fractionalization can invalidate the discrete anomaly matching constraints for Type II anomalies, but not for the Type I's. In the following we describe the consequences of the above effects on the discrete anomalies and then present the final form of the discrete anomaly matching conditions explicitly.

4.2.1 Decoupling of Heavy Fermions

As already mentioned at the end of Section 4.1, the decoupling of massive fermions can have non-trivial consequences on the discrete anomaly matching conditions [23]. The contribution of such particles to the continuous anomalies is always vanishing. This is not the case for discrete anomalies and the possible contributions of such particles must be enumerated. One can, for example, have a pair of different Weyl fermions pairing up and acquiring a Dirac mass. In this case, the charges of these fermions must obey $q_1 + q_2 = mN$, where q_1, q_2, and m are integers. These fermions contribute integer multiples of N to all anomalies, and since all the anomaly matching equations are modulo N anyway, such particles do not change the anomaly matching equations. However for even N, there is another possibility: a single fermion with Z_N charge $\frac{m}{2}N$ can acquire a Majorana mass. In this case, the contribution of this fermion to the $Z_N(\text{gravity})^2$ anomaly is $\frac{m}{2}N$, thus there could be a difference which is a half-integer multiple of N between the high-energy and the low-energy values of the $Z_N(\text{gravity})^2$ anomaly. Similarly, the contribution of such a Majorana fermion to the Z_N^3 anomalies is $N^3/8$. Thus the Z_N^3 anomalies can differ by $mN^3/8$ if N is even (as well as multiples of N). The $U(1)Z_N^2$ and $U(1)^2Z_N$ anomaly can not have a similar contribution, since the $U(1)$ charge of a massive Majorana particle must be zero. Similarly, it can not contribute to the $G_F^2Z_N$ anomaly either, since the Dynkin indices of real representations (as it is the case for Majorana fermions) are even.

Therefore, the possible existence of massive Majorana fermions leads to the weaker anomaly matching condition for the $Z_N(\text{gravity})^2$ and the Z_N^3 anomalies. On the other hand, this also means that we might gain some information (even if very limited) about the massive spectrum as well. If the anomalies do differ by the additional factors due to the Majorana fermions, we conclude that there must be odd number of massive Majorana fermions with Z_N charge $N/2$ present in the theory. In the case of anomaly matching for dual pairs, an $N/2$ difference in the $Z_N(\text{gravity})^2$ anomaly signals that the number of massive Majorana fermions with charge $N/2$ in the electric and magnetic theories differs by an odd integer.

Furthermore, we can check the consistency of this assumption by noting that there necessarily must be a difference of $N/2$ in the $Z_N(\text{gravity})^2$ anomaly if there is a difference of $N^3/8$ in the Z_N^3 anomaly.[‡] Thus in addition to the fact that the anomalies have to be matched, certain correlations among the anomalies have to be satisfied as well.

4.2.2 Normalization of the $U(1)$ Charges

For the case of continuous anomaly matching conditions, the overall normalization of the $U(1)$ charges is irrelevant. However for the discrete $U(1)^2Z_N$ and $U(1)Z_N^2$ anomalies, this normalization is important since an overall change in the $U(1)$ charges can make all equations mod N to be satisfied. This is a valid argument for the case of anomaly cancellation of gauged discrete symmetries [23]. In the case of anomaly matching, however, we do know the $U(1)$ charges of the high-energy theory, and thus their normalization in the low-energy theory is fixed. Then choosing a normalization in the high-energy theory such that all $U(1)$ charges (including the ones in the low-energy theory) are integers should result in valid $U(1)^2Z_N$ and $U(1)Z_N^2$ anomaly matching constraints. One needs to choose integer $U(1)$ charges; otherwise a shift of N in the Z_N charges will not result in shifts proportional to N in the anomaly matching conditions. The most stringent constraint arises, of course, if one chooses the normalization of the $U(1)$ charges such that the charge assignments are the smallest while they are still all integers.[§]

[‡]The converse is not true. An $N/2$ difference in the $Z_N(\text{gravity})^2$ anomalies does not necessarily mean that an $N^3/8$ difference must be present in the Z_N^3 anomalies. The reason is that if N is divisible by four, then $N^3/8$ is automatically an integer multiple of N, thus the effect of the decoupling Majorana fermion can not be distinguished from the usual mod N effects coming from the non-uniqueness of the Z_N charges.

[§]If there are irrational $U(1)$ charges, we would not obtain any useful constraints.

4.2.3 Charge Fractionalization

In the context of anomaly cancellation for gauged discrete symmetries, Banks and Dine argued that the Z_N^3 anomalies do not lead to any condition on the low-energy theory [25]. Their argument was that one can not decide whether one had really a Z_N or a Z_{NM} symmetry in the high-energy theory from the pure low-energy point of view. This could be the consequence of the fact that there are fractionally charged states in the high-energy theory but they decouple from the low-energy theory. In particular, they argued that if there were states with charge $1/N$ in the high-energy theory, then at high energies the Z_N symmetry is enlarged to Z_{N^2}. Since in the low-energy theory all particles have charge zero mod N under Z_{N^2}, the $Z_{N^2}^3$ anomaly cancellation equations are trivially satisfied and hence give no useful information. In our case, however, the situation is different. We know what the particle content of the high-energy theory is, thus we know what the correct high-energy discrete symmetry group is. Unless there are massive bound states or topological states in the theory which carry fractional charges under the high-energy Z_N symmetry, the Z_N^3 anomaly matching conditions must be satisfied as well. The situation is similar with the $U(1)Z_N^2$ and the $U(1)^2Z_N$ anomalies: if we assume that the decoupled states carry integer Z_N and $U(1)$ charges, then the $U(1)Z_N^2$ and $U(1)^2Z_N$ anomalies have to be matched mod N. In fact, in every example of anomaly matching that we considered and where all the other anomaly matching conditions were satisfied, all the matching conditions for Type II anomalies (Z_N^3, $U(1)^2Z_N$, $U(1)Z_N^2$, $U(1)_iU(1)_jZ_N$, $Z_N^2Z_M$ and $U(1)Z_NZ_M$) were satisfied as well. This supports our claim that the Type II anomaly matching conditions must be considered as valid constraints as well. However, we have to stress that the discrete anomaly matching constraints for these Type II anomalies could in principle be invalidated if charge fractionalization occurs for the massive bound states, which can not be excluded on general grounds. On the other hand, the Type I anomalies ($G_F^2Z_N$ and $Z_N(\text{gravity})^2$) are not affected by a possible charge fractionalization and have to be always matched.

One can turn the above reasoning around for theories where one finds that the Type I anomalies are matched while the Type II anomalies are not matched, but there is ample of evidence for the considered low-energy spectrum. In this case, the failure of the anomaly matching for the Type II constraints could be used to gain some (even if very limited) insight into the massive spectrum. We learn that there must be massive states with fractional charges under the given symmetry for which anomaly matching is not satisfied.

4.2.4 Mixed $Z_N - Z_M$ Anomalies

Finally, let us note that it is possible to have more than one discrete symmetry in a theory, and that the discrete symmetry group is $Z_N \times Z_M$. In this case, one has to consider the mixed $Z_N^2Z_M$, $Z_M^2Z_N$ and $Z_NZ_MU(1)_i$ anomalies as well. Since the Z_N charges are defined only modulo N, while the Z_M charges modulo M, the mixed anomalies can be shifted by any integer combination of N and M, $aN + bM$, where a, b are integers. If N and M are relatively prime, then $aN + bM$ can take on any integer value and thus the mixed $Z_N - Z_M$ anomalies do not lead to any constraints. However if N and M have a common divisor K, then $aN + bM$ is always a multiple of K. Thus in general the mixed $Z_N - Z_M$ anomaly matching conditions must hold modulo the greatest common divisor of N and M. If N and M are both even, then the decoupling of massive Majorana fermions with charges $N/2, M/2$ can yield an additional contribution of the form $N^2M/8$ to the $Z_N^2 Z_M$ anomalies, but there cannot be such contributions to the $Z_N Z_M U(1)$ anomalies.

4.3 The Discrete Anomaly Matching Conditions

To summarize this section, we found that the presence of non-anomalous discrete global symmetries does yield anomaly matching constraints for these theories. The anomalies have to match in the low-energy and high-energy descriptions up to certain multiples of N. These anomalies and the possible multiples of N for the different anomalies are given in Table 1.

In Table 1, m and m' are integers, K is the greatest common divisor (GCD) of N and M, q_i are Z_N charges, p_i are Z_M charges, Q_i and P_i are $U(1)$ charges, all the q_i, p_i, Q_i, P_i are integers, G denotes a non-Abelian global symmetry, μ_i are the Dynkin indices under this non-Abelian global symmetry. m' can be non-zero only for N, M even. The Type II anomaly matching conditions have to be matched as long as the massive spectrum carries integer charges. They may be evaded if there are massive fractionally charged states. The Type I constraints have to be always satisfied, regardless of charge fractionalization.

5 Discrete Anomalies and Seiberg Dualities

As an application of the discrete anomaly matching conditions derived above, we show in this section that the exact results [2, 3, 4] on $N = 1$ supersymmetric gauge theories indeed satisfy these anomaly matching constraints. First of all, note that SUSY $SU(N)$ QCD does not have a non-trivial discrete symmetry besides the discrete subgroups of the continuous symmetries, as discussed in Section 3.3. The Z_2 color conjugation (see Section 3.2) exchanges the quarks and antiquarks, and is hence not a useful symmetry for discrete anomaly matching either. The same statement is true for the $Sp(2N)$ theories with fundamentals, except that in these theories there is not even a color conjugation present. Thus for the SU and Sp theories with only fundamental representations, there is no discrete symmetry present to check anomaly matching. The situation is different for $SO(N)$ theories with vectors. Since a vector of $SO(N)$ has Dynkin index two (*i.e.* there are two zero modes for the vector in a one-instanton background), there is a global Z_{2F} symmetry which is not contained in the continuous flavor symmetries as a discrete subgroup. Thus the global symmetries of this

	Anomaly	Expression	Difference
Type I	$G^2 Z_N$:	$\sum_i \mu_i q_i$	mN
Type I	$Z_N(\text{gravity})^2$:	$\sum_i q_i$	$mN + \frac{m'}{2}N$
Type II	Z_N^3:	$\sum_i q_i^3$	$mN + \frac{m'}{8}N^3$
Type II	$U(1)^2 Z_N$:	$\sum_i Q_i^2 q_i$	mN
Type II	$U(1)_i U(1)_j Z_N$:	$\sum_i P_i Q_i q_i$	mN
Type II	$U(1) Z_N^2$:	$\sum_i Q_i q_i^2$	mN
Type II	$Z_N^2 Z_M$:	$\sum_i q_i^2 p_i$	$mK + \frac{m'}{8}N^2 M$
Type II	$U(1) Z_N Z_M$:	$\sum_i Q_i q_i p_i$	mK

Table 1: The discrete anomaly matching conditions. The second column displays the given anomaly involving a discrete Z_N symmetry. The third column gives the explicit expression how to evaluate this anomaly both in the high-energy and in the low-energy theories. All charges are integers. μ_i denotes the Dynkin index of the representation i under the non-Abelian group G. The fourth column gives the allowed difference between the discrete anomalies evaluated in the high-energy and low-energy theories. m, m' are integers, and m' can be non-vanishing only if N, M are even. K is the GCD of N and M. Type I anomaly matching constraints have to be satisfied regardless of the details of the massive spectrum. Type II anomalies have to be also matched except if there are fractionally charged massive states.

theory are:

$$\begin{array}{c|cccc} & SO(N) & SU(F) & U(1)_R & Z_{2F} \\ \hline Q & \square & \square & 1-\frac{N-2}{F} & 1 \end{array} \qquad (5.1)$$

In addition to these symmetries, there is an extra Z_2 outer automorphism for $N = 2n$ which is not part of the gauge or flavor symmetries, and the automorphism defines a color conjugation symmetry as discussed in Section 3 and in Appendix A. The color conjugation can be defined to be the internal parity-like transformation (color-parity $\mathcal{P}$), which acts on vectors by flipping the sign of one particular color.* This color-parity transformation for $N = 4k+2$ is equivalent to the usual charge conjugation defined in Eq. (3.7) up to a gauge transformation. On the other hand, for $N = 4k$ they are not equivalent since the charge conjugation is trivial up to a gauge transformation. Note that a similarly defined color-parity transformation for $SO(2n+1)$ theories is gauge equivalent to an overall $Z_2 \subset Z_{2F}$ global transformation and is hence of flavor-type; this is expected to be the case since there are no non-trivial outer automorphisms for $SO(2n+1)$. However, for evaluating the discrete anomalies, we will use the color-parity transformation $\mathcal{P}$ for all $SO(N)$ groups so that the discrete anomalies are given by the same expression regardless of N being even or odd.

We have seen that the discrete symmetries of the $SO(N)$ theory are $Z_{2F} \times \mathcal{P}$ for N even and Z_{2F} for odd N. If F is odd, then the Z_{2F} symmetry is equivalent to a $Z_2 \times Z_F$ symmetry.† However, the Z_F factor is nothing but the center of the $SU(F)$ flavor symmetry of the vectors, thus for odd F the non-trivial discrete symmetry of the theory is just the Z_2 sign flip of all vectors. If N is even, this symmetry is already contained in the gauge group, thus for odd F even N, the only discrete symmetry of the theory is $\mathcal{P}$. For odd F odd N, the Z_2 sign flip of all vectors is not contained in the gauge group, but there is no color conjugation (color-parity is gauge equivalent to a flavor Z_2), thus the final symmetry of the theory is just Z_2. Therefore we conclude that the independent discrete symmetries of the $SO(N)$ theory are:

$$\begin{array}{cccccc} N & \text{even} & F & \text{even}: & Z_{2F} \times \mathcal{P} \\ N & \text{odd} & F & \text{even}: & Z_{2F} \\ N & \text{even} & F & \text{odd}: & \mathcal{P} \\ N & \text{odd} & F & \text{odd}: & Z_2 \end{array}$$

We will write all anomaly matching conditions for the full Z_{2F} symmetry regardless of whether F or N is even or odd. Even if F is odd the Z_{2F} anomaly matching conditions do have to be satisfied; it is just that the Z_F part of it has to be automatically satisfied due to the anomaly matching of the continuous $SU(F)$ symmetries, which Intriligator and Seiberg have already checked.

*This is what Intriligator and Seiberg called "charge conjugation" $\mathcal{C}$ [3]. We, however, reserve the name "charge conjugation" only for the conventional ones given in Eq. (3.7) to avoid confusions.

†In general, Z_{NM} is equivalent to $Z_N \times Z_M$ if N and M are relatively prime.

5.1 $F > N - 2$

For $F > N - 2$, the theory at the origin has a dual magnetic description in terms of the gauge group $SO(F - N + 4)$ [3]. The global symmetries of the dual theory are:

$$
\begin{array}{c|cccc}
 & SO(F-N+4) & SU(F) & U(1)_R & Z_{2F} \\
\hline
q & \square & \overline{\square} & \frac{N-2}{F} & -1 \\
M & 1 & \square\!\square & 2 - 2\frac{N-2}{F} & 2
\end{array} , \tag{5.2}
$$

and there is also a superpotential $W_{mag} = Mq^2$ in the magnetic theory.[‡] The Z_{2F} charge of M is determined by the matching $Q^2 \leftrightarrow M$, while the Z_{2F} charge of q is determined by the requirement that the superpotential has zero Z_{2F} charge mod $2F$. Note that this does not completely fix the charge of q, since one could as well add F to it. This modification does indeed happen, but in a very subtle way. It has been already noted in [3] that the mapping of baryon operators implies a non-trivial mapping between the Z_{2F} symmetries of the electric and the magnetic theories. The baryon Q^N of the electric theory is mapped to the "exotic baryon" $\tilde{W}_\alpha \tilde{W}^\alpha q^{F-N}$ of the magnetic theory. (Both of them have all color indices contracted with the ϵ-tensor.) Comparing the phases of these operators under a Z_{2F} symmetry transformation, we see that there is an overall sign difference in the Z_{2F} transformation properties of the two operators. The resolution of this puzzle is mixing of the Z_{2F} symmetry with the color-parity transformation $\mathcal{P}$. The Z_{2F} symmetry of the electric theory is mapped to $\mathcal{P}Z_{2F}$ of the magnetic theory, $Z_{2F} \leftrightarrow \mathcal{P}Z_{2F}$. This takes care of the difference in the sign of the baryon under Z_{2F} transformation, since the baryon operators have all color indices contracted by ϵ-tensors, and the effect of the color-parity transformation is to flip the sign of one particular color.

To see the mapping of discrete symmetries more precisely, we have to separate the cases when N and F are even or odd. If both N and F are even, the electric theory has a $Z_{2F} \times \mathcal{P}$ discrete symmetry. Since in this case $\tilde{N}$ and F are both even as well ($\tilde{N} = F - N + 4$, the size of the dual gauge group), the magnetic theory also has a $Z_{2F} \times \mathcal{P}$ discrete symmetry, and the mapping of the discrete symmetries is as above: $Z_{2F} \leftrightarrow \mathcal{P}Z_{2F}$, while $\mathcal{P} \leftrightarrow \mathcal{P}$. If N is even and F is odd the electric theory has only the $\mathcal{P}$ symmetry, while $\tilde{N}$ and F in the magnetic theory are both odd, and thus the magnetic theory has only a Z_2 discrete symmetry, and the mapping is given by $\mathcal{P} \leftrightarrow Z_2$. If N is odd and F is even, the electric theory has a Z_{2F} symmetry. Since $\tilde{N}$ is odd and F is even in the magnetic theory, the magnetic theory also has a Z_{2F} symmetry, with the mapping $Z_{2F} \leftrightarrow Z_{2F}$ (the generator $\omega = e^{2\pi i/2F}$ of Z_{2F} is, however, mapped to $-\omega$). Finally, for both N and F odd, the electric theory has a Z_2 symmetry. Since $\tilde{N}$ is even and F is odd, the magnetic theory has the $\mathcal{P}$ color-parity symmetry, and the mapping of symmetries is given by $Z_2 \leftrightarrow \mathcal{P}$. Therefore, we find that the discrete global

[‡]For $F = N - 1$, there is an additional $W = \det M$ term in the magnetic superpotential. The presence of this extra term, however, does not affect anomaly matching.

symmetries of the electric and the magnetic theories match for every possible combination of parities of N and F, and the mapping of the discrete symmetries is given above.

In the following, we show that the anomalies involving the Z_{2F} discrete symmetries match using the above mapping of the discrete symmetries. Regardless of whether F and N are even or odd, we will calculate the anomalies for the full Z_{2F} group, and use the same mapping of symmetries. For some particular cases this symmetry may involve a piece for which anomaly matching follows from the continuous anomalies, but we never lose any information by considering the bigger group.

The effect of the color-parity transformation in the mapping of the Z_{2F} symmetry has to be taken into account when one compares the discrete anomalies of the electric and the magnetic theories. This can be done by adding an extra Z_{2F} charge F to every field that carries the first magnetic color: q^1 has Z_{2F} charge $F-1$, and the gluinos $\tilde{\lambda}^{1i}$ $(i \neq 1)$ have Z_{2F} charge F. With this knowledge at hand, we can check the discrete anomaly matching conditions. In the following list of anomalies, we write contributions to the anomalies in the magnetic theory in the order of q^1, q^i $(i \neq 1)$, M and $\tilde{\lambda}^{1i}$.

	Electric theory	Magnetic theory
$SU(F)^2 Z_{2F}$	N	$(F-1)-(F-N+3)+$ $2(F+2)=N+2F$
$Z_{2F}(\text{gravity})^2$	NF	$F(F-1)-F(F-N+3)+$ $F(F+1)+(F-N+3)F=$ $2F^2$
Z_{2F}^3	NF	$F(F-1)^3-F(F-N+3)+$ $4F(F+1)+F^3(F-N+3)=$ $NF-NF^3 \bmod 2F$
$U(1)_R^2 Z_{2F}$	$NF(N-2)^2=$ $N^3F \bmod 2F$	$F(N-2-F)^2(F-1)-$ $(F-N+3)F(N-2-F)^2+$ $F(F+1)(F-2N+4)^2+$ $F^3(F-N+3)=$ $N^3F \bmod 2F$

$U(1)_R Z_{2F}^2$	$-NF(N-2) =$ $-N^2F \bmod 2F$	$F(N-2-F)(F-1)^2 +$ $F(F-N+3)(N-2-F) +$ $2F(F+1)(F-2N+4) +$ $(F-N+3)F^3 =$ $-N^2F \bmod 2F$

The $SU(F)^2 Z_{2F}$, $U(1)_R^2 Z_{2F}$ and $U(1)_R Z_{2F}^2$ anomalies obviously match mod $2F$. The $Z_{2F}(\text{gravity})^2$ anomalies match only mod F for odd N, which signals that the difference of the number of massive Majorana fermions with Z_{2F} charge F in the electric and magnetic theories is odd. This is confirmed by the fact that there is a term $-NF^3$ appearing in the Z_{2F}^3 anomalies, which is of the form $m(\frac{2F}{2})^3$, and can again be attributed to the decoupling of Majorana particles with Z_{2F} charge F. Thus these anomalies obey the discrete anomaly matching conditions as well. Therefore, all the discrete anomaly matching conditions are satisfied in a rather non-trivial way. Note that we chose $U(1)_R$ charges that are F-times the charges in Table 5.1 for the anomalies involving $U(1)_R$, in order to obtain integer $U(1)_R$ charge assignments for all fields.

5.2 $F = N - 2$

In the case of $F = N - 2$, the theory is in the Abelian Coulomb phase, with F pairs of magnetic monopoles becoming massless at the origin [3]. Thus the field content of the low-energy theory is given by

$$\begin{array}{c|cccc} & U(1) & SU(N-2) & U(1)_R & Z_{2N-4} \\ \hline M & 0 & \square\!\square & 0 & 2 \\ q^+ & 1 & \overline{\square} & 1 & -1 \\ q^- & -1 & \overline{\square} & 1 & -1 \end{array}, \qquad (5.3)$$

and a superpotential $W = Mq^+q^-$. There are no baryons in either the high-energy or the low-energy theory, while the "exotic baryon" $W_\alpha Q^{N-2}$ of the original $SO(N)$ theory is mapped to the photon $\tilde{W}_\alpha$ of the low-energy $U(1)$ theory [3]. Since there is again a sign difference in the Z_{2N-4} transformation properties of these two operators, the Z_{2N-4} symmetry of the $SO(N)$ theory is mapped to $\mathcal{P}Z_{2N-4}$ in the $U(1)$ theory. This can be taken into account by adding $N-2$ to the Z_{2N-4} charge of one of the monopoles (say $q^1 = i(q^+ - q^-)/\sqrt{2}$ which corresponds to the first $SO(2)$ color) and of the photino $\tilde{\lambda}$. Now we can calculate the discrete anomalies in both the high-energy $SO(N)$ theory and the low-energy $U(1)$ description. In the following list of anomalies, we write contributions to the anomalies in the magnetic theory in the order of q^1, q^2, M and $\tilde{\lambda}$.

	UV	IR
$SU(N-2)^2 Z_{2N-4}$	N	$(N-3)-1+2N=3N-4$
$Z_{2N-4}(\text{gravity})^2$	$N(N-2)$	$(N-2)(N-3)-(N-2)+$ $(N-2)(N-1)+(N-2)=$ $(N-2)(2N-4)$
Z_{2N-4}^3	$N(N-2)$	$(N-2)(N-3)^3-(N-2)+$ $4(N-2)(N-1)+(N-2)^3=$ $N(N-2)^3+N(N-2) \bmod 2N-4$
$U(1)_R^2 Z_{2N-4}$	$N(N-2)^3$	$(N-2)(N-1)+(N-2)=$ $N(N-2)$
$U(1)_R Z_{2N-4}^2$	$-N(N-2)^2$	$-2(N-2)(N-1)+(N-2)^2=$ $(N-2)^2 \bmod 2N-4$

The $SU(N-2)^2 Z_{2N-4}$ anomalies obviously match mod $2N-4$. The $Z_{2N-4}(\text{gravity})^2$ anomalies match only modulo $N-2$ for odd N, which signals the presence of massive Majorana fermions with charge $N-2$. The difference in the Z_{2N-4}^3 anomalies is $N(N-2)^3$ modulo $2N-4$. The $(N-2)^3$ term is due to the presence of the massive Majorana fermions. The difference in the $U(1)_R^2 Z_{2N-4}$ anomalies is $(N-2)N[(N-2)^2-1]$, which is a multiple of $2N-4$ since $N[(N-2)^2-1]$ is even. Similarly, the difference $(N-2)^2(1+N)$ in the $U(1)_R Z_{2N-4}^2$ anomalies is a multiple of $2N-4$ since $(N-2)(1+N)$ is even. Thus all discrete anomaly matching conditions for the $F=N-2$ case are satisfied.

5.3 $F = N-3$

For $F=N-3$, there are two branches of the theory [3]. On one branch, there is a dynamically generated superpotential and we do not expect anomaly matching. On the other branch, however, the theory close to the origin is described by massless gauge singlet composites M and b, whose global symmetry properties are

$$\begin{array}{c|ccc} & SU(N-3) & U(1)_R & Z_{2N-6} \\ \hline M & \square\square & \frac{-2}{N-3} & 2 \\ b & \overline{\square} & 1+\frac{1}{N-3} & N-4 \end{array} \tag{5.4}$$

and there is a superpotential $W = Mb^2$ in the low-energy theory. The field b can be identified with the exotic baryon $W_\alpha W^\alpha Q^{N-4}$ of the original $SO(N)$ theory. The Z_{2N-6} charges of M and b have been chosen such that this mapping is exactly obeyed. In the following list of anomalies, we write contributions to the anomalies in the magnetic theory in the order of M and b. The discrete anomalies are:

	UV	IR
$SU(N-3)^2 Z_{2N-6}$	N	$2(N-1)+(N-4)=3N-6$
$Z_{2N-6}(\text{gravity})^2$	$N(N-3)$	$(N-2)(N-3)+(N-4)(N-3)=$ $(N-3)(2N-6)$
Z_{2N-6}^3	$N(N-3)$	$4(N-2)(N-3)+$ $(N-3)(N-4)^3=$ $(N-3)(N-4)^3 \bmod 2N-6$
$U(1)_R^2 Z_{2N-6}$	$N(N-2)^2(N-3)$	$(1-N)^2(N-3)(N-2)+$ $(N-3)(N-4)=$ $(N-3)(N-4) \bmod 2N-6$
$U(1)_R Z_{2N-6}^2$	$-N(N-2)(N-3)$	$2(N-2)(N-3)(1-N)+$ $(N-3)(N-4)^2=$ $(N-3)(N-4)^2 \bmod 2N-6$

The $SU(N-3)^2 Z_{2N-6}$ anomaly is obviously matched modulo $2N-6$. The $Z_{2N-6}(\text{gravity})^2$ anomaly is matched modulo $2N-6$ if N is even and modulo $N-3$ if N is odd. Thus in the odd N case we again have massive Majorana particles carrying half of the total Z_{2N-6} charge. However such particles will not affect the Z_{2N-6}^3 anomalies, since for odd N the term $(2N-6)^3/8$ is a multiple of $2N-6$. As expected from this argument, the Z_{2N-6}^3 anomalies match modulo $2N-6$, without extra cubic contributions. Finally, the difference $(N-3)[(N-4)-N(N-2)^2]$ in the $U(1)_R^2 Z_{2N-6}$ anomalies and $(N-3)[(N-4)^2+N(N-2)]$ for the $U(1)_R Z_{2N-6}^2$ anomalies are both multiples of $2N-6$. Thus we have again found that all discrete anomalies are matched in the $F=N-3$ case.

5.4 $F = N - 4$

For $F = N - 4$, there are again two branches [3]. On one branch there is a dynamically generated superpotential and this branch is not of our interest. On the other branch there is a moduli space of quantum vacua described by the meson field M and no superpotential. The global symmetry properties of M are given by

$$\begin{array}{c|ccc} & SU(N-4) & U(1)_R & Z_{2N-8} \\ \hline M & \square\square & -\frac{4}{N-4} & 2 \end{array}. \tag{5.5}$$

While all the continuous anomalies are matched by M, the anomalies involving the discrete Z_{2N-8} symmetries are not matched. For example, the $SU(N-4)^2 Z_{2N-8}$ anomaly is N in the original $SO(N)$ theory and $2N - 4$ in the infrared; the difference is not a multiple of $2N - 8$.

The resolution to this puzzle is that the discrete Z_{2N-8} symmetry is actually spontaneously broken to Z_{N-4}. To understand this breaking of the Z_{2N-8} symmetry, we have to consider the details of the dynamics of the theory. On a generic point in the moduli space, where all $N - 4$ Q's have expectation values, the $SO(N)$ gauge group is broken to $SO(4) \sim SU(2)_L \times SU(2)_R$. The matching of the scales is given by $\Lambda_L^6 = \Lambda_R^6 = \Lambda^{2(N-1)}/(\det M)$. Then gaugino condensation occurs in both $SU(2)$ gauge groups, producing the superpotential

$$\frac{1}{2}(\epsilon_L + \epsilon_R) \left(\frac{16\Lambda^{2(N-1)}}{\det M} \right)^{\frac{1}{2}}, \tag{5.6}$$

where $\epsilon_{L,R}$ are ± 1. The branch with $\epsilon_L \epsilon_R = 1$ corresponds to the theory with a dynamical superpotential. The branch with $\epsilon_L \epsilon_R = -1$ produces the theory with no superpotential and with a moduli space of vacua. Even though there is no superpotential generated, the existence of a gaugino condensate already suggests that some of the global symmetries might be spontaneously broken. A pure super Yang-Mills theory has only discrete symmetries, so we expect that the broken symmetry is only the Z_{2N-8} discrete symmetry.

To see the spontaneous breakdown of the discrete symmetry explicitly, we have to identify the symmetry properties of the $SU(2)$ gauginos. The glueball field $S = W_\alpha^{SU(2)} W^{\alpha, SU(2)}$ can be identified with the exotic composite baryon $S \leftrightarrow W_\alpha W^\alpha Q^{N-4}$ of the $SO(N)$ theory. The transformation properties of this operator under the global symmetries are:

$$\begin{array}{c|ccc} & SU(N-4) & U(1)_R & Z_{2N-8} \\ \hline S & 1 & 0 & N-4 \end{array} \tag{5.7}$$

Thus one can see that the effect of the expectation value to S is to leave all continuous global symmetries unbroken, but to break the discrete Z_{2N-8} symmetry to its Z_{N-4} subgroup. Therefore in this case, one only has to check the anomaly matching conditions with respect to this Z_{N-4} discrete group. These anomaly matching conditions, however, are all automatically satisfied, since this Z_{N-4} group can be identified with the center of the global $SU(N-4)$ symmetry whose anomaly matching is already checked. An explicit calculation of the anomalies confirms this result.

A method to verify the expectation value of the S field is to first add another flavor and decouple it with a mass term. On the branch in the $F = N - 3$ theory without a runaway superpotential, the superpotential is given by $W = M^{ij} b_i b_j - m M^{N-3,N-3}$. The equation of motion for $M^{N-3,N-3}$ gives $b_{N-3} = \sqrt{m}$. Recall that $b_{N-3} = W_\alpha W^\alpha Q^{N-4}$ and is nothing but the field S above.

To summarize this section, we have shown that the Seiberg results on $N = 1$ supersymmetric gauge theories all satisfy the discrete anomaly matching conditions of the previous section. In the case of SU and Sp theory this is not new, since all the discrete symmetries are subgroups of the continuous symmetries, and the anomaly matching conditions follow from those of the continuous symmetries. However for $SO(N)$ groups, the discrete symmetries are not all contained in the continuous global symmetries. We have seen that the anomaly matching conditions are rather non-trivial, and give us further confidence in both Seiberg's results as well as in the method of discrete anomaly matching described in the previous section.

6 More $N = 1$ Supersymmetric Examples

In this section we present several other examples of discrete anomaly matching conditions for $N = 1$ supersymmetric theories. First we present two s-confining SO theories [11]. In both of these examples, the origin of the discrete symmetry is the higher Dynkin index of the representation.

Then we present an $SU(6)$ example with a three-index antisymmetric tensor [11]. If there are three additional flavors of $SU(6)$ fundamentals present in the theory, the $SU(6)$ theory confines with a quantum deformed moduli space which breaks the global continuous symmetries spontaneously. On one point of the moduli space, the global continuous symmetries leave an unbroken Z_{12} discrete symmetry, and we show that the matching of this symmetry is satisfied. If there are no flavors present, the theory is claimed to have multiple branches [11], with one branch having a dynamical superpotential while the other branch with a moduli space of vacua. The matching of discrete anomalies does not appear to work on the moduli space. We show how this second branch arises and also show that the discrete Z_6 global symmetry is actually spontaneously broken to Z_2, and hence the puzzle is resolved.

Next we consider the "ISS-model": $SU(2)$ with a three-index symmetric tensor. This theory was argued to confine and break supersymmetry after an appropriate tree-level superpotential is added [30]. We show that in this theory the discrete anomaly conditions are satisfied as well, giving additional (weak) evidence in favor of the description of Ref. [30].

Finally, we present the most non-trivial example of discrete anomaly matching that we found. It is based on the Kutasov-type duality of Ref. [31] for $SO(N)$ with a symmetric tensor and additional vectors under $SO(N)$. This theory has two different discrete symmetries: one from the explicit breaking of a global $U(1)$ by the tree-level superpotential term, while the other from the breaking of the anomalous $U(1)$ due to instantons. We show how these symmetries are mapped to the dual theory and that all anomaly matching conditions are satisfied.

6.1 S-confining Theories

6.1.1 $SO(9)$ with Four Spinors

The first s-confining example we present is $SO(9)$ with four spinors [11]. Since the Dynkin index of the spinor is four, there is a discrete global Z_{16} symmetry in this theory. The global symmetries of the theory together with the conjectured confined low-energy bound states is given in the table below.

	$SO(9)$	$SU(4)$	$U(1)_R$	Z_{16}
S	16	$\square$	$\frac{1}{8}$	1
S^2		$\square\!\square$	$\frac{1}{4}$	2
S^4		$\boxplus$	$\frac{1}{2}$	4
S^6		$\square\!\square$	$\frac{3}{4}$	6

(6.1)

The anomaly matching conditions are:

	UV	IR
$SU(4)^2 Z_{16}$	16	$2\times 6+4\times 16+6\times 6=112$
$Z_{16}(\text{gravity})^2$	64	$2\times 10+4\times 20+6\times 10=160$
Z_{16}^3	64	$2^3\times 10+4^3\times 20+6^3\times 10=220\times 16$
$U(1)_R^2 Z_{16}$	196×16	140×16
$U(1)_R Z_{16}^2$	-28×16	-196×16

where the contributions to the first three anomalies in the magnetic theory are quoted in the order S^2, S^4, S^6. The $U(1)_R$ charges are multiplied by a factor of 8 to make all the charges integers. One can see that all anomalies match mod 16.

6.1.2 $SO(7)$ with Six Spinors

The global symmetries of the theory and the low-energy confining spectrum is given in the table below [11].

$$\begin{array}{c|c|ccc} & SO(7) & SU(6) & U(1)_R & Z_{12} \\ \hline S & 8 & \square & \frac{1}{6} & 1 \\ \hline\hline S^2 & & \square\square & \frac{1}{3} & 2 \\ S^4 & & \begin{smallmatrix}\square\\\square\end{smallmatrix} & \frac{2}{3} & 4 \end{array} \tag{6.2}$$

The anomaly matching conditions are:

	UV	IR
$SU(6)^2 Z_{12}$	8	$2\times 8+4\times 4=8+2\times 12$
$Z_{12}(\text{gravity})^2$	48	$2\times 12+4\times 15=8\times 12+6$
Z_{12}^3	48	$2^3\times 21+4^3\times 15=94\times 12$
$U(1)_R^2 Z_{12}$	1200	76×12
$U(1)_R Z_{12}^2$	$-5\times 8\times 6$	-68×12

where the contributions to the first three anomalies in the magnetic theory are quoted in the order S^2, S^4. The $U(1)_R$ charges are multiplied by a factor of 6 to make all the charges integers.

All anomalies match mod 12 except the $Z_{12}(\text{gravity})^2$ anomaly, which is matched mod 6, and signals the presence of massive Majorana fermions with charge 6. But we do not see the corresponding contribution to Z_{12}^3 anomaly because $12^3/8 = 216$ is a multiple of 12.

6.2 Quantum Modified Constraint and Moduli Space of Vacua

Next we present two examples using an $SU(6)$ theory with a three-index antisymmetric tensor and fundamental flavors [11]. The first example is $SU(6)$ with $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ and $3(\square+\overline{\square})$. This theory is confining with one quantum modified and one unmodified constraint. The matter fields, global symmetries and the confining spectrum of the theory are:

	$SU(6)$	$SU(3)_Q$	$SU(3)_{\bar{Q}}$	$U(1)_B$	$U(1)_A$	$U(1)_R$
A	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	1	1	0	1	0
Q	$\Box$	$\Box$	1	1	-1	0
$\bar{Q}$	$\bar{\Box}$	1	$\Box$	-1	-1	0
$M_0 = Q\bar{Q}$		$\Box$	$\Box$	0	-2	0
$M_2 = QA^2\bar{Q}$		$\Box$	$\Box$	0	0	0
$B_1 = AQ^3$		1	1	3	-2	0
$\bar{B}_1 = A\bar{Q}^3$		1	1	-3	-2	0
$B_3 = A^3Q^3$		1	1	3	0	0
$\bar{B}_3 = A^3\bar{Q}^3$		1	1	-3	0	0
$T = A^4$		1	1	0	4	0

The superpotential implementing the constraints is

$$\begin{aligned} W &= \lambda\left(B_1\bar{B}_1T + B_3\bar{B}_3 + M_2^3 + TM_2M_0^2 - \Lambda^{12}\right) + \\ &\quad \mu\left(M_2^2M_0 + TM_0^3 + \bar{B}_1B_3 + B_1\bar{B}_3\right), \end{aligned} \tag{6.3}$$

where λ and μ are Lagrange multipliers. The original high-energy theory does not have an interesting discrete symmetry. One might think that there is a Z_6 symmetry rotating only the A field. However, if one in addition to this Z_6 performs a discrete $U(1)_A$ transformation with phase $\pi/3$, one gets a Z_6 transformation which acts only on the $Q, \bar{Q}$ fields, and is just the Z_{2F} symmetry of Section 3.3 which was shown to be contained in the continuous global symmetries as a discrete subgroup. It seems that this theory does not have interesting discrete symmetries. This conclusion is changed by the presence of the quantum modified constraint.

Let us, for example, examine the case when the operators $B_1, \bar{B}_1$ and T acquire expectation values. In this case, the $SU(3) \times SU(3)$ non-Abelian global symmetries as well as $U(1)_R$ are left unbroken by the VEV's, while both $U(1)_A$ and $U(1)_B$ are broken. However, one can combine a discrete Z_6 subgroup of $U(1)_B$

$$B_1 \to e^{2\pi i\frac{3}{6}}B_1, \quad \bar{B}_1 \to e^{-2\pi i\frac{3}{6}}\bar{B}_1, \tag{6.4}$$

with the discrete Z_4 subgroup of $U(1)_A$

$$B_1 \to -B_1, \quad \bar{B}_1 \to -\bar{B}_1, \quad T \to T, \tag{6.5}$$

to find a Z_{12} transformation which leaves $B_1, \bar{B}_1$ and T invariant. Thus there is an unbroken Z_{12} with charges $q_{12} = 12(Q_A/6 + Q_B/4)$. These Z_{12} charges are:

$$A:3, \qquad Q:-1, \qquad \bar{Q}:-5, \qquad B_3:6, \qquad \bar{B}_3, M_0:-6, \tag{6.6}$$

and all other fields have zero Z_{12} charges. Because of the two constraints in the low-energy theory, one has to exclude, for example, the fields B_1 and B_3 from the low-energy spectrum, since the constraints give linear equations for them and are hence not independent degrees of freedom in the low-energy theory. Therefore these contribution of these fields to anomaly matching should not be taken into account. The discrete anomaly matching conditions are:

	UV	IR
$SU(3)^2_Q Z_{12}$	-6	-18
$SU(3)^2_{\bar{Q}} Z_{12}$	-30	-18
$Z_{12}(\text{gravity})^2$	-48	-60
Z_{12}^3	-144×12	-180×12
$U(1)^2_R Z_{12}$	-48	-60
$U(1)_R Z_{12}^2$	-54×12	-30×12

All anomaly matching conditions for Z_{12} are satisfied.

Next we consider the case with no fundamentals, *e.g.* $SU(6)$ with $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$. This theory has a discrete Z_6 symmetry. The low-energy theory has two branches. On one branch there is a dynamically generated superpotential. On the other branch there is a moduli space of vacua described by the VEV of the operator $T = A^4$ and no superpotential:

	$SU(6)$	$U(1)_R$	Z_6
A	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	-1	1
$T = A^4$	1	-4	4

(6.7)

This description matches the $U(1)_R(\text{gravity})^2$ and the $U(1)^3_R$ anomalies. However, checking for example the $Z_6(\text{gravity})^2$ anomalies, we find that they do not match ($Z_6(\text{gravity})^2$ is 20 in the UV and 4 in the IR, and the difference is not divisible by 3). We expect that, analogously to the case of $SO(N)$ with $N-4$ vectors discussed in Section 5.4, the reason for the failure of anomaly matching is the spontaneous breaking of the Z_6 discrete symmetry. In the following, we show that this is indeed the case: Z_6 is spontaneously broken to Z_2. To see this, we start with the $SU(6)$ theory with $\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 4(\Box + \overline{\Box})$. This theory is s-confining, with the confining spectrum [11]

	$SU(6)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_A$	$U(1)_R$
A	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	1	1	0	-4	-1
Q	$\square$	$\square$	1	1	3	1
$\bar{Q}$	$\bar{\square}$	1	$\square$	-1	3	1
$M_0 = Q\bar{Q}$		$\square$	$\square$	0	6	2
$M_2 = QA^2\bar{Q}$		$\square$	$\square$	0	-2	0
$B_1 = AQ^3$		$\bar{\square}$	1	3	5	2
$\bar{B}_1 = A\bar{Q}^3$		1	$\bar{\square}$	-3	5	2
$B_3 = A^3Q^3$		$\bar{\square}$	1	3	-3	0
$\bar{B}_3 = A^3\bar{Q}^3$		1	$\bar{\square}$	-3	-3	0
$T = A^4$		1	1	0	-16	4

and a confining superpotential

$$\begin{aligned} W_{dyn} &= \frac{1}{\Lambda^{11}}\Big(M_0B_1\bar{B}_1T + B_3\bar{B}_3M_0 + M_2^3M_0 + TM_2M_0^3 + \\ &\quad \bar{B}_1B_3M_2 + B_1\bar{B}_3M_2\Big). \end{aligned} \tag{6.8}$$

To obtain the $SU(6)$ theory with no flavors, we add a mass term $m_{ij}M_0^{ij}$ with $\det m \neq 0$ to the superpotential in Eq. (6.8). One can see that the effect of the mM_0 term is to break the global $SU(4) \times SU(4) \times U(1)_B \times U(1)_A \times U(1)_R$ symmetries to $U(1)_R \times Z_6$, because the quantum numbers of m under these global symmetries are $(\bar{\square}, \bar{\square}, 0, -6, 0)$. Examination of the solutions to the equations of motion obtained from the superpotential of (6.8) with the mM_0 mass term shows that there is a branch of solutions with $\langle M_0\rangle = \langle B_1\rangle = \langle \bar{B}_1\rangle = \langle B_3\rangle = \langle \bar{B}_3\rangle = 0$, $\langle T\rangle$ arbitrary and $M_2^3 = \Lambda^{11}m$. This solution with arbitrary value of $\langle T\rangle$ is the branch with the moduli space of vacua. We can see that this branch is characterized by a VEV for the operator M_2, which carries Z_6 charge two, and hence Z_6 is spontaneously broken to Z_2. One can easily check that the discrete anomaly matching conditions involving the Z_2 are all satisfied.

The order parameter M_2 of $Z_6 \to Z_2$ breaking involves extra flavors Q, $\bar{Q}$, which do not exist in the SU(6) theory with a three-index anti-symmetric tensor and no flavors of our interest. However, the expectation value of M_2 corresponds to that of the $A^2W_\alpha W^\alpha$ operator which does not involve heavy flavors. The chiral anomaly equation in the $SU(6)$ theory with the four massive flavors of mass m implies that [29]*

$$m\langle \bar{Q}T^aQ \text{ Tr } T^aA^2\rangle = -\frac{1}{32\pi^2}\langle \text{Tr } T^aW_\alpha W^\alpha \text{ Tr } T^aA^2\rangle = \Lambda^{11/3}m^{4/3} = \Lambda_{LE}^5, \tag{6.9}$$

*This is a supersymmetric generalization of the anomaly equation for gauge-covariant axial currents $\bar{\psi}T^a\gamma^\mu\gamma_5\psi$. A term in the anomaly equation with supercovariant derivatives $\bar{D}^2$ can be dropped because of the translational invariance and supersymmetry of the vacuum.

where T^a's are the $SU(6)$ generators, W_α the field-strength chiral superfield, and Λ_{LE} is the dynamical $SU(6)$ scale after we integrate out the four flavors Q, $\bar{Q}$. The $SU(6)$ group is broken to a pure $SU(3) \times SU(3)$ theory if A is given an expectation value. The glueball field of the pure $SU(3)$ theory can be identified with $A^2 W_\alpha W^\alpha$. The expectation value for M_2 signals that the field $A^2 W_\alpha W^\alpha$ has a non-vanishing VEV, that is gaugino condensation in the $SU(3) \times SU(3)$ theory.

6.3 The ISS Model

Our next example is the ISS model: $SU(2)$ with a three-index symmetric tensor t. It has been conjectured in Ref. [30] that this theory confines without generating a confining superpotential. The basis of this conjecture is that the single independent gauge invariant $X = t^4$ itself satisfies the 't Hooft anomaly matching conditions for the $U(1)_R$ which is the only continuous symmetry of the theory. However, there is also a discrete global symmetry in this theory. The symmetries are

	$SU(2)$	$U(1)_R$	Z_{10}
t	$\square\!\square\!\square$	$-\frac{1}{5}$	1
$X = t^4$	1	$-\frac{4}{5}$	4

(6.10)

The discrete anomaly matching conditions are:

	UV	IR
$Z_{10}(\text{gravity})^2$	4	4
Z_{10}^3	4	64
$U(1)_R^2 Z_{10}$	144	324
$U(1)_R Z_{10}^2$	-24	-144

All discrete Z_{10} anomaly matching conditions are satisfied mod 10. This seems to be a strong argument in favor of this theory. This is however not the case, because we will argue that most of the anomaly matching constraints follow from the anomaly matching for the $U(1)_R$. The reason is that one can combine the Z_{10} transformation with a discrete $U(1)_R$ transformation to get a Z_2 R-symmetry which together with the $U(1)_R$ symmetry is equivalent to the $U(1)_R \times Z_{10}$. Thus the non-trivial part is only a Z_2, under which the fermionic component of t, X and the $SU(2)$ gauginos switch sign. However, in the case of a Z_2 symmetry, neither the $Z_2(\text{gravity})^2$ nor the Z_2^3 anomalies yield any anomaly matching constraints because of the possible $N/2 = 1$ and $N^3/8 = 1$ terms in the matching equations. The only non-trivial piece of information is the correlation between the $Z_2(\text{gravity})^2$ and the Z_2^3 anomalies. If there is a contribution from a massive Majorana fermion with charge 1 to the $Z_2(\text{gravity})^2$ anomalies, there must a contribution to the Z_2^3 anomalies as well. Thus assuming that charge fractionalization does not occur in the heavy spectrum, either both $Z_2(\text{gravity})^2$ and Z_2^3 have to match or neither of them. But even this correlation is trivial, since in a Z_2 symmetry one can have only charges 0 or 1, which have the same contribution to $Z_2(\text{gravity})^2$ and Z_2^3. Thus one does not gain any information whatsoever from these two anomalies. The only non-trivial ones are the $U(1)_R^2 Z_2$ and the $U(1)_R Z_2^2$, both of which are Type II and thus even if they do not match we could not completely exclude the conjectured spectrum. These anomalies are:

	UV	IR
$U(1)_R^2 Z_2$	-37	-81
$U(1)_R Z_2^2$	-7	-9

Both anomalies are matched mod 2. We conclude that the ISS model can not be excluded using discrete anomaly matching, which gives a weak additional evidence for the conjecture of Ref. [30].

6.4 Kutasov-type Duality

Our final $N = 1$ supersymmetric example in this section is the Kutasov-type [5, 6] dual of $SO(N)$ with a traceless symmetric tensor and F vectors and a tree-level superpotential for the symmetric tensor. This theory has been first studied by Intriligator [31]. The reason we chose this theory is that it has two separate discrete symmetries. One discrete symmetry arises from the presence of the tree-level superpotential which explicitly breaks a global $U(1)$ symmetry to its discrete subgroup. The other source for a discrete symmetry is that we have an SO theory with vectors and thus there is a discrete Z_{2F} symmetry present. This theory will be an example for extremely non-trivial matching of discrete anomalies, including even mixed discrete anomalies.

The field content and the symmetries of the electric theory are:

	$SO(N)$	$SU(F)$	$U(1)_R$	Z_{2F}	$Z_{(k+1)F}$
X	$\square\!\square$	1	$\frac{2}{k+1}$	0	F
Q	$\square$	$\square$	$1 - \frac{2(N-2k)}{(k+1)F}$	1	$-(N+2)$

(6.11)

The superpotential of the electric theory is

$$W_{el} = \mathrm{Tr}\, X^{k+1}. \tag{6.12}$$

The field content and the symmetries of the dual magnetic theory is given by [31]

$$\begin{array}{c|c|cccc} & SO(\tilde{N}) & SU(F) & U(1)_R & Z_{2F} & Z_{(k+1)F} \\ \hline q & \square & \bar{\square} & 1-\frac{2(N-2k)}{(k+1)F} & -1 & N+2-kF \\ Y & \square\!\square & 1 & \frac{2}{k+1} & 0 & F \\ M_j & 1 & \square\!\square & \frac{2(j+k)}{k+1}-4\frac{N-2k}{(k+1)F} & 2 & (j-1)F-2(N+2) \end{array}, \tag{6.13}$$

where $\tilde{N} = k(F+4) - N$, $j = 1, 2, \ldots, k$ and the superpotential of the magnetic theory is

$$W_{magn} = \mathrm{Tr}\, Y^{k+1} + \sum_{j=1}^{k} M_j Y^{k-j} q^2. \tag{6.14}$$

The fields M_j match the gauge-invariant polynomials $X^{j-1}Q^2$ in the electric theory. The discrete charges in the magnetic theory have been assigned such that the discrete symmetries are anomaly free, the magnetic superpotential is invariant under the discrete symmetries and the gauge singlets M_j in the magnetic theory have the same charge as $X^{j-1}Q^2$ of the electric theory. As described in Section 3, the electric $SO(N)$ theory also has a $\mathcal{P}$ outer automorphism if N is even. Furthermore, depending on whether F and $(k+1)$ are even or odd, some or all of the discrete symmetries may be contained in the continuous global symmetries. Table 2 shows that the non-trivial discrete symmetries are always in one to one correspondence between the electric and the magnetic theories, and also gives the mapping of the discrete symmetries which is determined by comparing the baryon type $B_p^{(n_1,\ldots,n_k)} = W_\alpha^p Q_{(1)}^{n_1} \ldots Q_{(k)}^{n_k}$ operators with their magnetic analog $Y^{(k-1)(k-p)} \tilde{B}_{\tilde{p}}^{(\tilde{n}_1,\ldots,\tilde{n}_k)}$, where $Q_{(i)} = X^i Q$, $\tilde{p} = 2k - p$, and $\tilde{n}_l = F - n_{k+1-l}$ (for details of this mapping, see Ref. [31]). The values of N, F and $(k+1)$ in the Table 2 stand for N mod 2, F mod 2 and $(k+1)$ mod 2. In Table 2 we have used that for F odd Z_{2F} is equivalent to $Z_2 \times Z_F$, where the Z_F factor can be identified with the center of $SU(F)$. Furthermore, if $(k+1)$ and F are relatively prime, $Z_{(k+1)F}$ is equivalent to $Z_{k+1} \times Z_F$, and the Z_F factor can again be identified with the center of the $SU(F)$ symmetry. Since k and F are arbitrary integers in this theory, however, we quote the discrete symmetries for generic k and F in Table 2. Note that in the case when N is even and F, $(k+1)$ are odd (the fifth row in Table 2), the mapping of the Z_{2F} symmetries is non-trivial: the generator ω is mapped to $-\omega$.

From the point of view of anomaly matching, we do not have to check the discrete anomalies individually for every separate case. We will check the anomaly matching conditions for the full $Z_{2F} \times Z_{(k+1)F}$ symmetries, for any value of N, F and k. In some cases, part of these discrete symmetries is already contained in the continuous global symmetries;

N	F	$(k+1)$	Electric	Magnetic	Mapping
0	0	0	$\mathcal{P}\times Z_{2F}\times Z_{(k+1)F}$	$\mathcal{P}\times Z_{2F}\times Z_{(k+1)F}$	$\mathcal{P}\leftrightarrow\mathcal{P}, Z_{2F}\leftrightarrow\mathcal{P}Z_{2F},$ $Z_{(k+1)F}\leftrightarrow Z_{(k+1)F}$
0	0	1	$\mathcal{P}\times Z_{2F}\times Z_{(k+1)F}$	$\mathcal{P}\times Z_{2F}\times Z_{(k+1)F}$	$\mathcal{P}\leftrightarrow\mathcal{P}, Z_{2F}\leftrightarrow Z_{2F},$ $Z_{(k+1)}\leftrightarrow Z_{(k+1)F}$
0	1	0	$\mathcal{P}\times Z_{(k+1)F}$	$Z_2\times Z_{(k+1)F}$	$\mathcal{P}\leftrightarrow Z_2, Z_{(k+1)F}\leftrightarrow\mathcal{P}Z_{(k+1)F}$
0	1	1	$\mathcal{P}\times Z_{(k+1)F}$	$\mathcal{P}\times Z_{(k+1)F}$	$\mathcal{P}\leftrightarrow\mathcal{P}, Z_{(k+1)F}\leftrightarrow Z_{(k+1)F}$
1	0	0	$Z_{2F}\times Z_{(k+1)F}$	$Z_{2F}\times Z_{(k+1)F}$	$Z_{2F}\leftrightarrow Z_{2F},$ $Z_{(k+1)F}\leftrightarrow Z_{(k+1)F}$
1	0	1	$Z_{2F}\times Z_{(k+1)F}$	$Z_{2F}\times Z_{(k+1)F}$	$Z_{2F}\leftrightarrow Z_{2F},$ $Z_{(k+1)F}\leftrightarrow Z_{(k+1)F}$
1	1	0	$Z_2\times Z_{(k+1)F}$	$\mathcal{P}\times Z_{(k+1)F}$	$Z_2\leftrightarrow\mathcal{P}, Z_{(k+1)F}\leftrightarrow\mathcal{P}Z_{(k+1)F}$
1	1	1	$Z_2\times Z_{(k+1)F}$	$Z_2\times Z_{(k+1)F}$	$Z_2\leftrightarrow Z_2, Z_{(k+1)F}\leftrightarrow Z_{(k+1)F}$

Table 2: The mapping of discrete symmetries in the Kutasov-type duality of Ref. [31] depending on the values of N, F and k.

anomaly matching for that piece should follow from anomaly matching for the continuous symmetries, but it must still be satisfied. Thus we do not loose any information by checking anomaly matching for the bigger group. The effect of the mixing of the discrete symmetries with the color-parity transformation can be taken into account by adding Z_{2F} charge kF to every field carrying the first $SO(\tilde{N})$ color and $Z_{(k+1)F}$ charge $k(k+1)F^2/2$ to the same fields. The charge assignments for fermion fields used for checking the anomaly matching are given in Table 3.

Since the expressions for the anomalies are sometimes quite lengthy, we quote only the simplified forms of the differences between the anomalies of the magnetic and the electric theories. If a difference is given mod $2F$ or mod $(k+1)F$, the expression is given after removing terms that are manifestly multiples of $2F$ or $(k+1)F$ and thus are irrelevant to the anomaly matching conditions.

	$\text{Anomalies}_{electric} - \text{Anomalies}_{magnetic}$
$SU(F)^2 Z_{2F}$	$-2Fk$
$Z_{2F}(\text{gravity})^2$	$(5-F)Fk - 2F(4k^2+Fk^2-kN)$

One can see that the $Z_{2F}(\text{gravity})^2$ anomaly matches mod $2F$ only if k is even or if F is odd. If k is odd and F is even, there must be a massive Majorana fermion with Z_{2F} charge F.

	$SO(N,\tilde{N})$	$SU(F)$	$(k+1)F(R-1)$	Z_{2F}	$Z_{(k+1)F}$
X	$\frac{N(N+1)}{2}-1$	1	$-(k-1)F$	0	F
Q	N	$\square$	$-2(N-2k)$	1	$-(N+2)$
q^1	1	$\overline{\square}$	$-2(\tilde{N}-2k)$	$kF-1$	$N+2-kF+\frac{k(k+1)F^2}{2}$
q^i	$\tilde{N}-1$	$\overline{\square}$	$-2(\tilde{N}-2k)$	-1	$N+2-kF$
Y^{1i}	$\tilde{N}-1$	1	$-(k-1)F$	kF	$F+\frac{k(k+1)F^2}{2}$
Y^{ij}	$\frac{\tilde{N}(\tilde{N}-1)}{2}$	1	$-(k-1)F$	0	F
M_j	1	$\square\square$	$F(2j+k-1)-4(N-2k)$	2	$(j-1)F-2(N+2)$
$\tilde{\lambda}^{1i}$	$\tilde{N}-1$	1	$(k+1)F$	kF	$\frac{k(k+1)F^2}{2}$

Table 3: The charge assignments used for anomaly matching in the Kutasov-type $SO(N)$ duality with a symmetric tensor. The R-charges are for the fermionic components and are normalized in such a way that all fields carry integer charges, and $\tilde{\lambda}$ is the gaugino in the dual theory.

Since $2F$ is divisible by four in this case, we do not expect to see the effect of this fermion in the Z_{2F}^3 anomaly.

$$Z_{2F}^3 \qquad -F^3k^2(Fk-3) \bmod 2F$$

This difference is as expected always a multiple of $2F$. This is obvious if F or k is even. However, if both of them are odd then $Fk-3$ is even and the above expression is again a multiple of $2F$.

$$U(1)_R^2 Z_{2F} \qquad \frac{F^3(1+F)(1-k)k(1+k)}{3} \bmod 2F$$

This is always a multiple of $2F$ since $(k-1)k(k+1)$ contains at least one number divisible by three and one divisible by two.

$$U(1)_R Z_{2F}^2 \qquad -4F^2k\Big[4k+Fk+Fk^2-2N\Big]$$

This is obviously a multiple of $2F$.

$$SU(F)^2 Z_{(k+1)F} \qquad 3Fk(1+k)$$

This is obviously a multiple of $(k+1)F$.

$$Z_{(k+1)F}(\text{gravity})^2 \qquad -F(1+k)\frac{F(F-3)k}{4} \bmod (k+1)F$$

If k is even, then the anomalies are matched mod $(k+1)F$, since $F(F-3)$ is always even. If k is odd and F is 0 or 3 mod 4, then the anomalies are still matched mod $(k+1)F$. However if k is odd and F is 1 or 2 mod 4, then the anomalies are matched only mod $(k+1)F/2$, which signals the presence of odd number of massive Majorana fermions with charge $(k+1)F/2$. We expect to see the effect of these fermions in the $Z^3_{(k+1)F}$ anomaly for odd k and $F = 1$ mod 4; for odd k and $F = 2$ mod 4, $(k+1)F$ is divisible by four and hence $((k+1)F)^3/8$ term cannot be distinguished from mod $(k+1)F$ freedom.

$$Z^3_{(k+1)F} \qquad \begin{array}{l} \frac{F^3k(1+k)}{8}\big[61Fk + 23F^2k - 31Fk^2 + 35F^2k^2 - 2F^3k^2 + F^4k^2 \\ +4F^3k^3 + 4F^4k^3 + 14F^3k^4 + 5F^4k^4 + 2F^4k^5 - 76N \\ -2F^3k^2N - 4F^3k^3N - 2F^3k^4N - 12N^2\big] \bmod (k+1)F \end{array}$$

One can show that this expression indeed satisfies all the requirements. The terms $-76N - 12N^2$ give a multiple of 8 and can be dropped. Then all terms in the square bracket have a factor of F, and hence the difference is obviously a multiple of $(k+1)F$ if F is even. If $k = 2n$, the difference is simplified to $-\frac{1}{2}F^4n^2(1+2n)(5+7F)(1+N)$ modulo $(k+1)F$ and we again obtain a multiple of $(k+1)F$ because $F(5+7F)$ is even. The non-trivial case is when $F = 2m-1$, $k = 2n-1$. Then the difference reduces to $m((k+1)F)^3/8$ modulo $(k+1)F$; thus as expected, one can see the presence of the massive Majorana fermions only for odd k and $F = 1$ mod 4 (*i.e.*, odd m).

$$U(1)^2_R Z_{(k+1)F} \qquad \begin{array}{l} -\frac{F^2(k+1)}{12}\big[32Fk + 3F^2k + F^3k - 464Fk^3 - 123F^2k^3 - F^3k^3\big] \\ \bmod (k+1)F \end{array}$$

By adding multiples of $(k+1)F$, one can simplify the difference to the form $\frac{F^3}{12}(F-1)(F+4)(k-1)k(k+1)^2$. The last factor $(k-1)k(k+1)$ is a multiple of 6, and $(F-1)(F+4)$ is an even number. Therefore, the difference vanishes mod $(k+1)F$.

$$U(1)_R Z^2_{(k+1)F} \qquad \begin{array}{l} \frac{-F^2(1+k)}{12}\big[32Fk + 3F^2k + F^3k - 176Fk^3 - 123F^2k^3 - F^3k^3\big] \\ \bmod (k+1)F \end{array}$$

One can further simplify the difference to the form $\frac{F^3}{12}(F-1)(F+4)(k-1)k(k+1)^2$ by adding multiples of $(k+1)F$. This result is the same as the $U(1)_R^2 Z_{(k+1)F}$ and hence the anomalies match here as well.

For the mixed discrete anomalies $Z^2_{(k+1)F}Z_{2F}$, $Z_{(k+1)F}Z^2_{2F}$, and $U(1)_R Z_{(k+1)F}Z_{2F}$, either the charges or the multiplicity for both electric and magnetic degrees of freedom have a factor of F. If k is even, the greatest common divisor of $(k+1)F$ and $2F$ is F, and the anomalies need to be matched only mod F; therefore their anomaly matching is trivial for even k. If k is odd, then $(k+1)F$ is divisible by $2F$, thus the greatest common divisor of $2F$ and $(k+1)F$ is $2F$. Therefore we would like to show that the differences in anomalies are multiples of $2F$ for odd $k = 2n-1$.

$$Z^2_{(k+1)F}Z_{2F} \qquad \begin{aligned} &\frac{-F^2k}{12}\Big[14F + 2F^2 - 30Fk + 30F^2k - 3F^3k - 44Fk^2 + 76F^2k^2 \\ &-12F^3k^2 + 3F^4k^2 + 9F^3k^3 + 12F^4k^3 + 42F^3k^4 + 15F^4k^4 + 6F^4k^5 \\ &+168N - 6F^3k^2N - 12F^3k^3N - 6F^3k^4N\Big] \bmod 2F \end{aligned}$$

Substituting $k = 2n-1$ and leaving out multiples of $2F$, the difference becomes $\frac{1}{3}F^3(1+F)n(2n-1)(1+2n-3F^2n)$. The factor $F(1+F)$ is even, and hence the term $-3F^2n$ can be dropped modulo $2F$. Then the last three factors give $n(2n-1)(2n+1)$ which is a multiple of 3, thus the difference vanishes modulo $2F$.

$$Z_{(k+1)F}Z^2_{2F} \qquad \begin{aligned} &\frac{-F^2k}{2}\Big[-22 - F - 6k + 7Fk - 2F^2k + 8Fk^2 - 4F^2k^2 + F^3k^2 \\ &+6F^2k^3 + 3F^3k^3 + 8F^2k^4 + 2F^3k^4 - 8N - 2F^2k^2N - 2F^2k^3N\Big] \end{aligned}$$

By substituting $k = 2n-1$, the difference can be simplified to $F^3(9+F^2)n(2n-1) \bmod 2F$, and the factor $n(2n-1)$ is always even; thus the anomaly is matched modulo $2F$.

$$U(1)_R Z_{(k+1)F}Z_{2F} \qquad \begin{aligned} &\frac{F^3k}{6}\Big[1 + F + 66k + 3Fk + 3F^2k + 5k^2 - 7Fk^2 \\ &-9F^2k^2 - 3F^3k^2 - 9Fk^3 - 3F^2k^3 + 9F^2k^4 + 3F^3k^4 - 6N \\ &+30kN - 3FkN + 3F^2kN + 3Fk^3N - 3F^2k^3N\Big] \bmod 2F \end{aligned}$$

Substituting $k = 2n-1$ we get for the difference $\frac{-2F^3}{3}(1+F)(n-1)n(2n-1) \bmod 2F$. The last factor is a multiple of 3 and hence the $U(1)_R Z_{(k+1)F}Z_{2F}$ anomalies match modulo $2F$ as well.

Thus we have seen that this example with two different discrete symmetries have all anomalies matched between the electric and the magnetic theories in a highly non-trivial manner. Note that all Type II anomaly matching conditions including the correlation between the $Z_N(\text{gravity})^2$ and Z_N^3 anomalies are satisfied as well.

7 Excluded Models

We have seen several examples of discrete anomaly matching in the previous two sections. In this section we will show examples of theories where the conjectured low-energy spectrum does not satisfy the discrete anomaly matching conditions, which means that the given spectrum can not be the correct low-energy solution of the theory.

In the first example, we will consider the recently suggested chirally symmetric phase of $N = 1$ supersymmetric pure Yang-Mills theory [14]. We will show that the chirally symmetric vacuum described by the natural variable of the Veneziano–Yankielowicz Lagrangian [15] does not satisfy the discrete anomaly matching conditions and thus can be excluded. However, the concept of a chirally symmetric phase can not be completely excluded, since there may be another set of massless states which does satisfy the anomaly matching conditions. The next set of examples will deal with the non-supersymmetric confining examples conjectured in the early 80's [16, 17]. We will show that almost all examples in this category which have a non-trivial discrete symmetry can be excluded based on discrete anomaly matching. Finally, we consider the self-dual $N = 1$ supersymmetric theories based on exceptional and orthogonal groups [18, 19, 20, 32]. We show that the discrete symmetries of the electric and the magnetic theories can not be mapped to each other in the examples based on exceptional groups [18, 19, 20]. However, since both the electric and the magnetic theories are strongly coupled, one can not exclude the presence of accidental symmetries. Thus this category of theories can not be completely excluded based on discrete anomaly matching, but the evidence for duality is made much weaker than it is in other theories. The self-dual theories based on orthogonal groups [32, 20] do satisfy discrete anomaly matching once the maximal number of meson fields is included in the dual theory as elementary fields.

7.1 $N = 1$ Supersymmetric Pure Yang-Mills Theories

These theories do not have any continuous global symmetries. Their only symmetry is a discrete $Z_{\mu(G)}$ R-symmetry, where $\mu(G)$ is the Dynkin index of the adjoint representation of the gauge group G (twice the dual Coxeter number $h^\vee$). The λ_α gaugino (which is the only fermion in these theories) carries one unit of the discrete $Z_{\mu(G)}$ charge.

The canonical description of the low-energy dynamics [35] of this theory is that gaugino condensation occurs,

$$\langle \lambda_\alpha \lambda^\alpha \rangle = \omega_i \Lambda_G^3, \quad i = 1, 2, \ldots, \mu(G)/2, \tag{7.1}$$

where the ω_i's are the $\mu(G)/2$ roots of unity. This gaugino condensate breaks the discrete $Z_{\mu(G)}$ spontaneously to Z_2, and the fields from the vector multiplet W_α form massive bound states. The theory confines with chiral symmetry breaking. One does not get any useful information from a Z_2 discrete symmetry, since the massive Majorana fermions can modify both the $Z_2(\text{gravity})^2$ and the Z_2^3 anomalies by one.

algebra	group	dim	$\mu(G)$	$H^*(G;\mathbf{R})$ generators
A_n	$SU(n+1)$	$n(n+2)$	$2(n+1)$	$3, 5, \cdots, 2n+1$
B_n	$SO(2n+1)$	$n(2n+1)$	$2(2n-1)$	$3, 7, \cdots, 4n-1$
C_n	$Sp(2n)$	$n(2n+1)$	$2(n+1)$	$3, 7, \cdots, 4n-1$
D_n	$SO(2n)$	$n(2n-1)$	$2(2n-2)$	$3, 7, \cdots, 4n-5, 2n-1$
E_6	E_6	78	24	3, 9, 11, 15, 17, 23
E_7	E_7	133	36	3, 11, 15, 19, 23, 27, 35
E_8	E_8	248	60	3, 15, 23, 27, 35, 39, 47, 59
F_4	F_4	52	18	3, 11, 15, 23
G_2	G_2	14	8	3, 11

Table 4: Dimensions, Dynkin index of adjoint representations for semi-simple Lie algebras, and the degrees of forms which generate the cohomology ring of the group manifold.

However, it has been recently suggested by Kovner and Shifman [14] that there might be another branch of the theory on which spontaneous breaking of $Z_{\mu(G)}$ does not occur, but there are massless fermions at the origin. This conclusion has been made in Ref. [14] by examining the vacuum structure of a modified Veneziano–Yankielowicz (VY) Lagrangian [15] which is $Z_{\mu(G)}$ symmetric and reproduces all the Green's functions for the fields of $W_\alpha W^\alpha$. The modified VY Lagrangian suggests that there is a single massless fermion $\Phi = (W_\alpha W^\alpha)^{\frac{1}{3}}$ present in the low-energy theory. If there is indeed such a phase of the theory where $Z_{\mu(G)}$ is not spontaneously broken, the discrete anomaly matching conditions must be satisfied. In the following we show that the discrete anomaly matching conditions are satisfied neither with the field Φ nor the field $S = W_\alpha W^\alpha$ as the only massless composite field in the low-energy theory.

First we assume that the only massless field is $\Phi = (W_\alpha W^\alpha)^{\frac{1}{3}}$ as suggested by the modified VY Lagrangian. In this case, the R-charge of the fermionic component of Φ is $-\frac{1}{3}$, which signals the fractionalization of the $Z_{\mu(G)}$ charges. Therefore, it is convenient to rescale the discrete charges such that the gaugino of the high-energy theory has charge 3, and check the anomaly matching conditions for the resulting $Z_{3\mu(G)}$ symmetry. Values of $\mu(G)$ for semi-simple gauge groups are listed in Table 4.

The discrete anomalies for $G = SU(N)$ are ($\mu(G) = 2N$):

	UV	IR
$Z_{6N}(\text{gravity})^2$	$3(N^2-1)$	-1
Z_{6N}^3	$27(N^2-1)$	-1

The difference in the $Z_{6N}(\text{gravity})^2$ anomalies of the UV and the IR descriptions is 2 mod $3N$, which means that the discrete anomalies can not be matched for any value of N. Recall that the $Z_{6N}(\text{gravity})^2$ anomaly is Type I and must be matched irrespective of charge fractionalization. Therefore, this low-energy description of the pure $SU(N)$ YM theories is excluded.

Next we consider the case of $SO(N)$ groups ($\mu(G) = 2N - 4$). The discrete anomalies are:

	UV	IR
$Z_{6N-12}(\text{gravity})^2$	$3\frac{N(N-1)}{2}$	-1
Z_{6N-12}^3	$27\frac{N(N-1)}{2}$	-1

The difference in the $Z_{6N-12}(\text{gravity})^2$ anomalies of the UV and the IR descriptions is $3\frac{N(N-1)}{2} + 1$ which should be divisible at least by $3(N-2)$. However, $3N^2 - 3N + 2$ is never divisible by $3N - 6$, thus we conclude that anomaly matching is not satisfied for $SO(N)$ theories either.

For $Sp(2N)$ groups $\mu(G) = 2N + 2$. The anomaly matching conditions are:

	UV	IR
$Z_{6N+6}(\text{gravity})^2$	$3N(2N+1)$	-1
Z_{6N+6}^3	$27N(2N+1)$	-1

The difference in the $Z_{6N+6}(\text{gravity})^2$ anomalies is $(N+1)(6N-3)+4$, which is never divisible by $3(N+1)$. Thus the discrete anomaly matching constraints are not satisfied for the $Sp(2N)$ groups either. Furthermore, we have checked that none of the similarly constructed solutions for the exceptional groups $G_2, F_4, E_{6,7,8}$ satisfy the discrete anomaly matching conditions. Note that the $Z_{3\mu(G)}(\text{gravity})^2$ anomalies are Type I and should be satisfied under all circumstances. We conclude that the most natural implementation of a chirally symmetric vacuum of $N = 1$ pure Yang-Mills theories can be excluded based on discrete anomalies.

However, this does not completely exclude the idea of a chirally symmetric phase of $N = 1$ pure Yang-Mills theories. It excludes only a specific realization of it described above. One could, for example, try to match anomalies with the operator $S = W_\alpha W^\alpha$ instead of Φ. Here no charge fractionalization occurs, and hence anomalies should be matched mod $\mu(G)$.

The anomalies for $SU(N)$ are

	UV	IR
$Z_{2N}(\text{gravity})^2$	N^2-1	1
Z_{2N}^3	N^2-1	1

The differences in the anomalies are both N^2-2, which is divisible by N only for $N=1,2$. Performing a similar analysis we find that the field S matches the discrete anomalies for $SO(N)$ only if N is odd, while it matches always for $Sp(2N)$. None of the discrete anomalies for the exceptional groups are matched by S. Even though anomalies are matched for some special cases by S, generically it does not match the discrete anomalies and therefore we conclude that it is not a likely candidate for a low-energy solution.

As emphasized above, the failure of anomaly matching excludes only a particular low-energy spectrum. It is in fact possible to find a set of discrete R-charges that match the anomalies. However, we cannot identify natural interpolating fields as composite operators of the field strength superfield. The following construction is an example for a set of R-charges which satisfy discrete anomaly matching. This construction works for all semi-simple gauge groups.

As clear from the previous discussions, we would like to match $\text{Tr}R = \dim(G)$ modulo $\mu(G)/2$ and $\text{Tr}R^3 = \dim(G)$ modulo $\mu(G)$. A set of useful numbers for semi-simple Lie algebras is given in Table 4. The last column in Table 4 shows the degrees k of the forms which generate the cohomology ring on group manifolds;* they are k-forms which can be written as $\text{Tr}(g^{-1}dg)^k$ with group elements $g \in G$. All other elements of the cohomology ring are given by products of the generators (note that one cannot use the same generator more than once because they are all forms of odd degrees) and their linear combinations. In particular, the volume form is given by the product of all generators and hence the sum of the numbers in the last column must give the dimensions of the groups; this can be checked easily. Therefore, if one has a set of fermions whose R-charges are given by the degrees of cohomology generators, the $Z_{\mu(G)}(\text{gravity})^2$ anomalies are matched exactly.

Curiously enough, the $Z_{\mu(G)}^3$ anomalies are also matched modulo $\mu(G)$ with this set of R-charges. The problem is to find interpolating fields for such R-charges using gauge invariant composite operators of field strength superfield W_α^a with spin 1/2. The operators $\omega_k^{a_1,\ldots,a_k}\lambda_{\alpha_1}^{a_1}\ldots\lambda_{\alpha_k}^{a_k}$, where the ω_k is the cohomology generator of degree k and λ's are the gauginos, have the correct R-charges, but the spinor indices α_i are totally symmetric for these operators and they cannot produce spin 1/2 fermions. Since massless fields with higher spin cannot have consistent interactions, we exclude this choice of operators. If there are

*They coincide with $2e_i+1$ where the e_i's are the exponents of the Lie algebra.

operators which match the required R-charges, they necessarily need to involve derivatives and hence are bound states with higher relative orbital angular momenta. We find such composite spectrum to be highly unlikely.

7.2 Non-supersymmetric Theories

In this section, we examine several non-supersymmetric theories which were conjectured to be confining based on the 't Hooft anomaly matching conditions in the early 80's [16, 17]. We show that most of the examples which have a non-trivial discrete symmetry do not satisfy the discrete anomaly matching conditions and thus one can exclude these conjectured spectra. We briefly comment on the recently proposed duality for non-supersymmetric QCD [33] at the end of the section.

The first example we consider is based on a non-supersymmetric $SU(4)$ theory with two massless left-handed Weyl fermions in the antisymmetric tensor representation of $SU(4)$, and one in the adjoint representation [16]. This theory was conjectured to be confining. The global symmetries and the conjectured confining spectrum is given in the table below.

	$SU(4)$	$SU(2)$	$U(1)$	Z_{12}
A	$\ydiagram{1,1}$	$\square$	2	1
X	$\ydiagram{2,1,1}$	1	-1	1
(A^2X)	1	$\ydiagram{2}$	3	3

(7.2)

All the continuous global anomalies ($SU(2)^2U(1)$, $U(1)(\text{gravity})^2$ and $U(1)^3$) are matched between the high-energy and the confining spectrum. The discrete anomalies are:

	UV	IR
$SU(2)^2Z_{12}$	6	12
$Z_{12}(\text{gravity})^2$	27	9
Z_{12}^3	27	81
$U(1)^2Z_{12}$	63	81
$U(1)Z_{12}^2$	9	81

The $U(1)^2Z_{12}$ anomaly matching is satisfied mod 12 and the $Z_{12}(\text{gravity})^2$ anomaly matching is satisfied mod 6. However, while the $SU(2)^2Z_{12}$, the $U(1)^2Z_{12}$ and the Z_{12}^3

anomalies must match mod 12, they match only mod 6, and hence the discrete anomaly matching conditions are violated. In the absence of any dynamical explanation of spontaneous breaking of Z_{12}, and since $SU(2)^2 Z_{12}$ is a Type I anomaly, one has to consider this model excluded based on discrete anomaly matching.

The next example is an $SU(5)$ theory with the field content and the conjectured confining spectrum to be [16]

	$SU(5)$	$SU(6)$	$U(1)$	Z_{42}
A	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\Box$	4	1
Y	$\overline{\begin{smallmatrix}\Box\Box\\\Box\\\Box\end{smallmatrix}}$	1	-3	1
(A^2Y)	1	$\Box\Box$	5	3

(7.3)

The anomalies with respect to the continuous flavor symmetries are all matched, and so are all discrete anomalies except the $SU(6)^2 Z_{42}$ whose value is 10 in the UV and 24 in the IR, and thus the difference is 14. Since $SU(6)^2 Z_{42}$ is a Type I anomaly, this example is excluded as well.

Finally, we present two examples where all continuous anomalies are matched but almost all of the discrete anomaly matching conditions are violated. The first example is an $SU(9)$ gauge theory with massless fermions and the confining spectrum:

	$SU(9)$	$U(1)$	Z_{42}
A	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	5	1
B	$\overline{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}$	-1	1
$6 \times (A^2B)$	1	9	3

(7.4)

The conjectured spectrum contains six different copies of the (A^2B) bound state. The discrete anomalies are:

	UV	IR
$Z_{42}(\text{gravity})^2$	162	18
Z_{42}^3	162	162
$U(1)^2 Z_{42}$	1026	1458
$U(1) Z_{42}^2$	54	486

One can see that none of the discrete anomaly matching conditions (except the Z_{42}^3) are satisfied, thus this example is excluded as well.

Finally, we consider a theory based on an $SU(3)$ gauge group:

	$SU(3)$	$SU(2)$	$U(1)$	Z_{30}
S	$\ydiagram{2}$	$\square$	2	1
W	$\overline{\ydiagram{3,1}}$	1	-1	1
(S^2W)	1	$\ydiagram{2}$	3	3

(7.5)

The discrete anomalies are:

	UV	IR
$SU(2)^2Z_{30}$	6	12
$Z_{30}(\text{gravity})^2$	27	9
Z_{30}^3	27	81
$U(1)^2Z_{30}$	63	81
$U(1)Z_{30}^2$	9	81

In this example none of the discrete anomaly matching conditions are satisfied. Note that the global symmetries and charge assignments in this theory are exactly equal to those in the $SU(4)$ example at the beginning of this section because the dimensions of the representations and the ratios of the Dynkin indices are the same. However, the values of the Dynkin indices under the gauge groups are different (only their ratios are the same), and thus there is a different discrete symmetry in this theory. Even though the values of the discrete anomalies are exactly equal in the two theories, the discrete anomaly matching conditions are very different.

To close this section, we comment on the dual of non-supersymmetric QCD recently proposed by Terning [33]. It has been suggested that $SU(3)$ with F flavors of left- and right-handed quarks might have a dual in terms of a $G(F-6)$ group with F flavors as well and some composite baryons containing three quarks, and G could be SU, Sp or SO. Since the electric theory is an SU theory which contains fundamentals, it does not have any interesting discrete symmetries. If the dual gauge group is $SU(F-6)$ or $Sp(F-6)$ (for even number of flavors) then the same statement holds for the dual theory. However, if one assumes that $G = SO(F-6)$, then the dual theory does have a Z_{4F} non-trivial discrete symmetry, which can not be mapped to any non-trivial discrete symmetry of the electric theory. The lack of mapping of the discrete global symmetries makes the $SU(3) \leftrightarrow SO(F-6)$ duality much less plausible, even though it can not be completely excluded due to the potential presence of accidental symmetries in the strongly interacting electric theory for $F < 33/2$. However, for $F > 33/2$, the $SU(3)$ theory is infrared free and thus weakly coupled, and accidental symmetries cannot appear. Thus the $SU(3) \leftrightarrow SO(F-6)$ duality is certainly excluded for $F > 33/2$ and implausible for $F < 33/2$. We have to emphasize again, however, that the $SU(3) \leftrightarrow SU(F-6)$ duality is still a valid possibility about which we have nothing new to say.

7.3 Self-dual Theories

The final set of examples we will consider are the $N = 1$ supersymmetric self-dual examples based on certain exceptional groups and SO groups with spinors [18, 19, 20, 32]. (The self-dual theories of Refs. [34, 32] based on SU and Sp groups do not have any non-trivial discrete symmetries and thus one can not gain new information about them).

Let us consider, for example, the self-dual theory of Ref. [18] based on an E_6 gauge group. The conjectured electric and magnetic theories are:

	E_6	$SU(6)$	$U(1)_R$	Z_{36}
Q	27	$\square$	$\frac{1}{3}$	1
q	$\overline{27}$	$\overline{\square}$	$\frac{1}{3}$	-1
Z	1	$\square\square\square$	1	3

(7.6)

with a superpotential in the magnetic theory $W = Zq^3$. The Z_{36} charge of Z has been chosen such that the mapping $Z \leftrightarrow Q^3$ is obeyed, while that of q such that the magnetic superpotential is invariant under Z_{36}. Note that one could add a multiple of 12 to the q charge. The Type I discrete anomalies are:

	UV	IR
$SU(6)^2 Z_{36}$	27	81
$Z_{36}(\text{gravity})^2$	162	6

Neither of these anomaly matching conditions is satisfied. The ambiguity of a multiple of 12 in the Z_{36} charge assignments for q does not help the anomaly matching either.

The failure of anomaly matching could have been actually expected, since the Z_{36} symmetries of the electric and the magnetic theories can not be mapped to each other. This is because the E_6 theory contains at least one more independent flat direction corresponding to Q^6, which is supposedly matched to q^6 of the magnetic theory.† This is however impossible, since Q^6 carries Z_{36} charge 6, while q^6 charge -6. The difference of charges is 12, and there is no way to make up for this charge difference since there is no other non-trivial discrete symmetry in this theory. Thus we have to conclude that this duality does not satisfy the mapping of global symmetries, unless we assume that there are accidental Z_{36} symmetries appearing both in the electric and the magnetic theories (which is not impossible since both theories are strongly coupled). Therefore we conclude that the lack of the matching of discrete global symmetries makes this duality much less plausible even though this self-dual is not completely excluded. One possible way to cure the lack of matching of the discrete Z_{36} symmetries in the above E_6 example is to modify the electric theory by adding a tree-level superpotential $W = Q^6$, and regard this as a Kutasov-type duality.‡ The magnetic superpotential then becomes $W = q^6 + Zq^3$. The additional superpotential terms explicitly break the Z_{36} discrete symmetry to $Z_6 \subset SU(6)$ both in the electric and the magnetic theories (and also break part of the $SU(6)$ global symmetries). This way the constraints arising from the discrete symmetries are eliminated and all the other consistency conditions for this duality are satisfied.

One can show that the same statement holds for every self-dual theory based on exceptional groups presented in Refs. [18, 19, 20], that is without a tree-level superpotential term the mapping of discrete symmetries is not manifest, however after introducing tree-level superpotential terms one obtains consistent Kutasov-type dualities.

There is another set of self-dual theories which have non-trivial discrete symmetries: the SO series of Ref. [32] and an analogous SO series of Ref. [20]. Let us, for example, examine a self-dual theory from the SO series of Ref. [32]. Let us consider the $SO(12)$ theory with one spinor and eight vectors. The dual pair is described by

	$SO(12)$	$SU(8)$	$U(1)$	$U(1)_R$	Z_8
S	32	1	2	$\frac{1}{2}$	1
V	$\Box$	$\Box$	-1	0	0
s	32	1	2	$\frac{1}{2}$	-1
v	$\Box$	$\Box$	-1	0	0
(S^2V^2)	1	$\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$	2	1	2
(S^2V^6)	1	$\overline{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}$	-2	1	2

(7.7)

and a superpotential in the magnetic theory $W = (S^2V^2)s^2v^6 + (S^2V^6)s^2v^2$. One can see that it is possible to assign a Z_8 discrete symmetry in the dual theory such that the superpotential is Z_8 invariant and the gauge singlets (S^2V^2) and (S^2V^6) have the correct Z_8 charges. But this is not enough, since the mapping of all other independent gauge invariants

†A complete classification of gauge invariant polynomials is not known for exceptional groups.

‡We thank Philippe Pouliot for pointing this out to us.

has to preserve Z_8 as well. It turns out that this example does satisfy this additional requirement. The reason is that the additional independent gauge invariants involve only the fourth power of the spinor S, and thus the Z_8 charge of such operators will be ± 4 in the electric and the magnetic theory. Discrete anomalies match between the electric and magnetic theories almost trivially because of the high multiplicity of the fields (32 for the spinor).

One can also check that the other examples in the SO series in Ref. [32] do have the correct mapping of the discrete symmetries once the maximal number of gauge singlet mesons are included into the magnetic degrees of freedom. One way to see this is that most of the other self-dual models of the SO series can be derived from the above $SO(12)$ example by giving expectation values to vectors. The other way to see it is to note that the discrete symmetry is Z_8 in every case, and the gauge invariants contain only two or four powers of the spinor field. If one includes all gauge invariants containing two powers of spinors as elementary fields in the dual theory, the remaining operators with four powers of spinors can be matched by the similar construction as above. However, the self-duals in the SO series of Ref. [32] where not all of the invariants quadratic in spinors are included as elementary fields in the magnetic theory cannot have the required mapping of discrete symmetries. Thus the requirement of discrete anomaly matching favors a single dual rather than multiple duals. The remaining multiple self-duals can be interpreted only as Kutasov-type dualities after adding a tree-level superpotential term S^4 to the electric theory.

Similarly, one can show that the SO series of Ref. [20] satisfy the discrete anomaly matching. The reason is that the highest theory based on the $SO(14)$ group does have the correct mapping of discrete symmetries, and all other examples can be derived from this by giving an expectation value to one vector. To see this, let us investigate the $SO(14)$ example of [20]. The field content of the electric and the dual magnetic theories is given by:

	$SO(14)$	$SU(6)$	$U(1)$	$U(1)_R$	Z_{12}
S	64	1	3	$\frac{1}{7}$	0
V	$\Box$	$\Box$	-4	$\frac{1}{7}$	1
s	64	1	3	$\frac{1}{7}$	0
v	$\Box$	$\Box$	-4	$\frac{1}{7}$	-5
(S^2V^3)	1	$\begin{array}{c}\Box\\\Box\\\Box\end{array}$	-6	$\frac{5}{7}$	3
(S^6V^3)	1	$\begin{array}{c}\Box\\\Box\\\Box\end{array}$	6	$\frac{9}{7}$	3

(7.8)

with a superpotential in the magnetic theory $W = (S^2V^3)s^6v^3 + (S^6V^3)s^2v^3$. The Z_{12} charges of the additional gauge invariants S^4Q^2, S^4Q^4 and S^8Q^4 are also mapped between the electric and the magnetic theories. The only non-trivial discrete anomaly is the $SU(6)^2Z_{12}$ which is 14 in the electric and -70 in the magnetic theories; the difference is a multiple of 12. All other anomalies are multiples of 12 themselves.

In summary, the self-dual theories based on exceptional groups do not have the correct mapping of discrete symmetries between the electric and the magnetic theories, and hence have a much weaker foundation than dualities where one does not have to rely on accidental symmetries. However, they can have consistent interpretation as Kutasov-type dualities, once additional terms are included in the superpotential. The self-dual SO series do have the correct mapping of discrete symmetries and satisfy the discrete anomaly matching conditions, once the maximal number of meson fields is included in the magnetic theory.

8 Conclusions

We have shown that any conjectured low-energy bound state spectrum has to satisfy anomaly matching conditions for the discrete global symmetries. There are two types of discrete anomalies. Type I anomaly matching conditions ($G_F^2 Z_N$ and $Z_N(\text{gravity})^2$) have to be satisfied regardless of assumptions on the massive bound states. Type II constraints have to be satisfied except if there are fractionally charged massive states. We have given two separate arguments for discrete anomaly matching. The argument based on instantons is valid only for Type I anomalies, but it does not not involve any subtleties concerning charge fractionalization. The argument based on spurions is valid for all anomalies, but the issues of charge fractionalization have to be taken into account.

We have tested several conjectured low-energy solutions using discrete anomaly matching. All the results by Seiberg on $N = 1$ supersymmetric gauge theories satisfy these conditions, which in some cases are extremely non-trivial. However, certain solutions do not satisfy the discrete anomaly matching conditions. These include an explicit realization of a chirally symmetric phase of $N = 1$ pure Yang-Mills theories based on the Veneziano–Yankielowicz Lagrangian, several non-supersymmetric confining theories with large representations, and some self-dual $N = 1$ supersymmetric theories based on exceptional gauge groups. These theories should be considered excluded or at least highly implausible.

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Note Added

After completing this work a preprint appeared by M. Schwetz and M. Zabzine [37], in which the authors conclude that the chirally symmetric phase of pure $N = 1$ supersymmetric Yang-Mills theories based on the VY Lagrangian is excluded, using considerations different from ours.

Appendix A Outer Automorphism, Charge Conjugation, and Color Parity in $SO(N)$ Groups

We have seen in Section 3 that $SO(2n)$ groups have a non-trivial Z_2 outer automorphism. The definition we gave for this automorphism is a parity-like transformation $\mathcal{P}$ (color-parity) in the internal $2n$ dimensional space which flips the sign of one particular color. In this appendix, we show that color-parity defines an outer automorphism for all $SO(2n)$ groups. The usual charge conjugation $T^a \to -T^{a*}$ is equivalent to color-parity for $SO(4k+2)$ groups while it is trivial up to a gauge transformation for $SO(4k)$ groups.

We first explain our notation for $SO(2n)$ groups [36]. The 2^n by 2^n gamma matrices γ_i form a Clifford algebra

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \qquad i, j = 1, 2, \ldots, 2n. \tag{A.1}$$

These gamma matrices can be constructed by iteration starting with $\gamma_1 = \tau_1$ and $\gamma_2 = \tau_2$ for $n = 1$ and taking tensor products with τ_3 (the τ's are the Pauli matrices). The explicit form of the general γ matrices is then

$$\gamma_{2k-1} = \overbrace{\tau_3 \otimes \tau_3 \otimes \ldots \otimes \tau_3}^{n-k} \otimes \tau_1 \otimes \overbrace{1 \otimes 1 \otimes \ldots \otimes 1}^{k-1}, \tag{A.2}$$

$$\gamma_{2k} = \tau_3 \otimes \tau_3 \otimes \ldots \otimes \tau_3 \otimes \tau_2 \otimes 1 \otimes 1 \otimes \ldots \otimes 1 \tag{A.3}$$

with 1 appearing $k-1$ times in the product, τ_3 appearing $n-k$ times in the product and τ_2 or τ_1 appearing once. This way we can see that γ_k is antisymmetric for even k while γ_k is symmetric for odd k. The spinor representation of $SO(2n)$ is defined as the object transforming as

$$\psi \to e^{i\omega_{ij}\sigma_{ij}/4}\psi, \qquad \sigma_{ij} = \frac{i}{2}[\gamma_i, \gamma_j]. \tag{A.4}$$

The analog of γ_5 is γ_{2n+1} which is defined by

$$\gamma_{2n+1} = (-i)^n \gamma_1 \gamma_2 \ldots \gamma_{2n} = \tau_3 \otimes \tau_3 \otimes \ldots \otimes \tau_3, \tag{A.5}$$

which anticommutes with all γ_i's. Thus the spinors $\frac{1}{2}(1+\gamma_{2n+1})\psi$ and $\frac{1}{2}(1-\gamma_{2n+1})\psi$ transform separately and there are two inequivalent spinor representations for $SO(2n)$.

The usual definition of the charge conjugation matrix C in field theory is that $\psi_1^T C \psi_2$ is invariant under $SO(2n)$ transformations. This implies that

$$C^{-1}\sigma_{ij}^T C = -\sigma_{ij}, \tag{A.6}$$

which exactly coincides with the definition $T^a \to -T^{a*}$ (3.7) in the spinor basis. Note that this implies that

$$C^{-1}[\gamma_i^T, \gamma_j^T]C = [\gamma_i, \gamma_j], \tag{A.7}$$

for the γ matrices, which is satisfied if

$$C^{-1}\gamma_i^T C = \pm\gamma_i. \tag{A.8}$$

This equation can be satisfied either with

$$C_1 = \gamma_1\gamma_3 \dots \gamma_{2n-1} \tag{A.9}$$

or with

$$C_2 = \gamma_2\gamma_4 \dots \gamma_{2n}. \tag{A.10}$$

These are both good definitions of charge conjugation in the sense of Eq. (A.6), but their symmetry properties are different.* For C_1:

$$C_1^{-1}\gamma_i^T C_1 = (-1)^{n-1}\gamma_i. \tag{A.11}$$

To see this relation let us first assume that the subscript i is even. Then $\gamma_i^T = -\gamma_i$, and one has to anticommute γ_i with n different γ matrices, thus the sign $(-1)^{n-1}$. Similarly, γ_i is symmetric if i is odd, but one has to perform only $n-1$ exchanges, thus the above formula follows for i even or odd. Similarly, one can show that for C_2:

$$C_2^{-1}\gamma_i^T C_2 = (-1)^n\gamma_i. \tag{A.12}$$

From the above explicit construction of the C's in terms of γ matrices and from the iterative construction for the γ matrices, we obtain that

$$\begin{aligned} C_1 &= i\tau_2 \otimes \tau_1 \otimes i\tau_2 \otimes \tau_1 \otimes \dots \otimes i\tau_2 \otimes \tau_1 \\ C_2 &= -i\tau_1 \otimes \tau_2 \otimes -i\tau_1 \otimes \tau_2 \otimes \dots \otimes -i\tau_1 \otimes \tau_2 \end{aligned} \tag{A.13}$$

for $SO(4k)$, with k τ_1's and τ_2's appearing both in C_1 and in C_2, while for $SO(4k+2)$

$$\begin{aligned} C_1 &= \tau_1 \otimes i\tau_2 \otimes \tau_1 \otimes i\tau_2 \otimes \tau_1 \otimes \dots \otimes i\tau_2 \otimes \tau_1 \\ C_2 &= \tau_2 \otimes -i\tau_1 \otimes \tau_2 \otimes -i\tau_1 \otimes \tau_2 \otimes \dots \otimes -i\tau_1 \otimes \tau_2 \end{aligned} \tag{A.14}$$

*Usually, the matrix which satisfies $C^{-1}\gamma_i^T C = -\gamma_i$ is referred to as the charge conjugation matrix, while the other matrix which satisfies $T^{-1}\gamma_i^T T = \gamma_i$ as the time reversal matrix.

with $k+1$ factors of τ_1's and k factors of τ_2's appearing in C_1 while in C_2 there are $k+1$ factors of τ_2's and k factors of τ_1's.

Armed with this knowledge we can now proceed to show that the effect of charge conjugation is just equivalent to an $SO(4k)$ gauge transformation for $SO(4k)$ gauge groups. For this all we need is that $C \in SO(4k)$ for the above choice of the charge conjugation matrix C. Let us choose, for example, $C = C_1 = \gamma_1\gamma_3 \dots \gamma_{4k-1}$. Since we are considering $SO(4k)$ groups, there are even number of γ matrices appearing in C, thus C can also be written as $C = (\gamma_1\gamma_3)(\gamma_5\gamma_7)\dots(\gamma_{4k-3}\gamma_{4k-1})$. However, the products $\gamma_i\gamma_{i+2}$ are all $SO(4k)$ elements. This can be seen by considering the $SO(4k)$ element

$$O(i,j) = e^{-i\frac{\pi}{2}\sigma_{ij}} = \cos\frac{\pi}{2} - i\sigma_{ij}\sin\frac{\pi}{2} = -i\frac{i}{2}[\gamma_i,\gamma_j] = \gamma_i\gamma_j \qquad (i \neq j) \tag{A.15}$$

due to the anticommutation relations of the γ's. Thus for $SO(4k)$ groups, $C = O(1,3)O(5,7)$ $\cdots O(4k-3, 4k-1)$ is an element of the gauge group, and does not act as an outer automorphism on the Lie algebra (it is not an additional discrete symmetry of the theory). This proof shows at the same time that the spinors for $SO(4k)$ are self conjugates (real or pseudo-real), since charge conjugation is equivalent to a gauge transformation and does not interchange representations.

The situation is very different for $SO(4k+2)$, since there C contains odd number of γ matrices and therefore can not be a gauge transformation. This can be most easily seen by the fact that C anticommutes with γ_{2n+1} and thus interchanges the two kinds of spinor representations. The two kinds of spinors of $SO(4k+2)$, therefore, are charge conjugates of each other.

What remains to be seen is that the color-parity transformation $\mathcal{P}$ which we employed to define the outer automorphism for $SO(2n)$ coincides with the above charge conjugation up to a gauge transformation for $SO(4k+2)$, and is also a non-trivial automorphism for $SO(4k)$. We have defined the automorphism $\mathcal{P}$ of $SO(2n)$ as a parity-like transformation in the internal $2n$ dimensional space, which flips the sign of a particular color (*e.g.* 1). This means that the transformed spinor ψ' has to transform as

$$\psi' \to e^{i\omega'_{ij}\sigma_{ij}/4}\psi' \tag{A.16}$$

under an $SO(2n)$ transformation, where $\omega'_{1i} = -\omega_{1i}$ and $\omega'_{ij} = \omega_{ij}$ for $i,j \neq 1$. Since $\gamma_1\gamma_i\gamma_1 = -\gamma_i$ if $i \neq 1$, the spinor constructed by $\psi' = i\gamma_1\gamma_{2n+1}\psi$ will transform exactly the right way. Thus we conclude that parity-like transformation which changes the sign of the 1 direction is implemented on the spinors by multiplication by $i\gamma_1\gamma_{2k+1}$. Since γ_1 always anticommutes with γ_{2n+1}, it connects the two different spinor representations characterized by $\gamma_{2n+1} = \pm 1$, and thus cannot be an inner automorphism. Therefore this transformation is always a good definition for the outer automorphism of $SO(2n)$. Note that a similar definition of automorphism for $SO(2n+1)$ groups is gauge equivalent to the overall sign flip of the whole vector in the $SO(2n+1)$ group, and is of flavor-type. The transformation of the spinors under this flavor-type symmetry depends on the models.

Finally, we show that the two definitions for the Lie algebra automorphisms coincide for $SO(4k+2)$ up to gauge transformations. We have seen that the definition using the parity-like transformation acts as a multiplication by $i\gamma_1\gamma_{4k+3} = (-1)^k\gamma_2\gamma_3\cdots\gamma_{4k+2}$ on the spinor, while the charge conjugation acts like multiplication by C. The important point is that $i\gamma_1\gamma_{4k+3} = (\gamma_3\gamma_5)\cdots(\gamma_{4k-1}\gamma_{4k+1})C_2$. A pair of γ matrices can be thought of as an $SO(4k+2)$ transformation as we have shown above. Thus C_2 is nothing but a product of $SO(4k+2)$ transformation matrices multiplied by the $i\gamma_1\gamma_{4k+3}$ matrix, and is hence equivalent to the color-parity transformation up to a gauge transformation. The other charge conjugation matrix $C_1 = i\gamma_{4k+3}C_2^{-1}$ is also gauge equivalent because $i\gamma_{4k+3} = (-1)^k O(1,2)\cdots O(4k+1, 4k+2)$.

Let us add brief comments on $SO(2n+1)$ groups. We add γ_{2n+1} to the set of γ_i-matrices to represent the Clifford algebra (A.1) for $i = 1, \cdots, 2n+1$. The matrix C_2 (A.10) satisfies $C_2^{-1}\gamma_i^T C_2 = (-1)^n\gamma_i$ including $i = 2n+1$. There is, however, no consistent definition of the color-parity for $SO(2n+1)$ spinors. On the other hand, color-parity can be defined on the vectors as the sign flip of the first color. Together with a gauge transformation $O(2,3)O(4,5)\cdots O(2n, 2n+1)$, however, this color-parity flips the sign of a vector as a whole, and hence is of flavor-type (*i.e.*, commutes with the gauge group). This is because there is no outer automorphism for $SO(2n+1)$ groups. Therefore, the only possible discrete symmetries in $SO(2n+1)$ gauge theories are flavor-type symmetries.

Appendix B Centers of Simple Groups

We have seen in Section 3 that the correct identification of the additional discrete symmetries requires the knowledge of the centers of the continuous global symmetries. One can avoid unnecessary checks of anomaly matching when a discrete symmetry is a part of the continuous ones. The centers of semi-simple Lie groups have been classified (see e.g. [22]) and in the following we give a complete list of them:

$$\begin{aligned} SU(N):&\quad Z_N \\ Sp(2N):&\quad Z_2 \\ SO(2N+1):&\quad Z_2 \\ SO(4N):&\quad Z_2\times Z_2 \\ SO(4N+2):&\quad Z_4 \\ E_6:&\quad Z_3 \\ E_7:&\quad Z_2 \end{aligned}$$

The other semi-simple groups do not have a non-trivial center.

Let us give what the actions of the centers are. For $SU(N)$, the center is the Z_N phase rotation of the fundamental representation. The Z_2 center of $Sp(2N)$ is the sign flip of the fundamental of $Sp(2N)$. For $SO(2N+1)$ the center is the 2π rotation in the $SO(2N+1)$ gauge group which flips the sign of the spinor representation.

The case of $SO(2n)$ groups is more complicated. The center is different for $SO(4k)$ and $SO(4k+2)$, which has to do with the different definition of the analog of γ_5, $\gamma_{2n+1} = (-i)^n \gamma_1\gamma_2\ldots\gamma_{2n}$. For $SO(4k)$ groups ($n=2k$), $\gamma_{4k+1} = (-1)^k \gamma_1\gamma_2\ldots\gamma_{4k}$. Note that there is no i in the definition of γ_{4k+1}. As shown in Appendix A, a product of two γ-matrices $\gamma_i\gamma_j$ is always an SO group element $O(i,j)$, which is just a 180 degree rotation in the $i-j$ plane. We know however, how γ_{4k+1} acts on the spinors: $\gamma_{4k+1}S_1 = S_1$, $\gamma_{4k+1}S_2 = -S_2$. Since $\gamma_{4k+1} = (-1)^k O(1,2)O(3,4)\ldots O(4k-1,4k)$, we conclude that the $SO(4k)$ group element $g = O(1,2)O(3,4)\ldots O(4k-1,4k)$ acts as the above Z_2 transformation on the spinors. Note that this $SO(4k)$ element flips the overall sign of the vector. There is a separate Z_2 transformation: 2π $SO(4k)$ rotation that flips the signs of both spinors, $S_1 \to -S_1$, $S_2 \to -S_2$. Note that the vector of $SO(4k)$ switches sign under the $g = O(1,2)O(3,4)\ldots O(4k-1,4k)$ $SO(4k)$ transformation, but not under the 2π $SO(4k)$ rotation. We can combine the Z_2 of 2π rotation and the other Z_2 generated by g to obtain the $Z_2\times Z_2$ center of $SO(4k)$ as sign flips of any two of the spinors S_1, S_2 and the vector.

The case of $SO(4k+2)$ groups differs from the $SO(4k)$ because $\gamma_{4k+3} = i(-1)^k\, O(1,2)$ $O(3,4)\ldots O(4k+1,4k+2)$. Thus the effect of $g' = O(1,2)O(3,4)\ldots O(4k+1,4k+2)$ on the spinors is $S_1 \to iS_1$, $S_2 \to -iS_2$, which forms a Z_4 group that is the center of $SO(4k+2)$. The vector of $SO(4k+2)$ has charge two under the Z_4 center.

The center of E_6 is Z_3. We can find the action of the center on the representations by considering the embedding of $SO(10)\times U(1)$ into E_6. Under this subgroup $27 \to 16_1 + 10_{-2} + 1_4$ where the lower indices are the $U(1)$ charges. Let us consider the Z_3 subgroup of $U(1)$ under which

$$
\begin{aligned}
&16 \to e^{2\pi i \frac{n}{3}} 16, \\
&10 \to e^{2\pi i \frac{-2n}{3}} 10 = e^{2\pi i \frac{n}{3}} 10, \qquad\qquad n = 0,1,2 \\
&1 \to e^{2\pi i \frac{4n}{3}} 10 = e^{2\pi i \frac{n}{3}} 1.
\end{aligned}
\tag{B.1}
$$

This Z_3 acts uniformly on $16, 10$ and 1 and is a Z_3 phase rotation of the fundamental 27 of E_6. Therefore this Z_3 is the center of E_6.

The center of E_7 is Z_2. To identify the action of this Z_2, we note that E_7 has an $SU(8)$ subgroup under which the fundamental 56 decomposes as $56 \to 28 + \overline{28}$, where 28 is the rank two antisymmetric tensor of $SU(8)$. The center of $SU(8)$ is Z_8, whose action on 28 is effectively a Z_4. However, 28 and $\overline{28}$ transform with the opposite phase under Z_4, thus only the Z_2 subgroup of Z_4 acts uniformly on 28 and $\overline{28}$. Therefore the Z_2 center of E_7 is the sign flip of the fundamental 56 representation.

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