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Daniel Brace, Bogdan Morariu, and Bruno Zumino

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Daniel Brace, Bogdan Morariu, and Bruno Zumino

Department of Physics
University of California, Berkeley

and

Physics Division
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

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Duality Invariant Born-Infeld Theory

Daniel Brace*, Bogdan Morariu† and Bruno Zumino‡

Department of Physics
University of California
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

Abstract

We present an $Sp(2n, \mathbf{R})$ duality invariant Born-Infeld $U(1)^{2n}$ gauge theory with scalar fields. To implement this duality we had to introduce complex gauge fields and as a result the rank of the duality group is only half as large as that of the corresponding Maxwell gauge theory with the same number of gauge fields. The latter is self-dual under $Sp(4n, \mathbf{R})$, the largest allowed duality group. A special case appears for $n = 1$ when one can also write an $SL(2, \mathbf{R})$ duality invariant Born-Infeld theory with a real gauge field. We also describe the supersymmetric version of the above construction.

Keywords: Duality, Born-Infeld, Supersymmetry

*email address: brace@thsrv.lbl.gov
†email address: morariu@thsrv.lbl.gov
‡email address: zumino@thsrv.lbl.gov

The general theory of duality invariance of abelian gauge theory developed in [1, 2] was inspired by the appearance of duality in extended supergravity theories [3, 4]. For a theory with M abelian gauge fields and appropriately chosen scalars the maximal duality group is $Sp(2M, \mathbf{R})$. The only known example with an $Sp(2M, \mathbf{R})$ duality symmetry, where the Lagrangian is known in closed form is the Maxwell theory with M gauge fields and an M-dimensional symmetric matrix scalar field.

Without scalar fields the theory is self-dual only under the maximal compact subgroup of the duality group. Noncompact duality transformations relate theories at different values of the coupling constants. The equations of motion derived from the Born-Infeld Lagrangian with a $U(1)$ gauge group are invariant under a $U(1)$ duality group just like pure electromagnetism [5]. Introduction of scalar fields [6, 7, 8, 9], as described by the general theory developed in [1], results in an $SL(2, \mathbf{R})$ duality invariant Born-Infeld theory. However, the proof of duality is complicated by the appearance of the square-root.

In this paper, inspired by the use of auxiliary fields in [10, 11], we present an alternative form of the Born-Infeld action with scalars. The new form, without a square-root, simplifies the proof of duality invariance. Furthermore, using this form we extend the Born-Infeld theory to include more than one abelian gauge field. However, these gauge fields must be complex. To obtain an $Sp(2n, \mathbf{R})$ duality group, the gauge group must be $U(1)^{2n}$. The $n = 1$ case is special in that both real and complex gauge fields are allowed. For a single real gauge field, we give both the formulation in terms of the auxiliary fields and the square-root form obtained after eliminating the auxiliary fields.

For the bosonic Born-Infeld with arbitrary n we have calculated the first few terms in the square-root expansion and based on these we conjecture the general form of the action without auxiliary fields. It involves a symmetrized trace. We also present an $N = 1$ supersymmetrization of the constructions described above.

We now briefly review the general theory of duality invariance of an abelian gauge theory developed in [1]. However, we assume the gauge fields are complex, i.e. we start with an even number of gauge fields and pair them into complex fields. Consider an arbitrary Lagrangian

$$\mathcal{L} = \mathcal{L}(F^a, \bar{F}^a, \phi^i, \phi^i_\mu),$$

where ϕ^i are some scalar fields, F^a are n complex field strengths, and $\bar{F}^a$ their complex conjugate . We define the dual field $G^a_{\mu\nu}$ or rather its Hodge dual $\tilde{G}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{a\,\rho\sigma}$ as

$$\tilde{G}^a_{\mu\nu} \equiv 2\frac{\partial\mathcal{L}}{\partial\bar{F}^{a\,\mu\nu}}, \quad \tilde{\bar{G}}^a_{\mu\nu} \equiv 2\frac{\partial\mathcal{L}}{\partial F^{a\,\mu\nu}}.$$

The main result of the paper [1], and extended here to the case of complex gauge fields, was to find the general conditions such that the equations of motion derived from the Lagrangian $\mathcal{L}$ are invariant under the infinitesimal transformations

$$\begin{aligned} \delta\begin{pmatrix} G \\ F \end{pmatrix} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} G \\ F \end{pmatrix}, \\ \delta\phi^i &= \xi^i(\phi). \end{aligned} \tag{1}$$

In (1) we have combined the field strengths F and its dual G into a $2n$-dimensional column vector, and ξ^i are some unspecified transformations of the scalar fields. The equations of motion are invariant if the matrices A, B, C, and D are real and satisfy

$$A^T = -D, \quad B^T = B, \quad C^T = C, \tag{2}$$

and additionally the Lagrangian transforms as

$$\delta\mathcal{L} = \frac{1}{2}(\bar{F}B\tilde{F} + \bar{G}C\tilde{G}), \tag{3}$$

where all the space-time indices are contracted and a transposition with respect to the gauge index a is used when necessary but is not explicitly written.

The finite form of the transformation (1) is given by

$$\begin{pmatrix} G' \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}, \tag{4}$$

and must be an $Sp(2n, \mathbf{R})$ transformation. This is the group of $2n$-dimensional matrices where the n-dimensional blocks a, b, c and d satisfy

$$c^T a = a^T c, \quad b^T d = d^T b, \quad d^T a - b^T c = 1. \tag{5}$$

Using

$$a \approx 1 + A, \quad b \approx B, \quad c \approx C, \quad d \approx 1 + D,$$

in (5) and keeping only linear terms we obtain the infinitesimal relations (2).

One can check using (1) that the condition on the variation of the Lagrangian (3) is equivalent to the invariance of the following combination

$$\mathcal{L} - \frac{1}{4}\bar{F}\tilde{G} - \frac{1}{4}F\tilde{\bar{G}}. \tag{6}$$

The linear combination (6) must therefore also be invariant under finite transformations. In the known examples for real gauge fields one can write the Lagrangian as a sum of two pieces, the first invariant under $Sp(2n, \mathbf{R})$ and the second equal to $\frac{1}{4}F\tilde{G}$. Similarly, for complex gauge fields there is an invariant piece and a piece equal to $\frac{1}{4}\bar{F}\tilde{G} + \frac{1}{4}F\tilde{\bar{G}}$.

Now we are ready to describe the main result of this paper, a Born-Infeld Lagrangian with a $U(1)^{2n}$ gauge group, written in terms of the auxiliary fields, which is $Sp(2n, \mathbf{R})$ self-dual

$$\mathcal{L} = \mathrm{Re}\left[\mathrm{Tr}\left(i(\lambda - S)\chi - \frac{i}{2}\lambda\chi S_2\chi^\dagger + i\lambda(\alpha - i\beta)\right)\right]. \tag{7}$$

Here the auxiliary fields λ and χ are arbitrary complex n-dimensional matrices, S is a complex symmetric matrix with a positive definite imaginary part, and α and β are hermitian matrices defined in terms of the complex gauge field strengths by

$$\alpha^{ab} = \frac{1}{2}F^a\bar{F}^b, \quad \beta^{ab} = \frac{1}{2}\tilde{F}^a\bar{F}^b.$$

We can write the scalar fields as

$$S = S_1 + iS_2, \quad \lambda = \lambda_1 + i\lambda_2, \quad \chi = \chi_1 + i\chi_2,$$

where S_i, λ_i and χ_i are hermitian matrices. Note that since S is symmetric the S_i's are real symmetric. The Lagrangian (7) is invariant under a parity transformation, under which the fields transform as

$$\begin{aligned} \alpha' &= \bar{\alpha}, \\ \beta' &= -\bar{\beta}, \\ S' &= -\bar{S}, \\ \chi' &= \bar{\chi}, \\ \lambda' &= -\bar{\lambda}. \end{aligned} \tag{8}$$

The duality transformations of the scalar fields are given by

$$\begin{aligned} S' &= (aS + b)(cS + d)^{-1}, \\ \lambda' &= (a\lambda + b)(c\lambda + d)^{-1}, \\ \chi' &= (c\lambda + d)\chi(cS + d)^T. \end{aligned} \tag{9}$$

For convenience we also give the following transformation properties derived from (9)

$$\begin{aligned} S_2' &= (cS + d)^{-T} S_2 (c\bar{S} + d)^{-1}, \\ \lambda_2' &= (c\bar{\lambda} + d)^{-T} \lambda_2 (c\lambda + d)^{-1}, \\ \chi'^{\dagger} &= (c\bar{S} + d)\chi^{\dagger}(c\bar{\lambda} + d)^T. \end{aligned} \tag{10}$$

Explicit use of the symmetry of S was used to obtain the first relation.

Note that one can require a matrix transforming by fractional transformation of the symplectic group to be symmetric. However the transformation of χ is inconsistent with requiring χ to be symmetric. If we want consistent equations of motions derived from the Lagrangian (7) we cannot require a

symmetric λ either. For $n \geq 2$ this in turn is consistent with the transformations of the gauge field strengths and their duals (4) only if we take complex gauge fields.

The term $\mathrm{Re}\,[\mathrm{Tr}\,(i\lambda(\alpha - i\beta))]$ in the Born-Infeld Lagrangian (7) exactly cancels $\frac{1}{4}\bar{F}\tilde{G} + \frac{1}{4}F\tilde{\bar{G}}$ in (6). This is similar to the Maxwell theory where the noninvariant term can be written as $\frac{1}{4}F\tilde{G}$. The first two terms in the Born-Infeld Lagrangian (7) are $Sp(2n, \mathbf{R})$ invariant. To show the invariance it is convenient to use

$$\mathrm{Re}\left[\mathrm{Tr}\,(-\frac{i}{2}\lambda\chi S_2\chi^\dagger)\right] = \mathrm{Tr}\,(\frac{1}{2}\lambda_2\chi S_2\chi^\dagger)$$

to rewrite the second term, and then use the transformations (9) and (10) to show the invariance of $\mathrm{Tr}\,(\frac{1}{2}\lambda_2\chi S_2\chi^\dagger)$ and $\mathrm{Tr}\,(i(\lambda - S)\chi)$.

The equation of motion obtained by varying λ is

$$\chi - \frac{1}{2}\chi S_2\chi^\dagger + \alpha - i\beta = 0. \tag{11}$$

Using this in (7) the coefficient of λ vanishes and the Lagrangian simplifies to

$$\mathcal{L} = \mathrm{Re}\,[\mathrm{Tr}\,(-iS\chi)] = \mathrm{Tr}\,(S_2\chi_1 + S_1\chi_2). \tag{12}$$

For $n = 1$ we can solve explicitly the equation (11). It has two solutions and we chose

$$\chi = \frac{1 - \sqrt{1 + 2S_2\alpha - S_2^2\beta^2}}{S_2} + i\beta,$$

such that the kinetic term in the action has the correct sign. Using this in (12) we finally obtain

$$\mathcal{L} = 1 - \sqrt{1 + 2S_2\alpha - S_2^2\beta^2} + S_1\beta. \tag{13}$$

As mentioned before, for $n = 1$ we can also consider a single real gauge field. In this case we define $\alpha = \frac{1}{4}F^2$ and $\beta = \frac{1}{4}F\tilde{F}$. If $\mathcal{L}_{BI}$ is the standard Born-Infeld Lagrangian without scalar fields

$$\mathcal{L}_{BI}(F) = 1 - \sqrt{\det(\eta + F)},$$

we can also write (7) as

$$\mathcal{L} = \mathcal{L}_{BI}(S_2^{1/2}F) + \frac{S_1}{4}F\widetilde{F}. \tag{14}$$

As discussed in [9], this is the standard way of extending the self-duality group from its compact form to the maximally split noncompact duality group by introducing scalar fields.

For arbitrary n the equation (11) implies

$$\chi_2 = \beta$$

and using this in (11) we obtain an equation for χ_1

$$\chi_1 = \frac{1}{2}(\chi_1 S_2 \chi_1 + i\beta S_2 \chi_1 - i\chi_1 S_2 \beta + \beta S_2 \beta) - \alpha. \tag{15}$$

This equation simplifies using the following field redefinitions

$$\begin{aligned} \widehat{\chi_1} &= S_2^{1/2}\chi_1 S_2^{1/2}, \\ \hat{\alpha} &= S_2^{1/2}\alpha S_2^{1/2}, \\ \widehat{\beta} &= S_2^{1/2}\beta S_2^{1/2}. \end{aligned} \tag{16}$$

The hatted variables satisfy equation (15) with $S_2 = 1$. Then $X \equiv 1 - \widehat{\chi_1}$ satisfies

$$X^2 = 1 + 2\hat{\alpha} - \widehat{\beta}^2 + i[\widehat{\beta}, X], \tag{17}$$

and the Lagrangian takes the form

$$\mathcal{L} = n - \mathrm{Tr}\,(X) + \mathrm{Tr}\,(S_1\beta).$$

We can solve the equation (17) as an expansion in powers of F^2

$$X = \sum_{m\geq 0} \frac{1}{m!} X^m, \tag{18}$$

where m counts the number of times F^2 appears in X^m. One can show using (17) that X^m satisfies the following recursion relation

$$2X^m = -\sum_{j=1}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} X^j X^{m-j} + A^m + im[\widehat{\beta}, X^{m-1}],$$

where $X^0 = 1$, $A^1 = 2\widehat{\alpha}$, $A^2 = -2\widehat{\beta}$ and $A^m = 0$ for $m \geq 3$. We have calculated up to the $\mathrm{Tr}X^6$ term in the expansion for the trace of X using (18) and the result coincides with the symmetrized trace of the square root of $1 + 2\widehat{\alpha} - \widehat{\beta}^2$

$$\mathrm{Tr}\,(X) = \mathrm{Tr}_S \sqrt{1 + 2\widehat{\alpha} - \widehat{\beta}^2}. \tag{19}$$

Here Tr_S denotes the symmetrized trace where symmetrization is with respect to $\widehat{\alpha}$ and $\widehat{\beta}$. One has to first expand the square root, then symmetrize each monomial in the expansion, and finally take the trace. We conjecture that the relation (19) is true to all orders. Then the Born-Infeld Lagrangian takes the form

$$\mathcal{L} = n - \mathrm{Tr}_S \sqrt{1 + 2\widehat{\alpha} - \widehat{\beta}^2} + \mathrm{Tr}\, S_1\beta. \tag{20}$$

The appearance of only even powers of β in the second term of (20) is due to the discrete symmetry (8).

In general one also adds to (7) a nonlinear sigma model Lagrangian for the S field. A metric invariant under the transformations (9) is given by

$$\mathrm{Tr}\left[(\bar{S} - S)^{-1} d\bar{S} (S - \bar{S})^{-1} dS\right], \tag{21}$$

which is a generalization of the metric on the Poincare upper half plane.

Finally we briefly discuss the supersymmetric Born-Infeld action. Using the superfields $V^a = \frac{1}{\sqrt{2}}(V_1^a + iV_2^a)$ and $\check{V}^a = \frac{1}{\sqrt{2}}(V_1^a - iV_2^a)$ where V_1^a and V_2^a are vector superfields, we define

$$W_\alpha^a = -\frac{1}{4}\bar{D}^2 D_\alpha V^a, \quad \check{W}_\alpha^a = -\frac{1}{4}\bar{D}^2 D_\alpha \check{V}^a.$$

Note that both W^a and $\check{W}^a$ are chiral superfields. Let us also define

$$\mathcal{M}^{ab} = W^a \check{W}^b.$$

We can construct the supersymmetric version of the Lagrangian (7)

$$\mathcal{L} = \mathrm{Re} \int d^2\theta \left[\mathrm{Tr}\,(i(\lambda - S)\chi - \frac{i}{2}\lambda \bar{D}^2(\chi S_2 \chi^\dagger) - i\lambda\mathcal{M})\right]. \tag{22}$$

Here S, λ and χ are chiral superfields with the same symmetry properties as the corresponding bosonic fields. The bosonic fields S and λ appearing in (7) are the lowest component of the superfields denoted by the same letter. The field χ in the action (7) is the highest component of the superfield χ.

Just as in the bosonic case for $n = 1$ we can also consider a Lagrangian with a single real superfield. In this case one can integrate out the auxiliary superfields and obtain a supersymmetric version of (14)

$$\mathcal{L} = \int d^4\theta \frac{S_2^2 W^2 \bar{W}^2}{1 - A + \sqrt{1 - 2A + B^2}} + \mathrm{Re}\left[\int d^2\theta (-\frac{i}{2} S W^2)\right], \qquad (23)$$

where

$$A = \frac{1}{4}(D^2(S_2 W^2) + \bar{D}^2(S_2 \bar{W}^2)), \quad B = \frac{1}{4}(D^2(S_2 W^2) - \bar{D}^2(S_2 \bar{W}^2)).$$

For $S = i$ this reduces to the supersymmetric Born-Infeld action described in [12, 13, 14]. In the case of weak fields the first term of (23) can be neglected and the Lagrangian is quadratic in the field strengths. Under these conditions the combined requirements of supersymmetry and self-duality have been used in [15] to constrain the form of the weak coupling limit of effective supergravity Lagrangians describing the low energy limit of string theory.

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Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

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