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A DISCONTINUOUS DEPENDENCE OF LIQUID BRIDGE CONFIGURATIONS*

PAUL CONCUS†, ROBERT FINN‡, AND JOHN MCCUAN§

Abstract. It is shown that, with the exception of very particular cases, any tubular liquid bridge configuration joining parallel plates in the absence of gravity must change discontinuously with tilting of the plates, thereby proving a conjecture of Concus and Finn [Phys. Fluids, 10 (1998), pp. 39–43]. Thus the stability criteria that have appeared previously in the literature, which take no account of such tilting, are to some extent misleading.

Key words. capillary surface, stability, contact angle, liquid bridge, mean curvature

AMS subject classifications. Primary 76B45, 76D45, 53A10. Secondary 49Q05, 53C42, 76E10.

Abbreviated title. LIQUID BRIDGE CONFIGURATIONS

1. Introduction. There is a considerable literature on stability of liquid drops that form bridges joining parallel plates in the absence of gravity. Earlier literature focused on drops whose bounding surfaces were constrained to pass through prescribed circular rings on the plates; more recently papers have appeared concerning drops whose positions on the plates are not prescribed but whose configurations are determined by the natural boundary conditions of prescribed *contact angles* γ_1, γ_2 with the respective plates, arising from the underlying variational problem of minimizing mechanical energy. We mention the papers [1, 2, 5, 6, 8–13, 15–17] of which the sharpest (and in some ways definitive) results appear in [16]. In all cases, the drop surface is known to have a constant mean curvature.

A consequence of the results in [11, 16] is that given the plate separation, then for any γ_1 and γ_2 in $[0, \pi]$ and sufficiently large V, there is a stable drop of volume V forming a tubular bridge that meets the plates in angles γ_1, γ_2; this bridge is determined uniquely up to rigid translation parallel to the plates [16].

In [7], the problem was considered from a more general point of view in which the plates are not assumed parallel, but may meet in an angle 2α, $0 < 2\alpha < \pi$. There it was observed that a spherical bridge exists if and only if $\gamma_1 + \gamma_2 - 2\alpha > \pi$, and it was proved that *if this condition does not hold, then there exists no physically realizable bridge.*

From this theorem we see that the general existence result of [11, 16] fails when the plates intersect; further, we find immediately that all known stability criteria for bridges joining parallel plates are misleading when $\gamma_1 + \gamma_2 \leq \pi$, as they take no account of plate tilting; any bridge between parallel plates will simply disappear on infinitesimal tilting of one of the plates. Physically, the bridge presumably moves into the juncture of the plates, where it forms a disk-type surface or

disappears to infinity along the intersection line; see [3] for distinguishing conditions, and see [4] for conditions under which this surface must be spherical.

In [3] a discussion is given of the effect of plate tilting that includes the remaining case $\gamma_1+\gamma_2 > \pi$. (Although details are given there only for $\gamma_1 = \gamma_2$, only routine change is needed for $\gamma_1 \neq \gamma_2$.) For parallel plates and given γ_1, γ_2, the bridge will be spherical for exactly one value $V = V_s$; otherwise it will be a non-spherical nodoid, catenoid, or unduloid Delaunay surface (see, e.g., [5, 15]). Partly on the basis of numerical evidence, it is suggested in [3] that in all such configurations for which $V \neq V_s$, the dependence on plate tilting will be discontinuous. This result would be immediate from the results of [7], were it known that the spherical configuration is the only one that can occur in a wedge under the given boundary conditions; since that uniqueness has not yet been established, [3] leaves this point unsettled.

If $V = V_s$ then one sees directly that continuous change with tilt is always possible. Uniqueness for the spherical configuration when the plates are not parallel would imply that the change is in this case necessarily continuous.

2. Discontinuous dependence. In the present note we establish that *the dependence on plate tilting is always discontinuous when $V \neq V_s$, independently of the uniqueness question for the spherical bridges.* Our method applies to every configuration and provides in particular a new and much simpler proof in the case $\gamma_1 + \gamma_2 - 2\alpha \leq \pi$, not requiring the deep nonexistence theorem of [7]. We obtain our result as a consequence of necessary conditions satisfied by tubular surfaces in wedges, which need not hold for parallel plates.

By a *tubular bridge joining planes* Π_1, Π_2, we will understand an embedded surface $\mathcal{S}$ that is topologically a portion of a cylinder, bounded by two simple closed curves (contact lines) Σ_1, Σ_2 that lie on the respective planes. We assume the closure of $\mathcal{S}$ to be in the class $C^{(1)}$, and we suppose that $\Sigma_1 \cap \Pi_2 = \Sigma_2 \cap \Pi_1 = \emptyset$. We note that these definitions do not preclude surfaces that cross through the planes, although such surfaces would not physically be possible.

We consider such a surface $\mathcal{S}$ having constant mean curvature H in a configuration for which Π_1 and Π_2 intersect in an angle 2α, $0 < 2\alpha < \pi$, as indicated in the figure. We suppose additionally that $\mathcal{S}$ meets Π_1, Π_2 in the respective constant (contact) angles γ_1, γ_2 measured interior to the tube enclosed by $\mathcal{S}$ and that $0 \leq \gamma_1, \gamma_2 \leq \pi$. We denote by Ω_1, Ω_2 the portions of the planes Π_1, Π_2 enclosed by Σ_1, Σ_2, respectively.

In the absence of an external gravity field, the position vector $\mathbf{x}$ on $\mathcal{S}$ satisfies

$$\Delta \mathbf{x} = 2H\mathbf{N}, \tag{1}$$

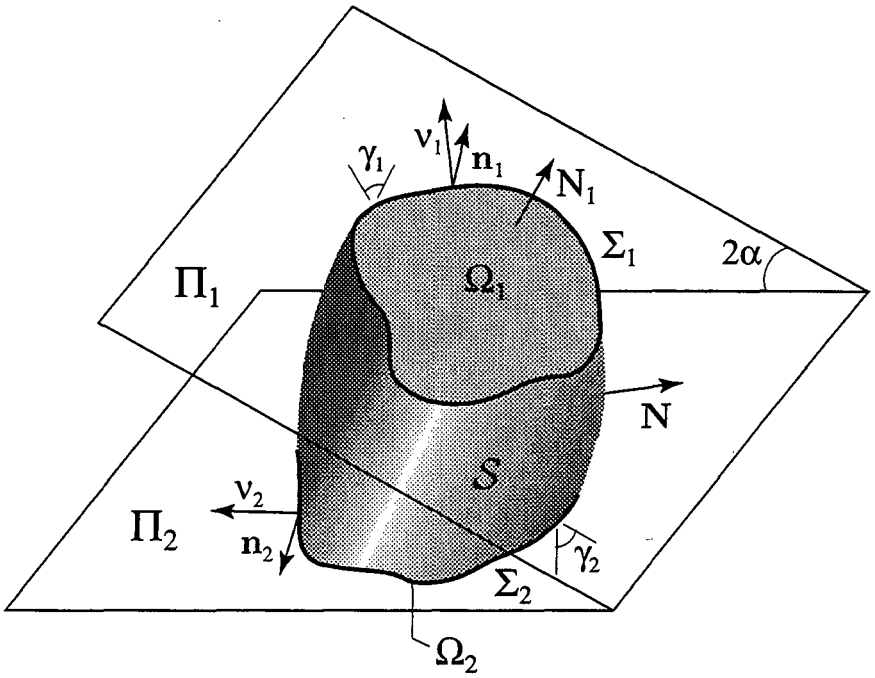

FIGURE. *Tubular liquid bridge between tilted plates.*

where Δ denotes the invariant Laplacian on $\mathcal{S}$, and $\mathbf{N}$ is the unit exterior normal to $\mathcal{S}$. Integrating $\Delta\mathbf{x}$ over $\mathcal{S}$ we obtain

$$2H \int_{\mathcal{S}} \mathbf{N} d\omega = \oint_{\Sigma} \frac{\partial \mathbf{x}}{\partial n} ds = \oint_{\Sigma} \mathbf{n} ds, \tag{2}$$

where $\Sigma = \Sigma_1 \cup \Sigma_2$, and $\mathbf{n}$ is the unit conormal to $\mathcal{S}$ on Σ. We find for each plane Π_j, $j = 1, 2$,

$$\mathbf{n}_j = (\mathbf{n}_j \cdot \mathbf{N}_j)\mathbf{N}_j + (\mathbf{n}_j \cdot \boldsymbol{\nu}_j)\boldsymbol{\nu}_j = \mathbf{N}_j \sin\gamma_j + \boldsymbol{\nu}_j \cos\gamma_j, \tag{3}$$

where $\mathbf{N}_j$ is the unit normal to Ω_j exterior to the volume enclosed by $\mathcal{S}$, Ω_1, and Ω_2, $\boldsymbol{\nu}_j$ is the exterior unit normal to Σ_j in Π_j, and $\mathbf{n}_j$ is the unit conormal to $\mathcal{S}$ on Σ_j (see the figure). We have

$$\oint_{\Sigma_j} \boldsymbol{\nu}_j ds = \int_{\Omega_j} \nabla(1) d\omega = 0. \tag{4}$$

Also, denoting the bridge interior by $\mathcal{V}$, we have

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{N} d\omega &= \int_{\mathcal{S}\cup\Omega_1\cup\Omega_2} \mathbf{N} d\omega - |\Omega_1|\mathbf{N}_1 - |\Omega_2|\mathbf{N}_2 \\ &= \int_{\mathcal{V}} \nabla(1) dV - |\Omega_1|\mathbf{N}_1 - |\Omega_2|\mathbf{N}_2 \\ &= -|\Omega_1|\mathbf{N}_1 - |\Omega_2|\mathbf{N}_2 \end{aligned} \tag{5}$$

by the divergence theorem. From (2), (3), and (5) we obtain

$$(|\Sigma_1|\sin\gamma_1 + 2H|\Omega_1|)\,\mathbf{N}_1 + (|\Sigma_2|\sin\gamma_2 + 2H|\Omega_2|)\,\mathbf{N}_2 = 0. \tag{6}$$

Since $\mathbf{N}_1$ and $\mathbf{N}_2$ are independent, we are led to

THEOREM 1. *A tubular bridge can exist in a wedge only if*

$$|\Sigma_1|\sin\gamma_1 + 2H|\Omega_1| = 0, \qquad |\Sigma_2|\sin\gamma_2 + 2H|\Omega_2| = 0. \tag{7}$$

In this result, we note that $H > 0$ if the curvature vector is directed along the exterior normal to $\mathcal{S}$, otherwise $H \leq 0$. The theorem yields immediately

COROLLARY 1. *There is no tubular bridge in a wedge, for which $H > 0$.*

COROLLARY 2. *There is no tubular bridge in a wedge, for which $H \neq 0$ and with contact angles 0 or π on the bounding walls.*

If the planes are parallel, then the identical proof yields

COROLLARY 3. *If a tubular liquid bridge meets parallel planes in simple curves Σ_1, Σ_2 and makes constant angles γ_1, γ_2 with the planes, then*

$$|\Sigma_1|\sin\gamma_1 + 2H|\Omega_1| = |\Sigma_2|\sin\gamma_2 + 2H|\Omega_2|. \tag{8}$$

Suppose now that the two planes Π_1, Π_2 are (initially) parallel. Then *any tubular liquid bridge of constant mean curvature that meets Π_1 and Π_2 with constant contact angle on each plane is rotationally symmetric and intersects any intermediate parallel plane Π transversally in a circle.* This follows from a procedure due to Wente [14], who adapted the Aleksandrov reflection method to the case of drops on a single planar surface. The procedure applies in the identical way to the case of two parallel planar surfaces considered here.

Let us denote by r the radius of the circle in the plane Π and by ψ the angle between Π and $\mathcal{S}$ along that circle. Applying Corollary 3 with Π in place of Π_2 we obtain $\sin\psi + Hr = \text{const}$ (independent of Π). But if (7) holds on either plate then the constant vanishes, and we conclude that the bridge is spherical. We have proved

THEOREM 2. *Suppose that a tubular bridge between two parallel plates is initially not a sphere. Then the configuration changes discontinuously on infinitesimal tilting of one of the plates, in the sense that either no tubular bridge exists for the tilt, or else that at least one of the quantities Σ_j, Ω_j, H must change discontinuously.*

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