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R. Kimmel and J.A. Sethian

Computing Sciences Directorate
Mathematics Department

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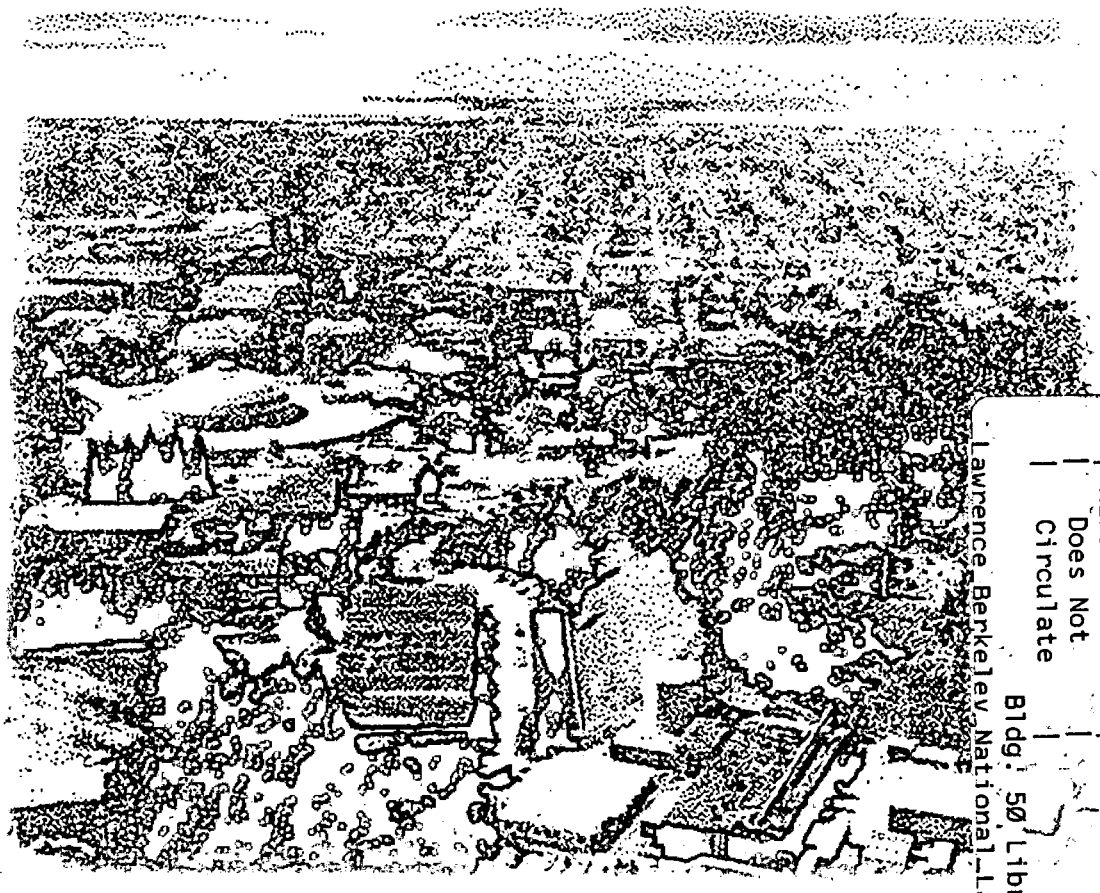

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AN OPTIMAL TIME ALGORITHM FOR SHAPE FROM SHADING[1]

R. Kimmel[||] and J.A. Sethian[2]

Mathematics Department
Computing Sciences Directorate
Lawrence Berkeley National Laboratory
Berkeley, CA 94720

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[||]Computer Science Department, Technion, Haifa 32000, Israel.

[2]Department of Mathematics, University of California, Berkeley, CA 94720.

An Optimal Time Algorithm for Shape from Shading *

R. Kimmel[||] **and** **J.A. Sethian**[**]

Abstract

An optimal numerical algorithm for the reconstruction of a surface from its shading image is presented. The algorithm solves the 3D reconstruction from a single shading image problem. The shading image is treated as a penalty function and the hight of the reconstructed surface is a weighted distance. A first order numerical scheme based on Sethian's Fast Marching Method [19, 18] is used to compute the reconstructed surface. The surface is a viscosity solution to an Eikonal equation for the vertical light source case. For the oblique light source case, the surface is the viscosity solution to a different partial differential equation. A small modification of the Fast Marching Method yields a numerically consistent fast algorithm for the general shape from shading problem.

1 Introduction

One of the earliest problems in the field of computer vision is the reconstruction of a three dimensional object from its single gray level image. The problem, for the case of a diffusive reflectance model of the surface, also known as Lambertian reflectance, is recognized as the 'shape from shading problem' [7, 8]. Various numerical schemes were proposed over the years, most of these methods were based on variational principles that require an additional smoothness or additional regularization terms that introduce second order derivatives into the minimization process, see [9]. Only two early direct models for the shape from shading did not incorporate extra smoothness terms, the first is the characteristic strips expansion method that Horn used when he first introduced the problem [7], the second is Bruckstein's equal hight contours tracking model [2]. Unfortunately, the first numerical implementations of these algorithms suffered from numerical instabilities.

New numerical algorithms based on recent results in curve evolution theory, control theory, and the viscosity framework [4], were applied to the shape from shading problem in [16, 5, 10, 12]. In these advanced numerical algorithms the smoothness assumption is embedded within the scheme without the need for an extra smoothness as a penalty.

Recently, Sethian [19, 18] introduced an $O(N \log N)$ computational steps sequential algorithm for solving the Eikonal equation. This algorithm known as the 'Fast Marching Method'

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[||]Computer Science Department, Technion, Haifa 32000, Israel.

[**]Department of Mathematics, University of California, Berkeley, CA 94720

relies on a systematic causality relationship based on upwinding, coupled with a heap structure for efficiently ordering the updated points. Using this approach, the computation of weighted distances, edge integration in images [3], volumetric segmentation [15], shape from shading with vertical light source [18], low dimensional path planning [13], simulating photolithography development [17], and many other problems can now be solved in an efficient, accurate, and consistent way.

An important property of the solution that distinguishes it from graph search based methods is its converges to the continuous physical (viscosity) solution as the rectangular numerical grid is refined. In a related effort, Tsitsiklis [21] was able to solve the Eikonal equation also on a rectangular grid, with the same computational complexity by iteratively solving a 'cost to go' optimization problem for the dynamically sorted grid points.

In this note we use Sethian's Fast Marching Method and slightly modify it to construct a numerical solution for the oblique light source shape from shading problem.

2 Shape from Shading

Let us first review the shading image formation model for a 3D Lambertian object. Assume, that the object we try to reconstruct is given as a function $z(x,y) : R^2 \to R$, whose surface normal at each point is given by $\vec{n}(x,y) : R^2 \to S^2$. Next, let the light source direction be given by $\vec{l} \in S^2$. Then, the intensity image $I(x,y) : R^2 \to R$ of an orthographic projection of the object is given by the inner product of the light source direction and the surface normal,

$$I(x,y) = \vec{l} \cdot \vec{n}(x,y).$$

For the simple vertical light source case $\vec{l} = (0,0,1)$, in which the light source is located near the viewer, the shading image is given by

$$I(x,y) = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}}.$$

The problem in hand is the reconstruction of $z(x,y)$ from its gradient magnitude at each point given by

$$|\nabla z(x,y)| = \sqrt{(I(x,y))^{-2} - 1}.$$

This equation is known at the *Eikonal equation.* It was shown in [11] that with a simple smoothness assumption, the reconstruction problem can be solved for surfaces with complicated topologies as long as the surface normals are known to be pointing outwards along the boundaries of a given domain (e.g. the image boundaries). In the following sections we deal with the problem of how to solve the problem in a computationally efficient and numerically consistent way.

3 Sethian's Fast Marching Method

The Fast Marching Method is an $O(N \log N)$ numerical algorithm for solving the Eikonal equation, e.g. $|\nabla z(x,y)| = f(x,y)$. Obviously, our shading image is usually given on a

rectangular pixels grid. Therefore, the Fast Marching Method can be directly applied to solve the shape from shading problem with a vertical light source. However, for the general oblique light source, the model to be solved, that can be reformulated by the selection of a new coordinate system, reads $|\nabla z(x,y)| = f(x,y,z(x,y))$ as shown in the following section. Observe that now the right hand side depends on $z(x,y)$. We will show how to include this partial differential equation, which is not an Eikonal equation anymore, within the Fast Marching framework. Full details on the Fast Marching Method are given in [19].

4 Shape from Shading: Oblique Light Source

Let us focus on the oblique light source case in which the light source direction is different than that of the viewer. Recall, that the shading image for this Lambertian case is given by

$$I(x,y) = \vec{l} \cdot \vec{n},$$

where $\vec{l} = (l_1, l_2, l_3)$ is the light source direction, and $\vec{n}$ the unit normal to the surface $z(x,y)$ that we want to reconstruct is given by

$$\vec{n} = \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}. \tag{1}$$

We use our freedom to choose the coordinate system so that $l_2 = 0$, this is done by rotating the (x,y) image plane, so that $\vec{l} \cdot (0,1,0) = 0$. The shading image is then given by

$$I(x,y) = (l_1, 0, l_3) \cdot \vec{n}, \tag{2}$$

where $l_1^2 + l_3^2 = 1$. Eq. (2) involves the term z_x. It requires some additional thought to construct a monotonic approximation to this term and an appropriate update rule.

If we would have had the brightness image in the light source coordinates $\tilde{I}(\tilde{x})$, then the problem would have become the vertical light source case, which is given by the Eikonal equation

$$\tilde{z}_{\tilde{x}}^2 + \tilde{z}_y^2 = \frac{1}{\tilde{I}(\tilde{x},y)^2} - 1, \tag{3}$$

see Figure 1

In the light source coordinate system, the equation to solve looks like the Eikonal equation, yet the right hand side depends on the surface itself via

$$\tilde{I}(\tilde{x},y) = I(l_3\tilde{x} + l_1\tilde{z}, y). \tag{4}$$

That is, we need to evaluate the value of the surface at a point in order to find the 'brightness' and only then plug it in Eq. (4) and use the Fast Marching Method to solve Eq. (3).

In order to overcome this dependence, we use the directional propagation and 'adopt' the smallest $\tilde{z}$ value from all the neighbors of the updated grid point. The numerical algorithm in this case is still consistent, one pass (since the smallest $\tilde{z}$ neighbor will never change its value), and is thus within the fast marching framework.

We have thus extended the method to the case of $|\nabla z| = F(z)$ relevant to the oblique light source shape from shading problem. A consistent solution can be computed with $O(N \log N)$.

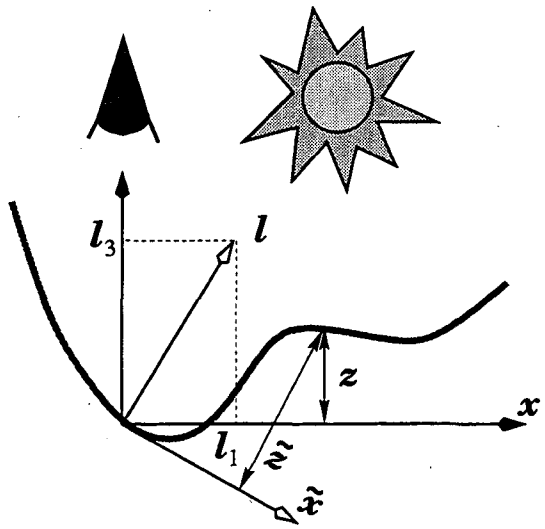

Figure 1: In the oblique light source case, the coordinate system is determined by the light source. The result is a partial differential equation that can be solved by the Fast Marching Method.

5 Experimental Results

We have tested the algorithm on a synthetic shading image of the simplest surface with the three basic types of local extremum points: a maximum, a minimum, and a saddle. The oblique light source is given by $\vec{l} = (0.2, 0, 0.96)$. Observe that we do not deal here with self casting shadows (see [14]), nor with solving the global topological structure (see [11, 20, 6]).

The local extremum points cause singularities at the right hand side of the equation since the intensity at their corresponding image locations is equal to zero. This fact should not cause any problem to our numerical algorithm, since one could set the intensity values that are smaller than $O(\Delta h)$ to some $O(\Delta h)$ without reducing the global order of accuracy. Where Δh is the grid spacing (the distance between two grid points). Figure 2 presents the surface, its shading image, the reconstructed surface, and the error, for the oblique light source case. The surface is the solution to Eq. (3) and (4) with a fixed value at the maximum point (one of the singular points).

6 Conclusion

We have presented an $O(N \log N)$ algorithm for surface reconstruction from its shading image. The computational complexity bound is data independent (unlike other iterative methods [1, 6]). It is the most efficient sequential algorithm for Horn's original formulation of the shape from shading problem and a natural application of the Fast Marching Method.

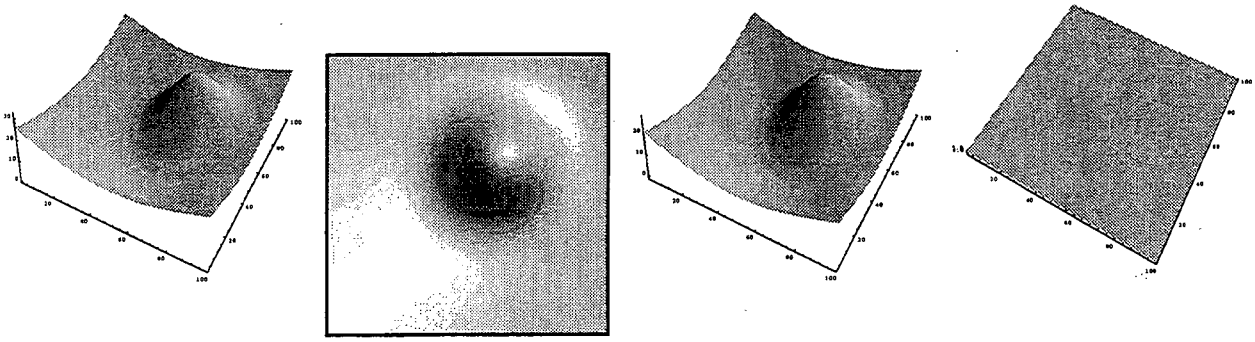

Figure 2: The reconstruction of the surface in the first (left) column, from its shading image in the second column is given in the third column. The forth (right) column is the difference between the original surface and its reconstruction.

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Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

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