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Super Yang-Mills on the Noncomutative Torus*†

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Abstract

After a brief review of matrix theory compactification leading to noncommutative supersymmetric Yang-Mills gauge theory, we present solutions for the fundamental and adjoint sections on a two-dimensional twisted quantum torus in two different gauges. We also give explicit transformations connecting different representations which have appeared in the literature. Finally we discuss the more mathematical concept of Morita equivalence of C^*-algebras as it applies to our specific case.

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1 Introduction

It was conjectured in [1] that the infinite momentum frame description of M-theory is given by the large n limit of supersymmetric quantum mechanics (SQM) [2, 3, 4], obtained as the dimensional reduction of the $9+1$ dimensional $U(n)$ supersymmetric Yang-Mills (SYM) gauge field theory. Shortly afterwards Susskind took this a step further [5], conjecturing that the discrete light cone quantization (DLCQ) of M-theory is equivalent to the finite n matrix theory.

Toroidal compactification of M-theory can then be obtained by first considering matrix theory on the covering space and then imposing a periodicity condition on the matrix variable [1, 6, 7], also known as the quotient condition. The result is a SYM field theory on a dual torus.

If we consider the DLCQ of M-theory and compactify on a torus T^d for $d \geq 2$ there are additional moduli coming from the three-form of 11-dimensional supergravity. For example, if we compactify on T^2 along X_1 and X_2 then C_{-12} cannot be gauged away, and is a modulus of the compactification. It was conjectured in [8] that turning on this modulus corresponds to deforming the SYM theory on the dual torus to a noncommutative SYM on a quantum torus [9] with deformation parameter θ given by

$$\theta = \int C_{-12}\, dX_- dX_1 dX_2.$$

Evidence for this conjecture comes from comparison of the BPS mass spectra of the two theories and of their duality groups. Further evidence and discussions of this conjecture followed in [10, 11, 12, 13, 14, 16, 15, 17, 18, 19].

In Section 2 we present a review of matrix theory compactification leading to noncommutative SYM gauge theory on trivial quantum bundles. We follow the elementary treatment of [17] with an emphasis on giving explicit formulae that closely resemble the commutative case. We further present an explicit realization of the algebra of the quantum torus $\mathcal{A}(T^2_\theta)$ in terms of quantum plane coordinates.

In Section 3 we introduce non-trivial quantum bundles as in [17] corresponding to compactified DLCQ of M-theory in the presence of transversely wrapped membranes. We also explain in some detail how to solve the boundary conditions for sections in the fundamental and adjoint quantum bundle. Finally using the special form of the transition functions in the given gauge we find an equivalent but simpler form of the general solution for fundamental sections.

In Section 4 we discuss the more abstract language of projective modules, as presented in [8] and references therein, and we then give the explicit map between this formulation and the more elementary formulation in [17]. We also explain the notion of Morita equivalence [20, 8, 18, 19, 23] applied to our specific case. For an expanded coverage of noncommutative geometry see [22] and for a brief description see [24].

Finally in Section 5 we discuss the general theory of gauge transformations on the noncommutative torus and find an explicit gauge transformation that trivializes one of the transition functions. With trivial transition functions T-duality transformations take the standard form, allowing us to interpret the gauge field as D-strings on the dual torus.

2 Review of Matrix Compactification

In this section we present a review of matrix theory compactification closely following the description given by Ho in [17]. The $P_- = n/R$ sector of the DLCQ of uncompactified M-theory is given by the $U(n)$ SQM [2, 3, 4] whose action in the temporal gauge is given by

$$S = \frac{1}{2R}\int dt\,\mathrm{Tr}\left(\dot{X}^\mu\dot{X}_\mu + \sum_{\mu>\nu}[X_\mu, X_\nu]^2 + i\Theta^T\dot{\Theta} - \Theta^T\Gamma_\mu[X^\mu,\Theta]\right) \qquad (1)$$

where $\mu,\nu = 0,\ldots,9$. We will compactify matrix theory on a rectangular 2-torus of radii R_1 and R_2. First let us consider matrix variables on the

covering space and impose the quotient condition

$$\begin{aligned} U_i^{-1} X_j U_i &= X_j + 2\pi R_j \delta_{ij}, \\ U_i^{-1} X_a U_i &= X_a, \\ U_i^{-1} \Theta\, U_i &= \Theta, \quad i,j = 1,2 \quad a = 2,\ldots,9. \end{aligned} \tag{2}$$

Here the U_i are unitary operators. The consistency of these equation requires

$$U_1 U_2 = e^{2\pi i\theta} U_2 U_1.$$

Before solving the quotient condition (2), it is convenient to introduce two more unitary operators $\widetilde{U}_i$, $i = 1,2$ which commute with the U_i's and satisfy the relation

$$\widetilde{U}_1 \widetilde{U}_2 = e^{-2\pi i\theta} \widetilde{U}_2 \widetilde{U}_1. \tag{3}$$

One way to realize this algebra is by using canonical variables σ_i satisfying

$$[\sigma_1, \sigma_2] = 2\pi i\theta. \tag{4}$$

Then $\widetilde{U}_i \stackrel{\text{def}}{=} e^{i\sigma_i}$ satisfy (3). The variables σ_i are noncommutative coordinates on the quantum plane which is the covering space of the quantum torus. The algebra of functions on the quantum torus denoted $\mathcal{A}(T^2_{-\theta})$ is generated by $\widetilde{U}_i$. Similarly the U_i operators generate the algebra denoted $\mathcal{A}(T^2_\theta)$. To realize them we introduce partial derivative operators on the quantum plane, satisfying the following algebra

$$[\partial_i, \sigma_j] = \delta_{ij}, \tag{5}$$

$$[\partial_i, \partial_j] = 0.$$

Then, we realize U_i as

$$U_1 = e^{i\sigma_1} e^{2\pi\theta\,\partial_2}, \quad U_2 = e^{i\sigma_2} e^{-2\pi\theta\,\partial_1}.$$

For $\theta = 0$ we have $U_i = \widetilde{U}_i = e^{i\sigma_i}$; all generators commute allowing us to use either U_i's or $\widetilde{U}_i$'s to generate the algebra of functions on the classical torus. It is then easy to check that

$$U_i^{-1} \frac{1}{i}\partial_j \, U_i = \frac{1}{i}\partial_j + \delta_{ij}. \tag{6}$$

This and many other formulae in this paper can be proven using the Campbell-Baker-Hausdorff formula which can be written in closed form since commutators like (4) and (5) are c-numbers. Equation (6) is very similar to the quotient condition (2) so one can write a solution as a sum of the partial derivative and a fluctuating part that commutes with the U_i's. However this is just the definition of the covariant derivative

$$\begin{aligned} X_i &= -2\pi i R_j \delta_{ij}\partial_j + A_i(\widetilde{U}_i), \\ X_a &= X_a(\widetilde{U}_i), \\ \Theta &= \Theta(\widetilde{U}_i), \end{aligned} \tag{7}$$

where A_i, X_a and each spinorial component of Θ are $n \times n$ hermitian matrices with operator valued entries. Note that since the partial derivatives already satisfy the cocycle condition, the gauge fields A_i and the scalar fields X_a must satisfy a homogeneous quotient condition like the second relation in (2). Hence A_i and X_a must depend only on $\widetilde{U}_i$. Hidden in this dependence is the fact that we are working on a trivial bundle over the quantum torus.

If one inserts (7) into the SQM action (1) the result is a noncommutative SYM gauge field theory in $2+1$ dimensions, with the space part given by the above quantum torus and a commutative time. For the commutative case, matrix compactification on T^d results in a SYM gauge theory in $d+1$ dimensions on the dual torus. In the limit when the size of the original torus vanishes the dual torus becomes $\mathbf{R}^d$, therefore we obtain the opposite of dimensional reduction. If one starts from a Euclidean 10-dimensional SYM and dimensionally reduces in all directions including the Euclidean time one obtains the IKKT [25] functional. Matrix compactification of one direction

in the IKKT functional results in the finite temperature action of the original theory (1).

3 Twisted Quantum Bundles

We can consider more general solutions of the quotient condition (2) which are connections on twisted bundles. They correspond to compactification of the DLCQ of M-theory in the presence of transversely wrapped membranes. Again the solution is a sum of two terms, a constant curvature connection D_i and a fluctuating part

$$\begin{aligned} X_i &= -2\pi i R_j \delta_{ij} D_j + A_i(Z_j), \\ X_a &= X_a(Z_i), \\ \Theta &= \Theta(Z_i). \end{aligned} \tag{8}$$

Here the Z_i's are $n \times n$ matrices with operator entries and, just like the $\widetilde{U}_i$'s for the trivial bundle, commute with the U_i's, but now are sections of the twisted bundle whose exact form will be discussed shortly. However, while for the trivial bundle A_i, X_a and the spinorial components of Θ are $n \times n$ matrix functions, in (8) A_i,X_a and the components of Θ are one-dimensional functions but with matrix arguments. Later, this will allow us to establish a relationship between a SYM on a twisted $U(n)$ bundle and one on a $U(1)$ bundle.

Following [17], up to a gauge transformation the constant curvature connection can be written as

$$D_1 = \partial_1, \; D_2 = \partial_2 - if\sigma_1, \tag{9}$$

where f is the constant field strength

$$[D_1, D_2] = -if.$$

Such a gauge field can only exist in a non-trivial bundle. One can introduce transition functions Ω_i such that the sections of the fundamental bundle satisfy the twisted boundary conditions

$$\begin{aligned} \Phi(\sigma_1 + 2\pi, \sigma_2) &= \Omega_1(\sigma_1, \sigma_2)\ \Phi(\sigma_1, \sigma_2), \\ \Phi(\sigma_1, \sigma_2 + 2\pi) &= \Omega_2(\sigma_1, \sigma_2)\ \Phi(\sigma_1, \sigma_2). \end{aligned} \tag{10}$$

Similarly the adjoint sections satisfy

$$\begin{aligned} \Psi(\sigma_1 + 2\pi, \sigma_2) &= \Omega_1(\sigma_1, \sigma_2)\ \Psi(\sigma_1, \sigma_2)\ \Omega_1(\sigma_1, \sigma_2)^{-1}, \\ \Psi(\sigma_1, \sigma_2 + 2\pi) &= \Omega_2(\sigma_1, \sigma_2)\ \Psi(\sigma_1, \sigma_2)\ \Omega_2(\sigma_1, \sigma_2)^{-1}. \end{aligned} \tag{11}$$

Consistency of the transition functions of the bundle requires that

$$\Omega_1(\sigma_1, \sigma_2 + 2\pi)\ \Omega_2(\sigma_1, \sigma_2) = \Omega_2(\sigma_1 + 2\pi, \sigma_2)\ \Omega_1(\sigma_1, \sigma_2). \tag{12}$$

This relation is known in the mathematical literature as the cocycle condition. The covariant derivatives transform just as the adjoint sections

$$D_i(\sigma_1 + 2\pi, \sigma_2) = \Omega_1(\sigma_1, \sigma_2)\ D_i(\sigma_1, \sigma_2)\ \Omega_1(\sigma_1, \sigma_2)^{-1},$$

$$D_i(\sigma_1, \sigma_2 + 2\pi) = \Omega_2(\sigma_1, \sigma_2)\ D_i(\sigma_1, \sigma_2)\ \Omega_2(\sigma_1, \sigma_2)^{-1}.$$

A particular solution for the transition functions compatible with the constant curvature connection (9) and satisfying the cocycle condition is given by

$$\Omega_1 = e^{im\sigma_2/n} U,\ \Omega_2 = V, \tag{13}$$

where $U,\ V$ are $n \times n$ unitary matrices satisfying

$$UV = e^{-2\pi i m/n} VU$$

and m is an integer. For simplicity, here we will only consider the case when n and m are relatively prime. For the general case see [8, 14, 28]. Using the representation given in [17] one has

$$U_{kl} = e^{2\pi i k m/n} \delta_{k,l}, \quad V_{kl} = \delta_{k+1,l},$$

where the subscripts are identified with period n.

We can express the above matrices in terms of the standard 't Hooft matrices [26] denoted here by U' and V' and satisfying

$$U'V' = e^{-2\pi i/n} V'U', \quad U'^n = V'^n = 1.$$

The relation is given by

$$U = e^{2\pi i m/n} U'^m, \quad V = V'. \tag{14}$$

The phase in (14) is due to the nonstandard definition of U used in [17]. This has certain advantages but similar phases will appear when comparing the results of [17] with similar results where the standard 't Hooft matrices were used. We also introduce a unitary matrix K which changes the representation so that V' is diagonal, and satisfies

$$KU'K^{-1} = V'^{-1}, \quad KV'K^{-1} = U'. \tag{15}$$

Note that n is quantized since we are considering a $U(n)$ gauge theory and m is quantized since the magnetic flux f through T_2 is quantized

$$2\pi f = \frac{m}{n - m\theta}.$$

In M-theory m is the transversal membrane wrapping number.

One can solve the boundary conditions (10) for the fundamental sections as in [17] generalizing a previous result for $m = 1$ in the commutative case presented in [7]. Using the ordered exponential explained below, the general solution has the form

$$\Phi_k(\sigma_1, \sigma_2) = \sum_{s \in \mathbf{Z}} \sum_{j=1}^{m} E\left(\frac{m}{n}\left(\frac{\sigma_2}{2\pi} + k + ns\right) + j, i\sigma_1\right) \widehat{\phi}_j \left(\frac{\sigma_2}{2\pi} + k + ns + \frac{nj}{m}\right).$$

The ordered exponential [17] is defined for two variables whose commutator is a c-number

$$E(A, B) = \frac{1}{1 - [A, B]} \sum_{l=0}^{\infty} \frac{1}{l!} A^l B^l.$$

The normalization is such that

$$E(-B, A)E(A, B) = 1$$

and it has the following desirable properties similar to the usual exponential

$$\begin{aligned} E(A + c, B) &= E(A, B)e^{cB}, \\ E(A, B + c) &= e^{cA}E(A, B). \end{aligned} \tag{16}$$

The $\widehat{\phi}_j$ functions are defined on the whole real axis and are unrestricted except for the behavior at infinity. They should be considered as vectors in a Hilbert space on which all the elements of the algebra are represented.

Next we explain in some detail how to obtain this result. First we define

$$\phi(\sigma_1, \sigma_2) \stackrel{\text{def}}{=} \Phi_k(\sigma_1, \sigma_2 - 2\pi(k-1)).$$

The second boundary condition (10) implies that the definition of ϕ is consistent, i.e. k-independent. Using $V^n = 1$ we also find that ϕ is a periodic function in σ_2

$$\phi(\sigma_1, \sigma_2 + 2\pi n) = \phi(\sigma_1, \sigma_2).$$

The other boundary condition gives

$$\phi(\sigma_1 + 2\pi, \sigma_2) = e^{im(\sigma_2+2\pi)/n}\phi(\sigma_1, \sigma_2).$$

It is convenient to separate out a factor to eliminate the above twist

$$\phi(\sigma_1, \sigma_2) = f(\sigma_1, \sigma_2)\check{\phi}(\sigma_1, \sigma_2)$$

and to require a simpler periodicity condition for $\check{\phi}$

$$\check{\phi}(\sigma_1 + 2\pi, \sigma_2) = \check{\phi}(\sigma_1, \sigma_2).$$

Then the function f must satisfy

$$f(\sigma_1 + 2\pi, \sigma_2) = e^{im(\sigma_2+2\pi)/n}f(\sigma_1, \sigma_2).$$

This is satisfied exactly for

$$f(\sigma_1, \sigma_2) = E\left(\frac{m}{n}\left(\frac{\sigma_2}{2\pi} + 1\right), i\sigma_1\right),$$

where in the right hand side we used the ordered exponential defined above. Now we can Fourier transform $\check{\phi}$ in σ_1

$$\check{\phi}(\sigma_1, \sigma_2) = \sum_{p \in \mathbf{Z}} e^{ip\sigma_1}\ \phi_p(\sigma_2)$$

and using the property (16) of the ordered exponential we obtain

$$\phi(\sigma_1, \sigma_2) = \sum_{p \in \mathbf{Z}} E\left(\frac{m}{n}\left(\frac{\sigma_2}{2\pi} + 1\right) + p, i\sigma_1\right) \phi_p(\sigma_2).$$

Let $p = ms + j$ with $j = 1, \ldots, m$ and s is an integer. Then the solution can be written as

$$\phi(\sigma_1, \sigma_2) = \sum_{s \in \mathbf{Z}} \sum_{j=1}^{m} E\left(\frac{m}{n}\left(\frac{\sigma_2}{2\pi} + 1\right) + ms + j, i\sigma_1\right) \phi_{s,j}(\sigma_2),$$

where $\phi_{s,j} \stackrel{\text{def}}{=} \phi_{ms+j}$. Periodicity in σ_2 then implies $\phi_{s-1,j}(\sigma_2 + 2\pi n) = \phi_{s,j}(\sigma_2)$ so that using this recursively we have $\phi_{s,j}(\sigma_2) = \phi_{0,j}(\sigma_2 + 2\pi n s)$. Finally after defining $\widetilde{\phi}_j(x) \stackrel{\text{def}}{=} \phi_{0,j}(2\pi(x-1))$ we obtain

$$\Phi_k(\sigma_1, \sigma_2) = \sum_{s \in \mathbf{Z}} \sum_{j=1}^{m} E\left(\frac{m}{n}\left(\frac{\sigma_2}{2\pi} + k + ns\right) + j, i\sigma_1\right) \widetilde{\phi}_j\left(\frac{\sigma_2}{2\pi} + k + ns\right).$$

This is the result mentioned above up to another redefinition

$$\widetilde{\phi}_j(x) = \widehat{\phi}_j(x + \frac{n}{m} j).$$

While the solutions for the sections of the fundamental bundle given in [17] are suitable for showing the equivalence to the projective modules of [8] as we will discuss in Section 4, the appearance of the ordered exponential is somewhat inconvenient. Using the special form of the transition functions we were able to rewrite the solution in an equivalent but simpler

form. The transition functions in this gauge do not contain σ_1 and it is convenient to order all σ_1 to the right in the solution. Using $V^n = 1$ in the second condition (10) one can express all n components of Φ in terms of a single function with period $2\pi n$ in σ_2. After Fourier transforming in σ_2 and imposing both boundary conditions (10) we obtain the general solution

$$\Phi_k(\sigma_1, \sigma_2) = \sum_{p \in \mathbf{Z}} e^{2\pi i(\sigma_2/2\pi + k)p/n} \; e^{2\pi i(\sigma_1/2\pi - p/m)m/n} \; \widehat{\psi}_p(\sigma_1/2\pi - p/m),$$

where only m of the $\widehat{\psi}_p$ functions are independent, since

$$\widehat{\psi}_{p+m}(x) = \widehat{\psi}_p(x).$$

Using the same technique one can show that an arbitrary adjoint section has the following expansion

$$\Psi(\sigma_1, \sigma_2) = \sum_{s,t \in \mathbf{Z}} c_{st} Z_1^s Z_2^{-t}. \tag{17}$$

Here $c_{s,t}$ are c-numbers and

$$Z_1 = e^{i\sigma_1/(n-m\theta)} V^b, \quad Z_2 = e^{i\sigma_2/n} U^{-b},$$

where b is an integer, such that we can find another integer a satisfying $an - bm = 1$. For n and m relatively prime one can always find integer solutions to this equation. Again, we emphasize that the Z_i's commute with the U_i's. They are generators of the algebra of functions on a new quantum torus

$$Z_1 Z_2 = e^{2\pi i\theta'} Z_2 Z_1,$$

where θ' is obtained by an $SL(2, \mathbf{Z})$ fractional transformation from $-\theta$

$$\theta' = \frac{a(-\theta) + b}{m(-\theta) + n}.$$

Now we outline how to obtain this result. Note first that

$$\Psi(\sigma_1 + 2\pi n, \sigma_2) = \Omega_1^n \; \Psi(\sigma_1, \sigma_2) \; \Omega_1^{-n} = \Psi(\sigma_1 + 2\pi\theta m, \sigma_2).$$

In the last equality we used the fact that $U^n = 1$, and we also used the exponential formula to shift σ_1. Using both boundary conditions we have

$$\Psi(\sigma_1 + 2\pi(n - m\theta), \sigma_2) = \Psi(\sigma_1, \sigma_2),$$

$$\Psi(\sigma_1, \sigma_2 + 2\pi n) = \Psi(\sigma_1, \sigma_2).$$

We can expand the section as

$$\Psi(\sigma_1, \sigma_2) = \sum_{s,t\in\mathbf{Z}} e^{is\sigma_1/(n-m\theta)} e^{-it\sigma_2/n} \Psi_{s,t},$$

where $\Psi_{s,t}$ is a $n \times n$ matrix and can be expanded as

$$\Psi_{s,t} = \sum_{i=i_0}^{n+i_0} \sum_{j=j_0}^{n+j_0} c_{s,t,i,j} V'^i U'^j. \tag{18}$$

Here i_0, j_0 are two arbitrary integers, allowing us to freely shift the summation limits assuming that $c_{s,t,i+n,j} = c_{s,t,i,j+n} = c_{s,t,i,j}$. Then one can obtain further restrictions on the $c_{s,t,i,j}$ coefficients using the boundary conditions (11). For example using the first equation (11) and comparing like coefficients in the Fourier expansion we have

$$c_{s,t,i,j} e^{2\pi is/(n-m\theta)} = c_{s,t,i,j} e^{-2\pi imi/n} e^{2\pi ism\theta/[n(n-,\theta)]}.$$

From this and the similar relation obtained by imposing the second equation (11) we have that $c_{s,t,i,j}$ vanish unless $(s + mi)/n = k$ and $(t + j)/n = s$ for k and s two integers. These equations have multiple solutions. However, if (i, j) and (i', j') are two solutions then $i - i' \in n\mathbf{Z}$ and $j - j' \in n\mathbf{Z}$. This ensures that only one term survives in the sum (18) over i and j. Choosing for later convenience $i_0 = sb$ and $j_0 = mbt$ we have

$$\Psi(\sigma_1, \sigma_2) = \sum_{s,t\in\mathbf{Z}} e^{is\sigma_1/(n-m\theta)} e^{-it\sigma_2/n} \sum_{i=sb}^{n+sb} \sum_{j=mbt}^{n+mbt} c_{s,t,i,j}\, V'^i U'^j.$$

Since n and m are relatively prime let $a, b \in \mathbf{Z}$ such that $an - bm = 1$. Then

$$k = as,\ l = at,\ i = bs,\ j = mbt$$

is an integer solution inside the i and j summation range. Dropping the i, j indices since they are determined by s and t we have

$$\Psi(\sigma_1, \sigma_2) = \sum_{s,t \in \mathbf{Z}} c_{s,t} \left(e^{i\sigma_1/(n-m\theta)} V'^b\right)^s \left(e^{i\sigma_2/n} U'^{-mb}\right)^{-t},$$

which is just (17) after an additional phase redefinition of $c_{s,t}$ to accommodate the phase difference between U and U'^m.

4 Projective Modules and Morita Equivalence

A classic mathematical result of Gel'fand states that compact topological spaces are in one to one correspondence with commutative C^*-algebras. In one direction, to a topological space X we associate the algebra of continuous functions $C(X)$. Conversely and rather nontrivially, the spectrum of a commutative C^*-algebra is equivalent to a compact topological space. This important result allows for a dual description of topological spaces and brings powerful algebraic methods into the realm of topology. On the other hand, if we drop the commutativity requirement, a C^*-algebra $\mathcal{A}$ describe what is called by correspondence a quantum space. To illustrate, consider the algebra of the quantum torus $\mathcal{A}(T^2_\theta)$ generated by the U_i's. An arbitrary element a has the form

$$a = \sum_{k,l \in \mathbf{Z}} a_{k,l} U_1^k U_2^l, \tag{19}$$

where some restrictions (which we do not discuss here) are imposed on the c-number coefficients $a_{k,l}$. For $\theta = 0$, formula (19) reduces to the Fourier expansion of functions on T^2. Thus we can read the compact space from the commutative algebra.

Using the same strategy one can describe other spaces of classical geometry in commutative algebraic terms and then remove the commutativity requirement. A quantum vector bundle is a projective $\mathcal{A}$-module $\mathcal{E}$. First consider the classical commutative picture. The set $\mathcal{E}$ of global sections of a

vector bundle over a base space X has the structure of a projective module over the algebra $C(X)$. Having a module essentially means that we can add sections and can multiply them by functions. Not all modules over a commutative algebra are vector bundles. For example the set of sections on a space consisting of a collection of fibers of different dimensions over a base space also form a module. However, projective modules over the algebra of functions on a topological space are in one to one correspondence with vector bundles over that space. By definition a projective module is a direct summand in a free module. A free module $\mathcal{E}_0$ over an algebra $\mathcal{A}$ is a module isomorphic to a direct sum of a finite number of copies of the algebra

$$\mathcal{E}_0 = \mathcal{A} \oplus \ldots \oplus \mathcal{A}.$$

Trivial bundles correspond to free modules since the description of their sections in terms of components is global, and each component is an element of $C(X)$. For every vector bundle we can find another one such that their direct sum is a trivial bundle. In dual language this implies that the module of sections $\mathcal{E}$ is projective

$$\mathcal{E}_0 = \mathcal{E} \oplus \mathcal{E}'.$$

Again it is nontrivial to show the converse, that every projective module is isomorphic to the set of sections of some vector bundle. Finally projective modules over noncommutative algebras are the quantum version of vector bundles.

In the noncommutative case we distinguish between left and right projective modules. Multiplying fundamental sections from the right with elements of $\mathcal{A}(T^2_{-\theta})$ preserves the boundary conditions (10) while multiplication on the left gives something that no longer is a global section. Thus the set of sections of the fundamental bundle form a right projective module over the $\mathcal{A}(T^2_{-\theta})$ algebra which we denote $\mathcal{F}^{\theta}_{n,m}$. This is no longer true for the adjoint sections since in (11) the transition functions multiply from both the left and right. However one can check that the fundamental and the adjoint are both left

and right projective modules over the $\mathcal{A}(T^2_\theta)$ algebra. This is because the exponents of the U_i's satisfy

$$[i\sigma_1 + 2\pi\theta\partial_2, \sigma_i] = 0, \quad [i\sigma_2 - 2\pi\theta\partial_1, \sigma_i] = 0; \tag{20}$$

thus the U_i's can be commuted over the transition functions in (10) and (11). Additionally, the fact that $\mathcal{F}^\theta_{n,m}$ is both a left $\mathcal{A}(T^2_\theta)$-module and a right $\mathcal{A}(T^2_{-\theta})$-module can be understood as follows. Since $[U_i, \sigma_j] = 0$ we have

$$U_i\Phi(\sigma_1, \sigma_2) = \Phi(\sigma_1, \sigma_2)\widetilde{U}_i,$$

where we dropped the derivatives when there was nothing to their right. Thus multiplying on the left with a is equivalent to multiplying on the right with $\widetilde{a}$

$$a\Phi = \Phi\widetilde{a}, \tag{21}$$

where $\widetilde{a} = \sum_{k,l\in\mathbf{Z}} a_{k,l}\widetilde{U}_2^l\widetilde{U}_1^k$ is the same function as a but with $\widetilde{U}_i$'s as arguments and with all the factors written in reversed order.

As mentioned in [17] the construction in Section 2 is equivalent to the projective modules discussed in [8]. By solving the boundary conditions we went from a local to a global description. Here we present explicit formulae for this equivalence. First one has to express the left actions on the fundamental sections as actions on the Hilbert space [17]. For example the action of the Z_i generators is given by

$$(Z_1\widehat{\phi})_j(x) = \widehat{\phi}_{j-a}(x - \frac{1}{m}), \quad (Z_2\widehat{\phi})_j(x) = e^{-2\pi ij/m}e^{2\pi ix/(n-m\theta)}\widehat{\phi}_j(x).$$

This can be written as

$$Z_1 = W_1^a V_1, \quad Z_2 = W_2 V_2,$$

where V_i and W_i are operators acting on the Hilbert space as

$$(V_1\widehat{\phi})_j(x) = \widehat{\phi}_j(x - \frac{1}{m}), \quad (V_2\widehat{\phi})_j(x) = e^{2\pi ix/(n-m\theta)}\widehat{\phi}_j(x),$$

$$(W_1\widehat{\phi})_j(x) = \widehat{\phi}_{j-1}(x), \quad (W_2\widehat{\phi})_j(x) = e^{-2\pi ij/m}\widehat{\phi}_j(x).$$

These operators satisfy the following relations

$$V_1V_2 = e^{-2\pi i/[m(n-m\theta)]}V_2V_1, \quad W_1W_2 = e^{2\pi i/m}W_2W_1, \quad [V_i, W_j] = 0$$

and can be used to express other operators acting in the Hilbert space. For example we have $U_1 = W_1V_1^{n-m\theta}$ and $U_2 = W_2^nV_2^{n-m\theta}$.

We can now present the correspondence between [8] and [17]. The two integers p and q and the angular variable $\theta_{\rm CDS}$ labeling the projective module $\mathcal{H}_{p,q}^{\theta_{\rm CDS}}$ of [8], and $\theta'_{\rm CDS}$ can be expressed in terms of the quantities used in this paper or in [17]

$$p = n, \quad q = -m, \quad \theta_{\rm CDS} = -\theta, \quad \theta'_{\rm CDS} = \theta'.$$

Then $\mathcal{F}_{n,m}^{\theta} \cong \mathcal{H}_{n,-m}^{-\theta}$. The Hilbert space representation of [8] written in terms of the function $f(s,k)$ with $s \in \mathbf{R}$ and $k \in \mathbf{Z}_q$ is linearly related to the $\widehat{\phi}_k(x)$ representation

$$\widehat{\phi}_k(x) = \sum_{l=1}^{m} \mathcal{K}_{kl}\ (\mathcal{S}_{(-\frac{m}{n-m\theta})}f)(x,l).$$

Here $\mathcal{K}$ is an $m \times m$ representation changing matrix defined as in (15) but for m-dimensional 't Hooft matrices, and $\mathcal{S}_\lambda$ is the rescaling operator $(\mathcal{S}_\lambda f)(x,k) = f(\lambda x, k)$ which can be expressed using the ordered exponential

$$\mathcal{S}_\lambda = \lambda E((\lambda-1)x, \partial_x).$$

Also, using lower case to distinguish them from our current notation which follows [17], the operators in [8] represented in the $\hat{\phi}_k(x)$ basis are given by

$$v_0 = V_2^{n-m\theta}, \quad v_1 = V_1^{n-m\theta}, \quad w_0 = e^{2\pi in/m}W_2^n, \quad w_1 = e^{2\pi i/m}W_1$$

$$z_0 = e^{2\pi i/m}Z_2, \quad z_1 = e^{-2\pi ia/m}Z_1^{-1}, \quad u_0 = e^{2\pi in/m}U_2, \quad u_1 = e^{2\pi i/m}U_1.$$

Next we introduce the Morita equivalence of two algebras [20, 21, 18, 19], which can be used to describe a subgroup of the T-duality group of the

M-theory compactification in the language of noncommutative SYM gauge theory.

Two C^*-algebras $\mathcal{A}$ and $\mathcal{A}'$ are Morita equivalent if there exists a right $\mathcal{A}$-module $\mathcal{E}$ such that the algebra $End_{\mathcal{A}}\mathcal{E}$ is isomorphic to $\mathcal{A}'$. Here $End_{\mathcal{A}}\mathcal{E}$ denotes the set of endomorphisms of the $\mathcal{A}$-module $\mathcal{E}$. It consists of linear maps T on $\mathcal{E}$ where linearity is not only with respect to c-numbers but also with respect to right multiplication by elements of $\mathcal{A}$

$$T(\Phi f) = T(\Phi)f, \quad \Phi \in \mathcal{E}, \quad f \in \mathcal{A}.$$

An example of Morita equivalent algebras is $\mathcal{A}(T^2_{-\theta})$ and $\mathcal{A}(T^2_{\theta'})$. As discussed above, the projective module associated to the quantum fundamental bundle $\mathcal{F}^{\theta}_{n,m}$ is a right $\mathcal{A}(T^2_{-\theta})$-module. One can prove that $End_{\mathcal{A}(T^2_{-\theta})}\mathcal{F}^{\theta}_{n,m}$ is isomorphic to $\mathcal{A}(T^2_{\theta'})$. Here we just show that the two algebras have the same generators. Using (21) we have $T(\Phi\tilde{a}) = T(a\Phi)$ and $T(\Phi)\tilde{a} = aT(\phi)$ and since T is an endomorphism we obtain $T(a\Phi) = aT(\phi)$, which can also be written as $[T, a] = 0$. But the Z_i's were found exactly by requiring that they commute with U_i's so $T \in \mathcal{A}(T^2_{\theta'})$.

The physical interpretation of Morita equivalence is that a $U(n)$ SYM gauge theory on the twisted bundle with magnetic flux m is equivalent to a $U(1)$ gauge theory on a dual quantum torus $T^2_{\theta'}$. This can be seen as a consequence of the discussion following equation (8). The gauge field components A_i, the scalar fields X_a, and the components of Θ are not matrix valued, rather they are one-dimensional. The final result is a matrix because the Z_i's are matrices. On the other hand, we can ignore the internal structure of the Z_i's and just regard them as the generators of $\mathcal{A}(T^2_{\theta'})$, thus allowing us to reinterpret the original theory as a noncommutative $U(1)$ gauge theory on the quantum torus $\mathcal{A}(T^2_{\theta'})$.

Generally, two theories with parameters (n_1, m_1, θ_1) and (n_2, m_2, θ_2) and appropriately chosen compactification radii are equivalent if they are on the

same orbit of the $SL(2,\mathbf{Z})$ duality group

$$-\theta_2 = \frac{A(-\theta_1)+B}{C(-\theta_1)+D},$$

$$\begin{pmatrix} n_2 \\ -m_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n_1 \\ -m_1 \end{pmatrix}.$$

Since θ is a continuous variable, we can interpolate continuously, through noncommutative SYM theories, between two commutative SYM theories with gauge groups of different rank and appropriate magnetic fluxes. This $SL(2,\mathbf{Z})$ duality subgroup has a nice geometric interpretation in the T-dual picture of [10] where it corresponds to a change of basis of the dual torus lattice [14, 28].

5 Gauge Transformations

In this section we consider a gauge equivalent formulation of the previous results closely following the treatment of Taylor in [27] of the corresponding commutative case. In that paper a gauge transformation was considered so as to change the standard 't Hooft transition function into trivial transition function in the X_2 direction. When the transition functions are trivial T-duality has the standard form, i.e. the gauge field translates directly into the position of a D-string on the dual torus. Here we show that a similar gauge transformation can be performed in the noncommutative case. See also [28] for further discussions of this including a relation to the three-string vertex of Douglas and Hull introduced in [10].

First let us consider a general gauge transformation $g(\sigma_1,\sigma_2)$. Just as in the classical case the covariant derivatives transform as $D_i' = g^{-1}D_i g$ resulting in the following transformation for the gauge fields

$$A_i' = g^{-1}A_i g + ig^{-1}\partial_i g. \tag{22}$$

As a result the new transition functions are given by

$$\Omega_1'(\sigma_1,\sigma_2) = g^{-1}(\sigma_1+2\pi,\sigma_2)\Omega_1(\sigma_1,\sigma_2)g(\sigma_1,\sigma_2), \tag{23}$$
$$\Omega_2'(\sigma_1,\sigma_2) = g^{-1}(\sigma_1,\sigma_2+2\pi)\Omega_2(\sigma_1,\sigma_2)g(\sigma_1,\sigma_2).$$

Again all this is just as in the classical case except that one has to take into account the noncommutativity of the σ_i's.

It will be useful to consider first the $\theta = 0$ commutative case. Then we know both the original gauge fields (9) and the transformed ones

$$A_1' = 0, \quad A_2' = \frac{m}{n}\frac{\sigma_1}{2\pi} + Q,$$

where $Q = \frac{1}{n}\mathrm{diag}(0,1,\ldots,n-1)$, and we use primes for all variables in the new gauge. We can write a differential equation for the gauge transformation

$$\partial_1 g = 0, \quad i\partial_2 g = gA_2' - A_2 g = gQ$$

which can be integrated to give

$$g = Ke^{-i\sigma_2 Q}, \tag{24}$$

where the integration constant K is the $n\times n$ matrix (15). It was fixed by requiring a trivial Ω_2' as given by (23). Using the gauge transformation (24) we can now calculate both transition functions

$$\Omega_1' = e^{2\pi i m/n}e^{i\sigma_2 T_m}V^m, \quad \Omega_2' = 1 \tag{25}$$

where $T_k = \mathrm{diag}(0,\ldots,0,1,\ldots,1)$, $k = 1,\ldots,n$ with the first $n-k$ entries vanishing and the last k equal to unity.

Next we discuss the noncommutative case. The first thing to notice is that the original quantum transition functions (13) are θ independent and only contain the σ_2 variable. Similarly the classical gauge transformation (24) only depends on σ_2 so that the classical computation of the new transition

functions is also valid in the quantum case. Using (22) we can compute the new gauge fields

$$A_1' = 0, \quad A_2' = \frac{m}{n - m\theta}\frac{\sigma_1}{2\pi} + \frac{n}{n - m\theta}\, Q.$$

Since (20) implies $[U_i, g] = 0$ we see that the gauge transformation is compatible with the quotient conditions (2). We can use the gauge transformation to obtain the generators of the sections of the adjoint bundle

$$Z_1' = e^{i\sigma_1/(n-m\theta)} e^{-2\pi i n\theta' Q}, \quad Z_2' = e^{2\pi i/n} V e^{i\sigma_2(1-T_{n-1})}.$$

We can also write explicit formulae for the fundamental sections in the new gauge

$$\Phi_k'(\sigma_1, \sigma_2) = \sum_{r\in\mathbf{Z}} e^{i\sigma_2 r} \chi_{k-nr}\left(\frac{\sigma_1}{2\pi} + \frac{k - nr}{m}\right).$$

The χ_s functions are defined over the real axis and must satisfy

$$\chi_{s+m}(x) = e^{-2\pi i m/n} \chi_s(x),$$

so that only m of them are independent. Again, we note that since the transition functions only contain σ_2 and all were ordered to the left of σ_1 in the solution for the sections of the fundamental bundle, they have the same form in the noncommutative and in the classical case.

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