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Shamit Kachru,[1] Albrecht Klemm,[2] and Yaron Oz[1]

[1]Department of Physics
University of California, Berkeley
and
Theoretical Physics Group
Physics Division
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

[2]Enrico Fermi Institute
University of Chicago
5460 S. Ellis Avenue
Chicago, Illinois 60637

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Calabi-Yau Duals for CHL Strings

Shamit Kachru[1], Albrecht Klemm[2], and Yaron Oz[1]

[1]Department of Physics
University of California at Berkeley
and
Theoretical Physics Group
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

[2] Enrico Fermi Institute
University of Chicago
5460 S. Ellis Avenue
Chicago, IL 60637

We find M-theory (Type IIA) duals for compactifications of the 9d CHL string to 5d (4d) on $K3$ ($K3 \times S^1$). The IIA duals are Calabi-Yau orbifolds with nontrivial RR $U(1)$ backgrounds turned on.

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1. Introduction

In ten dimensions, the consistent critical string theories with (at least) sixteen supercharges have been known since the 1980s. There are (after accounting for the S-duality of the two $SO(32)$ theories) four. In $D = 9$, in addition to the compactifications of the $D = 10$ theories on S^1 we find a new theory with 16 supercharges – the CHL string [1,2]. This theory can be obtained by compactifying the $E_8 \times E_8$ heterotic string on a circle with radius R^9 and orbifolding by the Z_2 which identifies the two E_8s and shifts $x^9 \to x^9 + \pi R^9$. The resulting heterotic string has E_8 current algebra at level 2 on its worldsheet. The moduli space of vacua in nine-dimensions is

$$SO(9,1;\mathbb{Z})\backslash SO(9,1)/SO(9) \times SO(1) \ . \tag{1.1}$$

The low-energy M-theoretic description of this theory involves the compactification of 11 dimensional supergravity on a Möbius strip [3].

In the case of the $E_8 \times E_8$ string, dual Type IIA descriptions after compactification to 6d on a torus or 4d on $K3 \times T^2$ were given in [4,5,6,7]. For the CHL string, the dual of the maximally supersymmetric compactification to 6d was given by Schwarz and Sen [8]. Other aspects of the maximally supersymmetric CHL compactifications were recently discussed in [9,10]. It is the purpose of this paper to begin the task of finding duals of CHL compactifications with less supersymmetry, by finding the IIA and M-theory duals of the CHL compactifications to 4 and 5 dimensions with 8 supercharges.

In investigations of vacua of the $E_8 \times E_8$ heterotic string with 8 supercharges, a proper understanding of *singular* points in the moduli space has led to the discovery of many new nontrivial renormalization group fixed points in $d = 4, 5, 6$ (in [11,12] and much subsequent work). Analysis of the dual Calabi-Yau models has been a powerful tool for exploring these field theories. One motivation for our work is to provide a similar framework for studying novel theories without gravity which may arise in $d = 4, 5$ from CHL compactifications. Results in this direction will appear in a companion paper [13]. It would also be interesting to determine whether or not the web of 4d $N = 2$ string vacua discussed here is connected, through phase transitions, to the web of conventional 4d $N = 2$ Calabi-Yau vacua [14].

2. The Heterotic Theories

Starting from the 9d CHL string with $E_8 \times U(1)^2$ gauge symmetry, one can compactify on $K3 \times S^1$ and obtain a 4d $N = 2$ supersymmetric low-energy theory. From the perturbative heterotic string Bianchi identity, one should in addition require a background gauge bundle V on the $K3$ with

$$c_2(V) = 12 \ . \tag{2.1}$$

Said differently, there should be 12 instantons embedded in the E_8.

One argument that (2.1) is the correct condition is the following. Start with the $E_8 \times E_8$ string on $K3 \times T^2$. Imagine there are bundles $V_{1,2}$ embedded in the two E_8s. Then, one can do a Z_2 orbifold to obtain the CHL string as long as V_1 and V_2 are identical. The Bianchi identity for the $E_8 \times E_8$ theory in this case is

$$c_2(V_1) + c_2(V_2) = 24 \tag{2.2}$$

so in particular to do the CHL orbifold, one requires $c_2(V_{1,2}) = 12$. After orbifolding, one is left with the diagonal E_8 and a bundle of instanton number 12.

A more microscopic description of the same vacuum comes from M-theory on a Möbius strip M times $K3 \times S^1$. In this description, the E_8 gauge fields and the bundle V live on ∂M. One can now imagine more general configurations where N instantons shrink and leave the boundary of the world in their avatar as fivebranes wrapping the base of M and the circle.[1] Then, the Bianchi identity (2.1) is modified and becomes

$$c_2(V) + N = 12 \ . \tag{2.3}$$

This can be argued as in the previous paragraph, now by using the general configurations studied by Duff, Minasian, and Witten [15]. We will mostly concentrate on finding duals in the case that $c_2(V) = 12$, though we also find Calabi-Yau duals for some cases with wrapped fivebranes present (notably, the maximal case $N = 12$).

For a fixed choice of instanton and fivebrane numbers satisying (2.3), we can still find a whole web of vacua by considering bundles with different structure groups, yielding different unbroken non-Abelian gauge groups $G \subset E_8$. By passing to a generic point on the Coulomb branch of G, one obtains an Abelian gauge theory characterized by the number

[1] By the base of M we mean a representative of the nontrivial class in $H_1(M)$, which looks like a base if one locally views M as a fibration of the interval over S^1.

of vector and hypermultiplets. Some of the expected gauge groups and matter contents are presented below in Table 1, in terms of

$$n \equiv 8 - N = c_2(V) - 4 \,. \tag{2.4}$$

One should consider

$$n \geq 0 \tag{2.5}$$

in the table, since for $n < 0$ there are no stable bundles with the right instanton number on $K3$. The number of hypermultiplets n_H at a generic point in the Coulomb branch of G is given below, while the number of vector multiplets at a generic point is given by

$$n_V = \mathrm{rank(G)} + N + 3 \,. \tag{2.6}$$

It is important to emphasize that we only list the unbroken subgroups of the perturbative E_8 in Table 1; the table ignores the omnipresent $U(1)^3$ which appears in (2.6), as well as the N vector multiplets on the (wrapped) fivebrane worldvolumes.

In addition to listing the gauge group G, we have presented the relevant singularity type expected to produce G in the Calabi-Yau dual, and the expected G charged matter content (which becomes massless at the origin of the G Coulomb branch, in the *classical* heterotic string theory). The charged matter content is simply computed using $c_2(V)$ and the index theorem, as in [5]. Decompose the adjoint of E_8 under $G \times H$ (where H is the commutant of G in E_8) as

$$\mathbf{248} = \sum_i (M_i, R_i) \tag{2.7}$$

where M_i is the G representation and R_i is the H representation. Then it follows from the index theorem that the number of left-handed spinor multiplets transforming in the M_i representation of G is given by

$$N_{M_i} = \mathrm{dim(R_i)} - \frac{1}{2}\int_{K3} c_2(V)\ \mathrm{index(R_i)} \,. \tag{2.8}$$

(2.8) is normalized to properly count numbers of hypermultiplets.

The detailed explanation of the singularity types (and in particular the occurence of "split," "non-split," and "semi-split" singularities, denoted with superscript s, ns, and ss) can be found in [16], from which Table 1 has been lifted with suitable modifications. The geometrical realization of non-simply laced gauge groups was first explained in [17].

Roughly speaking, the threefold singularities can be understood as elliptic surface singularities fibered over an additional curve. There can be monodromies which orbifold the naive (A-D-E) gauge group coming from the surface singularity by an outer automorphism as one goes around singular points on the additional curve, yielding a non-simply laced group. In the "split" cases this does *not* occur, while in the other cases such an outer automorphism *does* act.

In all cases in Table 1 where a multiplicity becomes negative, the corresponding branch with gauge group G does not exist (there aren't enough instantons to break E_8 to G). In addition, the complete Higgsing of E_8 is only possible for $n \geq 6$.

Table 1: Some Strata in the Moduli Space

Type	Group	Matter content	n_H
E_7	E_7	$(\frac{n}{2})\mathbf{56}$	$n+33$
E_6^s	E_6	$(n-2)\mathbf{27}$	$2n+32$
E_6^{ns}	F_4	$(n-3)\mathbf{26}$	$3n+30$
D_5^s	$SO(10)$	$(n-4)\mathbf{16}+(n-2)\mathbf{10}$	$3n+29$
D_5^{ns}	$SO(9)$	$(n-3)\mathbf{9}+(n-4)\mathbf{16}$	$5n+24$
D_4^s	$SO(8)$	$(n-4)(\mathbf{8}_c+\mathbf{8}_s+\mathbf{8}_v)$	$5n+24$
D_4^{ss}	$SO(7)$	$(n-5)\mathbf{7}+(2n-8)\mathbf{8}$	$7n+15$
D_4^{ns}	G_2	$(3n-14)\mathbf{7}$	$11n-2$
A_3^s	$SU(4)$	$(n-6)\mathbf{6}+(4n-16)\mathbf{4}$	$7n+15$
A_3^{ns}	$SO(5)$	$(n-7)\mathbf{5}+(4n-16)\mathbf{4}$	$9n+2$
$A_1\times A_1$	$SO(4)$	$(n-8)(\mathbf{2},\mathbf{2})+(4n-16)[(\mathbf{1},\mathbf{2})+(\mathbf{2},\mathbf{1})]$	$9n+2$
A_2^s	$SU(3)$	$(6n-30)\mathbf{3}$	$11n-2$
A_1	$SU(2)$	$(6n-32)\mathbf{2}$	$17n-33$
D_6^s	$SO(12)$	$\frac{r}{2}\mathbf{32}+(\frac{n-r-4}{2})\mathbf{32}'+(n)\mathbf{12}$	$n+30$
D_6^{ns}	$SO(11)$	$(\frac{n}{2}-2)\mathbf{32}+(n-1)\mathbf{11}$	$3n+29$
A_5^s	$SU(6)$	$\frac{r}{2}\mathbf{20}+(r+2n)\mathbf{6}+(n-r-6)\mathbf{15}$	$2n-r+25$
A_5^{ns}	$Sp(3)$	$(2n+\frac{3}{2}r)\mathbf{6}+(n-r-7)\mathbf{14}+\frac{1}{2}r\mathbf{14}'$	$3n+19-2r$
A_4^s	$SU(5)$	$(3n-8)\mathbf{5}+(n-6)\mathbf{10}$	$4n+24$
None	None	(*only possible for* $n\geq 6$)	$29n-100$

In the case $G = SO(12)$, an additional integer r is required to specify the heterotic vacuum. That is because the commutant of $SO(12)$ in E_8 is $SO(4) \simeq SU(2)\times SU(2)$, and we can embed $r+4$ instantons in one $SU(2)$ and $n-r$ instantons in the other $SU(2)$.

In order to keep at least 4 instantons in each $SU(2)$ (which is the minimum number for which a suitable $SU(2)$ bundle exists), we must require $r \geq 0$ and $(n-r) \geq 0$. The cases $G = SU(6), Sp(3)$ are also r-dependent, because they can be obtained by Higgsing $SO(12)$ for suitable r, but not by Higgsing E_7 with **56**s.

3. Calabi-Yau Duals

3.1. Method for Finding Calabi-Yau Duals

There is a well known string-string duality relating the heterotic string on T^4 to type IIA compactifications on $K3$. Schwarz and Sen found an analogous statement for the compactification of the 9d CHL string to six dimensions on a T^3 [8]. Namely, they found that this 6d CHL string is dual to Type IIA on a Z_2 orbifold of $K3$ which preserves precisely *twelve* of the twenty $(1,1)$ forms. The other eight are projected out of the untwisted sector. This exactly reproduces the rank reduction (by eight) of the CHL string.

Normally, the 8 "missing" $(1,1)$ forms would be resurrected in the twisted sector. However, by also embedding a Z_2 action in a RR $U(1)$ gauge group, they were able to remove the twisted sector contributions. The Z_2 gauge flux is concentrated at the orbifold fixed points of the geometrical Z_2 action on the $K3$, and removes the blow-up modes. The conclusion is that IIA on this particular $K3/Z_2$ is dual to the CHL string.

As was discussed in [8], a more geometrical formulation of the same Z_2 action can be given in M-theory. The $E_8 \times E_8$ heterotic string on T^4 is dual to M-theory on $K3 \times S^1$. If we go to the $E_8 \times E_8$ point in moduli space and do the CHL orbifold, the exchange of the two E_8s maps in M-theory to a Z_2 which exchanges the two E_8 singularities of the $K3$. If we take the $K3$ to be an elliptic fibration with coordinate z on the P^1 base and a defining equation of the form

$$y^2 = x^3 + xf(z) + g(z) \qquad (3.1)$$

then the Z_2 acts by

$$z \to \frac{1}{z}, \; y \to -y \; . \qquad (3.2)$$

The two E_8 singularities naturally arise at $z = 0, \infty$ and are identified by (3.2). The shift on the heterotic circle maps, in M-theory, to a shift on the M-theory S^1. Therefore, the Z_2 symmetry is *freely acting* in M-theory, and the possibility of fixed point contributions to

the spectrum does not arise. In the IIA picture, the $U(1)$ RR gauge field arises via Kaluza-Klein reduction along the M-theory circle, and the shift on the S^1 therefore embeds a Z_2 action in the $U(1)$.

We are interested in using a similar technique to find Calabi-Yau duals for 9d CHL strings compactified to 4d on $K3 \times S^1$, or to 5d on $K3$. We can use the above duality and the adiabatic argument [18] to find duals as follows.

Consider the heterotic $E_8 \times E_8$ string on $K3 \times T^2$ with (12,12) instantons in the two E_8s. After maximal Higgsing, it is known to be dual to the IIA compactification on the Calabi-Yau X given by an elliptic fibration over $P^1 \times P^1$ with hodge numbers $(3, 243)$. If we Higgs "symmetrically," then we can still perform the CHL Z_2 by exchanging the two E_8s and shifting one of the circles of the T^2. This will leave us with a CHL string compactification with 132 hypermultiplets and 3 vector multiplets. The corresponding Calabi-Yau dual X_{CHL} should have hodge numbers $(3, 131)$. More accurately, these should be the contributions to the hodge numbers from the "untwisted" cohomology classes, ignoring any modes which originate at the Z_2 fixed loci on X_{CHL}.

Using the adiabatic argument, we can construct X_{CHL} from X. X is a $K3$ fibration, and we can implement the action (3.2) on the $K3$ fibers of X, while at the same time acting on the P^1 in a way that preserves 4d $N = 2$ supersymmetry (the precise details are provided in §3.2). If we consider M-theory on $X \times S^1$ and at the same time act with a shift on the S^1, then the overall Z_2 action will be free. This is why the massless modes should come from the "untwisted sector" of $X_{CHL} = X/Z_2$. In fact, one finds agreement with the expected $(3, 131)$.

Given that M-theory on $X_{CHL} \times S^1$ is dual to the 4d $N = 2$ CHL compactification, we can now make the circle very large and obtain an approximate 5d duality between M-theory on X_{CHL} (with the prescription that the Z_2 fixed points cannot be resolved) and the compactification of the 9d CHL string on $K3$. More precisely, we need to take a double-scaling limit to go to the M-theory description, as in [19]. The Kahler classes of X_{CHL} as measured in Type IIA and M-theory, which we will call K_{IIA} and K_M, are related by

$$K_M = \frac{1}{T^{1/3}R} K_{IIA} \tag{3.3}$$

where T is the two-brane tension and R is the radius of the circle. If we take R to infinity but wish to keep the Kahler classes of the Calabi-Yau fixed in M-theory units, we must take $K_{IIA} \to \infty$ as well. Note that although X also admits an "F-theory limit," which is

dual to an $E_8 \times E_8$ compactification on $K3$, X_{CHL} does not. The F-theory limit involves shrinking the elliptic fibers of X, and the Z_2 action which turns X into X_{CHL} destroys the relevant fibration structure. This is not surprising, since there is no obvious way to get theories with 6d $(1,0)$ supersymmetry starting with the 9d CHL string.

Alternatively, as in [8], we can view the S^1 shift as a Z_2 RR flux in IIA string theory. Then we see that IIA on X_{CHL}, with suitable Z_2 fluxes at the orbifold fixed points, is dual to the maximally Higgsed 4d $N = 2$ CHL vacuum.

More generally, one can first "unHiggs" some gauge group G by going through suitable extremal transitions starting with X. Of course, one should unHiggs symmetrically in the two E_8s. Then, using the same fibre-wise argument to find the correct Z_2 action, we can find Calabi-Yau duals for the general CHL compactifications on $K3$ discussed in §2. We have done this in many cases, and find complete agreement with the CHL expectations.

Generally, $N = 2$ supergravity in 4d requires a local integrability structure on the vector moduli space known as special geometry (globally, it requires the positivity of the kinetic terms). Both requirements are naturally satisfied for the moduli spaces of Calabi-Yau threefolds. A natural generalisation is to consider the moduli space of an invariant sector under a group action on the Calabi-Yau. This is familiar for rigid $N = 2$ theories, where one has to consider quite generally certain subsectors of the moduli space of a Riemann surface and not the full abelian variety, but a Prym variety. However, unlike the Riemann surfaces, the CY is not an auxiliary surface for decribing the moduli space. Hence, in general twisted sectors have to be considered, and one ends up with the moduli space of another CY. The CHL string gives a rationale for dealing only with invariant states under particular Z_2 actions. It would be very interesting to see whether other invariant sectors of Calabi-Yau moduli spaces can also arise in special string constructions.

3.2. Toric Description of the Calabi-Yau Duals

As described in §3.1, to construct a dual description of the $N = 2$ CHL vacua in four dimensions, we start with the non-perturbative equivalence between the $E_8 \times E_8$ heterotic string on $K3 \times T^2$ with symmetric embedding in the E_8s and Type IIA (IIB) on Calabi-Yau spaces X (X^*). To find simple tests of the duality we shall consider perturbative CHL compactifications. Then, X will be a $K3$ fibration [20], and the IIA dual will be a Z_2 orbifold of X. Using the adiabatic argument, we expect the Z_2 to act as in [8] on the $K3$ fibre of the CY manifold.

The relevant CY manifolds on which such Z_2 actions are to be expected are constructed as follows. We consider the most general elliptic fibre $X_6(3,2,1)$ in affine coordinates

$$y^2 + x^3 + a_1 xy + a_2 x^2 + a_3 y + a_4 x + a_6 = 0. \tag{3.4}$$

Here we view the a_i as functions of the coordinate t of $\mathbb{C}$. Locally, the Kodaira types I of minimal singularities of the fibre over $(x,y,t) = (0,0,0)$ were analysed using a generalized Tate's algorithm in [16]. The type is specified essentially[2] by the degree of a_i in t, see Table 2 of [16].

To get a compact Calabi Yau model we instead view the a_i as sections of line-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ of type $\mathcal{O}(k_i) \otimes \mathcal{O}(k_i)$ with $k_i = 2,4,6,8,12$ respectively. If we forget about one of the $\mathbb{P}^1$s, this yields a $K3$. If $(t:t')$ are homogeneous coordinates on the remaining $\mathbb{P}^1$, the singular fibres which occur at the north and south pole are determined by the lowest degrees of the homogenous polynomials a_i of degree k_i in t and t' respectively (as in Table 2 of [16]).

Now, let $(t:t')$ and $(s:s')$ be the homogeneous parameters of the first and the second $\mathbb{P}^1$. We denote the global model by $I_w \, {}^{I_n}_{I_s} \, I_e$ were we understand that singular fibres of type (I_n, I_s) appear at $t=0$ and $t'=0$ with (s,s') generic and (I_e, I_w) appear $s=0$ and $s'=0$ with (t,t') generic. The Newton polyhedron Δ of (3.4) is the convex hull of

$$\begin{aligned} &(0,0,0,2),\ (0,0,3,0) \\ &(a_i^{(w)}, a_i^{(n)}, v_i),\ (a_i^{(w)}, k_i - a_i^{(s)}, v_i), \\ &(k_i - a_i^{(e)}, a_i^{(n)}, v_i),\ (k_i - a_i^{(e)}, k_i - a_i^{(s)}, v_i), \end{aligned} \tag{3.5}$$

with $i = 1,2,3,4,6$ and $v_i = (1,1),(2,0),(0,1),(1,0),(0,0)$ shifted by $(-1,-1,-1,-1)$, and always contains the origin[3]. Let Λ be the coarsest lattice, which contains all integral points inside Δ, $V = \Lambda_{\mathbb{R}}$ the real extension, Λ^* and V^* the dual lattice and vector space respectively [21]. The polyhedra are reflexive if

$$\Delta^* = \{x \in V^* \| \langle x, y \rangle \geq -1, \forall y \in \Delta\} \tag{3.6}$$

[2] Up to some factorisation conditions, which have to be imposed in a few cases as extra constraints.

[3] For the fibre $K3$ the corresponding polyhedra are obtained by deleting the first or second entry of these vectors.

is a lattice polyhedron in Λ^* and in this case Δ defines a Calabi-Yau or $K3$ space [21]. For affine patches reflexivity implies the condition for having canonical hypersurface singularities given in [22],[23] and the toric description gives a straightforward prescription to resolve them [23]. Of course reflexivity is stronger, i.e. it is not always possible to compactify the affine patches to a $c_1 = 0$ manifold.

Points in Δ^* which are not at codimension one correspond to divisors in $\mathbb{P}_{\Delta^*}$, which intersect with the hypersurface and give rise to divisors in the CY. Most commonly in the examples below the intersection with the hypersurface yields one irreducible divisor on the CY. i.e. there is one $(1,1)$-form for each point in codim 2 and 3 in Δ^*. In general the number of irreducible divisors is given by the number of points interior to the dual face plus one, see [21] for details. There are κ homology relations in the full set of divisors, where $\kappa = \dim_{\mathbb{C}}(X) + 2$. So in order to count the independent $(1,1)$-forms in the CY (K_3), one must subtract 5(4) from the number of these divisors.

If we write the resolved (3.4) in Batyrev-Cox coordinates its specialization to any affine patch of the toric variety can be neatly displayed. In particular it is easy to identify the exceptional divisors, which are needed to resolve the singularities of the affine equations (3.4). The form of the polynomial is

$$p = \sum_i b_i \prod_j x_j^{\langle \nu_i, \nu_j^* \rangle + 1}, \tag{3.7}$$

where the sum runs over the relevant points ν_i of Δ and the product over the relevant points ν_j^* of Δ^*. E.g. the affine patch with the (3.4) singularity in one patch of the $\mathbb{P}^1$ is obtained by setting all variables in (3.7) to one except of the ones associated to $\nu_x^*, \nu_y^*, \nu_t^*$ (see the Appendix). The mirror polynomial has the analogous definition with Δ and Δ^* exchanged.

Points in Δ correspond to monomials in the defining equation of the hypersurface (3.7) and their coefficients correspond generically to independent complex structure deformations. They in turn are in one to one correspondence with the $(2,1)$ forms of the CY. But also in Δ the codimension one points do not contribute to the complex structure deformations and hence to $(2,1)$-forms. The reason here is that one can use the projective invariance group of the toric ambient space $\mathbb{P}GW(\Delta^*, \mathbb{C})$ to set (gauge) all their coefficients to constant values (zero). We shall call points not on codimension one relevant points. As (3.7) is quasihomogeneous, i.e. scale invariant under a overall rescaling of the

x_i, we shall rather consider[4] the $GW(\Delta^*,\mathbb{C})$, which contains a $(\mathbb{C}^*)^\kappa$ action of independent rescalings of the variables x_j. This leaves (3.7) invariant upon a $(\mathbb{C}^*)^\kappa$ rescaling of the coefficients a_i. We can (gauge) fix this freedom of rescaling by setting κ additional coefficients a_i to one. So generically the number of $(2,1)$-forms is the number of relevant points in Δ minus κ. Analogous to the situation with Δ^*, certain points in Δ can correspond to several $(2,1)$-forms. These points correspond to the deformation modes of curves (of genus g) of singular loci, with a $\mathbb{C}^2/Z_r$ action on the normal bundle. The resolution of the curves supports $g(r-1)$ additional $(2,1)$-forms, which do not correspond to deformation parameters in (3.7). Again the number of additional (2,1) forms is given by the number of interior points in the dual face cff. [21]. The gauge fixing of the coefficients determines the dimension of the moduli space. There remains however generically a (huge) discrete subgroup of $GW(\Delta^*,\mathbb{C})$ of invariances of (3.7), the so called R symmetries. As we shall see, the Z_2 symmetry on the coordinates lifts part of the R symmetries of the moduli space of the invariant sector.

According to [20], we can identify the complexified volume of e.g. the (w,e) $\mathbb{P}^1$ with the heterotic dilaton and as long as we are interested only in generic perturbative gauge enhancements we keep the fibre over this $\mathbb{P}^1$ generic, i.e. $I_w = I_e = I_0$, which implies $a_i^{(w)} = a_i^{(e)} = 0$. Moreover as we want to have cases with symmetric unhiggsing of the gauge group to perform the Z_2 modding we look at models $X(I) := I_0{}^I_I I_0$. We list in Table 2 various cases.

The $\mathbb{Z}_2$ symmetry acts by by exchanging

$$\sigma : t \to t', \quad t' \to t, \quad y \to -y. \tag{3.8}$$

It is obvious that (3.5) has two symmetry planes associated with the exchange of (t,t') and (s,s'). To act with (3.8), we must tune the complex structure parameters b_i in front of the monomials of p, which corresponds to points above and below the symmetry plane, to symmetric or antisymmetric values (depending on whether they multiply y). We must take the normal subgroup N of $GW(\Delta^*,\mathbb{C})$ w.r.t. to (3.8) to subtract the reparametrisation invariance. If we can globally diagonalize (3.8) , as e.g. in the example below, it is easy to

[4] E.g.for the elliptic curve $X_6(1,2,3)$, $GW(\Delta^*,\mathbb{C})$ is generated by the weight compatible transformation $t \mapsto at$, $x \mapsto b_1 x + b_2 t^2$ and $y \mapsto c_1 y + c_2 tx + c_3 t^3$ with complex coefficients. Three parameters correspond to the $(\mathbb{C}^*)^3$, while three can be used to set the coefficients of the three codim 1 points k_1, k_2, k_3 (cff. Appendix A) to zero.

see that the dimension of N is the number of invariant points in codimension one. However for general actions, that is not the case and we have to explicitly determine N to count the independent σ-invariant $(2,1)$ forms.

In fact due to (3.6) the dual polyhedron also has two such symmetry planes and we have to enforce symmetric values of the vector moduli in the mirror polynomial as well. The counting of invariant $(1,1)$-forms proceeds in a way similar to the counting in the Δ polyhedron, and is summarized in Table 2.

For all entries except the E_8^2 case, one should compare to the $n=8$ compactification of §2 (with no small instantons). For the E_8^2 case, all of the instantons are small and one is in the $N=12$ case of §2. Then, $G=E_8$ so there are $20+12=32$ hypermultiplets coming from the $K3$ moduli and positions of the (wrapped) fivebranes, and $12+8+3=23$ vector multiplets at generic points in the Coulomb branch. In all cases, using

$$n_V = h^{1,1}_{inv}, \quad n_H = h^{2,1}_{inv} + 1 \tag{3.9}$$

we find agreement between the CHL expectations and the numbers of Z_2 invariant cohomology classes.

Table 2: Symmetric $n=8$ models and Z_2 invariant sector. The numbers in the brackets correspond to non-toric deformations associated to curve singularities in X.

$X(I)$	Group	$h^{1,1}$	$h^{1,1}$	$h^{1,1}_{inv}$	$h^{2,1}_{inv}$
$X(II^*)$	$(E_8)^2$	43(22)	43 (0)	23(0)	31(10)
$X(III^*)$	$(E_7)^2$	17(0)	61(0)	10(0)	40(0)
$X(IV^{*s})$	$(E_6)^2$	15(0)	75(0)	9(0)	47(0)
$X(IV^{*ns})$	$(F_4)^2$	11(0)	107(20)	7(0)	53(10)
$X(I_2^{*ns})$	$(SO(11))^2$	13(0)	85(14)	8(0)	52(7)
$X(I_1^{*s})$	$(SO(10)^2$	13(0)	85(0)	8(0)	52(0)
$X(I_1^{*ns})$	$(SO(9))^2$	11(0)	107(5)	7(0)	63(5)
$X(I_0^{*ns})$	$(SO(7))^2)$	9(0)	121(6)	6(0)	70(3)
$X(I_0^{*ns})$	$(G_2)^2$	7(0)	151(20)	5(0)	85(10)
$X(I_5)^s)$	$(SU(5))^2$	11(0)	91(0)	7(0)	55(0)
$X(I_4^s)$	$(SU(4))^2$	9(0)	121(0)	6(0)	70(0)
$X(I_3^s)$	$(SU(3))^2$	7(0)	151(0)	5(0)	85(0)
$X(I_2)$	$(SU(2))^2$	5(0)	185(0)	4(0)	102(0)
$X(I_0)$	no gen.	3(0)	243(0)	3(0)	131(0)

Let us discuss the $X(I_0)$ case in some detail. From (3.5) we see that Δ has six corners at $\nu_1 = (-1,-1,-1,1)$, $\nu_2 = (-1,-1,2,-1)$, $\nu_3 = (-1,11,-1,-1)$, $\nu_4 = (11,-1,-1,-1)$, $\nu_5 = (11,11,-1,-1)$ and $\nu_6 = (-1,-1,-1,-1)$. The lattice Λ is spanned by standard unit vectors e_i in $\mathbb{R}^4$. We can diagonalize the action of (3.8) without changing the shape of Δ. On the new coordinates $s, s', \tilde{t}, \tilde{t}', z, x, y$, the Z_2 acts by $x_i \mapsto \exp(2\pi i r_i)x_i$ with $r = \frac{1}{2}(0,0,0,1,0,0,1)$. This Z_2 orbifoldisation does commute now with the action of the algebraic torus defining the toric ambient space and hence can be taken by considering the quotient lattice $\hat{\Lambda} = \Lambda/\mathbb{Z}$ of Λ spanned by

$$\hat{e}_1 = e_1,\ \hat{e}_2 = e_2 + e_4,\ \hat{e}_3 = e_1,\ \hat{e}_4 = 2e_4 \tag{3.10}$$

with $\hat{\Lambda}^*$ the dual to $\hat{\Lambda}$, as in [21].

The resolved orbifold Calabi-Yau $\hat{X} = \widehat{X/\mathbb{Z}_2}$ is defined by the old polyhedra (Δ, Δ^*) in the coarser lattice $\hat{\Gamma} = \Gamma/\mathbb{Z}_2$ and the finer dual lattice $\hat{\Lambda}^*$. Note that $\Lambda^* = \hat{\Lambda}^*/\mathbb{Z}_2$ and this defines an action of the $\mathbb{Z}_2$ on $\hat{\Lambda}^*$, which in turn can be used to define the dual orbifoldization $\widehat{X/\mathbb{Z}_2}/\mathbb{Z}_2 = X$. The invariant $(2,1)$-forms ($(1,1)$-forms) correspond to those points of Δ (Δ^*), which are on the coarser lattices $\hat{\Lambda}$ (Λ^*)[5].

Points in $\hat{\Lambda}^*$, but not in Λ^* correspond to the twisted $(1,1)$-forms of the original, and points in Λ but not in $\hat{\Lambda}$ correspond to the twisted $(2,1)$-forms of the dual orbifold.

In the particular case of $X(I_0)$, $h^{inv}_{1,1}(\hat{X}) = 3$ and $h^{inv}_{2,1}(\hat{X}) = 131$ while $h_{1,1}(\hat{X}) = 9$, $h_{2,1}(\hat{X}) = 153$. Keeping the invariant modes, we find agreement with the expectation from the CHL construction.

The vector moduli space of the CHL string is described by the deformations of the mirror polynomial. The mirror manifold of $X(I_0)$ can be itself obtained by a quotient of a group G of order 72 on X. The quotient lattice Λ_M is spanned by $e_1^M = e_1 + e_2 + e_3$, $e_2^M = 12e_2$, $e_3^M = 3e_3$ and $e_4^M = 2e_4$. The mirror polynomial is defined by (3.7) with i

[5] Hence it seems difficult to find such group actions which diminish both the numbers of $(1,1)$ and $(2,1)$ forms. However frequently one can deform the (vector) moduli space to a point, where a sufficient number of (vector) moduli become non-toric, so that their number now indeed depends on the points in the dual lattice. E.g. if we set for the $X(I_{A_1})$ model the perturbations which corresponds to ν_t^* and $\nu_{t'}^*$ to zero, we get the cohomology $h^{1,2} = 185(0)$ and $h^{1,1} = 5(1)$. Then under (3.10) we get the CHL cohomology in the invariant sector: $h^{1,2}_{inv} = 102(0)$ and $h^{1,1}_{inv} = 4(0)$. Similarly the $X(I_{E_8})$ example is at a point in the moduli space where (3.10) gives the CHL spectrum.

running over the relevant points of Δ, which are in Λ^M and j running over the relevant points in Δ^*

$$p = x_0(a_1y^2+a_2x^3+z^6\{a_3(ss'tt')^6+a_4(st)^{12}+a_5(st')^{12}+a_6(s't)^{12}+a_7(s't')^{12}\}+a_0xyss'tt'), \tag{3.11}$$

where the coordinates (s, s', t, t', z, x, y) for the CHL mirror are identified by the action of G, which is generated by $r_1 = \frac{1}{12}(-1,0,0,1,0,0,0)$, $r_2 = \frac{1}{3}(0,0,0,1,-1,0,0)$ but not by $r_3 = \frac{1}{2}(0,0,0,1,0,0,1)$ as it would be for the mirror of the $X(I_0)$ model. As a consequence the CHL moduli space is a double covering of the one of the $X(I_0)$ model.

Another interesting example [5], which is not in Table 1, comes from the $E_8 \times E_8$ heterotic theory on $K3\times T^2$ with symmetric $SU(2)$ instanton embedding $(n_1, n_2) = (10, 10)$ in the $E_8 \times E_8$ and $n = 4$ in the "stringy" $SU(2)$ of the T^2, which we take to be at an enhanced symmetry point. It has $2(20-3)$ hypermultiplets from the instantons in the E_8s, $8-3$ from the ones in the $SU(2)$, 20 from the gravitational sector and $2(3\cdot 56-133)$ from higgsing the E_7. Orbifolding by the CHL Z_2, we find that the hypermultiplet counting for the CHL string should be $17+5+20+35$, while there should be 2 vector multiplets. The polyhedron for the CHL dual is spanned by $\nu_1 = (11,-1,-1,-1)$, $\nu_2 = (-1,5,-1,-1)$, $\nu_3 = (-1,-1,5,-1)$, $\nu_4 = (-1,-1,-1,1)$, and $\nu_5 = (-1,-1,-1,-1)$. After the quotient by (3.10) we get $h^{inv}_{1,1}(\hat{X}) = 2$ and $h^{inv}_{2,1}(\hat{X}) = 76$ and the resolved cohomology is $h_{1,1}(\hat{X}) = 5$ and $h_{2,1}(\hat{X}) = 101$. Again, the invariant cohomology is in accord with the expectation for the CHL spectrum.

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4. Appendix

Although (3.5) and (3.6) define Δ^* for all cases, here we give a more concrete description in a convenient basis (see also [16][24])[6] Let $k_y = (0,1)$, $k_x = (1,0)$, $k_0 = (0,0)$,

[6] In this basis one can easily visualize the $K3$ polyhedron, see [24]. This basis is related to the one which comes out of direct application of (3.6) by the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$

$k_1 = (0,-1)$, $k_2 = (-1,-1)$, $k_3 = (-1,-2)$, $k_z = (-2,-3)$ be the Newton polyhedron of the $X_6(1,2,3)$ elliptic curve and $\nu^n_{k_i} = (0,n,k_i)$. Then Δ^* always involves the relevant points $\nu_0 = (0,0,0,0)$, $\nu^*_s = (1,0,-2,-3)$, $\nu^*_{s'} = (-1,0,-2,-3)$, $\nu^*_t = (0,1,-2,-3)$, $\nu^*_{t'} = (0,-1,-2,-3)$, $\nu^*_z = (0,0,-2,-3)$, $\nu^*_x = (0,0,1,0)$, $\nu^*_y = (0,0,0,1)$, which describe the dual polyhedron and hence the vector moduli space of $X(I_0)$. The unhiggsing of Δ^* adds the following points.

Table 3: Dual Polyhedra for the symmetric cases

$X(I)$	group	additional points
$X(II^*)$	$(E_8)^2$	$\nu^{\pm 6}_{k_z} \ldots \nu^{\pm 2}_{k_z}, \nu^{\pm 4}_{k_3}, \nu^{\pm 3}_{k_2}, \nu^{\pm 2}_{k_1}, \nu^{\pm 1}_{k_0}$
$X(III^*)$	$(E_7)^2$	$\nu^{\pm 4}_{k_z} \ldots \nu^{\pm 2}_{k_z}, \nu^{\pm 3}_{k_3}, \nu^{\pm 2}_{k_2}, \nu^{\pm 2}_{k_1}, \nu^{\pm 1}_{k_0},$
$X(IV^{*s})$	$(E_6)^2$	$\nu^{\pm 3}_{k_z}, \nu^{\pm 2}_{k_z}, \nu^{\pm 2}_{k_3}, \nu^{\pm 2}_{k_2}, \nu^{\pm 1}_{k_1}, \nu^{\pm 1}_{k_0},$
$X(IV^{*ns})$	$(F_4)^2$	$\nu^{\pm 3}_{k_z}, \nu^{\pm 2}_{k_z}, \nu^{\pm 2}_{k_3}, \nu^{\pm 1}_{k_1},$
$X(I_2^{*ns})$	$(SO(11))^2$	$\nu^{\pm 2}_{k_z}, \nu^{\pm 2}_{k_3}, \nu^{\pm 1}_{k_2}, \nu^{\pm 2}_{k_1},$
$X(I_1^{*s})$	$(SO(10)^2$	$\nu^{\pm 2}_{k_z}, \nu^{\pm 2}_{k_3}, \nu^{\pm 1}_{k_2}, \nu^{\pm 1}_{k_1}, \nu^{\pm 1}_{k_0}$
$X(I_1^{*ns})$	$(SO(9))^2$	$\nu^{\pm 2}_{k_z}, \nu^{\pm 2}_{k_3}, \nu^{\pm 1}_{k_2}, \nu^{\pm 1}_{k_1},$
$X(I_0^{*ns})$	$(SO(7))^2)$	$\nu^{\pm 2}_{k_z}, \nu^{\pm 1}_{k_2}, \nu^{\pm 1}_{k_1}, \nu^{\pm 1}_{k_1},$
$X(I_0^{*ns})$	$(G_2)^2$	$\nu^{\pm 2}_{k_z}, \nu^{\pm 1}_{k_1},$
$X(I_5)^s)$	$(SU(5))^2$	$\nu^{\pm 1}_{k_3}, \nu^{\pm 1}_{k_2}, \nu^{\pm 1}_{k_1}, \nu^{\pm}_{k_0},$
$X(I_4^s)$	$(SU(4))^2$	$\nu^{\pm 1}_{k_3}, \nu^{\pm 1}_{k_2}, \nu^{\pm 1}_{k_1}$
$X(I_3^s)$	$(SU(3))^2$	$\nu^{\pm 1}_{k_3}, \nu^{\pm 1}_{k_2}$
$X(I_2)$	$(SU(2))^2$	$\nu^{\pm 1}_{k_3}$
$X(I_0)$	no gen.	-

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Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

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