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A 3-DIMENSIONAL TOUSCHEK SCATTERING THEORY

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ABSTRACT

Touschek scattering theory using a 3-dimensional Gaussian velocity distribution is derived. The new theory is better for beams with large velocity spreads or large dispersion. We demonstrate that we can use the functional dependencies of the beam lifetime on beam current and the rf bucket size to measure the gas scattering time, coupling, emittance growth rate, and the dynamic momentum aperture. We observed a good agreement between the new theory and the lifetime measurements in the Advanced Light Source.

KEYWORDS

Touschek Lifetime, Touschek Scattering, Dynamic Momentum Aperture, Gaussian Distribution, High Energy Tail, Intrabeam Scattering

1. INTRODUCTION

Oscillations of electrons in a storage ring are confined in their transverse dimensions (betatron oscillations) through the focusing effects of the magnet lattice, and longitudinally (synchrotron oscillations) through the focusing caused by the radio-frequency (RF) accelerating fields. Binary Coulomb collisions between pairs of electrons in the beam can transfer their usually larger transverse momenta into longitudinal momenta, and the electrons are then lost to the beam if the longitudinal momenta are larger than the momentum acceptance of the accelerator. Bruno Touschek first recognized this loss mechanism at the ADA electron storage ring in Frascati in 1963. [1] The effect is known throughout the accelerator physics community as the "Touschek effect", and the component of beam lifetime associated with this beam loss mechanism is known as the Touschek lifetime. In modern day synchrotron radiation sources the Touschek effect is often the dominant mechanism in determining the beam lifetime. Many authors have reviewed the original theory, subsequently known as the "flat beam model" [2], since the velocity distribution was assumed to have only a horizontal component. In this paper we develop a fully three dimensional description, following the formalism developed for small angle binary Coulomb collisions (also known as intra-beam scattering – IBS) by Piwinski [3] and Bjorken and Mtingwa [4].

In section 3, we derive 2-D and the 1-D approximations from the theory, and compare the 1-D approximation with the flat beam model. In section 4 we compare the results of the different approximations with measurements carried out at the Advanced Light Source (ALS), one of the new lower-energy third generation light sources whose beam lifetime is dominated by the Touschek effect.

velocities before and after the collision. For elastic collisions $|\vec{V}| = |\vec{V}'|$. Let the z-axis be in the direction of beam propagation, y vertically upward, and x radially outward. Let the scattering angle (angle between $\vec{V}$ and $\vec{V}'$) be θ, the polar angle (angle to the z-axis) for $\vec{V}$ and $\vec{V}'$ be χ and χ', and the azimuth (angle around the z-axis) for $\vec{V}$ and $\vec{V}'$ be ϕ, and ϕ'.

The particle is lost if the longitudinal velocity after collision is larger than the maximum positive velocity allowed in the accelerator, $\gamma\, v' \cos\chi' > dv_{max}$. The theory can be easily extended to the case where the value of dv_{max} may have different values for negative velocity deviation. We assume that this is not the case. The effect of the longitudinal velocity before collision is negligible if the distribution is symmetric. Then the particle loss condition can be written as:

$$0 < \chi' < \cos^{-1}\mu, \text{ and } |v| > \frac{dv_{max}}{\gamma}. \tag{6}$$

The beam lifetime, $\frac{1}{\tau} = -\frac{1}{N}\frac{dN}{dt}$, can be obtained by integrating equation (5) over the limit given by equation (6). We now change variables from (v_{2x}, v_{2y}, v_{2z}) to the variables (v_{1x}, v_{1y}, v_{1z}) minus the center of mass variables, (v, χ, ϕ), according to the following relationships.

$$v_{2x} = v_{1x} - 2v_x,\ v_{2y} = v_{1y} - 2v_y,\ v_{2z} = v_{1z} - 2v_z \tag{7.1}$$

$$d^3\vec{v}_2 = -d^3\vec{V} = -8d^3\vec{v} \tag{7.2}$$

$$v_x = |v|\sin\chi\cos\phi,\ v_y = |v|\sin\chi\sin\phi,\ v_z = |v|\cos\chi \tag{7.3}$$

$$v'_x = |v| \sin\chi' \cos\phi',\ v'_y = |v| \sin\chi' \sin\phi',\ v'_z = |v| \cos\chi' \tag{7.4}$$

$$d\Omega' = \sin\chi' \, d\chi' \, d\phi' \tag{7.5}$$

The differential cross-section for elastic Coulomb scattering is according to Moller:

$$\frac{d\sigma}{d\Omega'} = \frac{4 r_e^2 c^4}{V^4} \left[\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta} \right], \tag{8.1}$$

where r_e is the classical electron radius, and c is the velocity of light. The scattering angle θ is related to the new variables by the relationship:

$$\cos\theta = \cos\chi \cos\chi' + \sin\chi \sin\chi' \cos(\phi - \phi') \ , \tag{8.2}$$

which can be readily obtained by considering the triangle formed by the vectors $\vec{v}$ and $\vec{v}'$, and the fact that $|v| = |v'|$.

Equation (5) is integrated in x, y, z, v_{1x}, v_{1y}, and v_{1z} as shown in Appendix A. The beam lifetime at a given point in the accelerator is then:

$$\frac{1}{\tau(s)} = \frac{N_0 \, r_e^2 \, c \, K(\zeta)}{32\pi^3 \tilde{\varepsilon}_x \varepsilon_y \sigma_l (\beta\gamma)^3 \, \delta_A{}^2}, \tag{9.1}$$

where

$$\tilde{\varepsilon}_x{}^2 = \varepsilon_x{}^2 + H_x(\eta_x, \eta'_x) \, \varepsilon_x \left(\sigma_p / p\right)^2, \tag{9.2}$$

$$\zeta = \left(\frac{\delta_A}{\gamma} \right)^2, \tag{9.3}$$

$$K(\zeta) = \zeta \int_{u=\zeta}^{\infty} \int_{\xi=-\xi_0}^{\xi_0} \exp\left\{-u\xi^2\right\}$$

$$\int_{\phi=0}^{2\pi} \exp\left\{-u\left[\left(\frac{\cos^2\phi}{\sigma_{x'}^{\,2}}+\frac{\sin^2\phi}{\sigma_{y'}^{\,2}}\right)\sin^2\chi\right.\right.$$

$$\left.\left.+\frac{2(\alpha_x\eta_x+\beta_x\eta'_x)}{\varepsilon_x}\cos\chi\sin\chi\cos\phi\right]\right\}K_4\,d\phi\,d\xi\frac{du}{u}\ , \tag{9.4}$$

$$K_4(u,\chi)=\int_{\chi'=0}^{\cos^{-1}\sqrt{\zeta/u}}\int_{\phi'=0}^{2\pi}\left[\frac{4}{\left\{1-\left[\cos\chi\cos\chi'+\sin\chi\sin\chi'\cos(\phi-\phi')\right]^2\right\}^2}\right.$$

$$\left.-\frac{3}{1-\left[\cos\chi\cos\chi'+\sin\chi\sin\chi'\cos(\phi-\phi')\right]^2}\right]d\phi'\sin\chi'\,d\chi', \tag{9.5}$$

$$\xi_0=\left(\frac{\gamma}{\delta_p}\right)\left(\frac{\tilde{\varepsilon}_x}{\varepsilon_x}\right)\text{, and} \tag{9.6}$$

$$\cos\chi=\frac{\xi}{\xi_0}. \tag{9.7}$$

The total beam lifetime can be obtained by averaging $1/\tau(s)$ in equation (9.1), over s around the entire storage ring.

A computer code is written that integrates equations (9.1) through (9.7) numerically. It is well known that the Coulomb integral, equation (9.5), does not converge unless one limits the small angle scattering, by limiting the impact parameter to less than the beam size or the Debye length, whichever is smaller. [4] In the ALS, the beam size (0.1 mm) is smaller than the Debye length (1 meter). Therefore, the domain of the $\phi'-\chi'$ integral is limited to the region: $\sin^2\chi\,(\phi-\phi')^2+(\chi-\chi')^2>\sin^2\theta_{\min}$. The value of the integral is quite insensitive to the exact value of $\theta_{\min}$.

3. 2-D AND 1-D APPROXIMATIONS AND THE FLAT BEAM MODEL

If we assume that the momentum spread in equation (9.4) is very small, then, $\chi = \pi/2$, and $\xi_0 = \infty$, and we can perform the ξ integral to obtain the 2-D approximation for the lifetime:

$$\frac{1}{\tau(s)} = \frac{r_e^2 c N_0 K(\varepsilon)}{8\pi\tilde{\varepsilon}_x(\varepsilon_y\beta_y)^{1/2}\sigma_l(\beta\gamma)^3\delta_A{}^2}, \tag{10}$$

where :

$$K(\varepsilon) = \pi^{1/2}\frac{\varepsilon}{(c\beta\gamma)^2}\left(\frac{\varepsilon_x/\beta_x}{\varepsilon_y/\beta_y}\right)^{1/2}\int_\varepsilon^\infty du\frac{1}{u^{3/2}}e^{-Z_+}I_0(-Z_-)\left(\frac{u}{\varepsilon}-\frac{1}{2}\ln\frac{u}{\varepsilon}-1\right), \tag{11.1}$$

$$\varepsilon = \frac{\beta_x}{\varepsilon_x}\left(\frac{dp_{max}}{p_0\gamma}\right)^2, \tag{11.2}$$

$$Z_+ = \frac{u}{2}\left(1+\frac{\varepsilon_x/\beta_x}{\varepsilon_y/\beta_y}\right), \tag{11.3}$$

$$Z_- = \frac{u}{2}\left(1-\frac{\varepsilon_x/\beta_x}{\varepsilon_y/\beta_y}\right), \text{ and} \tag{11.4}$$

$I_0(-Z_-)$ is the modified Bessel function.

If we further assume that the vertical emittance in equation (11) is very small, and $I_0(z) \rightarrow e^z/\sqrt{2\pi z}$, and simplify equation (10) to obtain the 1-D approximation for beam lifetime:

$$\frac{1}{\tau(s)} = \frac{r_e^2 c N_0 C(\varepsilon)}{8\pi\tilde{\varepsilon}_x(\varepsilon_y\beta_y)^{1/2}\sigma_l(\beta\gamma)^3\delta_A{}^2}, \tag{12}$$

where $C(\varepsilon) = \frac{\varepsilon}{(c\beta\gamma)^2}\int_{\varepsilon}^{\infty} du \frac{e^{-u}}{u^2}\left(\frac{u}{\varepsilon} - \frac{1}{2}\ln\frac{u}{\varepsilon} - 1\right)$. (13)

The 1-D approximation as shown in equation (12) is identical to the flat beam theory [2] only when the dispersion is equal to zero. If the dispersion is finite, the 1-D approximation is the same as the flat beam theory if $\tilde{\varepsilon}_x$ in the denominator of equation (12) is replaced by the quantity $\sqrt{(\beta_x\varepsilon_x + \sigma_\delta^2\eta_x^2)\,(\varepsilon_x/\beta_x + \sigma_\delta^2\eta_x'^2)}$.

Therefore, we expect that the difference between the 3-D theory and the flat beam theory increases: (1) if the momentum spread is large, (2) if the vertical velocity spread is large, or (3) dispersion is large. Further illustration of the differences between the two models will be shown in section 4.

4. ALS LIFETIME DATA

Unless otherwise specified, ALS beam parameters used for the present study are as shown in Table I. Beam intensity is varied by varying the number of bunches while keeping the total current constant at 8 mA.

We use the following fitting procedure to make the multi-parameter fitting as simple as possible. First, we consider the functional dependency of the current-lifetime product on beam current as shown in figure 1. We have reduced the rf power for these measurements (synchrotron tune = 0.0055, rf bucket size = 0.018) such that the lifetime is independent of the dynamic momentum aperture.

Beam current and lifetime are measured by measuring beam current using a DCCT at the rate of about once per second for several minutes. Short-term accuracy of current and

lifetime measurements is about $\pm 3\%$ as shown in figure 1. Long-term variations of beam lifetime occurs because of (1) the variations of coupling due to changes in accelerator alignments; (2) variation of momentum aperture due to addition of new insertion devices; and (1) the status of tuning of the beam feedback stabilization system.

Theoretical predictions of beam lifetime is expected to have some systematic errors because of (1) uncertainties in beam emittances and beam sizes used in the calculation, and (2) the high energy tails in the momentum distribution which was ignored in deriving the 3-D theory. For instance, the vertical beam size is about 3×10^{-5} meters which is difficult to measure accurately in the present ALS setup. We introduce a form factor, F, to take care of the systematic errors in fitting data with theory.

For currents below 0.5 mA per bunch, gas scattering is important and intrabeam scattering is negligible. A two-parameter fitting is adequate in this regime:

$$\frac{1}{\tau} = \frac{1}{\tau_{gas}} + \frac{1}{F\tau_{Touschek}}. \tag{14}$$

We find a good fitting for $\tau_{gas} = 55$ hours and F=0.92 as shown in figure 1. The fitted gas scattering time of 55 hours is consistent with measurements using the scraper. The value of F is smaller than unity possibly because the coupling is less than the assumed value of 0.8 %. However, we believe that the required coupling of 0.5 %, which will give F=1, is inconsistent with the vertical beam size measurements. High energy tails, which is not considered in the present theory, may also decrease the predicted beam lifetime. [5, 6]

For currents above 0.5 mA per bunch, emittance growth due to intrabeam scattering and instabilities are important. We use the following model to calculate beam emittances:

$$\frac{d\varepsilon_x}{dt} = -\frac{1}{\tau_x^{SR}}\left(\varepsilon_x - \frac{1}{1+\kappa}\varepsilon_x^{nat}\right) - \frac{1}{\beta\gamma}\frac{d(\beta\gamma)}{dt}\varepsilon_x + F_\varepsilon \gamma_x^{BM}\varepsilon_x \tag{15.1}$$

$$\frac{d\varepsilon_y}{dt} = -\frac{1}{\tau_y^{SR}}\left(\varepsilon_y - \frac{\kappa}{1+\kappa}\varepsilon_x\right) - \frac{1}{\beta\gamma}\frac{d(\beta\gamma)}{dt}\varepsilon_y + F_\varepsilon \gamma_y^{BM}\varepsilon_y \tag{15.2}$$

$$\frac{d\varepsilon_L}{dt} = -\frac{1}{\tau_L^{SR}}\left(\varepsilon_L - \varepsilon_L^{nat}\right) - \frac{1}{\beta\gamma}\frac{d(\beta\gamma)}{dt}\varepsilon_L + F_\varepsilon \gamma_L^{BM}\varepsilon_L \tag{15.3}$$

where $\tau_{x,y,L}^{SR}$ (the radiation damping time), ε_x^{nat} (the natural emittance), and κ (the coupling coefficient) are the input variables. The equilibrium solutions can be obtained either by setting the left-hand-sides of the equations to zero or, in practice, simply calculating the time dependence for long enough times. The adiabatic damping terms, $\frac{1}{\beta\gamma}\frac{d(\beta\gamma)}{dt}\varepsilon_{x,y,L}$, are negligible for storage rings.

The quantity, $F_\varepsilon \gamma_{x,y,L}^{BM}$, are the total emittance growth rates which may include the intrabeam scattering, microwave instabilities, coupled bunch instabilities, etc. In the present study, emittance growth is likely to be due to intrabeam scattering since the beam intensity is below the threshold of the coupled bunch and the microwave instability. The fitting parameter, F_ε, is the ratio of the measured emittance growth rate to the theoretical growth rate predicted by Bjorken and Mtingwa [4]. The discrepancy may be due to : (1) the presence of instabilities, and (2) presence of high energy tails. Raubenheimer [5] has shown that the growth rate of the core emittance may be significantly smaller (about 0.5) than the rms emittance including the high energy tail. We also find $F_\varepsilon = 0.5$ gives a good fitting as shown in figure 1. The implied emittance growth may be due to intrabeam scattering as

well as some possible instabilities. [9] One concludes, however, that the emittance growth rate due to intrabeam scattering is "less" than 0.5 times the value predicted by Bjorken and Mtingwa.

Figure 2 shows the dependency of the current-lifetime product on the rf bucket size. The lifetime increases initially with the rf bucket size, but eventually levels off or "saturates" when the rf bucket size becomes larger than about 2.2 %. The leveling off of the beam lifetime is expected when the rf bucket size becomes larger than the dynamic momentum aperture. The beam lifetime starts to decrease with further increasing of the rf bucket size because the bunch length becomes shorter.

Linear theories of the dynamic momentum aperture, similar to the one in the ZAP [7] code, only qualitatively agrees with measurements, but gives a physical insight. The dynamic aperture, $A(\delta_p)$, at a given location, s*, is a function of the momentum deviation, δ_p, and is usually calculated by particle tracking or measured experimentally. In this case the dynamic aperture is determined by the higher order magnetic fields and lattice misalignments. The dynamic momentum acceptance of the accelerator, δ_A(s) at an other location, s, in the accelerator can be calculated by equating the emittance growth due to a collision at s and the ensuing momentum change to the dynamic acceptance of the accelerator at s*:

$$\mathrm{H_x}(\eta,\eta')\big|_s \delta_A{}^2 = \left[A(\delta_A)\right]^2 / \beta_x^* \tag{15}$$

where $\mathrm{H_x}(\eta,\eta')$ as defined in equation (2) is evaluated at s, and $\beta_x{}^*$ is the horizontal beta function at s*. If dispersion is zero, then $\mathrm{H_x}(\eta,\eta') = 0$, and the momentum acceptance is

determined by the smaller of the rf acceptance or the momentum deviation which gives an amplitude equal to the physical aperture at the location where dispersion is maximum. The resulting dynamic momentum aperture can be approximated by a square function, a constant in the nondispersive region and another constant in the dispersive region.

In reality the particles at the edge of the rf bucket have nonlinear trajectories with significant tune shifts [8,10] as a result of large betatron amplitudes and momentum deviations. Similar to the linear method, the resulting dynamic momentum aperture, δ_A(s), can be approximated as a square function with a constant value (δ_0) at non-dispersive regions and another constant (δ_1) at dispersive regions. We have the best fitting for δ_1=2.2 % and δ_0 = 8% for the ALS data which is shown as a solid line in figure 2. We use the same fitting parameters τ_{gas} , F, and F_ε, found in the previous paragraphs in the present fitting.

The same fitting procedure can be performed using the flat beam theory which will give different values for the fitting parameters. We find that the functional dependency of the current-lifetime product on rf bucket size is qualitatively different at large rf levels and gives lower values for the dynamic momentum aperture.

3. CONCLUSIONS

A generalized Touschek scattering theory using a 3-D Gaussian particle distribution is derived. The new theory agrees well qualitatively and quantitatively with the measured ALS data if one assumes a form factor of 0.92. The procedure used for fitting the measured beam lifetime with the new theory also gives the gas scattering time, coupling, upper limit of the emittance growth rate due to intrabeam scattering, and dynamic momentum aperture.

In order to be able to predict beam lifetime of an accelerator in the design stage, one needs to know how to calculate the dynamic momentum aperture of the accelerator.

Emittance growth rates in the ALS are about one half of the values predicted by the intrabeam scattering theory by Bjorken and Mtingwa. Emittances may also grow due to microwave instabilities [9] as well as intrabeam scattering.

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APPENDIX A

The phase space distribution in the laboratory coordinate system is shown in equation (1) in the main text. For further calculations it is convenient to rewrite the distribution function in the beam frame and in velocity variables. We assume that the beam is non-relativistic transversely and the transverse beam size does not change by the transformation. In the beam frame:

$$\Psi_b(\vec{x},\vec{v}) = \Psi_{b0} \exp\left[-\frac{H_X(x + v_z e_x,\ v_x + v_z e'_x)}{2\varepsilon_x} - \frac{H_Y(y,\ v_y)}{2\varepsilon_y} - \frac{z^2}{2\sigma_{zb}^2} - \frac{v_z^2}{2\sigma_{v_z b}^2} \right] \quad \text{(A1)}$$

where: $v_x = (c\beta\gamma)x'$, $v_y = (c\beta\gamma)y'$, $v_z = (c\beta\gamma)\delta_p/\gamma$, $\sigma_{v_z} = (c\beta\gamma)\sigma_\delta/\gamma$, $\sigma_{zb} = \gamma\,\sigma_z$,

$g_i = \gamma_i$, $a_i = \dfrac{\alpha_i}{c\beta\gamma}$ $b_i = \dfrac{\beta_i}{(c\beta\gamma)^2}$, $e_i = \dfrac{\eta_i\gamma}{c\beta\gamma}$, $e'_i = \eta'_i\,\gamma$ (i = x or y) , σ_{v_z} is the

longitudinal velocity spread, $H_x(x, v_x) = g_x x^2 + 2a_x x v_x + b_x v_x^2$, etc. We have the

identity, $b_x g_x - a_x^2 = 1/(c\beta\gamma)^2$. The normalization constant is

$\Psi_{b0} = N_0/(2\pi)^3 \varepsilon_x \varepsilon_y \varepsilon_{zb} (c\beta\gamma)^2$, where $\varepsilon_{zb} = \sigma_{zb}\sigma_{v_z}$.

The particle loss rate is given by equation (5). Substituting equation (A1) to equation (5), one obtains the lifetime in the beam frame:

$$\frac{1}{\tau_b} = \frac{1}{N_0} \iint_{\vec{r},\vec{v}_1} \iint_{\vec{v}_2,\Omega'} \Psi_{b0}^{\ 2}$$

$$\exp\left[-\frac{H_X(x + v_{1z} e_x,\ v_{1x} + v_{1z} e'_x)}{2\varepsilon_x} - \frac{H_Y(y,\ v_{1y})}{2\varepsilon_y} - \frac{z^2}{2\sigma_z^{\ 2}} - \frac{v_{1z}^{\ 2}}{2\sigma_{v_z}^{\ 2}} \right]$$

$$\exp\left[-\frac{H_X(x+v_{2z}e_x,\ v_{2x}+v_{2z}e'_x)}{2\varepsilon_x}-\frac{H_Y(y,\ v_{2y})}{2\varepsilon_y}-\frac{z^2}{2\sigma_z{}^2}-\frac{v_{2z}{}^2}{2\sigma_{v_z}{}^2}\right]$$

$$\left\{V\frac{d\sigma}{d\Omega}\right\}d^3\bar{x}d^3\bar{v}_1d^3\bar{v}_2d\Omega' \tag{A2}$$

We assume there is no vertical dispersion. Changing variables as defined in equations (7.1) – (7.5) and using the following identity

$$\int e^{ax^2+2bx}dx = e^{-b^2/a}\int e^{a(x+b/a)^2}dx \tag{A3}$$

repeatedly, one can show the following relationship:

$$H_x(u,u')+H_x(u+2a,u'+2b)=2H_x(u+a,u'+b)+2H_x(a,b), \tag{A4}$$

and the lifetime:

$$\frac{1}{\tau_b}=\frac{8\Psi_{b0}^2}{N_0}\iint_{\bar{r},\bar{v}_1}\iint_{\bar{v},\Omega'}\exp\left\{-\frac{H_x(x+\delta_z e_x,\delta_x+\delta_z e'_x)}{\varepsilon_x}-\frac{H_y(y,\delta_y)}{\varepsilon_y}-\left(\frac{\delta_z}{\sigma_{v_z}}\right)^2\right\}$$

$$\exp\left\{-\frac{H_x(-v_z e_x,-v_x-v_z e'_x)}{\varepsilon_x}-\frac{H_y(0,-v_y)}{\varepsilon_y}-\frac{z^2}{\sigma_{zb}^2}-\left(\frac{v_z}{\sigma_{v_z}}\right)^2\right\}$$

$$\frac{4r_e^2c^4}{8v^3}\left\{\frac{4}{\sin^4\theta}-\frac{2}{\sin^2\theta}\right\}dxdydzd\delta_xd\delta_yd\delta_zdvv^2d\Omega d\Omega' \tag{A5.1}$$

where $\delta_{1x}=v_{1x}-v_x$, $\delta_{1y}=v_{1y}-v_y$, $\delta_{1z}=v_{1z}-v_z$, and the factor 8 comes from equation (7.2). Using equation (A3) repeatedly we can integrate equation (A5.1) over variables x, y, z, $d\delta_x$, $d\delta_y$, and $d\delta_z$ as shown below:

$$\int \exp\left\{-\frac{H_x(x+\delta_z e_x, \delta_x+\delta_z e'_x)}{\varepsilon_x}-\left(\frac{\delta_z}{\sigma_{v_z}}\right)^2\right\} d\delta_z d\delta_x dx = \pi^{3/2}\varepsilon_x \sigma_{v_z} c\beta\gamma\,, \tag{A5.2}$$

$$\int \exp\left\{-\frac{H_y(y,\delta_y)}{\varepsilon_y}\right\} d\delta_y dy = \pi\varepsilon_y c\beta\gamma\,, \tag{A5.3}$$

$$\int \exp\left\{-\frac{z^2}{\sigma_{zb}^2}\right\} dz = \sqrt{\pi}\sigma_{zb}\,. \tag{A5.4}$$

Equation (A5.1) becomes:

$$\frac{1}{\tau_B} = \frac{N_0 F r_e^2 c^4}{16\pi^3 \varepsilon_x \varepsilon_y \sigma_{zb} \sigma_{v_z b} (c\beta\gamma)^2} \int_{v=\frac{dv_{max}}{\gamma}}^{\infty} K_3(v)\frac{dv}{v} \tag{A6.1}$$

where

$$K_3 = \int_{\chi=0}^{\pi} \exp\left\{-\frac{v^2\gamma^2\cos^2\chi}{(c\beta\gamma)^2}\left(\frac{H_x(\eta_x,\eta'_x)}{\varepsilon_x}+\frac{1}{\sigma_{\delta L}{}^2}\right)\right\}$$

$$\int_{\phi=0}^{2\pi} \exp\left\{-\frac{v^2}{(c\beta\gamma)^2}\left[\left(\frac{\beta_x}{\varepsilon_x}\cos^2\phi+\frac{\beta_y}{\varepsilon_y}\sin^2\phi\right)\sin^2\chi\right.\right.$$

$$\left.\left.+\frac{2(\alpha_x\eta_x+\beta_x\eta'_x)\gamma}{\varepsilon_x}\sin\chi\cos\chi\cos\phi\right]\right\} d\phi\, K_4(v,\chi)\sin\chi\, d\chi\,, \tag{A6.2}$$

$$K_4(u,\chi) = \int_{\chi'=0}^{\cos^{-1}\sqrt{\zeta/u}} \int_{\phi'=0}^{2\pi}\left[\frac{4}{\sin^4\theta}-\frac{2}{\sin^2\theta}\right] d\phi' \sin\chi' d\chi'\,, \tag{A6.3}$$

and

$$F = \frac{2\sqrt{\pi}\sigma_z}{N_0{}^2}\int N^2(z)dz \tag{A6.4}$$

is the form factor for the longitudinal charge distribution N(z) which is 1 for Gaussian distribution.

We define the following new variables

$$\tilde{\varepsilon}_x{}^2 = \varepsilon_x{}^2 + H_x(\eta,\eta')\varepsilon_x\,\sigma_{p\,lab}{}^2, \tag{A7.1}$$

$$\xi_0 = \frac{\tilde{\varepsilon}_x\gamma}{\varepsilon_x\sigma_{\delta L}}, \tag{A7.2}$$

$$\xi = \xi_0\cos\chi, \tag{A7.3}$$

$$u = \frac{v^2}{(c\beta\gamma)^2}, \tag{A7.4}$$

$$\zeta = u_{v=\frac{dv_{max}}{\gamma}} = \frac{1}{(c\beta\gamma)^2}\left(\frac{dv_{max}}{\gamma}\right)^2, \tag{A7.5}$$

$$\mu^2 = \left(\frac{dv_{max}}{\gamma\,v}\right)^2 = \frac{(c\beta\gamma)^2\zeta}{v^2} = \frac{\zeta}{u}. \tag{A7.6}$$

Finally equation (A6.1) for the lifetime in the beam frame becomes:

$$\frac{1}{\tau(s)} = \frac{N_0 F\, r_e^2\, c\, K(\zeta)}{32\pi^3\tilde{\varepsilon}_x\varepsilon_y\sigma_l(\beta\gamma)^3\,\delta_A{}^2}, \tag{A8.1}$$

where,

$$K(\zeta) = \int_\zeta^\infty \frac{1}{\xi_0\,\zeta}K_3(u)\frac{du}{u} \tag{A8.2}$$

$$K_3 = \frac{1}{\xi_0}\int_{\chi=-\xi_0}^{\xi_0}\exp\{-u\xi^2\}\int_{\phi=0}^{2\pi}\exp\left\{-u\left[\left(\frac{\cos^2\phi}{\sigma_{x'}{}^2}+\frac{\sin^2\phi}{\sigma_{y'}{}^2}\right)\sin^2\chi\right.\right.$$

$$+\frac{2(\alpha_x\eta_x+\beta_x\eta'_x)\gamma}{\varepsilon_x}\cos\chi\sin\chi\cos\phi\Bigg]\Bigg\}\,K_4(u,\xi)\mathrm{d}\phi\, d\xi \tag{A8.3}$$

and

$$K_4(u,\chi)=\int_{\chi'=0}^{\cos^{-1}\sqrt{\zeta/u}}\int_{\phi'=0}^{2\pi}\left[\frac{4}{\left[1-\{\sin\chi\sin\chi'\cos\phi'+\cos\chi\cos\chi'\}^2\right]^2}-\right.$$

$$\left.\frac{3}{\left[1-\{\sin\chi\sin\chi'\cos\phi'+\cos\chi\cos\chi'\}^2\right]}\right]\mathrm{d}\phi'\sin\chi'\mathrm{d}\chi' \tag{A8.4}$$

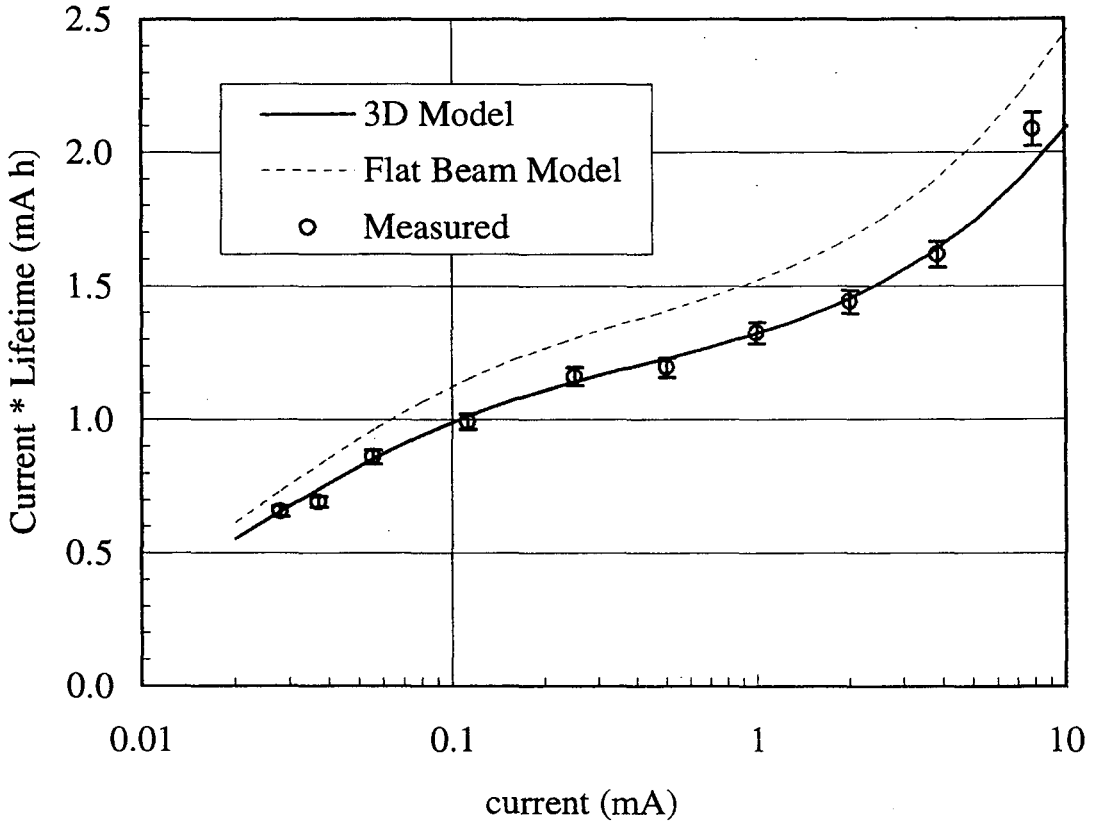
Table I. Typical beam parameters of the ALS used for the lifetime calculations are shown in this table, unless specified otherwise

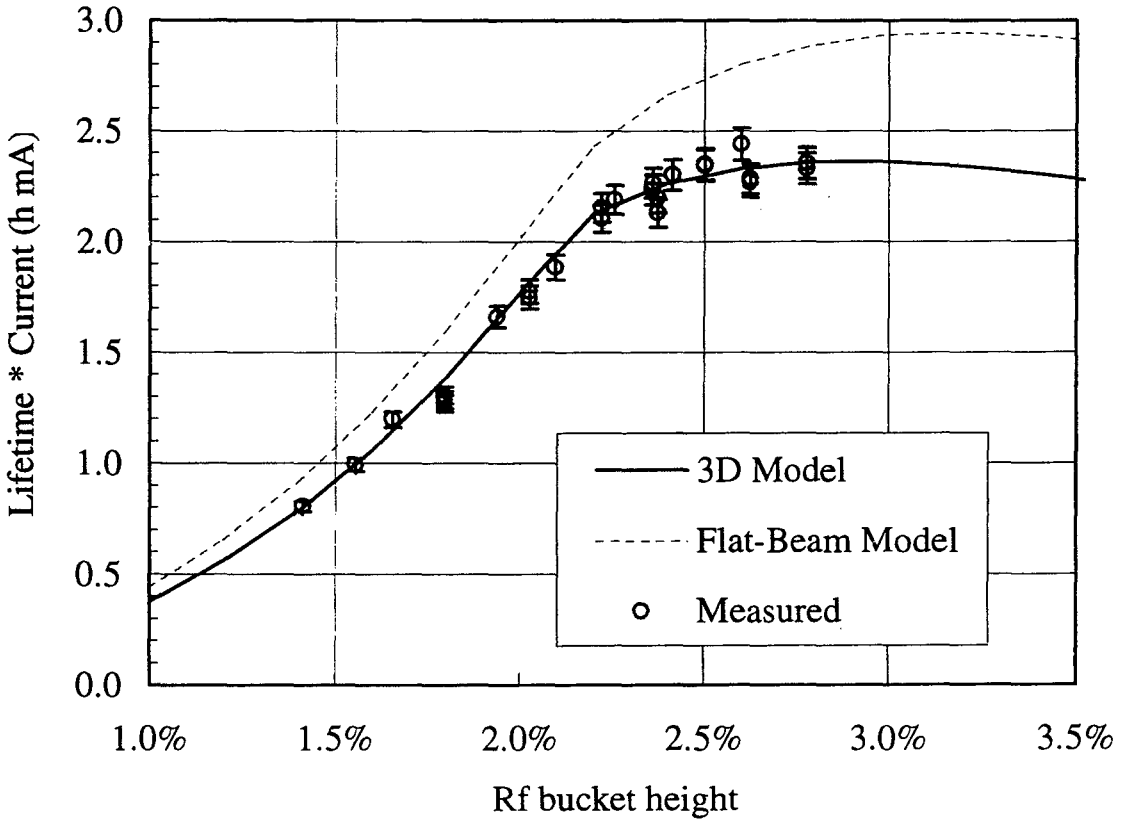
$\nu_s = f_s/f_o$	0.0045 - 0.0079
rf bucket height (δ_{RF}/p)	0.014 - 0.028
beam energy (GeV)	1.522
single bunch current (mA)	0.03 – 8.0
natural emittance (n m rad)	3.4
natural momentum spread (σ_p/p)	6.45×10^{-4}
natural bunch length (mm)	5.92
momentum compaction factor	1.6×10^{-3}
averaged β_x (m)	7.85
averaged β_y (m)	8.34
circumference(m)	196.8
natural emittance ratio (%)	0.8 ± 0.2

FIGURE CAPTIONS

Figure 1. The dependency of the current-lifetime product on beam current. Measured values, the 3-D model and the flat beam model are shown.

Figure 2. The dependency of the current-lifetime product on rf bucket size. The solid line represents the 3-D model for δ_1=2.2 % and δ_0 = 8%, and the dotted line (3) represents the flat beam model for δ_1=2.2 % and δ_0 = 8%.


2.5
2.0
1.5
1.0
0.5
0.0
Current * Lifetime (mA h)
3D Model
Flat Beam Model
Measured
0.01
0.1
1
10
current (mA)


3.0
2.5
2.0
1.5
1.0
0.5
0.0
Lifetime * Current (h mA)
1.0%
1.5%
2.0%
2.5%
3.0%
3.5%
Rf bucket height
3D Model
Flat-Beam Model
Measured

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