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New Physics and the Landau Pole

Michael S. Chanowitz

Physics Division

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New Physics and the Landau Pole [1]

Michael S. Chanowitz[2]

Theoretical Physics Group
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

Abstract

In scalar field theories the Landau pole is an ultraviolet singularity in the running coupling constant that indicates a mass scale at which the theory breaks down and new physics must intervene. However, new physics at the pole will in general affect the running of the low energy coupling constant, which will in turn affect the location of the pole and the related upper limit ("triviality" bound) on the low energy coupling constant. If the new physics is strongly coupled to the scalar fields these effects can be significant even though they are power suppressed. We explore the possible range of such effects by deriving the one loop renormalization group equations for an effective scalar field theory with a dimension 6 operator representing the low energy effects of the new physics. As an independent check we also consider a renormalizable model of the high-scale physics constructed so that its low energy limit coincides with the effective theory.

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[2]Email: chanowitz@lbl.gov

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(1) Introduction

In a classic paper Dashen and Neuberger[1] showed in perturbation theory at one loop that the location of the Landau pole in scalar field theory implies an upper limit on the mass of the Higgs boson. The Landau pole indicates the mass scale at which the running coupling constant, λ_Q, diverges. In the elegantly simple DN (Dashen-Neuberger) analysis it implies an upper bound on the scale where new physics supplants the scalar field theory, which is regarded as an effective low energy description of the Higgs sector. By requiring a minimal hierarchy between the new physics scale Λ and the Higgs boson mass, e.g., $\Lambda \geq 2m_H$, DN obtained an upper bound on m_H from the perturbative relation between the low energy coupling constant λ and the ratio Λ/m_H. They proposed a space-time lattice "experiment" to confirm the bound and make it quantitative. Lattice calculations[2] have established the bound on m_H at about 700 GeV, not far from the $\simeq 1$ TeV estimate of DN.

The purpose of this paper is to explore, in a similarly transparent way using one loop perturbation theory, the extent to which the new physics that must occur at or below the Landau pole can affect the relationship between the pole location and the low energy coupling constant. In this paper we consider the simplest case: O(N) ϕ^4 field theory in the symmetric phase, for which the DN analysis implies an upper bound on the coupling constant. The broken symmetry phase will be considered elsewhere.

Since new physics must exist at the Landau pole, it is not optional but essential to consider its possible effect on the analysis. The obvious method is to introduce higher dimension operators to represent the power-suppressed, low energy effects of the new physics. For instance, effects of dimension 6 operators are suppressed by $\mu^2/\Lambda^2_{\text{Landau}}$ where μ is the low energy renormalization scale and Λ_{Landau} is the scale of the Landau pole.[3] For the minimal hierarchy used to obtain the upper bound this suppression is only a factor 1/4, which could be overcome if the new physics is strongly coupled to the low energy scalar sector. We will compute the effect of such an operator on the running of the scalar coupling constant and the position of the Landau singularity.

Most lattice studies of triviality (e.g., [2]) considered renormalizable scalar field theories without higher dimension operators representing the possible effects of new physics and would apply literally if the new physics at the pole were actually the assumed space-time lattice. Some lattice simulations[3] (of the Higgs phase) have explored the effects of new physics by introducing higher dimension operators, as we do here, but with a different focus. Their results agree qualitatively with ours but are not directly comparable for two reasons (in addition to the fact that different phases are considered). First, a precise comparison would require studying the same operators with carefully matched normalizations. Second, the

[3]Away from the triviality limit the new physics could lie below the pole, $M_{\text{New}} < \Lambda_{\text{Landau}}$, in which case the effects of the new physics would be larger, $\propto \mu^2/M^2_{\text{New}}$.

goal in [3] is to establish an upper limit on m_H such that corrections to various Higgs sector quantities (e.g., the Higgs decay width) from the higher dimension operators are limited to a few percent, whereas the focus in this paper is on the maximum allowed value, independent of the size of other corrections, which are not known experimentally and could actually be large.

The coupling constants must be defined by renormalization conditions. We define the scalar coupling constant λ in terms of the $2 \to 2$ scattering amplitude, for off-shell external momenta, as is customary in the RG (renormalization group) analysis in order to avoid mass singularities.[4] Then the first dimension 6 operator that comes to mind, $\propto \phi^6/\Lambda^2$, does not contribute to the running of λ_Q, since its one loop contribution to the scattering amplitude, shown in figure 1, is a (divergent) constant, independent of the external scale Q. For the off-shell renormalization condition adopted below there is just one other independent O(N) symmetric dimension 6 operator, which we choose to write in the form $\kappa(\partial(\phi)^2)^2$. Here $\kappa = C/\Lambda^2$ where C is a dimensionless constant and Λ is the mass scale of the high energy theory, which we identify with the position of the Landau pole. Using the off-shell renormalization condition, κ is also defined in terms of the $2 \to 2$ amplitude. Operator mixing occurs, resulting in coupled renormalization group equations for λ and κ which we compute to order λ^2 and $\lambda\kappa$. Solving the coupled RGE's (renormalization group equations) we find fractional corrections of order $\kappa\mu^2/\lambda$ to the Landau pole postion, Λ_{Landau}, and to the upper limit on the low energy coupling λ, where μ is the low energy renormalization scale, chosen to be the scalar mass.

While effective Lagrangians were first used strictly in tree approximation, it has long been realized that it makes sense to consider them at the quantum level.[5] Though technically "nonrenormalizable" in the sense that they cannot be defined to all orders by a finite number of renormalization conditions, they can be renormalized to any finite order. The quantum effects of chiral effective Lagrangians have been thoroughly analyzed at the one loop level[6] and one loop quantum corrections from dimension 6 operators have been used to study the possible consequences of new physics in electroweak gauge theories.[7] These calculations can be carried out to useful approximations, though arbitrary precision would require arbitarily many renormalization constants. This is not a concern, since arbitrary precision is in any case not the goal in applications of effective theories. See [8] for an excellent review with interesting examples and additional references.

We have verified the renormalization of the effective theory considered here, first by checking explicitly that the result is independent of the choice of regulator (for dimensional regularization, Pauli-Villars regularization, or Euclidean space cutoff) and second by obtaining the same result from a renormalizable model with an additional, heavy O(N) singlet scalar field, constructed so that its low energy limit corresponds to the effective theory. Because of

the dimension 6 operator the effective theory has quadratic and logarithmic divergences at one loop. The quadratic divergences are constants, independent of the renormalization scale and are absorbed into the $\delta\lambda$ counterterm without affecting the running of λ. Furthermore, as the renormalizable model makes clear, the quadratic divergences are in any case artifacts of the effective theory, dominated by the scale of the cutoff where the effective theory breaks down. In contrast the logarithmic divergences reflect the domain in which the effective theory is valid and may be reliably extracted from the effective theory. They give rise to the renormalization scale dependence from which the RGE's follow.

Section 2 presents a brief review of the DN analysis, modified slightly to apply to the symmetric phase. In section 3 we derive the one loop coupled RGE's for the effective theory. In section 4 the results are rederived from the renormalizable model. The coupled RGE's are solved in section 5. In section 6 we use the solutions to estimate the corrections to the Landau pole position and to the triviality bound in the strong coupling regime. We conclude with a brief discussion in section 7.

(2) The DN analysis

We review the DN analysis, considering both the broken symmetry phase of O(4) scalar field theory (the SM Higgs sector) as considered by DN and also the unbroken phase which is the focus of this paper. For renormalizable ϕ^4 theory the ultraviolet RG behavior of the two phases is the same and the DN analysis applies also to the symmetric phase. However we must modify the renormalization conditions slightly, since DN used the Higgs boson mass m_H to specify the low energy coupling λ. In order to have a method that applies also to the symmetric phase we will define the low energy coupling constant in terms of the $2 \to 2$ scattering amplitude.

Where ϕ is an N component scalar field the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}(\phi^2)^2 - \frac{\mu^2}{2}\phi^2. \tag{2.1}$$

We first consider the symmetric phase, $\mu^2 > 0$. At the quantum level, with the RG analysis in mind, we define the renormalized coupling constant, $\lambda = \lambda_\mu$, by an off-shell scattering amplitude[4] chosen for convenience at a symmetric, space-like scale

$$s = t = u = -\mu^2, \tag{2.2}$$

that is,

$$\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1)_{s=t=u=-\mu^2} = -6i\lambda. \tag{2.3}$$

The one loop amplitude (see figure 2a) at an arbitrary space-like scale $Q^2 < 0$ is

$$\mathcal{M}^{(1)}_{s=t=u=Q^2} = 3\left((-6i\lambda)^2 + (N-1)(-2i\lambda)^2\right)\frac{i^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\mu^2)((k+Q)^2-\mu^2)} \tag{2.4}$$

Rewriting the integrand as a parametric integration $\int dx$ and performng the k integration by dimensional regularization in n dimensions the integral in (2.4) becomes

$$\frac{i}{16\pi^2}\Gamma(\epsilon)\int_0^1 dx\left(\frac{(x^2-x)Q^2-\mu^2}{\mu_D^2}\right)^{-\epsilon} \tag{2.5}$$

where Γ is the Gamma function, $\epsilon = 2-\frac{n}{2}$ and μ_D is the regularization scale. The x integration is then evaluated for $-Q^2 \gg \mu^2$, with the result

$$\mathcal{M}_Q^{(1)} = \frac{3i(N+8)\lambda^2}{8\pi^2}\left(\Gamma(\epsilon)-\log\frac{Q^2}{\mu_D^2}\right)+\ldots \tag{2.6}$$

where the omitted terms in (2.6) are finite constants that will be absorbed by counterterms without affecting the running of λ_Q.

Using the method of "renormalized perturbation theory", [9] we introduce counterterms,

$$\mathcal{L}_{\rm CT} = \frac{\delta Z}{2}(\partial\phi)^2 - \frac{\delta\lambda}{4}(\phi^2)^2 - \frac{\delta\mu^2}{2}\phi^2, \tag{2.7}$$

so that the amplitude through one loop is

$$\mathcal{M}_Q = -6i(\lambda+\delta\lambda)+\mathcal{M}_Q^{(1)}. \tag{2.8}$$

The counterterm $\delta\lambda$ is then determined from the definition of λ, equation (2.3), to be

$$\delta\lambda = -\frac{i}{6}\mathcal{M}_\mu^{(1)}. \tag{2.9}$$

Defining the running coupling constant at scale Q as

$$\lambda_Q = \frac{i}{6}\mathcal{M}_Q \tag{2.10}$$

we then find

$$\lambda_Q = \lambda + \frac{i}{6}(\mathcal{M}_Q-\mathcal{M}_\mu) = \lambda + \frac{N+8}{8\pi^2}\lambda^2\log\frac{Q}{\mu}. \tag{2.11}$$

The wave function and mass renormalizations can be neglected because they are trivial in the ϕ^4 model at one loop: the wave function renormalization vanishes and the mass is renormalized by a Q independent constant that is absorbed in the mass counterterm.

From (2.11) it is easy to determine the Landau pole and the upper bound on $\lambda = \lambda_\mu$. Differentiating (2.11) we have the RGE

$$\frac{d\lambda_Q}{d\log Q} = b_N\lambda^2 = b_N\lambda_Q^2 + O(\lambda^3) \tag{2.12}$$

where

$$b_N = \frac{N+8}{8\pi^2}. \tag{2.13}$$

Integrating (2.12) from μ to Q we have

$$\lambda_Q = \frac{\lambda}{1 - b_N \lambda \log\left(\frac{Q}{\mu}\right)} \tag{2.14}$$

which exhibits the pole at

$$\log\left(\frac{\Lambda_{\text{Landau}}}{\mu}\right) = \frac{1}{b_N \lambda} \tag{2.15}$$

The upper bound on the coupling constant then follows by requiring a minimal hierarchy between Λ_{Landau} and μ. For instance,

$$\Lambda_{\text{Landau}} > 2\mu \tag{2.16}$$

implies

$$\lambda < \frac{1}{b_N \log 2}. \tag{2.17}$$

In the broken symmetry phase, $\mu^2 < 0$, the analysis proceeds as above with the low energy renormalization specified at $-m_H^2$ instead of $-\mu^2$.[4] Since the Higgs boson mass is proportional to the coupling,

$$m_H^2 = 2\lambda v^2, \tag{2.18}$$

where $v^2 = 4m_W^2/g^2$ is determined from the W boson mass and SU(2) gauge coupling constant, the upper bound on λ becomes an upper bound on m_H. Setting $b_N = 3/2\pi^2$ for N=4 we obtain the DN bound,

$$m_H^2 < \frac{4\pi^2 v^2}{3 \log \frac{\Lambda_{\text{Landau}}}{m_H}} \tag{2.19}$$

For $\Lambda_{\text{Landau}} > 2m_H$ this implies $m_H \lesssim 1.08$ TeV.

(3) The effective theory

The effective theory is defined by

$$\mathcal{L}_{\text{EFF}} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda^E}{4}(\phi^2)^2 - \frac{\mu^2}{2}\phi^2 + \frac{\kappa}{4}(\partial\phi^2)^2. \tag{3.1}$$

where ϕ is an N component scalar field, and the superscript E, for "effective," distinguishes λ^E from the coupling λ^R of the renormalizable theory defined in the next section. The coupling κ is dimensionful, $\kappa = C/M^2$, where M is the mass scale of the "new physics" that gives rise to the dimension 6 operator and C is a dimensionless constant characterizing the strength of the interaction between the new physics and the scalar sector.

There is another independent dimension 6 operator that is quadratic in momentum, which may be written as $\phi^2(\partial\phi)^2$. On mass-shell it can be expressed as a linear combination

[4] Since we have neglected $\mu^2 << Q^2$ as noted above, the fact that we now have Goldstone boson loops in addition to the Higgs boson loop has no effect on the quoted results.

of the dimension 6 operator in equation (3.1) plus the ϕ^4 interaction. Off-shell it is in general an independent operator. However for the symmetric off-shell renormalization condition specified below in equation (3.3), its contribution is proportional to the dimension 6 operator in (3.1) and it is not considered separately in our analysis.

We define the renormalized couplings in terms of the diagonal elastic scattering amplitude $\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1)$ so that the definition can be used for all $N \geq 1$. The tree approximation amplitude from $\mathcal{L}_{\text{EFF}}$ is

$$\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1) = -6i\lambda^E + 2i\kappa(s+t+u). \tag{3.2}$$

Since $s+t+u = \Sigma p_i^2 = 4\mu^2$ for on-shell scattering, the on-shell amplitude is indistinguishable from the amplitude due to the $\lambda\phi^4$ interaction alone with λ replaced by $\lambda^E - \frac{4}{3}\kappa\mu^2$. The $(\partial\phi^2)^2$ and ϕ^4 interactions can however be distinguished by other means, for instance, with the off-shell four-point function or the on-shell six-point function. Since we wish in any case to consider an off-shell configuration to avoid mass singularities in the RG analysis[4], we will use the off-shell four-point function to define κ and λ.

In this section we will renormalize the effective Lagrangian at one loop order and to leading order in κ, retaining terms of order λ^2 and $\lambda\kappa$. We compute the running coupling constants λ_Q^E and κ_Q, where Q is the renormalization scale defined below. Wave function and mass renormalization can be ignored to this order, since both contribute constants, independent of Q. In the renormalizable $O(N)$ ϕ^4 field theory in the symmetric phase, considered in section 2 above, the wave function renormalization vanishes and the mass renormalization is accomplished by just a Q independent counterterm. With the addition of the dimension 6 operator in equation (3.1), the wave function renormalization does not vanish but is constant so that, as in the renormalizable ϕ^4 theory, no anomalous dimension is induced for the field ϕ. The mass renormalization in $\mathcal{L}_{\text{EFF}}$ also involves only a Q independent counterterm. Neither has any effect on the running of the coupling constants and so can be ignored. These conclusions follow because the one loop integral represented by figure 3 has no dependence on the external scale other than the multiplicative factor of Q^2 from the vertex of the dimension 6 operator.

To specify the renormalization conditions for λ^E and κ it is convenient to choose a symmetric, spacelike, off-shell point

$$s = t = u = Q^2 < 0, \tag{3.3}$$

corresponding to spacelike external 4-momenta with $p_i^2 = \frac{3}{4}Q^2$ for each external leg, $i = 1,2,3,4$. Then the low energy coupling constants $\lambda^E = \lambda_\mu^E$ and $\kappa = \kappa_\mu$ are defined by

$$\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1)_{Q^2=-\mu^2} = -6i(\lambda^E + \kappa\mu^2), \tag{3.4}$$

and the renormalized running couplings λ_Q^E and κ_Q for arbitrary Q are defined by

$$\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1)_Q = -6i(\lambda_Q^E - \kappa_Q Q^2). \tag{3.5}$$

Actually we must vary Q^2 by a small amount around each given central value since at least two measurements are needed to determine both λ_Q^E and κ_Q, e.g., $Q^2 = Q^2_{\text{Central}} \pm \epsilon$ where $\epsilon \ll Q^2_{\text{Central}}$. This is a difficult task, but our excellent, highly paid gedanken experimenters have the necessary skills to carry out the measurements (e.g., using dispersion relations in the external off-shell masses).

We now compute the one loop renormalized couplings to order λ^{E^2} and $\lambda^E\kappa$. The relevant Feynman diagrams are shown in figure 2. We will regularize the loop integrals with a momentum space cutoff, since it provides the most physical description of the loop amplitudes in the effective theory. We have the luxury of this choice since we are not concerned here with gauge invariance. We have checked that the same results are obtained from dimensional and Pauli-Villars regularization.

Since regularization by cutoff is becoming a lost art (see however [9]), we will warm up by evaluating the three $O(\lambda^{E^2})$ diagrams, which were computed by dimensional regularization in section 2. Together they contribute

$$\delta\mathcal{M}_Q^{(\lambda^2)} = 6(N+8)\lambda^{E^2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\mu^2)((k+Q)^2-\mu^2)} \tag{3.6}$$

or, introducing the parametric integral over x,

$$\delta\mathcal{M}_Q^{(\lambda^2)} = 6(N+8)\lambda^{E^2}\int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-X)^2} \tag{3.7}$$

where

$$X = (x^2-x)Q^2 + \mu^2 \tag{3.8}$$

To define the integral with a cutoff we first rewrite it as a Euclidean space integral by continuing the integrand into the complex k_0 plane. By contour integration we relate the integral along the real k_0 axis to one along the imaginary axis:

$$\int_{-\infty}^{\infty} dk_0 f(k_0) = -\int_{i\infty}^{-i\infty} dk_0 f(k_0) = -i\int_{-\infty}^{\infty} dk_0' f(-ik_0'), \tag{3.9}$$

where we define $k_0' = -ik_0$. The arcs in the first and third quadrants may be neglected since they only contribute constants that are absorbed in the counterterms. The Minkowski space 4-vector k then becomes negative definite within the domain of the k_0' integration,

$$k^2 = -{k_0'}^2 - \vec{k}^2. \tag{3.10}$$

We define a Euclidean 4-vector k_E whose components are

$$k_E = (k_0', \vec{k}) \tag{3.11}$$

so that

$$k^2 = -k_E^2. \tag{3.12}$$

The Minkowski space integral, equation (3.7), is then replaced by a Euclidean space integral,

$$\delta\mathcal{M}_Q^{(\lambda^2)} = -6i(N+8)\lambda^{E^2}\int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+X)^2} \tag{3.13}$$

which we will regulate with the O(4) symmetric cutoff $k_E^2 \leq \Lambda^2$. Equation (3.13) exhibits the advantage of choosing off-shell, spacelike external momenta: for $Q^2 < 0$ we have $X > 0$ so that $k_E^2 + X$ never vanishes and the integrand has no singularities.

The integrand is spherically symmetric so the angular integration yields a factor $2\pi^2$ and the remaining integration over $|k_E^2| \leq \Lambda^2$ is easily done. The result is

$$\delta\mathcal{M}_Q^{(\lambda^2)} = \frac{-3i\lambda^{E^2}(N+8)}{8\pi^2}\int_0^1 dx \log\left(\frac{X}{\Lambda^2}\right) \tag{3.14}$$

where we omit terms of order $1/\Lambda^2$. Finally, for $Q^2 \gg \mu^2$, we approximate $\log(X) \simeq \log(Q^2)$, obtaining the usual result,

$$\delta\mathcal{M}_Q^{(\lambda^2)} = \frac{-3i\lambda^{E^2}(N+8)}{4\pi^2}\log\left(\frac{Q}{\Lambda}\right) \tag{3.15}$$

where $Q = \sqrt{-Q^2}$. The terms we have neglected by approximating $\log(X) \simeq \log(Q^2)$ are either constants that would be absorbed in counterterms or are suppressed by μ^2/Q^2. The logarithmic term in (3.15) agrees with the dimensional regularization result (2.6) if the cutoff Λ is identified with the dimensional regularization scale μ_D.

It is straightforward to apply the same method to the $O(\lambda\kappa)$ diagrams shown in figure 2b, of which there are six, each contributing equally due to the symmetric kinematics, equation (3.3). Including a factor 6 for the number of diagrams, the Feynman rules yield

$$\delta\mathcal{M}_Q^{(\lambda\kappa)} = -36\lambda^E\kappa\int\frac{d^4k}{(2\pi)^4}\frac{(k+p_1)^2+(k+p_2)^2+\frac{1}{3}(N+2)Q^2}{(k^2-\mu^2)((k+Q)^2-\mu^2)} \tag{3.16}$$

Introducing the Feynman parameter integration and symmetrizing the k integration, we have

$$\delta\mathcal{M}_Q^{(\lambda\kappa)} = -36\lambda^E\kappa\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{2k^2+(2x^2-2x+\frac{N}{3}+\frac{13}{6})Q^2}{(k^2-X)^2} \tag{3.17}$$

where X is defined in equation (3.8). Just as in equations (3.9 - 3.13), the k integration is expressed as a Euclidean space integral, which after the angular integration is

$$\delta\mathcal{M}_Q^{(\lambda\kappa)} = \frac{-9i\lambda^E\kappa}{4\pi^2}\int_0^1 dx\int_0^{\Lambda^2} dk_E^2 k_E^2\frac{-2k_E^2+(2x^2-2x+\frac{N}{3}+\frac{13}{6})Q^2}{(k_E^2+X)^2} \tag{3.18}$$

The term proportional to Q^2 diverges logarithmically. The term proportional to k^2 is rewritten to isolate the quadratic divergence,

$$\int_0^{\Lambda^2} dk_E^2 \frac{(k_E^2)^2}{(k_E^2+X)^2} = \Lambda^2 - \int_0^{\Lambda^2} dk_E^2 \frac{2k_E^2 X + X^2}{(k_E^2+X)^2}. \tag{3.19}$$

The remaining log divergent term in (3.19) is then combined with the log divergent term proportional to Q^2 in (3.18). The finite term proportional to X^2 is neglected since it contributes $O(\Lambda^{-2})$ to the amplitude. The result is

$$\delta\mathcal{M}_Q^{(\lambda\kappa)} = \frac{-9i\lambda^E\kappa}{4\pi^2}\left(-\Lambda^2 + \int_0^1 dx \int_0^{\Lambda^2} dk_E^2 k_E^2 \frac{(6x^2-6x+\frac{N}{3}+\frac{13}{6})Q^2+4\mu^2}{(k_E^2+X)^2}\right) \tag{3.20}$$

Performing the integral over k_E^2 we are left with

$$\delta\mathcal{M}_Q^{(\lambda\kappa)} = \frac{9i\lambda^E\kappa}{4\pi^2}\left(\Lambda^2 + \int_0^1 dx \log\left(\frac{X}{\Lambda^2}\right)\left\{(6x^2-6x+\frac{N}{3}+\frac{13}{6})Q^2+4\mu^2\right\}\right) \tag{3.21}$$

If the cutoff Λ is chosen to be equal to the scale of the new physics, $\Lambda \simeq M$, then the Λ^2 term in (3.21) is an artifact since it is dominated by the region of the k integration near M where the effective theory fails. In any case, since it is independent of Q, it does not affect the running of the couplings. By contrast the $\log(\Lambda)$ term samples the entire hierarchy between M and Q: e.g., for large $\log(\Lambda/Q)$ the region between Λ and $\Lambda/2$ only contributes a small fraction, $\log(2) \ll \log(\Lambda/Q)$ to the logarithm while contributing 3/4 of the quadratic term. This is seen again in the renormalizable model presented in the next section.

The running couplings λ_Q^E and κ_Q receive contributions from the terms in (3.21) proportional to $\mu^2\log(Q)$ and $Q^2\log(Q)$ respectively. However, the contribution proportional to $\mu^2\log(Q)$ is not given simply by the coefficient of the term proportional to μ^2 in (3.21), since an additional μ^2 contribution is hidden in the term proportional to Q^2. Evaluating the integrals over x through order μ^2/Q^2, we have

$$\int_0^1 dx \log\left(\frac{X}{\Lambda^2}\right) = \left(1-2\frac{\mu^2}{Q^2}\right)\log\left(\frac{Q^2}{\Lambda^2}\right) + O\left(\frac{\mu^2}{Q^2}\right)^2 + \ldots \tag{3.22}$$

and for integer $n \geq 1$

$$\int_0^1 dx\, x^n \log\left(\frac{X}{\Lambda^2}\right) = \frac{1}{n+1}\left(1-(n+1)\frac{\mu^2}{Q^2}\right)\log\left(\frac{Q^2}{\Lambda^2}\right) + O\left(\frac{\mu^2}{Q^2}\right)^2 + \ldots \tag{3.23}$$

where we also omit constant terms that are independent of $\log(Q)$ and do not affect the running of λ and κ. Substituting (3.22 - 3.23) into (3.21) we have

$$\delta\mathcal{M}_Q^{(\lambda\kappa)} = \frac{-3i\lambda^E\kappa}{4\pi^2}\log\left(\frac{Q}{\Lambda}\right)\left\{2(2N+1)\mu^2 - (2N+7)Q^2\right\} + \ldots \tag{3.24}$$

where $Q = |Q| = \sqrt{-Q^2}$ and we again omit all terms (including the Λ^2 term) that do not contribute to the running of λ and κ.

We can now derive the renormalized couplings and the RGE's. The counterterm Lagrangian for the effective theory is

$$\mathcal{L}_{\rm CT} = \frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta\kappa}{4}(\partial\phi^2)^2 - \frac{\delta\lambda^E}{4}(\phi^2)^2 - \frac{\delta\mu^2}{2}\phi^2, \tag{3.25}$$

As noted above, δZ and $\delta\mu$ have no Q dependence and can be ignored. Combining the relevant tree, counterterm, and one loop contributions, the amplitude is

$$\mathcal{M}_Q = -6i(\lambda^E + \delta\lambda^E) + 6i(\kappa + \delta\kappa)Q^2 + \delta\mathcal{M}_Q^{(1)} \tag{3.26}$$

where

$$\delta\mathcal{M}_Q^{(1)} = \delta\mathcal{M}_Q^{(\lambda^2)} + \delta\mathcal{M}_Q^{(\lambda\kappa)}. \tag{3.27}$$

It is convenient to write $\delta\mathcal{M}_Q^{(1)}$ as

$$\delta\mathcal{M}_Q^{(1)} = (A\mu^2 + BQ^2)\log\left(\frac{Q}{\Lambda}\right) + \ldots \tag{3.28}$$

where we omit terms that are constant or small. The coefficients A and B are given by

$$A = \frac{-3i}{4\pi^2}[(N+8)\lambda^{E^2} + 2(2N+1)\lambda^E\kappa\mu^2] \tag{3.29}$$

and

$$B = \frac{-3i}{4\pi^2}(2N+7)\lambda^E\kappa Q^2. \tag{3.30}$$

The counterterms are determined at $Q^2 = -\mu^2$ (or, more precisely, in a small neighborhood around $Q^2 = -\mu^2$) from equation (3.4), which implies

$$-6i(\delta\lambda^E - \delta\kappa Q^2)|_{Q^2=-\mu^2} = -\delta\mathcal{M}^{(1)}_{Q^2=-\mu^2}. \tag{3.31}$$

Equating powers of Q^2 the counterterms are then

$$\delta\lambda^E = -\frac{i}{6}A\mu^2\log\left(\frac{\mu}{\Lambda}\right) \tag{3.32}$$

and

$$\delta\kappa = +\frac{i}{6}B\log\left(\frac{\mu}{\Lambda}\right) \tag{3.33}$$

Then from (3.3) and (3.26) the running couplings are

$$\lambda_Q^E = \lambda^E + \frac{i}{6}A\mu^2\log\left(\frac{Q}{\mu}\right) \tag{3.34}$$

and

$$\kappa_Q = \kappa - \frac{i}{6} B \log\left(\frac{Q}{\mu}\right). \tag{3.35}$$

or explicitly

$$\lambda_Q^E = \lambda^E + \frac{1}{8\pi^2}\log\left(\frac{Q}{\Lambda}\right)\left\{(N+8)\lambda^{E^2} + 2(2N+1)\lambda^E\kappa\mu^2\right\} \tag{3.36}$$

and

$$\kappa_Q^E = \kappa^E + \frac{1}{8\pi^2}\log\left(\frac{Q}{\Lambda}\right)(2N+7)\lambda^E\kappa. \tag{3.37}$$

To one loop and to leading order in κ, the coupled RGE's are then

$$\frac{d\lambda_Q^E}{d\log Q} = \frac{1}{8\pi^2}\left\{(N+8)\lambda_Q^{E^2} + 2(2N+1)\lambda_Q^E\kappa_Q\mu^2\right\} \tag{3.38}$$

and

$$\frac{d\kappa_Q^E}{d\log Q} = \frac{1}{8\pi^2}(2N+7)\,\lambda_Q^E\kappa_Q. \tag{3.39}$$

(4) A Renormalizable Model

In this section we construct a renormalizable model with a heavy scalar of mass $M \gg \mu$, which replicates the effective theory defined in equation (3.1) at low energy, $\mu < Q \ll M$, and then use the renormalizable model to verify the RGE's obtained in the previous section. In the renormalizable model the one loop, $\log(Q)$ dependent terms arise both from the low energy limit of finite Feynman diagrams as well as from log divergent diagrams reflecting the divergences of the original ϕ^4 theory. The results are of course regulator independent. For convenience we use dimensional regularization here. The renormalizable model provides a very useful check on the calculation because the result arises in a rather different way from the effective theory — in particular, the order μ^2/Q^2 terms from equations (3.22 - 3.23) contribute differently, so that the results disagree if those terms are overlooked (as the author learned the hard way).

In addition to the O(N) vector field ϕ we now add an O(N) singlet scalar field σ. The relevant terms in the renormalizable Lagrangian are

$$\mathcal{L}_R = \frac{1}{2}\left((\partial\phi)^2 + (\partial\sigma)^2\right) - \frac{\lambda^R}{4}(\phi^2)^2 - \frac{G}{2}\sigma\phi^2 - \frac{\mu^2}{2}\phi^2 - \frac{M^2}{2}\sigma^2, \tag{4.1}$$

where $M \gg \mu$ and G is a coupling constant with the dimension of a mass. Interaction terms involving more than a single heavy field σ, such as $\sigma^2\phi^2$, σ^3 or σ^4 are neglected since they give rise to diagrams that are suppressed by additional powers of M^2, inducing corrections beyond the leading order in κ in the effective low energy theory. We will see that even in tree approximation the coupling λ^R is not equal to the analogous coupling λ^E of the effective theory.

Despite our misleading notation, the renormalizable model is not a "sigma model" since in general the interactions do not have an $O(N+1)$ symmetry. In order for $\mathcal{L}_R$ to be embedded in a sigma model (i.e., one with its symmetry softly broken by the explicit "pion" mass μ) the parameters would have to be related by $\lambda^R = G^2/2M^2$. This is not a relevant limit for us since it implies $\lambda^E = 0$ (see equation (4.4) below), as required by the low energy theorem for "$\pi\pi$" scattering.

In tree approximation the $\phi_1\phi_1 \to \phi_1\phi_1$ scattering amplitude is

$$\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1) = -6i\lambda^R - iG^2\left(\frac{1}{s-M^2} + \frac{1}{t-M^2} + \frac{1}{u-M^2}\right) \tag{4.2}$$

Expanding for $M^2 >> |s|, |t|, |u|$ this is

$$\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1) = -6i\lambda^R + 3i\frac{G^2}{M^2} + i\frac{G^2}{M^4}(s+t+u) \tag{4.3}$$

which is equivalent to the tree approximation amplitude of the effective theory, equation (3.2), if we identify

$$\lambda^E = \lambda^R - \frac{G^2}{2M^2} \tag{4.4}$$

and

$$\kappa = \frac{G^2}{2M^4}. \tag{4.5}$$

The first term in the expansion of the σ propagator induces a tree level shift in λ while the second term reproduces the dimension 6 operator in $\mathcal{L}_{\text{EFF}}$.

We now consider the one loop corrections to the $\phi_1\phi_1 \to \phi_1\phi_1$ scattering amplitude in the renormalizable model. Each of the six Feynman diagrams in the effective theory, shown in figure 2b, is replaced by three diagrams in the toy model. The first of these, shown in figure 4a, contains precisely the same logarithmically divergent integral that renormalizes lambda in the original ϕ^4 theory, shown in figure 2a. In figure 4a it corresponds to a $\sigma\phi\phi$ vertex correction. Using the $\overline{MS}$ prescription and the symmetric, off-shell external momenta defined in equation (3.3), the six diagrams of type figure 4a together contribute

$$\mathcal{M}_a = \frac{-3i\lambda^R G^2}{8\pi^2(Q^2-M^2)}(N+2)\int_0^1 dx\, \log\left(\frac{X}{\mu_D^2}\right) \tag{4.6}$$

where μ_D is the regulator scale. Using equation (3.22) and expanding for $M^2 \gg Q^2 \gg \mu^2$ this becomes

$$\mathcal{M}_a = \frac{3i\lambda^R G^2}{4\pi^2}(N+2)\log\left(\frac{Q}{\mu_D}\right)\left\{\frac{1}{M^2} - 2\frac{\mu^2}{M^4} + \frac{Q^2}{M^4}\right\} \tag{4.7}$$

In addition to $\mathcal{M}_a$ there are 12 finite diagrams of the type shown in figure 4b which together contribute

$$\mathcal{M}_b = 36\lambda^R G^2 \int_k \frac{1}{((k-p_1)^2-\mu^2)(k^2-M^2)((k+p_2)^2-\mu^2)} \tag{4.8}$$

or, introducing Feynman parameters and symmetrizing,

$$\mathcal{M}_b = 72\lambda^R G^2 \int_0^1 dx \int_0^{1-x} dy \int_k \frac{1}{(k^2 - Y)^3} \tag{4.9}$$

where

$$Y = yM^2 + (1-y)\mu^2 + \left[\frac{3}{4}(y^2 - y) + x^2 - x + xy\right] Q^2 \tag{4.10}$$

After the d^4k integration the result is

$$\mathcal{M}_b = \frac{-9i\lambda^R G^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{1}{Y} \tag{4.11}$$

Performing the y integration and retaining only the terms proportional to $\log(Q)$ for $M \gg Q$, we find

$$\mathcal{M}_b = \frac{9i\lambda^R G^2}{4\pi^2 M^2} \int_0^1 dx \left\{ \left[1 + \frac{\mu^2}{M^2} + \frac{3}{4}\frac{Q^2}{M^2}\right] - x\frac{Q^2}{M^2} \right\} \log\left(\frac{X}{M^2}\right) \tag{4.12}$$

where X is defined in equation (3.8). The terms that are omitted in (4.12) are either constants that are absorbed in counterterms or are of higher order in small ratios. Applying equations (3.22 - 3.23) the final result for $\mathcal{M}_b$ is

$$\mathcal{M}_b = \frac{3i\lambda^R G^2}{4\pi^2 M^2} \log\left(\frac{Q}{M}\right) \left\{ 6 + 3\frac{\mu^2}{M^2} + \frac{3}{2}\frac{Q^2}{M^2} \right\} \tag{4.13}$$

The complete one loop amplitude is given by combining (4.7) and (4.13),

$$\delta\mathcal{M}_R^{(1)} = \frac{3i\lambda^R G^2}{4\pi^2 M^2} \log\left(\frac{Q}{\mu}\right) \left\{ (N+8) - (2N+1)\frac{\mu^2}{M^2} + \left(N + \frac{7}{2}\right)\frac{Q^2}{M^2} \right\} \tag{4.14}$$

where again we drop terms that do not vary as $\log(Q)$ and will be absorbed in counterterms.

The term proportional to (N+8) in (4.14) provides an interesting consistency check. Since it is proportional to $G^2/M^2 \propto \kappa M^2$ it seems to imply a nondecoupling contribution to the low energy theory which would invalidate the effective Lagrangian of section 3. But it actually provides just the appropriate renormalization of the tree level shift encountered in equation (4.4),

$$\delta\lambda_{\text{Tree}} = -\frac{G^2}{2M^2}, \tag{4.15}$$

to give $\lambda_Q^E = \lambda_Q^R + \delta\lambda_{\text{Tree}}$ the correct order λ^{E^2} renormalization. That is, in the effective theory the order λ^{E^2} renormalization, from (3.30), is[5]

$$\delta\lambda_Q^E = \frac{N+8}{8\pi^2}\lambda^{E^2} \log\left(\frac{Q}{\mu}\right). \tag{4.16}$$

[5] The shift in the scale of the logarithm from Λ to μ is absorbed by the counterterm in the renormalization procedure.

Substituting

$$\lambda^{E^2} \simeq \lambda^{R^2} + 2\lambda^R \delta\lambda_{\text{Tree}} \tag{4.17}$$

we see, using (4.15), that the contribution of the second term in (4.17) to (4.16) is

$$-\frac{N+8}{8\pi^2}\frac{\lambda^R G^2}{M^2}\log\left(\frac{Q}{\mu}\right). \tag{4.18}$$

But this is precisely equal to $i/6$ times the first term in (4.14), as required for consistency with equation (2.10).

The remaining two terms in (4.14), proportional to $G^2\mu^2/M^4$ and G^2Q^2/M^4, agree precisely with the corresponding terms proportional to $\kappa\mu^2$ and κQ^2 in the effective theory, as may be seen by comparing (4.14) with equations (3.14) and (3.24) using the expression for κ in (4.5). Therefore the renormalization of λ^E and κ computed in the effective theory in section 3 is verified by the results obtained here from the renormalizable model. We see that renormalization effects which arise in the effective theory from a combination of quadratically and logarithmically divergent diagrams arise in the renormalizable model in a rather different way, with some $\log(Q)$ terms arising from diagrams with the usual ultraviolet logarithmic divergences of the original ϕ^4 theory (figure 4a) while others arise from the low energy limit of finite diagrams (figure 4b).

(5) Solution of the RGE's

We now solve the coupled RGE's of the effective theory, first rewriting the RGE's, equations (3.32 - 3.33), in a more compact notation with the superscript E suppressed,

$$\lambda'_Q = b_N\left(\lambda_Q^2 + \frac{2(N+2)}{N+8}\lambda_Q\gamma_Q\right) \tag{5.1}$$

and

$$\gamma'_Q = \frac{2N+7}{8\pi^2}\lambda_Q\gamma_Q \tag{5.2}$$

where b_N was given in equation (2.13), and we have defined

$$\gamma_Q = \kappa_Q\mu^2. \tag{5.3}$$

The quantity γ_Q has the same $\log(Q)$ dependence as κ_Q since μ has no $\log(Q)$ dependence at one loop order.

We solve the coupled equations by constructing a quantity f_Q that obeys the usual one loop RGE,

$$f'_Q = \lambda_Q + c_N\gamma_Q, \tag{5.4}$$

where c_N is determined by requiring

$$f'_Q = b_N f_Q^2. \tag{5.5}$$

Working to first order in γ_Q we find

$$c_N = \frac{2}{9}(2N+1). \tag{5.6}$$

Integrating (5.5) from $-\mu^2$ to Q^2 we have the familiar solution

$$f_Q = \frac{f_\mu}{1 - b_N f_\mu \log\left(\frac{Q}{\mu}\right)}. \tag{5.7}$$

To solve for λ_Q and γ_Q we replace λ_Q in (5.2) by f_Q, valid to first order in γ_Q, obtaining

$$\gamma_Q' = \frac{2N+7}{8\pi^2} f_Q \gamma_Q. \tag{5.8}$$

Substituting (5.7) into (5.8) and integrating we then find

$$\gamma_Q = \gamma_\mu \left[\frac{1}{1 - b_N f_\mu \log\left(\frac{Q}{\mu}\right)}\right]^{\frac{2N+7}{N+8}} \tag{5.9}$$

and, from (5.4), (5.6), and (5.9),

$$\lambda_Q = \frac{\lambda_\mu}{1 - b_N f_\mu \log\left(\frac{Q}{\mu}\right)} + \frac{c_N \gamma_\mu}{1 - b_N f_\mu \log\left(\frac{Q}{\mu}\right)} \left\{1 - \left[\frac{1}{1 - b_N f_\mu \log\left(\frac{Q}{\mu}\right)}\right]^{\frac{N-1}{N+8}}\right\}. \tag{5.10}$$

We consider two special cases. For $N = 1$ the solutions take a particularily simple form,

$$\gamma_Q|_{N=1} = \frac{\gamma_\mu}{1 - b_1 f_\mu \log\left(\frac{Q}{\mu}\right)}, \tag{5.11}$$

and

$$\lambda_Q|_{N=1} = \frac{\lambda_\mu}{1 - b_1 f_\mu \log\left(\frac{Q}{\mu}\right)}. \tag{5.12}$$

We also consider $N = 4$ which would correspond to the Higgs sector of the Standard Model if we were to consider the broken symmetry phase. Then $c_4 = 2$ and the running couplings are

$$\gamma_Q|_{N=4} = \gamma_\mu \left[\frac{1}{1 - b_4 f_\mu \log\left(\frac{Q}{\mu}\right)}\right]^{\frac{5}{4}}. \tag{5.13}$$

and

$$\lambda_Q|_{N=4} = \frac{\lambda_\mu}{1 - b_4 f_\mu \log\left(\frac{Q}{\mu}\right)} + \frac{2\gamma_\mu}{1 - b_4 f_\mu \log\left(\frac{Q}{\mu}\right)} \left\{1 - \left[\frac{1}{1 - b_4 f_\mu \log\left(\frac{Q}{\mu}\right)}\right]^{\frac{1}{4}}\right\}. \tag{5.14}$$

From these solutions to the RGE's we see that "new physics" represented by the dimension 6 operator has two effects on the Landau pole. First, as we will discuss in the

next section, it changes the relationship between the coupling constant λ and the position of the pole. Second it also affects the strength of the singularity at the pole, making the leading singularity stronger for all $N > 1$. From (5.10) we see that since c_N is proportional to N, the correction proportional to $c_N \gamma_\mu$ would dominate λ_Q for sufficiently large N. Our perturbative approximation then breaks down in the large N limit, which would require a separate analysis.

(6) The Landau pole and the low energy coupling constant

We now consider the effect of the dimension 6 operator on the relationship between the low energy coupling constant and the location of the Landau pole. Without the dimension 6 operator, i.e., for $\kappa = 0$, we see from equation (2.15) that the pole location is fixed by the low energy coupling constant and mass,

$$\Lambda_{\text{Landau}} = \mu \exp\left(\frac{1}{b_N \lambda_\mu}\right). \tag{6.1}$$

Equivalently, the low energy coupling is determined by the ratio $\Lambda_{\text{Landau}}/\mu$,

$$\lambda_\mu = \frac{1}{b_N \log\left(\frac{\Lambda_{\text{Landau}}}{\mu}\right)}. \tag{6.2}$$

New physics must intervene at or below the pole. Defining M to be the new physics mass scale, (6.1) implies

$$M \leq \mu \exp\left(\frac{1}{b_N \lambda_\mu}\right). \tag{6.3}$$

In order that the low energy theory have some domain of validity we require a minimal hierarchy R between M and the mass scale of the low energy theory,

$$\frac{M}{\mu} \geq R, \tag{6.4}$$

which implies the upper limit on the low energy coupling constant,

$$\lambda_\mu \leq \frac{1}{b_N \log R}. \tag{6.5}$$

These are simple but powerful relations. They are accurate in the perturbative domain, for small coupling λ_μ and large hierarchy R. At strong coupling and small R, the domain of the triviality bound, lattice simulations have found them to be qualitatively correct and, beyond that, accurate to about $\simeq 30\%$.[2]

The dimension 6 operator considered in section 3 modifies these relations, due to the effect of the high scale physics on the running of the scalar coupling constant. The result is

simply to replace λ_μ in the above equations with f_μ defined in (5.5). Then (6.1) and (6.2) become

$$\Lambda_{\text{Landau}} = \mu \exp\left(\frac{1}{b_N(\lambda_\mu + c_N\kappa_\mu\mu^2)}\right). \tag{6.6}$$

and

$$\lambda_\mu = \frac{1}{b_N \log\left(\frac{\Lambda_{\text{Landau}}}{\mu}\right)} - c_N\kappa_\mu\mu^2. \tag{6.7}$$

The upper bound on the low energy coupling constant becomes

$$\lambda_\mu \leq \frac{1}{b_N \log R} - c_N\kappa_\mu\mu^2. \tag{6.8}$$

The sign of the new physics correction depends then on the sign of κ_μ. If $\kappa_\mu > 0$ the Landau pole position is lowered for fixed λ_μ and μ, and the upper bound on λ_μ becomes stronger for given hierarchy R. Conversely for $\kappa_\mu < 0$ the Landau pole moves to higher energy and the upper bound on λ_μ is weakened. In the renormalizable model considered in section 4, in which the new physics arises from the exchange of a heavy singlet scalar, equation (4.5) implies $\kappa_\mu > 0$. In general the sign may be positive or negative.

If the dimension 6 operator arises from a dimension 4 interaction with dimensionless coupling g between high-scale quanta of mass M and the light scalar fields ϕ, then κ will be of order $O(g^2/M^2)$ where for strong coupling g^2 would be of order $O(4\pi)$. If the underlying interaction has dimension 3 with dimensionful coupling G, then $\kappa \simeq O(G^2/M^4)$, as in equation (4.5). For strong coupling we would then expect $G^2/M^2 \simeq O(4\pi)$, and, again, $\kappa \simeq O(4\pi/M^2)$.

In general the strength of the dimension 6 operator is characterized by a dimensionless quantity C, defined by

$$\kappa_\mu = \frac{C}{M^2}. \tag{6.9}$$

Assuming now that M is as heavy as it can be, $M \simeq \Lambda_{\text{Landau}}$, and that the hierarchy inequality (6.4) is also saturated, i.e.,

$$\frac{M^2}{\mu^2} = \frac{\Lambda^2_{\text{Landau}}}{\mu^2} = R \qquad (6.10),$$

then (6.7) becomes

$$\lambda_\mu = \frac{1}{b_N \log R} - c_N \frac{C}{R^2}. \tag{6.11}$$

Defining r_λ as the ratio of the value of λ_μ determined from (6.11) to the corresponding value, equation (6.2), for $C = \kappa_\mu = 0$, we have

$$r_\lambda = 1 - b_N c_N C \frac{\log R}{R^2}. \tag{6.12}$$

Similarly, we define r_Λ as the ratio of Λ_{Landau} determined for $\kappa_\mu \neq 0$ from (6.6) relative to the value for $\kappa_\mu = 0$ from (6.1), for the same values of λ_μ and μ, and find

$$r_\Lambda = \exp\left(-\frac{c_N C}{b_N R^2}\frac{1}{\lambda_\mu(\lambda_\mu + c_N \frac{C}{R^2})}\right). \tag{6.13}$$

Expanding to leading order in $\kappa_\mu \mu^2/\lambda_\mu$, r_Λ is approximately

$$r_\Lambda \simeq 1 - b_N c_N C \left(\frac{\log R}{R}\right)^2, \tag{6.14}$$

which is enhanced by an additional factor of $\log R$ relative to the correction to r_λ in (6.12).

The value of λ_μ from equation (6.11) for $N = 4$ is plotted in figure 5 for $C = 0, \pm 4\pi$ as a function of the hierarchy R. Numerical values for the ratios r_λ and r_Λ from equations (6.12) and (6.13) are given in tables 1 and 2 for $N = 4$ and $N = 1$ respectively with $C = \pm 4\pi$. For $N = 1$ the corrections are approximately four times smaller than for $N = 4$. For $N = 4$ the corrections to the coupling constant are sizeable, reaching 66% for the hierarchy $R = 2$ at which the triviality bound is customarily obtained in lattice calculations. The value of such a large correction cannot be taken literally since it exceeds the domain of validity of the perturbative approximation, but it suggests that large corrections, potentially even of order one, are possible. At larger values of R the corrections become smaller and are therefore more reliably known. We may for instance consider where the approximation (6.14) becomes a good description of (6.13). From table 1 we find that this occurs at $R = 5$, where (6.14) implies a 40% correction to r_Λ in good agreement with the result shown in the table, and for which the correction to the triviality bound is 25%.

(7) Conclusion

We have studied the effect of new physics on the RG analysis of the Landau pole and the triviality bound in the unbroken phase of O(N) ϕ^4 theory. Including a dimension 6 operator to represent the low energy effects of the new physics that must exist at the Landau pole, we find that the pole position and the upper bound on the coupling constant can be modified by substantial amounts, if the new physics is strongly coupled to the O(N) scalars and if the O(N) scalars are themselves strongly coupled. The analysis is performed in the spirit of the original Dashen-Neuberger analysis, to explore the possible order of magnitude of the effects in a simple approximation.

Quantitative results for the strong coupling regime would require lattice simulations. As discussed in the introduction, the related lattice simulations carried out in [3] had a different goal than ours — to see the effect of dimension 6 operators subject to the constraint that the low energy Higgs sector resemble the SM Higgs sector to within a few percent — and is not directly comparable to the calculation presented here since different phases are considered

and the required operator matching has not been done. Here, within the limitations of one loop perturbation theory, we explored the maximum effect on the bound without regard to the size of other corrections to the low energy physics. It would be interesting to study this regime with lattice simulations.

To check the RG analysis of the effective theory we also considered a simple renormalizable model of the new physics, consisting of a heavy O(N) singlet scalar field which in its low energy limit gives rise to the dimension 6 operator. The model provides a computational check and a measure of physical insight. We saw that in addition to giving rise to the dimension 6 operator, the exchange of the heavy scalar causes a tree-level shift $\delta\lambda_{\text{Tree}}$ in the ϕ^4 coupling constant λ and that apparently dangerous one loop corrections, proportional to κM^2 (where M is the mass of the heavy scalar) conspire to give $\delta\lambda_{\text{Tree}}$ precisely the usual O(λ^2) renormalization.

The sign of κ determines whether the triviality limit is increased or decreased by new physics. For the model considered here with a heavy O(N) singlet scalar, κ is positive, in which case the Landau pole position and the triviality bound on the coupling are both lowered. For negative κ the opposite occurs. It is then very interesting to exhibit theories with $\kappa < 0$ or to prove that none exist. The lattice studies of dimension 6 operators in the Higgs phase reported possible increases of the triviality bound[3], but they too introduced dimension 6 operators by hand and so also did not address this issue. Since κ is a parameter of an effective theory with a limited domain of validity, which might furthermore arise from the low energy limit of another effective theory with a still limited (though higher energy) domain of validity, it is likely that no general theorem exists.

In the renormalizable O(N) ϕ^4 scalar theory it is easy to see that the RG flow of the coupling constant is the same in the symmetric and Higgs phases of the theory. Because the dimension 6 ϕ^4 interaction gives rise to dimension 5 ϕ^3 interactions in the Higgs phase, it is not immediately apparent that the same is true of the effective theory. This question is under study and will be reported elsewhere.

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Table 1. Tabulation of r_λ and r_Λ, from equations (6.12 - 6.13), for the $N = 4$ theory with $C = \pm 4\pi$.

	$C = 4\pi$		$C = -4\pi$	
R	r_λ	r_Λ	r_λ	r_Λ
2	0.34	0.26	1.66	1.32
e	0.48	0.34	1.52	1.41
5	0.75	0.59	1.25	1.37
7	0.85	0.71	1.15	1.29
10	0.91	0.80	1.09	1.20

Table 2. Tabulation of r_λ and r_Λ, from equations (6.12 - 6.13), for the $N = 1$ theory with $C = \pm 4\pi$.

	$C = 4\pi$		$C = -4\pi$	
R	r_λ	r_Λ	r_λ	r_Λ
2	0.84	0.87	1.17	1.10
e	0.87	0.86	1.13	1.12
5	0.94	0.90	1.06	1.10
7	0.96	0.93	1.04	1.07
10	0.98	0.95	1.02	1.05

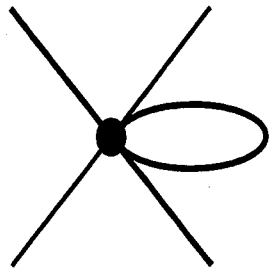

Figure 1: One loop correction to the four point function from the operator ϕ^6/Λ^2.

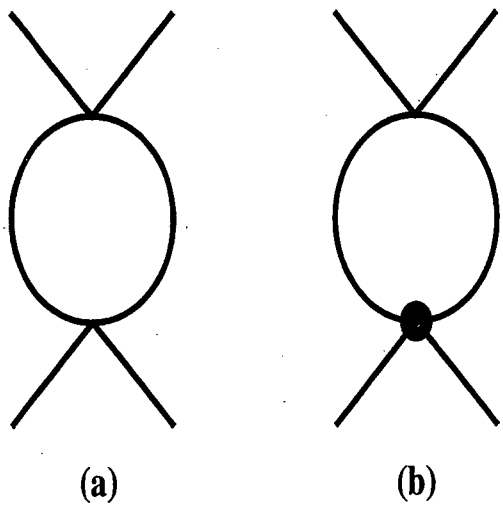

Figure 2: Feynman diagrams for (a) the order λ^2 and (b) the order $\kappa\lambda$ corrections to the scattering amplitude. The black dot denotes the $(\partial\phi^2)^2$ interaction.

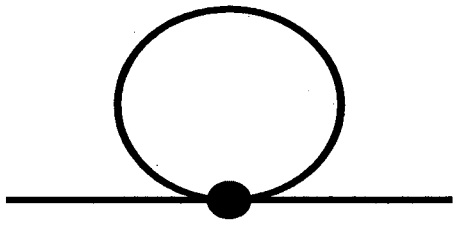

Figure 3: The one loop correction to the self energy from the $(\partial\phi^2)^2$ interaction.

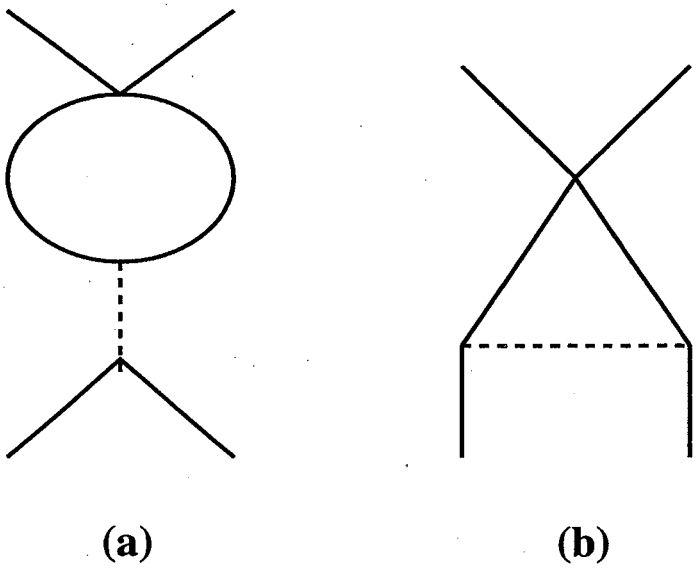

Figure 4: The leading one loop corrections to $\phi\phi$ scattering from exchange of the heavy scalar σ, denoted by the dashed lines.

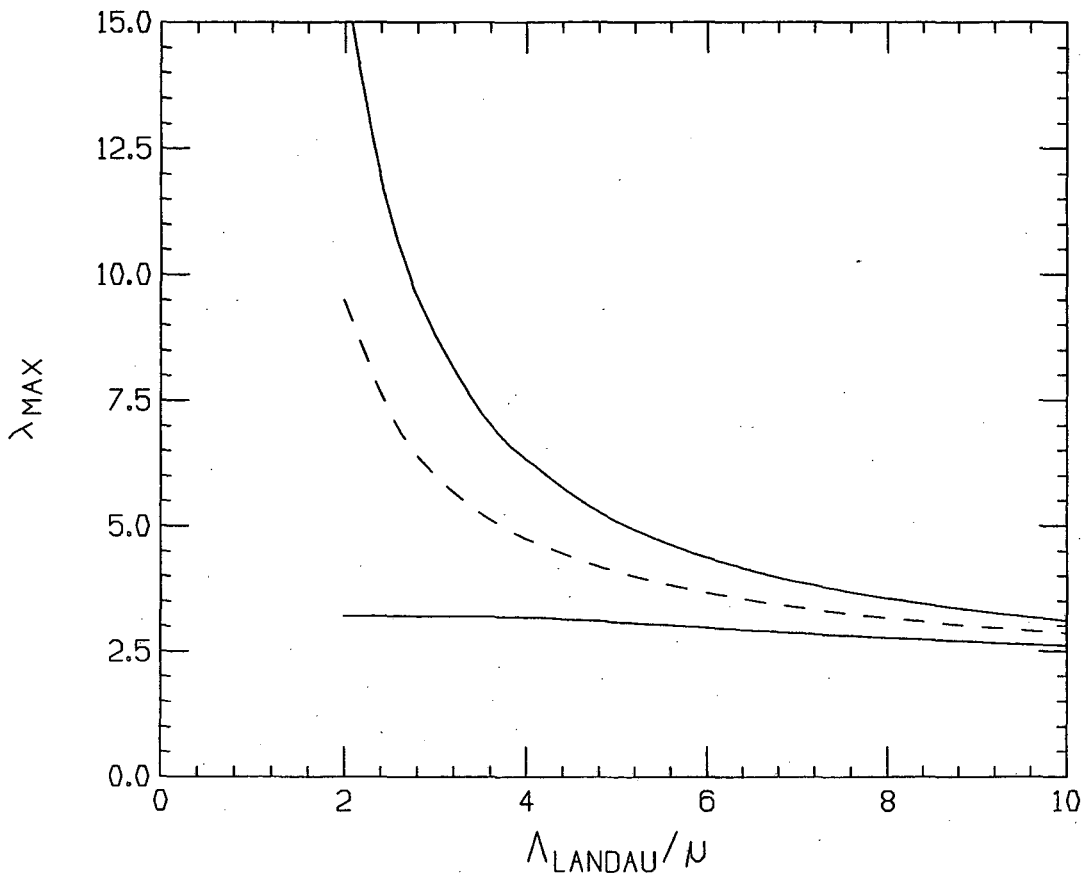

Figure 5: The triviality upper limit on the coupling constant λ_μ for $N = 4$ as a function of the hierarchy $\Lambda_{\text{Landau}}/\mu$. The dashed line shows the limit in the absence of the dimension 6 operator, while the upper and lower solid lines are from equation (6.11) with $C = -4\pi$ and $C = +4\pi$ respectively.

Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

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