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Earth Sciences Division

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Probability Density Functions for Solute Transport in Random Fields

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ABSTRACT

Effective description of flow and transport in irregular porous media, adequate understanding and prediction of dispersion phenomenon and reliable estimation of the uncertainty, all require stochastic approach. The primary problem is finding the relations between the non-random functionals of the unknown and the given random fields, i.e., means, variations, probability distribution densities, etc. The present paper considers the process of transport of non-reactive admixture in random porous media with non-random sources of flow and solute. The paper attempts to develop the methods of finding the probability density function of the concentration of the solute. We introduce the random functional $p(\mathbf{x},t;c)$, the density of distribution function (*DDF*) of the local random concentration $c(\mathbf{x},t)$ in the one-dimensional phase space of possible values of c, where $\mathbf{x}$ can be multi-dimensional vector, and t is time. By using the stochastic transport equation in the $(\mathbf{x},t)$ space, we can write the stochastic equation for the functional $p(\mathbf{x},t;c)$ in the $(\mathbf{x},t;c)$ space. We show that this equation bears the mathematical form of the transport equation in the $(\mathbf{x},t,c)$ space, with $p(\mathbf{x},t;c)$ in place of concentration. In this fictitious transport equation, the c axis velocity is non-random and depends on the densities of flow and solute sources in the $(\mathbf{x},t)$ space. It is also shown that the velocity field in the $(\mathbf{x},t;c)$ space is non-divergent, and that the $p(\mathbf{x},t;c)$ field, the fictitious solute, has no sources and is reactive. The averaging of the new transport equation in the $(\mathbf{x},t;c)$ space with special initial conditions that are dependent on the variable c (in the case where the closed-form averaging is possible) leads to describing $P(\mathbf{x},t;c) = < p(\mathbf{x},t;c) >$ - the probability density function (*PDF*) for $c(\mathbf{x},t)$ in c phase space and the corresponding power moments. We

present and analyze a number of example *PDFs* for the concentration $c(\mathbf{x},t)$ in several different cases of flow velocity fields and initial concentration distribution.

1. DENSITY DISTRIBUTION FUNCTION (*DDF*) OF CONCENTRATION $c(\mathbf{x},t)$ IN ONE-DIMENSIONAL PHASE SPACE

It is clear that merely describing the behavior of the mean concentration in space and time is not enough to understand the transport process in details. One must at least also examine the probability density function (*PDF*) of the local concentration $c(\mathbf{x},t)$ in space for its possible values of c or the set of highest momentum of the concentration $c(\mathbf{x},t)$.

It should be noted that these characteristics are not yet enough for a full description of the random concentration field $c(\mathbf{x},t)$. It is well known that for this purpose it is necessary to obtain the characteristic functional of the field $c(\mathbf{x},t)$ or its infinite-dimensional probability density function.

Following the works by *Klyatskin*[1980], *Shvidler*[1985], *Indelman and Shvidler*[1985], *Shvidler and Karasaki*[1995, 1996 and 1997], we will study the probability density function (*PDF*) of the concentration $c(\mathbf{x},t)$ in one-dimensional phase space of its possible values of c and the power-moments of the random function $c(\mathbf{x},t)$.

Let us now assume that the field of local concentration in infinite D-dimensional space is described by the equations:

$$\phi(\mathbf{x})\frac{\partial c(\mathbf{x},t)}{\partial t}+\frac{\partial}{\partial x_i}\left[v_i(\mathbf{x},t)c(\mathbf{x},t)\right]=\varphi(\mathbf{x},t) \tag{1}$$

$$c(\mathbf{x},t_0)=f(\mathbf{x}). \tag{2}$$

The density of solute sources $\varphi(\mathbf{x},t)$ and the density of liquid sources $\psi(\mathbf{x},t)$, where

$$\psi(\mathbf{x},t) = \frac{\partial v_i(\mathbf{x},t)}{\partial x_i} \tag{3}$$

are considered as non-random functions.

We introduce the *DDF* -the density of distribution function of the local concentration $c(\mathbf{x},t)$ in one-dimensional phase space C:

$$p(\mathbf{x},t;c) = \delta\left[c(\mathbf{x},t) - c\right] \tag{4}$$

Differentiating the equation (4) over time t and over x_i we can write:

$$\frac{\partial p(\mathbf{x},t;c)}{\partial t} = -\frac{\partial p(\mathbf{x},t;c)}{\partial c}\frac{\partial c(\mathbf{x},t)}{\partial t}, \tag{5}$$

$$\frac{\partial p(\mathbf{x},t;c)}{\partial x_i} = -\frac{\partial p(\mathbf{x},t;c)}{\partial c}\frac{\partial c(\mathbf{x},t)}{\partial x_i}. \tag{6}$$

And after combining the equations (1), (5) and (6), we obtain the so-called Liouville equation- a closed form of the stochastic equation for the density $p(\mathbf{x},t;c)$:

$$\phi(\mathbf{x})\frac{\partial p(\mathbf{x},t;c)}{\partial t} + v_i(\mathbf{x},t)\frac{\partial p(\mathbf{x},t;c)}{\partial x_i} + v_c(\mathbf{x},t;c)\frac{\partial p(\mathbf{x},t;c)}{\partial c} - \psi(\mathbf{x},t)p(\mathbf{x},t;c) = 0 \tag{7}$$

where the non-random function $v_c(\mathbf{x},t;c)$ is

$$v_c(\mathbf{x},t;c) = \varphi(\mathbf{x},t) - c\psi(\mathbf{x},t). \tag{8}$$

The initial condition for the density $p(\mathbf{x},t;c)$ can be found from (5) and (2):

$$p(\mathbf{x},t_0;c) = \delta\left[f(\mathbf{x}) - c\right]. \tag{9}$$

Clearly, the following interpretation can be made: The Liouville equation (7) is the transport equation of the reactive "solute" in the D+1 dimensional space $(\mathbf{x};c)$ with the field of velocity $\mathbf{V}(\mathbf{x},t;c) = \{[v_i(\mathbf{x},t)], v_c(\mathbf{x},t;c)\}$. The field $\mathbf{V}(\mathbf{x},t;c)$ in the fictitious space depends on $v_i(\mathbf{x},t)$ -

field of velocity in the real space and the densities $\varphi(\mathbf{x},t),\psi(\mathbf{x},t)$ and the parameter c. We can see that the "flow" in D+1 space does not have sources because:

$$div_{x,c}\mathbf{V}(\mathbf{x},t;c) = \partial v_i(\mathbf{x},t)/\partial x_i - \psi(\mathbf{x},t) = 0, \tag{10}$$

and the reactivity of the "solute" depends on $\psi(\mathbf{x},t)$ -the density of flow in real space. If the real flow does not have sources, the "solute" in $D+1$ space is non-reactive and it travels by the field $\mathbf{V}(x,t;c)$. In this case we have the transport equation:

$$\phi(\mathbf{x},t)\frac{\partial p(\mathbf{x},t;c)}{\partial t} + \mathbf{V}(\mathbf{x},t;c)\nabla_{x,c}p(\mathbf{x},t;c) = 0. \tag{11}$$

It should be noted that in contrast to (11), the transport equation (1) in the real space has sources, $\varphi(\mathbf{x},t)$, distributed in space when $\psi(\mathbf{x},t) = 0$ and $\varphi(\mathbf{x},t) \neq 0$. And only when $\psi(\mathbf{x},t) = 0$ and $\varphi(\mathbf{x},t) = 0$, are the transport equations (11) and (1) identical, and we can write:

$$\phi(\mathbf{x})\frac{\partial p(\mathbf{x},t;c)}{\partial t} + v_i(\mathbf{x},t)\frac{\partial p(\mathbf{x},t;c)}{\partial x_i} = 0 \tag{12}$$

$$p(\mathbf{x},t_0;c) = \delta\left[f(\mathbf{x}) - c\right] \tag{13}$$

2. PROBABILITY DENSITY FUNCTION (*PDF*) OF CONCENTRATION

Given the different variants of the equation for the random density $p(\mathbf{x},t;c)$, we can study the PDF-the probability density function of the local random concentration $c(\mathbf{x},t)$ in the phase space c. According to the definition, the probability density function (*PDF*) is:

$$P(\mathbf{x},t;c) =< p(\mathbf{x},t;c) >, \tag{14}$$

where the symbol $< >$ denotes the ensemble averaging of the realization of the random fields $\mathbf{v}(\mathbf{x},t)$ and $\phi(\mathbf{x})$. The primary problem is averaging the system described by (7) and (9) and finding a closed-form equation for $P(\mathbf{x},t;c)$. Obviously the averaging of this one system is

similar to averaging general transport equations, which is a well-known, very difficult problem. We assume that the averaged system exists and has the form:

$$LP(\mathbf{x},t;c) = 0 \tag{15}$$

$$P(\mathbf{x},t_0;c) = \delta\left[f(\mathbf{x}) - c\right] , \tag{16}$$

where L is a non-random linear operator defined in the space of functions of variables $\mathbf{x}$, t and c. Let $G(\mathbf{x},t;c|\ \overline{\mathbf{x}};\overline{c})$ be the solution of the problem:

$$LG(\mathbf{x},t;c|\ \overline{\mathbf{x}},\overline{c}) = 0 \tag{17}$$

$$G(\mathbf{x},t_0;c|\ \overline{\mathbf{x}};c) = \delta(\mathbf{x}-\overline{\mathbf{x}})\delta(c-\overline{c}) \ . \tag{18}$$

Then obviously the PDF is:

$$P(\mathbf{x},t;c) = \int G(\mathbf{x},t;c|\ \overline{\mathbf{x}};\overline{c})\ \delta\left[f(\mathbf{x}) - \overline{c})\right] d\overline{\mathbf{x}}\ d\overline{c} , \tag{19}$$

or after integration over $\overline{c}$:

$$P(\mathbf{x},t;c) = \int G(\mathbf{x},t;c|\ \overline{\mathbf{x}};f(\overline{\mathbf{x}}))\ d\overline{\mathbf{x}} . \tag{20}$$

If for example the operator L does not depend on c, and if $\varphi(\mathbf{x},t) = 0$ and $\psi(\mathbf{x},t) = 0$, the function G also does not depend on the parameter c, and we have the problem for G reduced to:

$$LG(\mathbf{x},t,\overline{\mathbf{x}}) = 0 \tag{21}$$

$$G(\mathbf{x},t_o,\overline{\mathbf{x}}) = \delta(\mathbf{x}-\overline{\mathbf{x}}) . \tag{22}$$

Then we can write:

$$P(\mathbf{x},t;c) = \int G(\mathbf{x},t,\overline{\mathbf{x}})\delta\left[f(\overline{\mathbf{x}}) - c\right] d\overline{\mathbf{x}} . \tag{23}$$

It should be remembered that the δ-function in the integrals (19) and (23) are defined in one-

dimensional space c and $\bar{c}$ respectively. If the $\mathbf{x}$-space is also one-dimensional, and suppose that $\bar{x}_i^c$ is the i-th root of the equation:

$$f(\bar{x}) = c, \tag{24}$$

we can write:

$$\delta[f(\bar{x}) - c] = \sum_i^n \left|f'(\bar{x}_i^c)\right|^{-1} \delta(\bar{x} - \bar{x}_i^c). \tag{25}$$

From (23) we find the PDF,

$$P(x,t;c) = \sum_i^n \left|f\left(\bar{x}_i^c\right)\right|^{-1} G(x,t,\bar{x}_i^c). \tag{26}$$

If $D > 1$, we can solve the equation (24) with respect to any variable $\bar{x}_j^c$ where $j \le D$. In this case the $i-th$ root $\bar{x}_{ji}^c$ depends not only on c, but also on all variables $\bar{x}_k$ except x_j.

$$\bar{x}_{ji}^c = \bar{x}_{ji}^c(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_{j+1},...,\bar{x}_D) \quad , i = 1,...,k \tag{27}$$

and we have the expression:

$$\delta\left|f(\bar{\mathbf{x}}) - c\right| = \sum_{i=1}^{k_j(\bar{\mathbf{x}})} \left|f'_{\bar{x}_j}(\bar{x}_1,...,\bar{x}_{ji}^c,...,\bar{x}_D)\right|^{-1} \delta(\bar{x}_j - \bar{x}_{ji}^c). \tag{28}$$

After using (28) and (23) we can finally write:

$$P(\mathbf{x},t;c) = \int \sum_{i=1}^{k_j(\bar{\mathbf{x}})} G\left((x_1,...,x_D),t,(\bar{x}_1,...,\bar{x}_{ji}^c,...,\bar{x}_D)\ \right)\left|f'_{\bar{x}}(\bar{x}_1,...,\bar{x}_{ji}^c,...,\bar{x}_D)\right|^{-1} \prod_{l,l\neq j}^{D} dx_l\,. \tag{29}$$

3. POWER MOMENTS OF THE CONCENTRATION $c(\mathbf{x},t)$

If the operator L does not depend on c, after multiplying equations (15) and (16) by c^n and integrating them over c, we can write for any power moment:

$$u_n(\mathbf{x},t) = \langle c^n(x,t)\rangle = \int c^n P(\mathbf{x},t;c)dc\,, \tag{30}$$

the equation and the initial conditions for power moment u_n are :

$$Lu_n = 0 \tag{31}$$

$$u_n(\mathbf{x},t_0) = f^n(\mathbf{x})\,. \tag{32}$$

4. ONE-DIMENSIONAL TRANSPORT

Now we consider one-dimensional transport problems where the flow velocity $v(t)$ is Gaussian or the telegraph processes. *Shvidler and Karasaki*, [1995 and 1996] derived the exact equations for the mean concentration and their exact solutions. Later [*Shvidler and Karasaki* ,1997] they presented the exact expressions for the probability density functions (PDF) and the exact power moments of the random concentration $c(x,t)$.

In the one-dimensional case for the random density $p(x,t;c)$, when $\phi(x) = \phi = const$, we have:

$$\phi\frac{\partial\, p(x,t;c)}{\partial t} + v(t)\frac{\partial\, p(x,t;c)}{\partial x} = 0 \tag{33}$$

$$p(x,t_0,c) = \delta\left[f(x) - c\right]\,. \tag{34}$$

Let $f(x) = C_0 \exp\left[-\frac{|x-x_o|}{a}\right]$. Thus for $0 < c < C_0$ the equation $f(x) = c$ has two roots x_c^1, x_c^2 and the initial condition (34) can be rewritten as:

$$p(x,t_0;c) = |f'(x_c^1)|\delta(x - x_c^1) + |f'(x_c^2)|\delta(x - x_c^2)\,. \tag{35}$$

for the exponential initial function we find $|f'(x_c)| = c/a$ and we have the form:

$$p(x,t_0;c)=\frac{a}{c}\left[\delta(x-x_0-a\ln\frac{c}{C_o})+\delta(x-x_0+a\ln\frac{c}{C_0})\right]. \tag{36}$$

If the velocity $v(t)$ is a Gaussian random process with the exponential autocorrelation function

$$B(t-t_0)=\alpha_0^2\exp\left[-\frac{|t-t_0|}{2\upsilon}\right] \tag{37}$$

- so called Ornstein-Ulenbeck random process [*Yaglom*,1987], we can use the expression of the mean concentration derived by *Shvidler and Karasaki* [1995], (see also the formula (53) from *Shvidler and Karasaki* [2000]) and find $\bar{\mathrm{P}}(\bar{y},\bar{\tau};\bar{c})=C_0\mathrm{P}(\bar{y},\bar{\tau};\bar{c})$ – the dimensionless PDF :

$$\bar{P}_G(\bar{y},\bar{\tau};\bar{c})=\frac{\bar{a}}{\bar{c}\sqrt{\pi\lambda(\bar{\tau})}}\left\{\exp\left[-\frac{(\bar{y}+\bar{a}\ln\bar{c})^2}{\lambda(\bar{\tau})}\right]+\exp\left[-\frac{(\bar{y}-\bar{a}\ln\bar{c})^2}{\lambda(\bar{\tau})}\right]\right\} \tag{38}$$

where $\bar{c}=c/C_0$, $\bar{\tau}=\upsilon(t-t_0)$, $\bar{y}=\upsilon\phi\alpha_0^{-1}\left[x-x_0-V(t-t_0)\right]$, $V=<v(t)>=const$, $\bar{a}=a\upsilon\phi\alpha_0^{-1}$,

$\lambda(\bar{\tau})=e^{-2\bar{\tau}}+2\bar{\tau}-1$

For the velocity *v(t)* – the telegraph process with the same mean V and correlation function *B(t)* in (37), the appropriate solution derived by *Shvidler and Karasaki* [1995], (see also the formulas (48)-(51) from *Shvidler and Karasaki* [2000]) should be used again , and after transformation we can write

$$\bar{P}_T(\bar{y},\bar{\tau};\bar{c})=\frac{\bar{a}}{\bar{c}}\left[\Phi(\bar{y}_1,\bar{\tau})+\Phi(\bar{y}_2,\bar{\tau})\right], \tag{39}$$

where for $|\bar{y}_i|\le\bar{\tau}$, $i=1,2$:

$$\Phi(\bar{y}_i,\bar{\tau})=\frac{e^{-\bar{\tau}}}{2}\left[\delta(\bar{y}_i+\bar{\tau})+\delta(\bar{y}_i-\bar{\tau})+\mathrm{I}_0(\bar{z}_i)+\frac{\bar{\tau}}{\bar{z}_i}\mathrm{I}_1(\bar{z}_i)\right].$$

Here $\mathrm{I}_0(\bar{z}_i)$ and $\mathrm{I}_1(\bar{z}_i)$ are the modified Bessel`s function of the first kind of order zero and one,

respectively and $\bar{y}_1 = \bar{y} - \bar{a}\ln\bar{c}$, $\bar{y}_2 = \bar{y} + \bar{a}\ln\bar{c}$, and $\bar{z}_i = \sqrt{\bar{\tau}^2 - \bar{y}_i^2}$.

For $|\bar{y}_i| > \bar{\tau}$, we have:

$$\Phi(\bar{y}_i, \bar{\tau}) = 0. \tag{41}$$

The *PDF's* $\bar{P}_G(\bar{y},\bar{\tau};\bar{c})$ and $\bar{P}_T(\bar{y},\bar{\tau};\bar{c})$ are clearly even for the variable $\bar{y}$. Also obvious is that the *PDF* $\bar{P}_G(\bar{y},\bar{\tau};\bar{c})$ is continuous in space $\bar{c}$. On the other hand $\bar{P}_T(\bar{y},\bar{\tau};\bar{c})$ can have discontinuities depending on the variables $\bar{y}$ and $\bar{\tau}$. Rearranging the δ −functions in (39) for $\bar{\tau} < \bar{y}$ and $\bar{y} > 0$, we have the expressions:

$$\bar{P}_T(\bar{y},\bar{\tau};\bar{c}) = \frac{e^{-\bar{\tau}}}{2}\left[\delta(\bar{c}-\bar{c}_1) + \delta(\bar{c}-\bar{c}_2) + \frac{\bar{a}}{\bar{c}}\Psi(\bar{\tau},\hat{z}_2)\right], \qquad \bar{c}_1 \le \bar{c} \le \bar{c}_2, \tag{42}$$

$$\bar{c}_1 = \exp(-\frac{\bar{y}+\bar{\tau}}{\bar{a}}), \qquad \bar{c}_2 = \exp(-\frac{\bar{y}-\bar{\tau}}{\bar{a}}), \qquad \hat{z}_2 = \sqrt{\bar{\tau}^2 - \bar{y}_2^{\,2}} \tag{43}$$

$$\Psi(\bar{\tau},\bar{z}) = \mathrm{I}_0(\bar{z}) + \frac{\bar{\tau}}{\bar{z}}\mathrm{I}_1(\bar{z}) \tag{44}$$

If $\bar{c} < \bar{c}_1$ *or* $\bar{c} > \bar{c}_2$, we have

$$\bar{P}_T(\bar{y},\bar{\tau};\bar{c}) = 0 \tag{45}$$

Thus if $\bar{y} > \bar{\tau}$, the *PDF* in the phase space outside the interval $(\bar{c}_1,\bar{c}_2)$ is equal to zero, at points $\bar{c}_1$ and $\bar{c}_2$ the *PDF* has δ −function components, and inside the interval the PDF is continuous.

If $\bar{\tau} > \bar{y}$, the *PDF* has the form:

$$\bar{P}_T(\bar{y},\bar{\tau};\bar{c}) = \frac{e^{-\bar{\tau}}}{2}\left\{\delta(\bar{c}-\bar{c}_1) + \delta(\bar{c}-\bar{c}_3) + \frac{\bar{a}}{\bar{c}}\left[\Psi_1(\bar{\tau},\hat{z}_1) + \Psi_2(\bar{\tau},\hat{z}_2)\right]\right\}, \tag{46}$$

where $\bar{c}_3 = \exp(\frac{\bar{y}-\bar{\tau}}{\bar{a}}) \ge \bar{c}_1$ and for $\bar{c}_1 \le \bar{c} \le 1$ we have $\hat{z}_1 = \sqrt{\bar{\tau}^2 - \bar{y}_1^{\,2}}$ and $\Psi_1(\bar{\tau},\bar{z}_1) = \Psi(\bar{\tau},\bar{z}_1)$

If $0 < \bar{c} < \bar{c}_1$, we find

$$\bar{P}_T(\bar{y},\bar{\tau};\bar{c}) = 0 \tag{47}$$

For $\bar{c}_3 \le \bar{c} \le 1$ we have $\hat{z}_2 = \sqrt{\bar{\tau}^2 - \bar{y}_2^{\,2}}$ and $\Psi_2(\bar{\tau},\bar{z}_2) = \Psi(\bar{\tau},\bar{z}_2)$

And if $0 < \bar{c} < \bar{c}_3$, we find

$$\Psi_2(\bar{\tau},\bar{z}_2) = 0 \tag{48}$$

Thus if $\bar{\tau} > \bar{y}$, the *PDF* in the interval $(0,\bar{c}_1)$ is equal to zero. At points $\bar{c}_1$ and $\bar{c}_3$, the PDF has δ-function components and inside the interval $(\bar{c}_1,1)$, the finite part of the *PDF* has a finite step at $\bar{c} = \bar{c}_3$. To note, the existence of the δ-components is associated with the fact that for any $\bar{\tau}$ the part of realization (the rate is $e^{-\bar{\tau}}$) of the velocity *v(t)* has no jumps. One half part of them has a velocity $(V - \alpha_0)$ and the other half has $-(V + \alpha_0)$. Thus at point $\bar{y}$ among the concentration transport realization with velocity jumps, we find two concentrations that transport with the constant velocity $(V - \alpha_0)$ and $(V + \alpha_0)$, respectively. In the initial moment these concentration are located at points $(\bar{y} - \bar{\tau})$ and $(\bar{y} + \bar{\tau})$, and their values are $\exp[-|\bar{y} - \bar{\tau}|/\bar{a}]$ and $\exp[-|\bar{y} + \bar{\tau}|/\bar{a}]$, respectively. Obviously if $\bar{y} < \bar{\tau}$ these concentrations are $\bar{c}_1$ and $\bar{c}_2$, if $\bar{y} > \bar{\tau}$ they are $\bar{c}_1$ and $\bar{c}_3$.

Now we consider the behavior of the higher moments of the concentration $c(x,t)$. Earlier we showed that when $\varphi(x,t) = 0$ *and* $\psi(x,t) = 0$, the equation for any power moment does not depend on the power and coincides with the equation for the probability density function, or with equation for mean concentration $u(x,t)$. The difference is contained only in the initial

conditions for the moment and the n-th moment has the form shown in (32). If the initial concentration is exponential, so will the $u_n(x,t_0)$ be also exponential:

$$u_n(x,t_0) = [f(x)]^n = C_0^n \exp\left[-\frac{|x-x_0|}{a}n\right] \tag{49}$$

Therefore, to compute the higher moments of $c(x,t)$ it is possible to utilize the explicit expressions for the mean concentration derived for the Gaussian and telegraph velocity random processes by *Shvidler and Karasaki*[1995],(see also the formulas (56) and (57) from *Shvidler and Karasaki* [2000]). All it is necessary is to change the parameter a to a/n .

For the Gaussian velocity we have:

$$\tilde{u}_{nG}(\bar{y},\bar{\tau}) = \frac{1}{2}\exp\left(\frac{\bar{g}_n}{2}\right)\left\{\exp\left(-\frac{\bar{y}}{\bar{a}}\right)\left[1-erf\left(\sqrt{\frac{\bar{g}_n}{2}}-\frac{\bar{y}}{\bar{a}\sqrt{2\bar{g}_n}}\right)\right]+\exp\left(\frac{\bar{y}}{\bar{a}}\right)\left[1-erf\left(\sqrt{\frac{\bar{g}_n}{2}}+\frac{\bar{y}}{\bar{a}\sqrt{2\bar{g}_n}}\right)\right]\right\} \tag{50}$$

where $\bar{g}_n = gn^2/2\bar{a}^2$, $g = e^{-2\bar{\tau}} + 2\bar{\tau} - 1$ *and* $erf\zeta = \frac{2}{\sqrt{\pi}}\int_0^{\zeta} e^{-u^2}du$

In the case of the telegraph random velocity and for the same exponential initial condition we can write the following expression :

$$\tilde{u}_{nT}(\bar{y},\bar{\tau}) = \frac{1}{2}e^{-\bar{\tau}}\left\{\exp\left(-\frac{|\bar{y}+\bar{\tau}|n}{\bar{a}}\right)+\exp\left(-\frac{|\bar{y}-\bar{\tau}|n}{\bar{a}}\right)+\int_{-\bar{\tau}}^{\bar{\tau}}\left[\mathrm{I}_0(z)+\frac{\bar{\tau}}{z}\mathrm{I}_1(z)\right]\exp\left(-\frac{|\bar{y}+\bar{\lambda}|n}{\bar{a}}\right)d\bar{\lambda}\right\} \tag{51}$$

where $z = \sqrt{\bar{\tau}^2 - \bar{\lambda}^2}$

Now we study the behavior of the *PDF* by examining the following important parameters of the distributions: dimensionless mean concentration $\tilde{u}(\bar{y},\bar{\tau})$, dimensionless mean square deviation

of the concentration $s(\bar{y},\bar{\tau})$, coefficients of asymmetry $k^a(\bar{y},\bar{\tau})$ and excess $k^e(\bar{y},\bar{\tau})$ for the *PDF*. These parameters are defined [*H.Cramer,* 1963] as:

$$\tilde{u}(\bar{y},\bar{\tau}) = \tilde{u}_1, \quad s(\bar{y},\bar{\tau}) = \left[\tilde{u}_2 - \tilde{u}_1^2\right]^{1/2} \tag{52}$$

$$k^a(\bar{y},\bar{\tau}) = s^{-3}\left(\tilde{u}_3 - 3\tilde{u}_2\tilde{u}_1 + 2\tilde{u}_1^3\right), \quad k^e(\bar{y},\bar{\tau}) = s^{-4}\left(\tilde{u}_4 - 4\tilde{u}_3\tilde{u}_1 + 6\tilde{u}_2\tilde{u}_1^2 - 3\tilde{u}_1^4\right) - 3 \tag{53}$$

In Figures.1~16 we show the *PDFs* for the Gaussian and telegraph velocity for various sets of parameters $\bar{a}, \bar{y}$ *and* $\bar{\tau}$. For each of these figures we calculate the moments $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ *and* $\tilde{u}_4$, the mean square deviation *s,* and the coefficients k^a *and* k^e. All this information is in Table 1.

Figures1~5 show the *PDF's* evaluated with $\bar{a} = 1$ and $\bar{y} = 1$ at different values of $\bar{\tau}$. It can be seen in Fig.1 that the densities $\bar{P}_G(1,0.1;\bar{c})$ and $\bar{P}_T(1,0.1;\bar{c})$ are very different. For the Gaussian random velocity the distribution is almost symmetric and bell-shaped. For the telegraph velocity random process the probability density outside the interval (0.3329, 0.4066) is equal to zero. On the borders of this interval the density is a δ - function multiplied by 0.4524. Inside the interval the density is decreased and it is almost linear. It is interesting that despite these strong differences in the distributions, the mean concentration $\tilde{u}_1 = \bar{u}(1,0.1) -$ (the first moment of concentration $\bar{c}(1,0.1)$) and other moments for both distributions are practically equal. The asymmetry for the telegraph velocity is less than that for the Gaussian velocity because at small time the essential part of $\bar{P}_T$ is the two symmetric spikes.

For $\bar{\tau} = 0.3$ and $\bar{\tau} = 0.5$ (Fig.2 and 3) the behavior of the *PDF's* are similar to those when $\bar{\tau} = 0.1$. For example, when $\bar{\tau} = 0.5$ the probability density function $\bar{P}_T(1,0.5)$ is non-zero inside the interval (0.2231, 0.6065). On the borders of the interval the density is a δ – function multiplied on 0.3033. Inside the interval the *PDF* is decreased. As before the mean concentration

$\tilde{u}_1 = \bar{u}(1,0.5)$ for both distributions of velocity is practically the same. The difference between higher moments is a little more.

When $\bar{\tau} = 0.99$ (Fig.4) the *PDF* is non-zero inside the interval (0.1367, 0.99). The coefficient of the delta-function on the borders of the interval is 0.1858 and the mean concentration $\tilde{u}_1 = \bar{u}(1,0.99)$ does moderate depend on the velocity process. The higher moments are more sensitive to the velocity distribution.

For $\bar{\tau} > \bar{y}$ in accordance with the previous analysis, the behavior of the *PDF* for the telegraph velocity process is different from that of the Gaussian. For example when $\bar{\tau} = 1.2$ (Fig.5) the PDF is non-zero for $\bar{c} \geq 0.1108$. At the points $\bar{c} = 0.1108$ and $\bar{c} = 0.8187$ the distribution is a delta-function with a coefficient 0.1506 and at these points the function undergoes finite jumps. For $\bar{c} > 0.8187$ the two probability densities $\bar{P}_G(1,1.2;\bar{c})$ and $\bar{P}_T(1,1.2;\bar{c})$ are relatively close.

When $\bar{\tau} = 2$ (Fig.6) we can see that these tendencies intensify. The coefficient of the delta – function is decreased to 0.0677, for $\bar{c} > 0.3679$ both distributions are practically equal. Although for small $\bar{c}$ the difference between the distributions is still significant, all moments are close.

Finally when $\bar{\tau} = 5$ (Fig.7) and the more so when $\bar{\tau} = 10$ (Fig.8), both *PDF's* are congruent in the entire interval of the variation parameter $\bar{c}$. The moments, the mean square deviation and the coefficients of asymmetry and excess are practically the same.

In summary, for $\bar{a} = 1$ the densities $\bar{P}_G$ and $\bar{P}_T$ are strongly different at small times, although the first moments are nearly identical. For larger times the tendency is noticeable for these *PDF* to come together. (They are practically identical for $\bar{\tau} = 5$ and more so when $\bar{\tau} = 10$.) It should be pointed out that for the case discussed above when $\bar{a} = 1$, the moments of the concentration are weakly sensitive to the distribution of the random velocity. Now we consider the behavior the probability distributions when the parameter $\bar{a} = 0.1$. In this case if $\bar{y} = 0$ (Fig.9-13) and *v(t)* is a

telegraph process, the points $\bar{c}_1$ and $\bar{c}_2$ coincide. Therefore, the probability density function $\bar{P}_T(0,\bar{\tau};\bar{c})$ is zero in the interval $0<\bar{c}<\bar{c}_1=\bar{c}_3$, at point $\bar{c}=\bar{c}_1=\bar{c}_3$ the *PDF* contains two delta-functions with each coefficient being $e^{-\bar{\tau}}/2$, and for $\bar{c}_1=\bar{c}_3<\bar{c}<1$ the *PDF* is continuous.

When $\bar{\tau}=0.1$ (Fig.9) we have $\bar{c}_1=\bar{c}_3=0.3679$, the coefficient of each delta-function is 0.4524 and hence the fraction of the velocity realization without jumps is 0.9048. For all these realizations $\bar{c}(0,0.1)=0.3679$ and therein lies the reason for the appearance of the delta – functions in the *PDF* at $\bar{c}=0.3679$. The functions $\bar{P}_G$ *and* $\bar{P}_T$ are essentially different and in contrast to the case where $\bar{a}=1$ the moments are sensitive to distributions of random velocity.

When $\bar{\tau}=0.2$ (Fig.10) the behaviors of the functions $\bar{P}_G(0,0.2;\bar{c})$ and $\bar{P}_T(0,0.2;\bar{c})$ are similar to the case when $\bar{\tau}=0.1$ but the tendency for the two curves to come together is noticeable, which is well expressed by the times $\bar{\tau}=0.5$ (Fig.11) and $\bar{\tau}=1$ (Fig.12). When $\bar{\tau}=5$ (Fig.13) the *PDFs* for the Gaussian and telegraph velocity, the moments and mean square deviation are all very close. Of interest to compare is the behavior of the probability density functions at different points of $\bar{y}$ for the same time $\bar{\tau}$. Fig.10 and Figures.14~16 show the distributions $\bar{P}_G(\bar{y},0.2;\bar{c})$ and $\bar{P}_T(\bar{y},0.2;\bar{c})$ for $\bar{y}=0,\bar{y}=0.1,\bar{y}=0.2$ and $\bar{y}=0.4$, respectively. For the Gaussian velocity the *PDF's* for different $\bar{y}$ are similar, whereas for the telegraph process the *PDF's* are essentially different, because the type (shape) of *PDF's* depends on the relation between $\bar{y}$ and $\bar{\tau}$.

Thus when $\bar{y}=0$ and $\bar{y}=0.1$, that is $\bar{y}<\bar{\tau}=0.2$, the density $\bar{P}_T(\bar{y},0.2;\bar{c})$ is equal to zero when $\bar{c}<\bar{c}_1=e^{-(\bar{y}+\bar{\tau})/\bar{a}}$. At points $\bar{c}_1$ and $\bar{c}_3=e^{(\bar{y}-\bar{\tau})/\bar{a}}$ the distribution has a delta-component with a factor $e^{-\bar{\tau}}/2$ and inside the interval $(\bar{c}_1,1)$ the distribution has a jump at $\bar{c}_3$. Therefore when

$\bar{y}=0$ (Fig.10) the points $\bar{c}_1$ and $\bar{c}_3$ coincide. In fact the distribution has only one delta-component with the factor $e^{-\bar{\tau}}$ and for $\bar{c}>\bar{c}_1$ it is continuous.

When $\bar{y}=0.1$ (Fig.14) the *PDF* for the telegraph velocity has two delta-function components at $\bar{c}=\bar{c}_1=0.04978$ and $\bar{c}=c_3=0.3679$ and at these points the *PDF* has a final jump.

When $\bar{y}=0.2$ (Fig.15) we have $\bar{y}=\bar{\tau}$ and $\bar{c}_1=0.01831$ and $\bar{c}_3=1$. The factor for each delta-component is 0.4094, but does not have the final jump because $\bar{c}_3=1$.

When $\bar{y}=0.4$ (Fig.16), that is $\bar{y}>\bar{\tau}$, for $\bar{c}<\bar{c}_1=0.006738$ and for $\bar{c}>\bar{c}_2=0.1353$ the probability density $\bar{P}_T(0.4,0.2;\bar{c})$ is equal to zero. At the points $\bar{c}_1$ and $\bar{c}_2$ this *PDF* has a delta-component and inside the interval $(\bar{c}_1,\bar{c}_2)$ it is continuous.

In Figs.17-28 we present the distributions of the dimensionless mean square deviation of concentration and dimensionless mean concentration $U(\bar{y},\bar{\tau})=u(\bar{y},\bar{\tau})/C_0$ for exponential initial distribution of concentration for $\bar{a}=1$ at different time $\bar{\tau}$.

For small time $\bar{\tau}=0.2$ with *v(t)* be Gaussian random process (Fig.17), the distribution $U(\bar{y},\bar{\tau})$ is close to the initial distribution . The mean square deviation $s(\bar{y},\bar{\tau})$ is relatively small. At point $\bar{y}=0$ the minimum is evident, which is a consequence of relatively small deviation of the initial concentration near the point $\bar{y}=0$.

For telegraph process *v(t)* (Fig.18) the maximum of $U(\bar{y},0.2)$ has a plateau, but the minimum of $s(\bar{y},0.2)$ at $\bar{y}=0$ is sharper.

Similar behavior is still retained for the larger times $\bar{\tau}$. As this takes place the mean concentration decreases monotonically and the mean square deviation increases as time goes on. For $\bar{\tau}=2$ (Fig.19, 20) and $\bar{y}\geq 2$ the standard deviation $s(\bar{y},2)$ is larger than $U(\bar{y},2)$ for both

distributions of the velocity *v(t)* .The minimum $s(\bar{y},2)$ at point $\bar{y} = 0$ is weakly expressed, more likely for $|\bar{y}| < 2$ the standard deviation has the plateau.

For $\bar{\tau} = 5$ (Fig.21, 22) the functions $U(\bar{y},5), s(\bar{y},5)$ depend only slightly on the behavior of the velocity *v(t)* that attained the preasymptotic phase of the transport process. For both cases except area $|\bar{y}| < 2$ the standard deviation $s(\bar{y},5)$ is larger than mean the concentration $U(\bar{y},5)$.

Furthermore we consider the case when the initial concentration $c(x,t_0)$ is exponential but parameter $\bar{a} = 0.1$. Fig.23, 24 show values for $U(\bar{y},0.1)$ and $s(\bar{y},0.1)$. It should be mentioned that for both distributions of velocity, when $|\bar{y} \geq 0.1|$ the mean concentration practically coincides with the standard deviation. However, when $|\bar{y}| < 0.1$ the behavior of the distributions is different. For the Gaussian random velocity at point $\bar{y} = 0$ the mean concentration is maximal but the standard deviation has a weak minimum. For the velocity-telegraph random process the functions $U(\bar{y},0.1), s(\bar{y},0.1)$ at point $\bar{y} = 0$ are minimal and at points $|\bar{y}| = 0.1$ both functions are maximal.

For $\bar{\tau} = 1.0$ (Fig.25, 26) when velocity *v(t)* is Gaussian random process, the behavior of the distributions $U(\bar{y},1), s(\bar{y},1)$ are similar but for any $\bar{y}$ the standard deviation of the concentration is larger than the mean concentration. The difference between $s(\bar{y},1)$ and $U(\bar{y},1)$ by $\bar{y} = 0$ is maximal, but the relative difference - $[s(\bar{y},1) - U(\bar{y},1)]/\bar{u}(\bar{y},1)$ grows for larger values of $|\bar{y}|$ and has reached a large value.

For the velocity-telegraph process (Fig.26)both distributions are nonmonotone .At points $|\bar{y}| = 1$ they are maximal and for any $\bar{y}$ the standard deviation is larger than the mean concentration.

For time $\bar{\tau} = 5$ (Fig.27, 28) both distributions of the velocity *v(t)* are near-identical because they reach the preasymptotic phase of the process. For any $\bar{y}$ the function $s(\bar{y},5)$ is larger than $\bar{u}(\bar{y},5)$. The distributions for the telegraph process in the neighborhood of $\bar{y} = 0$ are maximal at points $|\bar{y}| = 5$.

Thus in all above cases examined we can see the convergence of *PDF* for random concentration, mean concentration and mean square concentration that are functionals of random concentration, derived for different (Gaussian and telegraph) random velocities $v(t)$. This phenomenon discussed by *Shvidler and Karasaki* [ ] for mean concentration and have the same explanation for PDF of random concentration and the mean square concentration.

It is clear that the random density function $p(x,t;c)$ is a non-random function with the random argument $\xi(t) = x - \phi^{-1}\int_{t_0}^{t} v(\theta)d\theta$. If $v(t)$ is Gaussian random process, $\xi(t)$ is also Gaussian for any time t, as a sum of Gaussian summonds. If $v(t)$ is telegraph random process the argument $\xi(t)$ for $\upsilon(t-t_0) >> 1$ is the sum of a large number of uncorrelated summonds and tend to a Gaussian by the central limit theorem. The asymptotical convergence of $p(x,t;c)$ for different velocity fields leads to the convergence of functionals like *PDF* and moments.

It is interesting to note that for both cases of $\bar{a} = 1$ and $\bar{a} = 0.1$ the *PDF* and functionals $U(\bar{y},\bar{\tau})$ and $s(\bar{y},\bar{\tau})$ practically converge to the same limit at numbers $\upsilon\bar{\tau} = 5-10$.

SUMMARY

1. We examined in detail the behavior of local random concentration $c(x,t)$ by studying its density distribution– a random functional

$$p(x,t;c) = \delta[c(x,t) - c]$$

where the Dirak`s δ-function is defined in c -one dimensional phase space of possible values the random concentration $c(x,t)$.

2. The functional $p(x,t;c)$ satisfies so called stochastic Liouville equation in space of variables $(x,t;c)$ .This equation may be interpreted as transport equation for "reactive solute" $p(x,t;c)$ in $D+1$ space $(x;c)$.

The "flow" in fictitious $D+1$ space is free of sources in spite of the fact that the real flow in D space have sources. The component of "flow" velocity along the c -axis depends on the real flow sources, real solute sources and the variable c .

3. The problem of finding the *PDF*-probability density function for random concentration $c(x,t)$, i.e., $P(x,t;c) = \langle p(x,t;c) \rangle$, reduced to averaging the stochastic Liouville equation and initial condition for $p(x,t;c)$ and solving the averaged system.

4. The method of computation the *PDF* and power moments of random concentration $c(x,t)$ is illustrated for the case of one-dimensional transport with flow-velocity as Gaussian or telegraph processes of time. In these cases the exact averaging of the Liouville stochastic equation is possible and the equation for *PDF* is solved exactly.

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Table 1. Statistical parameters of the PDFs in corresponding figures.

$\bar{a}$	$\bar{y}$	Fig. #	$\bar{\tau}$	V	$\tilde{u}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{u}_4$	s	k^a	k^e
1.0	1.0	1	0.1	G	3.70E-01	1.38E-01	5.19E-02	1.97E-02	4.81E-01	2.92E-01	1.52E-01
				T	3.70E-01	1.38E-01	5.19E-02	1.97E-02	4.81E-01	7.73E-03	-1.95E+00
		2	0.3	G	3.82E-01	1.57E-01	6.95E-02	3.32E-02	4.74E-01	8.41E-01	1.21E+00
				T	3.82E-01	1.56E-01	6.76E-02	3.05E-02	4.75E-01	6.45E-02	-1.84E+00
		3	0.5	G	4.00E-01	1.89E-01	1.04E-01	6.45E-02	4.59E-01	9.62E-01	7.84E-01
				T	4.02E-01	1.89E-01	9.87E-02	5.48E-02	4.62E-01	1.65E-01	-1.72E+00
		4	1	G	4.18E-01	2.33E-01	1.55E-01	1.15E-01	4.30E-01	6.23E-01	-5.85E-01
				T	4.78E-01	3.35E-01	2.79E-01	2.51E-01	3.79E-01	5.17E-01	-1.30E+00
		5	1.2	G	4.14E-01	2.35E-01	1.60E-01	1.20E-01	4.23E-01	5.72E-01	-7.17E-01
				T	4.49E-01	2.89E-01	2.17E-01	1.73E-01	4.00E-01	3.38E-01	-1.45E+00
		6	2	G	3.84E-01	2.20E-01	1.51E-01	1.15E-01	4.05E-01	5.82E-01	-7.78E-01
				T	3.79E-01	2.16E-01	1.48E-01	1.12E-01	4.04E-01	5.71E-01	-7.58E-01
		7	5	G	2.96E-01	1.62E-01	1.10E-01	8.33E-02	3.66E-01	9.03E-01	-3.03E-01
				T	2.86E-01	1.55E-01	1.05E-01	7.94E-02	3.62E-01	9.61E-01	-1.86E-01
		8	10	G	2.27E-01	1.20E-01	8.10E-02	6.10E-02	3.27E-01	1.26E+00	5.38E-01
				T	2.22E-01	1.17E-01	7.86E-02	5.92E-02	3.24E-01	1.30E+00	6.47E-01
0.1	0.0	9	0.1	G	5.32E-01	3.45E-01	2.50E-01	1.95E-01	4.33E-01	8.62E-02	-1.04E+00
				T	3.93E-01	1.64E-01	7.52E-02	3.99E-02	4.79E-01	4.26E+00	1.81E+01
		10	0.2	G	3.53E-01	2.00E-01	1.38E-01	1.05E-01	3.91E-01	6.60E-01	-6.98E-01
				T	1.89E-01	5.96E-02	3.23E-02	2.30E-02	3.60E-01	3.27E+00	1.04E+01
		11	0.5	G	1.77E-01	9.18E-02	6.16E-02	4.64E-02	2.92E-01	1.62E+00	1.66E+00
				T	8.34E-02	4.01E-02	2.67E-02	2.00E-02	2.08E-01	2.96E+00	8.55E+00
		12	1	G	1.04E-01	5.27E-02	3.52E-02	2.65E-02	2.27E-01	2.45E+00	5.47E+00
				T	6.71E-02	3.37E-02	2.25E-02	1.68E-02	1.83E-01	3.27E+00	1.07E+01
		13	5	G	3.75E-02	1.88E-02	1.25E-02	9.40E-03	1.37E-01	4.59E+00	2.24E+01
				T	3.47E-02	1.74E-02	1.16E-02	8.69E-03	1.32E-01	4.80E+00	2.46E+01
	0.1	14	0.2	G	3.18E-01	1.76E-01	1.20E-01	9.11E-02	3.77E-01	8.06E-01	-4.77E-01
				T	2.43E-01	9.86E-02	4.99E-02	3.00E-02	3.80E-01	8.44E-01	7.81E-01
	0.2	15	0.2	G	2.34E-01	1.21E-01	8.05E-02	6.03E-02	3.37E-01	1.27E+00	6.10E-01
				T	4.61E-01	4.32E-01	4.24E-01	4.21E-01	1.71E-01	2.22E-01	-1.89E+00
	0.4	16	0.2	G	7.36E-02	2.69E-02	1.62E-02	1.16E-02	2.16E-01	3.50E+00	1.36E+01
				T	6.24E-02	7.91E-03	1.05E-03	1.41E-04	2.34E-01	2.22E-01	-1.89E+00

*G and T denote Gaussian and telegraph velocity process, respectively.

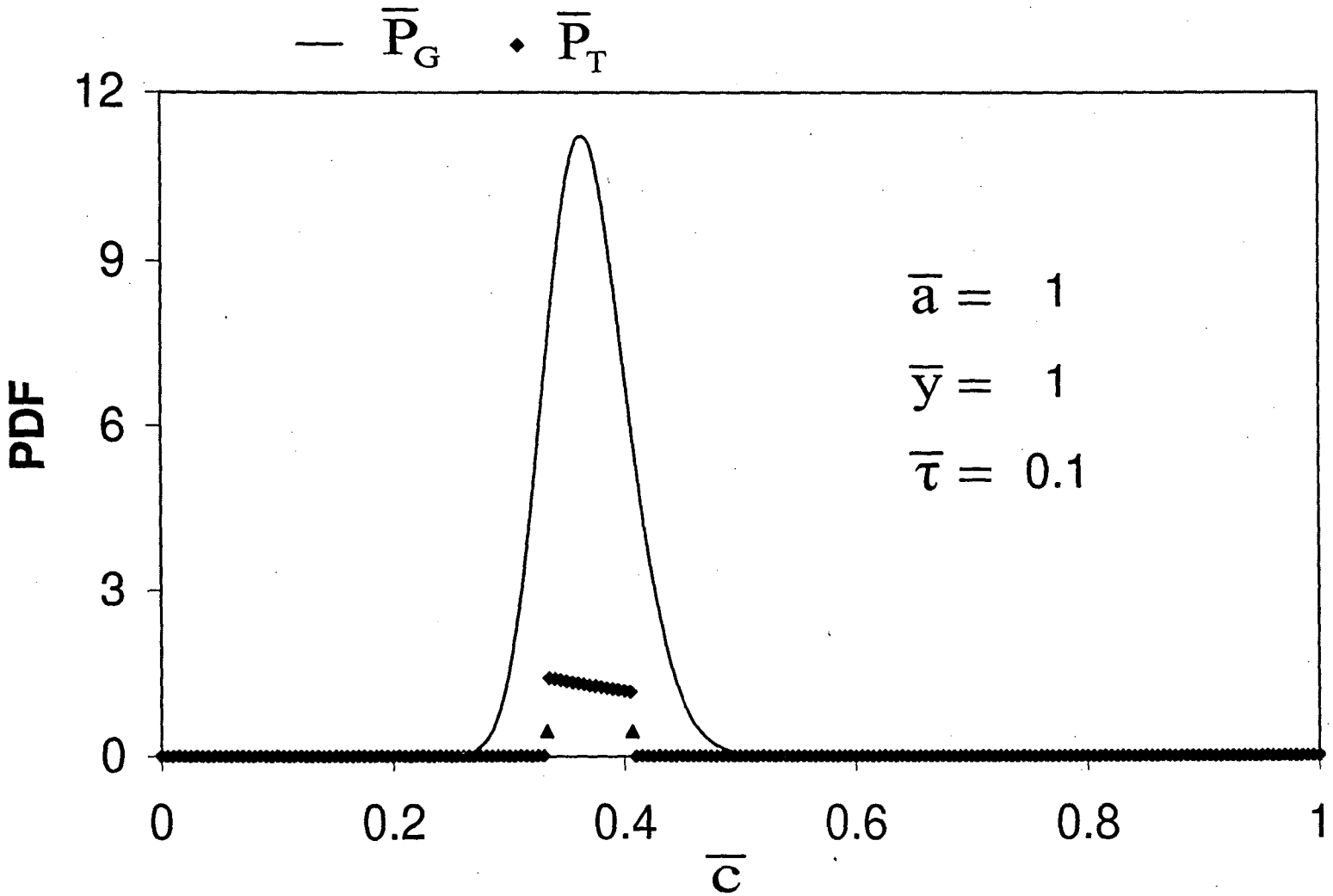

Figure 1. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=0.1$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.3229$ *and* $\overline{c}_2=0.4066$.

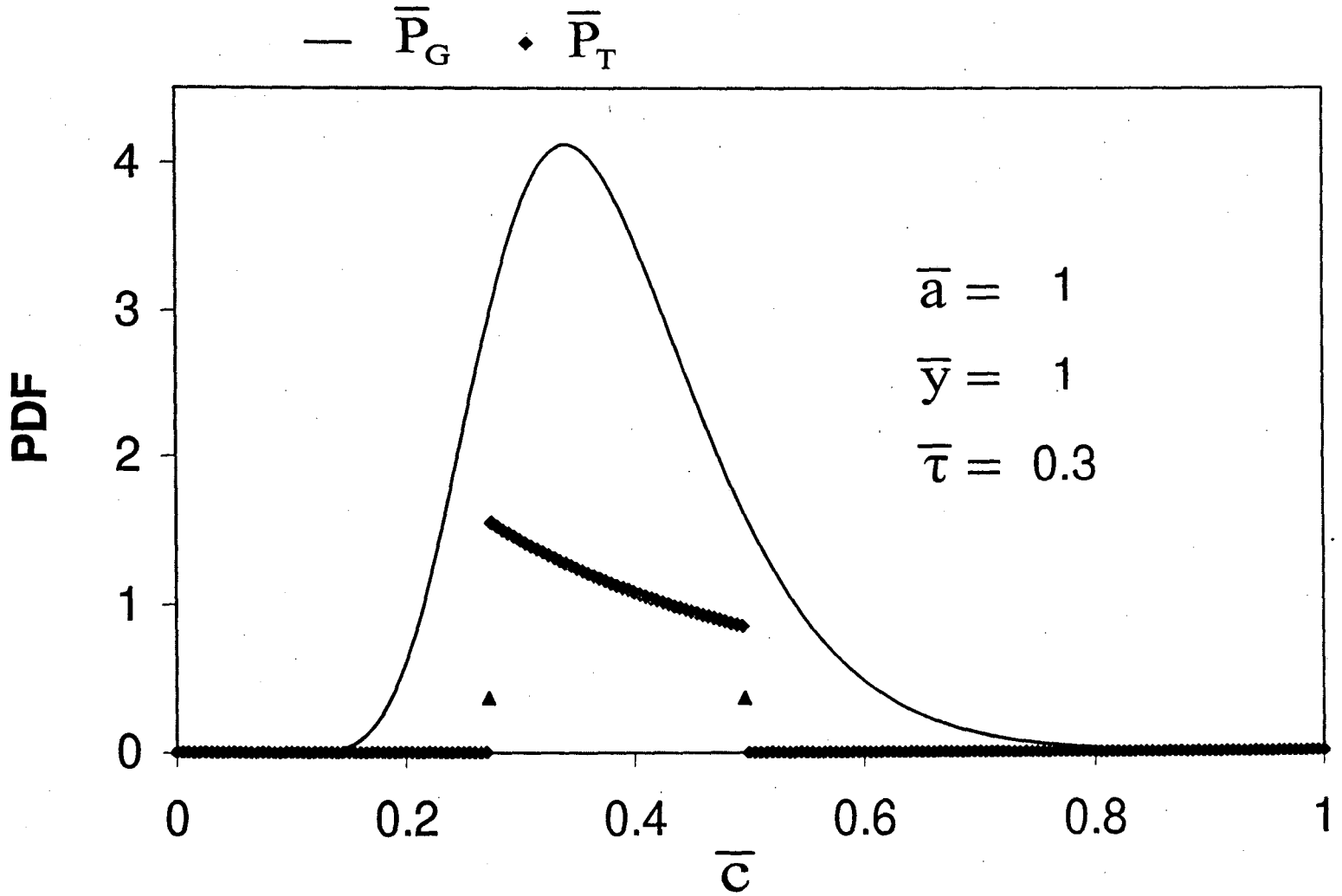

Figure 2. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=0.3$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.2725$ *and* $\overline{c}_2=0.4966$.

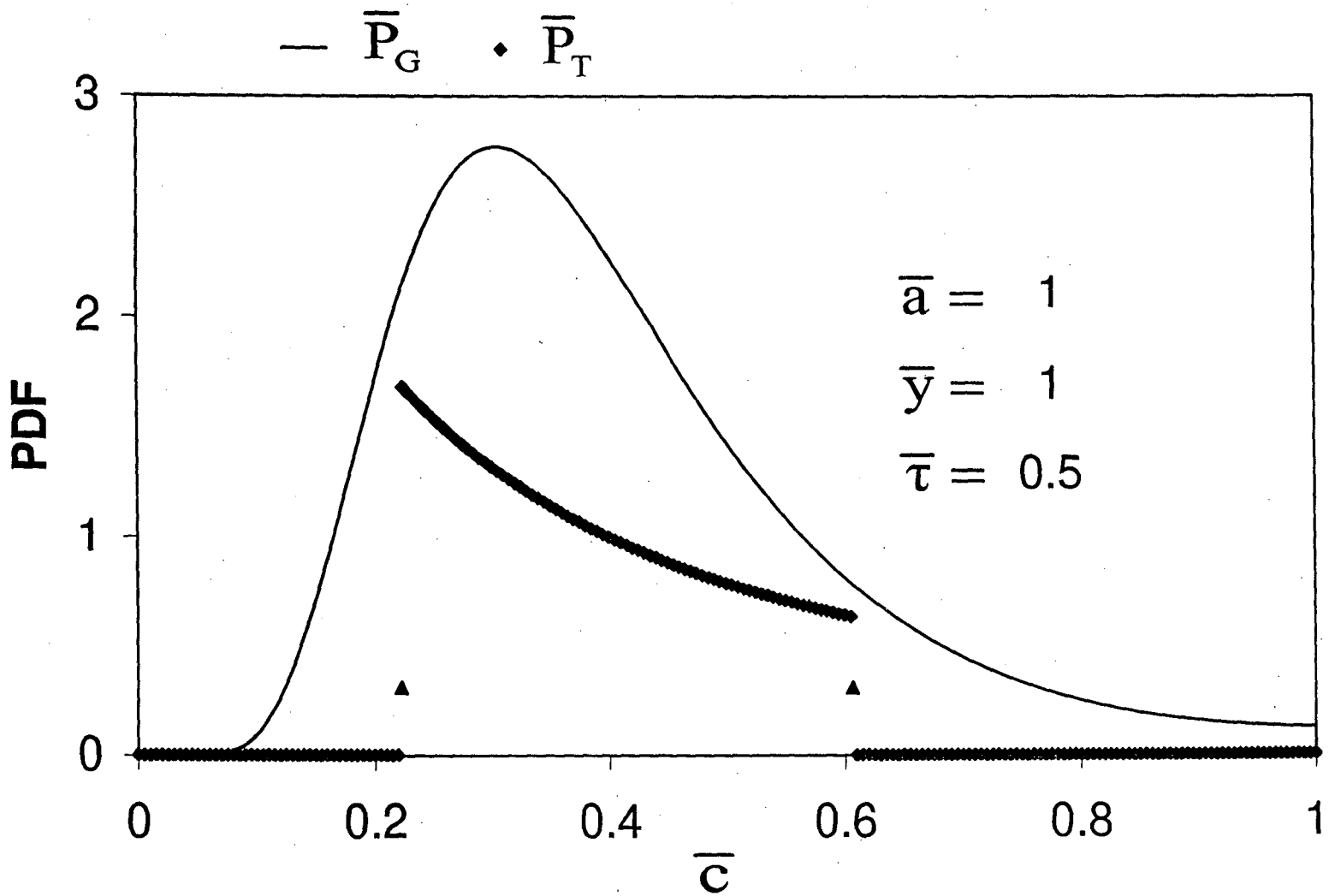

Figure 3. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=0.5$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.2231$ *and* $\overline{c}_2=0.6065$.

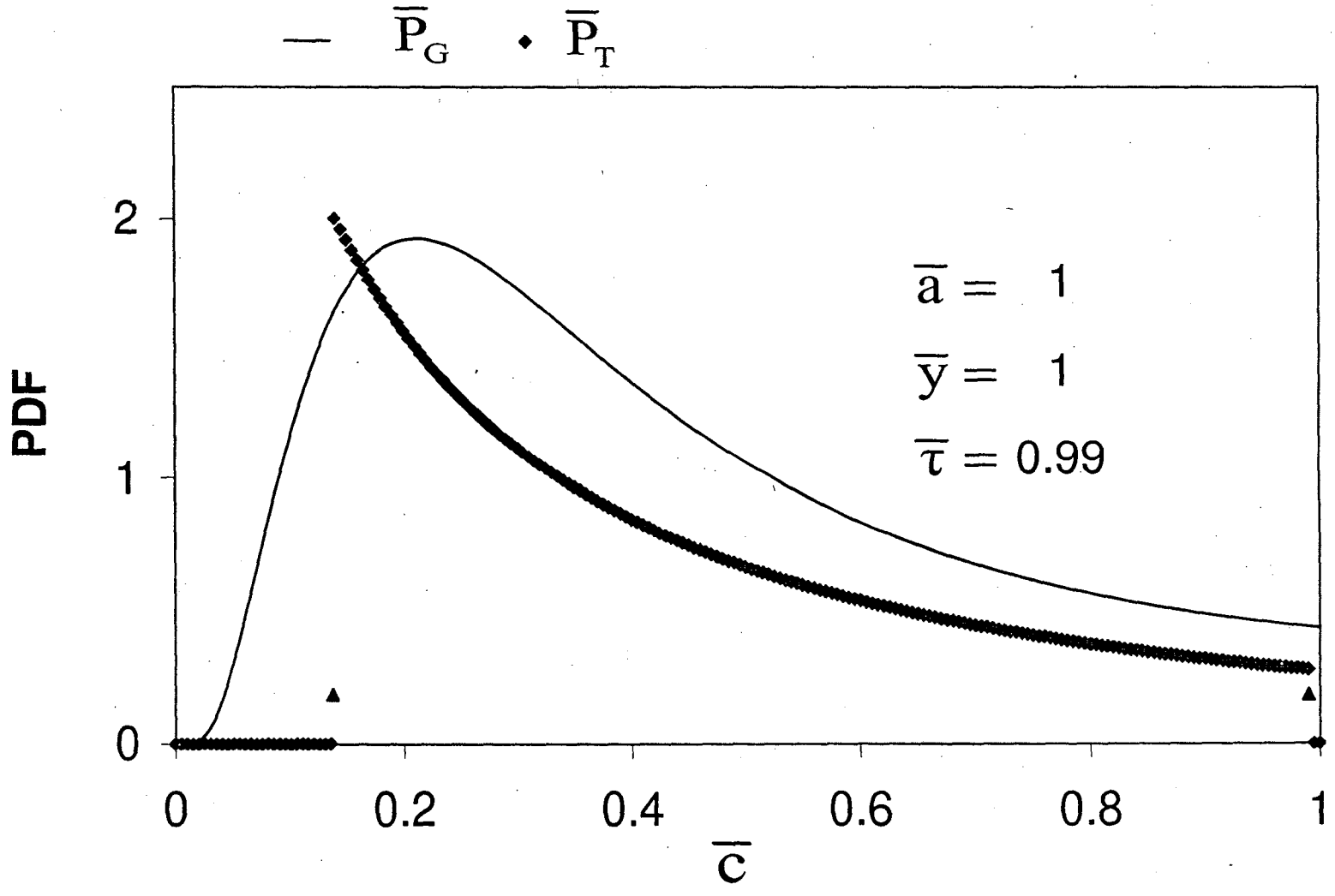

Figure 4. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for the Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=0.99$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.1367$ *and* $\overline{c}_2=0.99$.

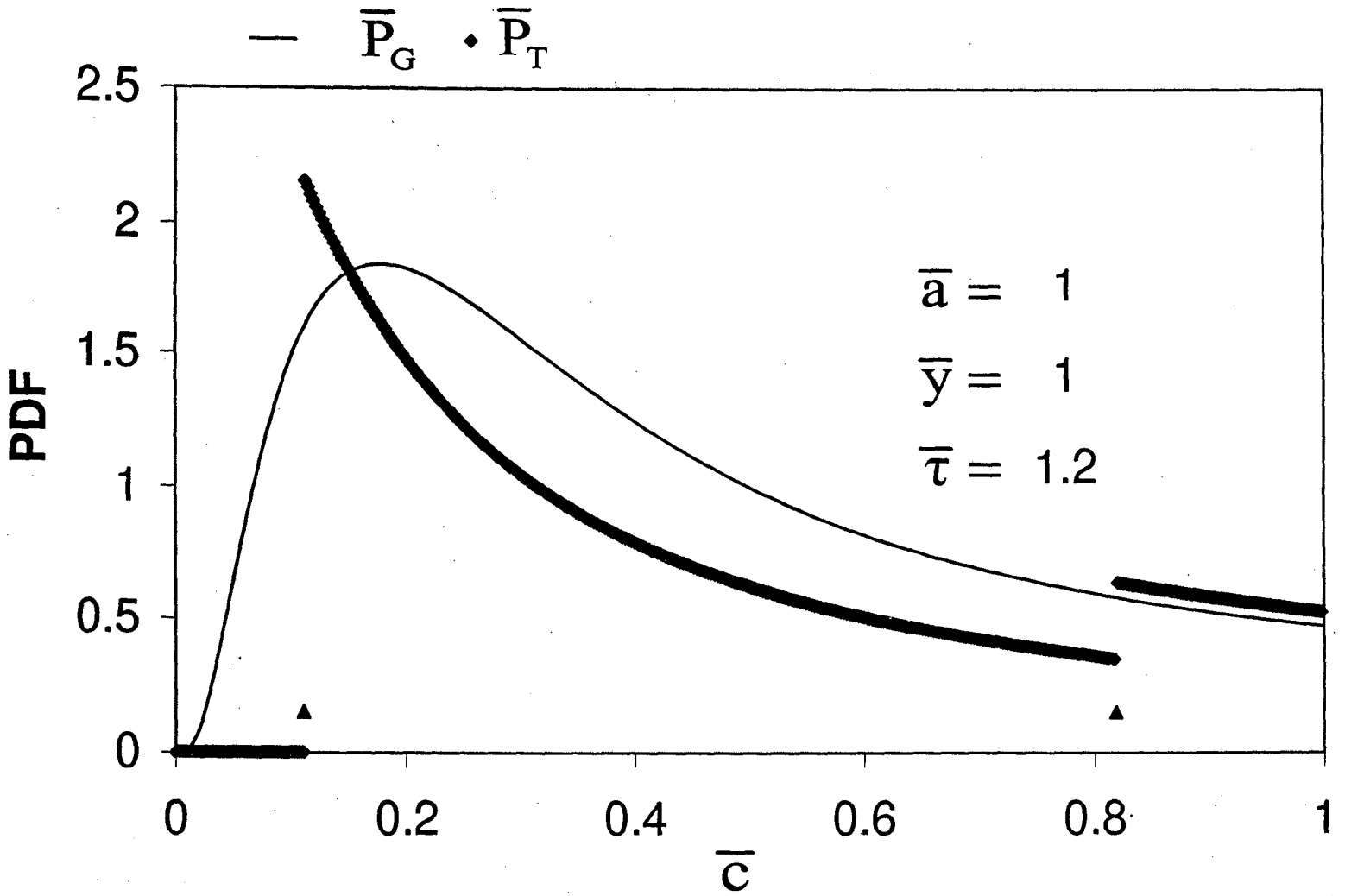

Figure 5. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=1.2$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.1108$ *and* $\overline{c}_2=0.8187$.

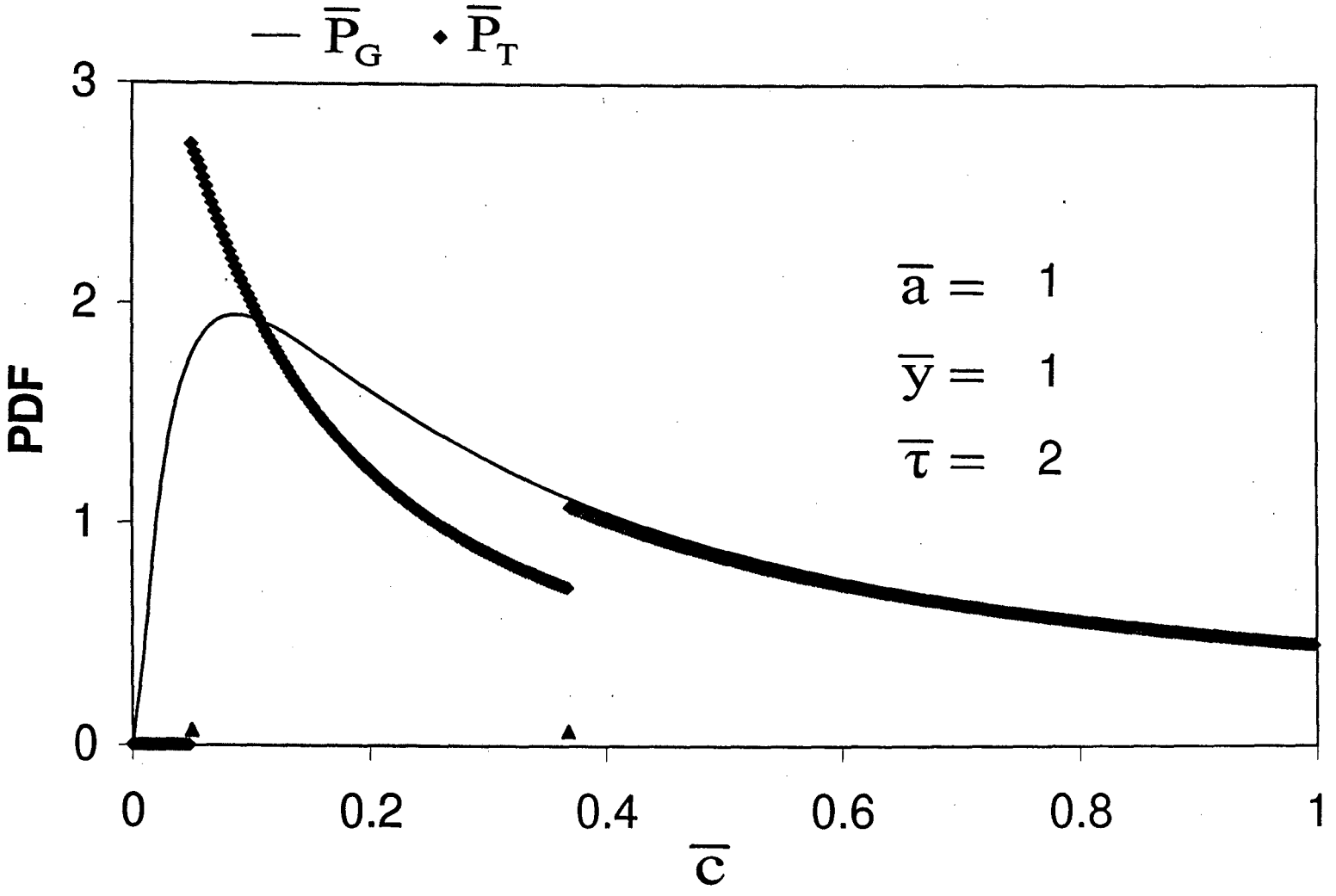

Figure 6. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=2$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.0498$ *and* $\overline{c}_3=0.3679$.

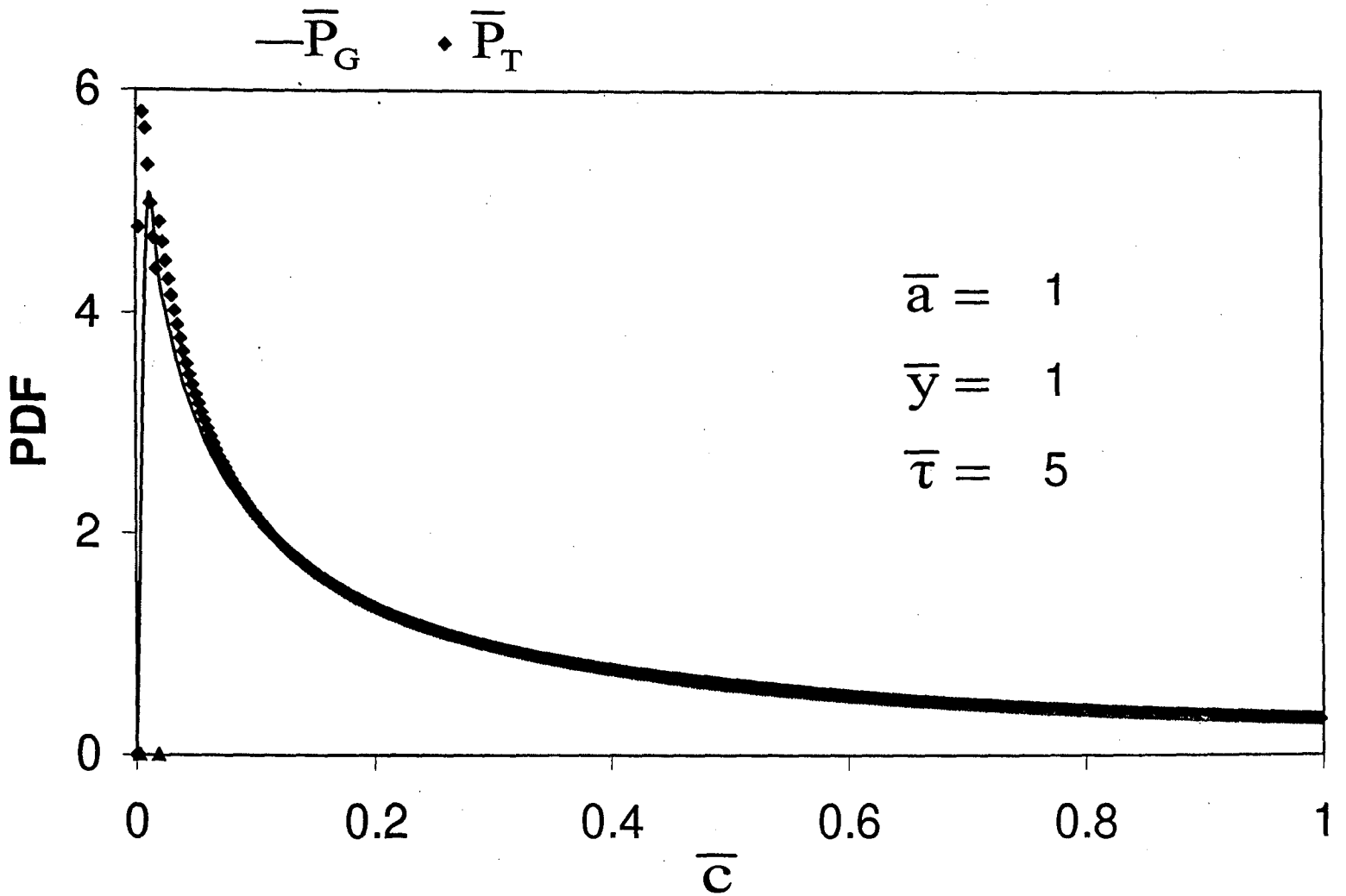

Figure 7. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=5$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.0025$ *and* $\overline{c}_2=0.0183$.

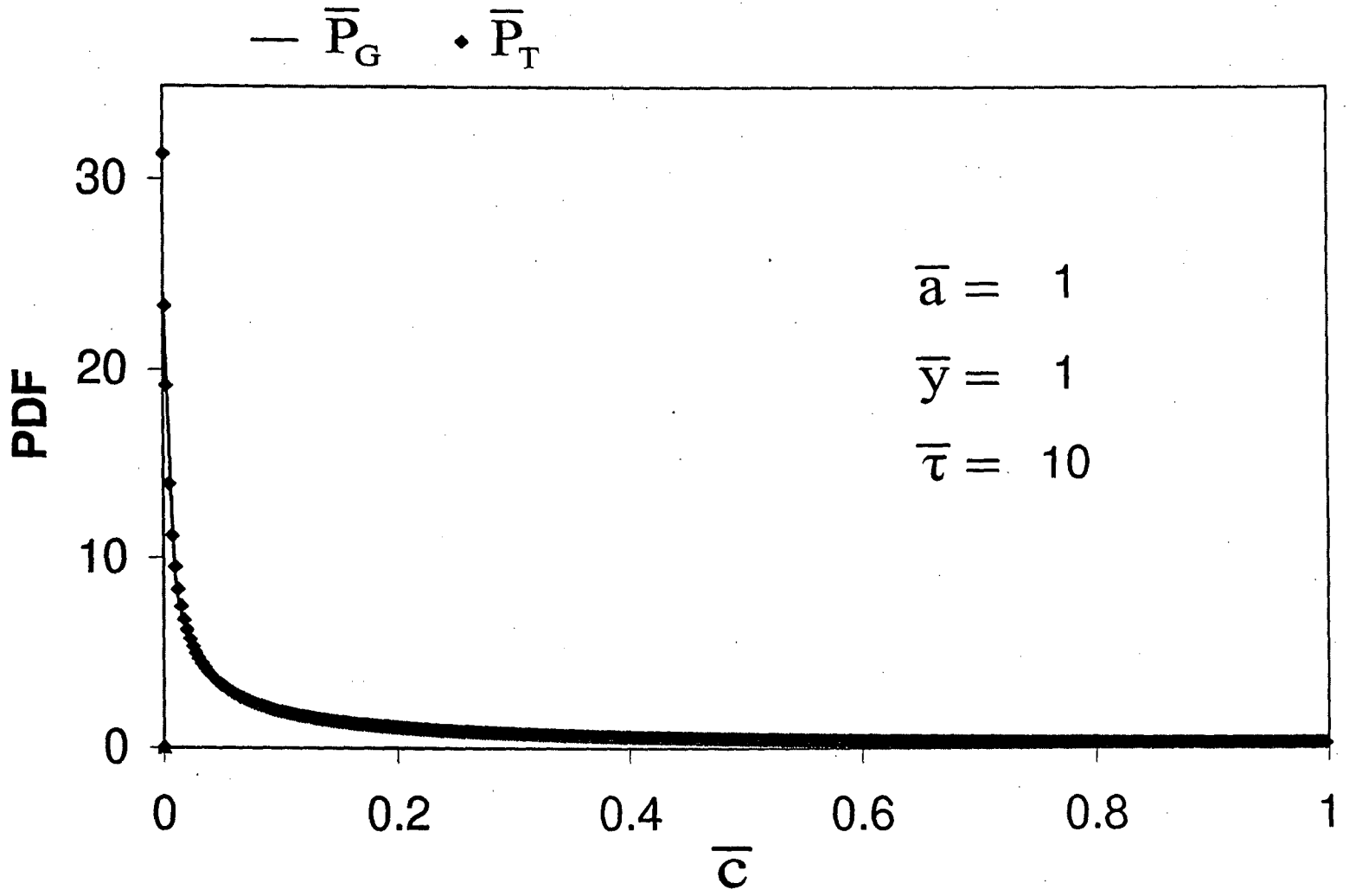

Figure 8. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=1$ and $\overline{\tau}=10$ for $\overline{a}=1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1 = 2\times10^{-5}$ *and* $\overline{c}_2 = 1\times10^{-4}$.

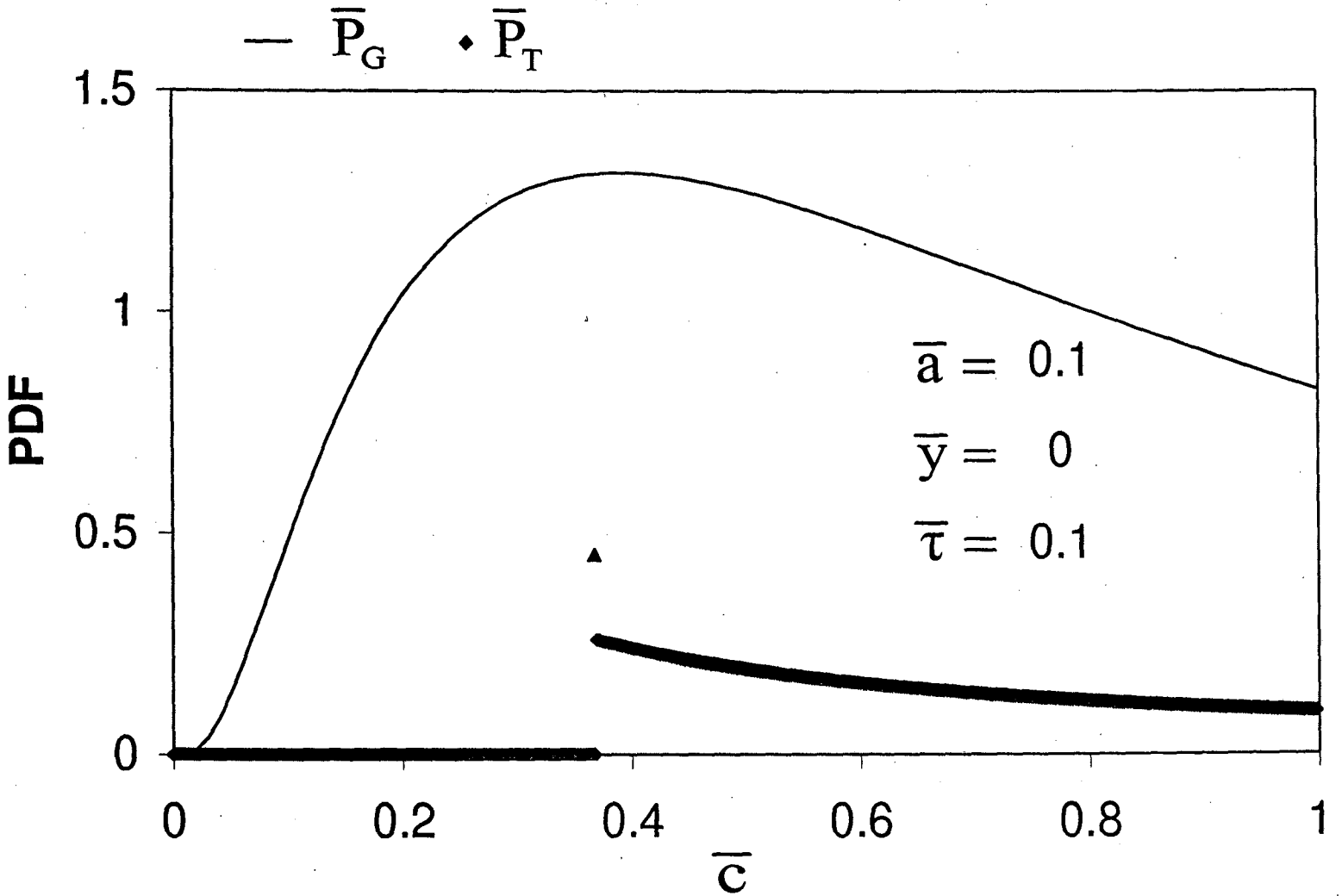

Figure 9. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0$ and $\overline{\tau}=0.1$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=\overline{c}_2=0.3679$.

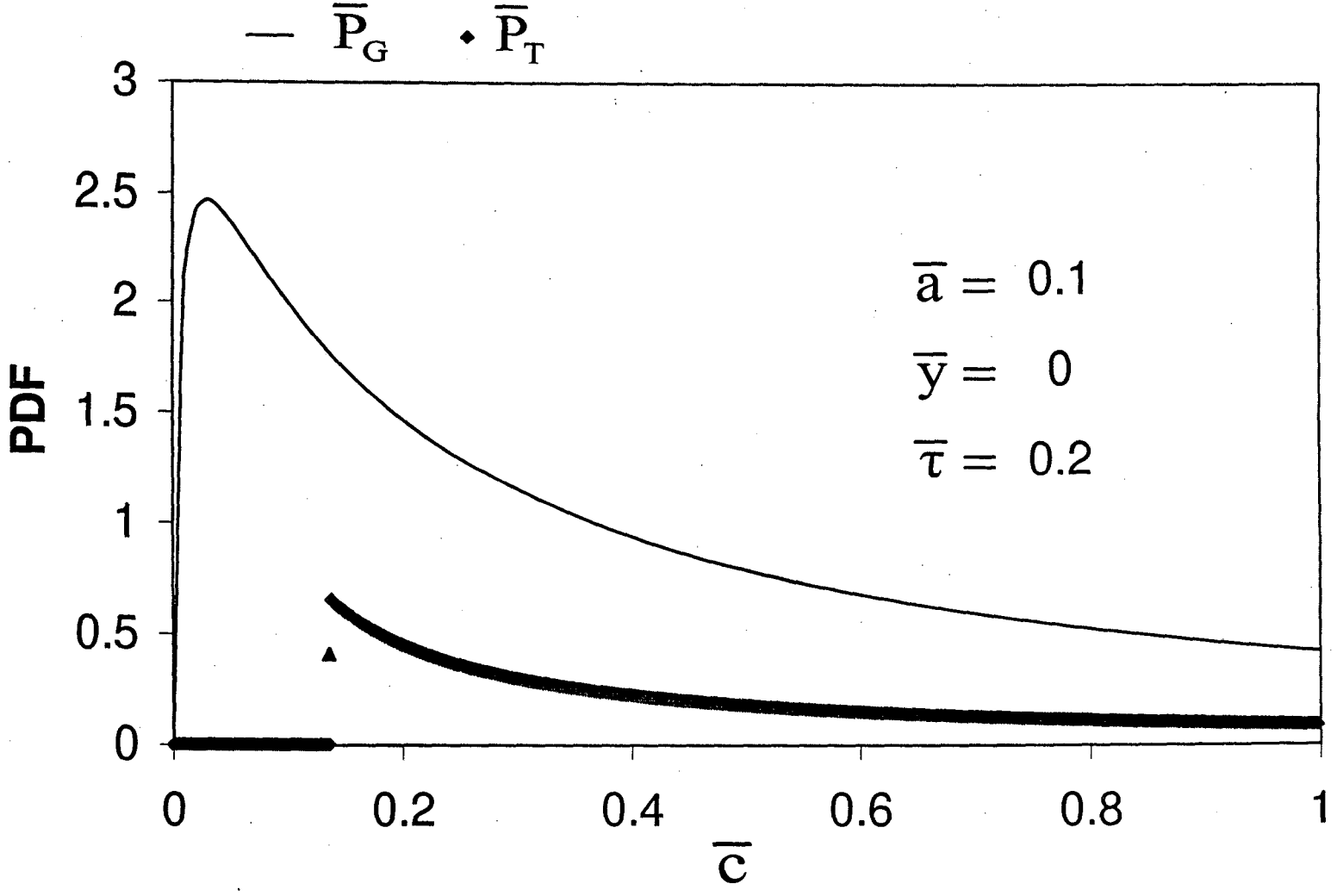

Figure 10. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0$ and $\overline{\tau}=0.2$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=\overline{c}_2=0.13534$.

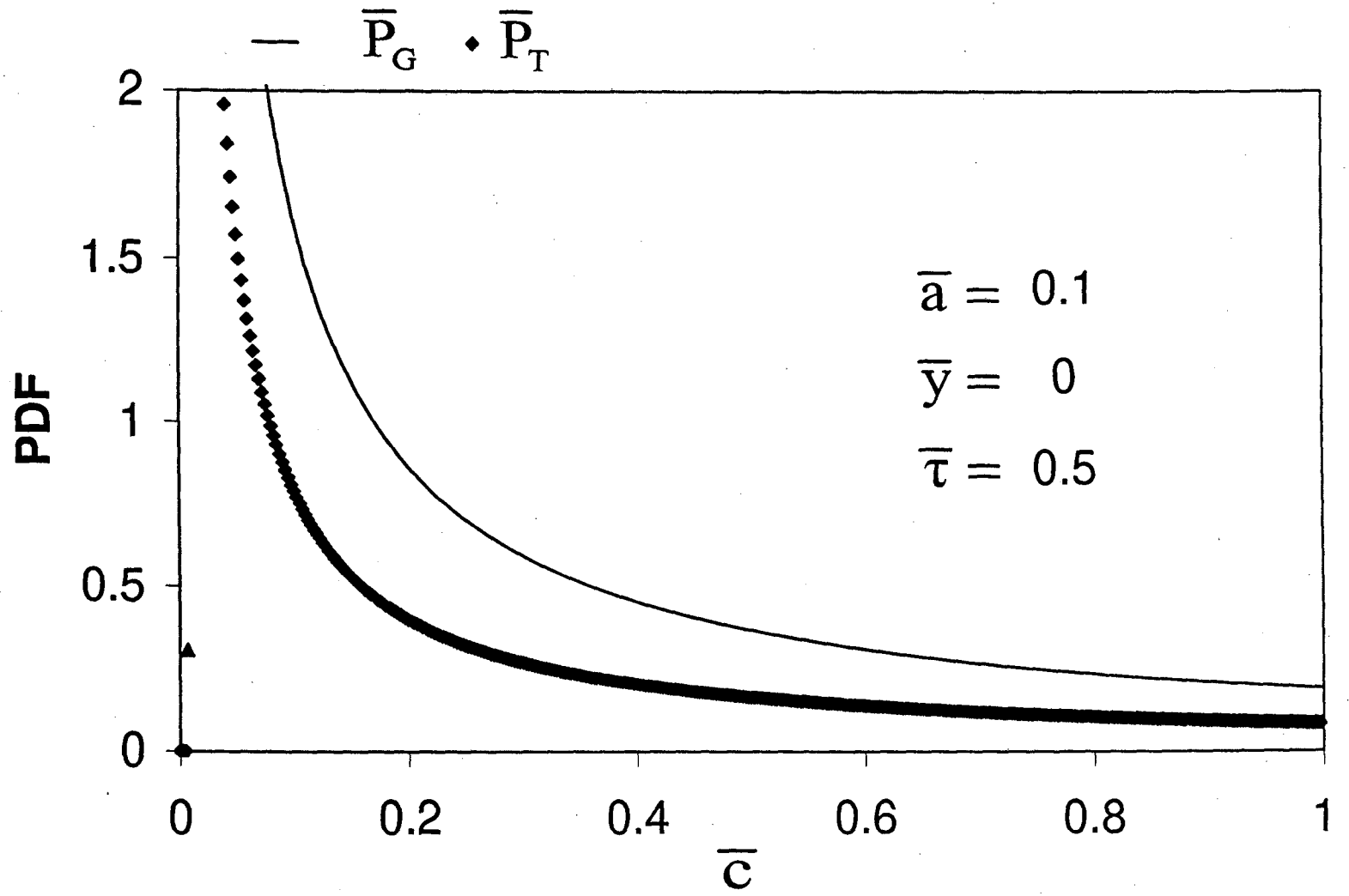

Figure 11. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0$ and $\overline{\tau}=0.5$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=\overline{c}_2=0.00674$.

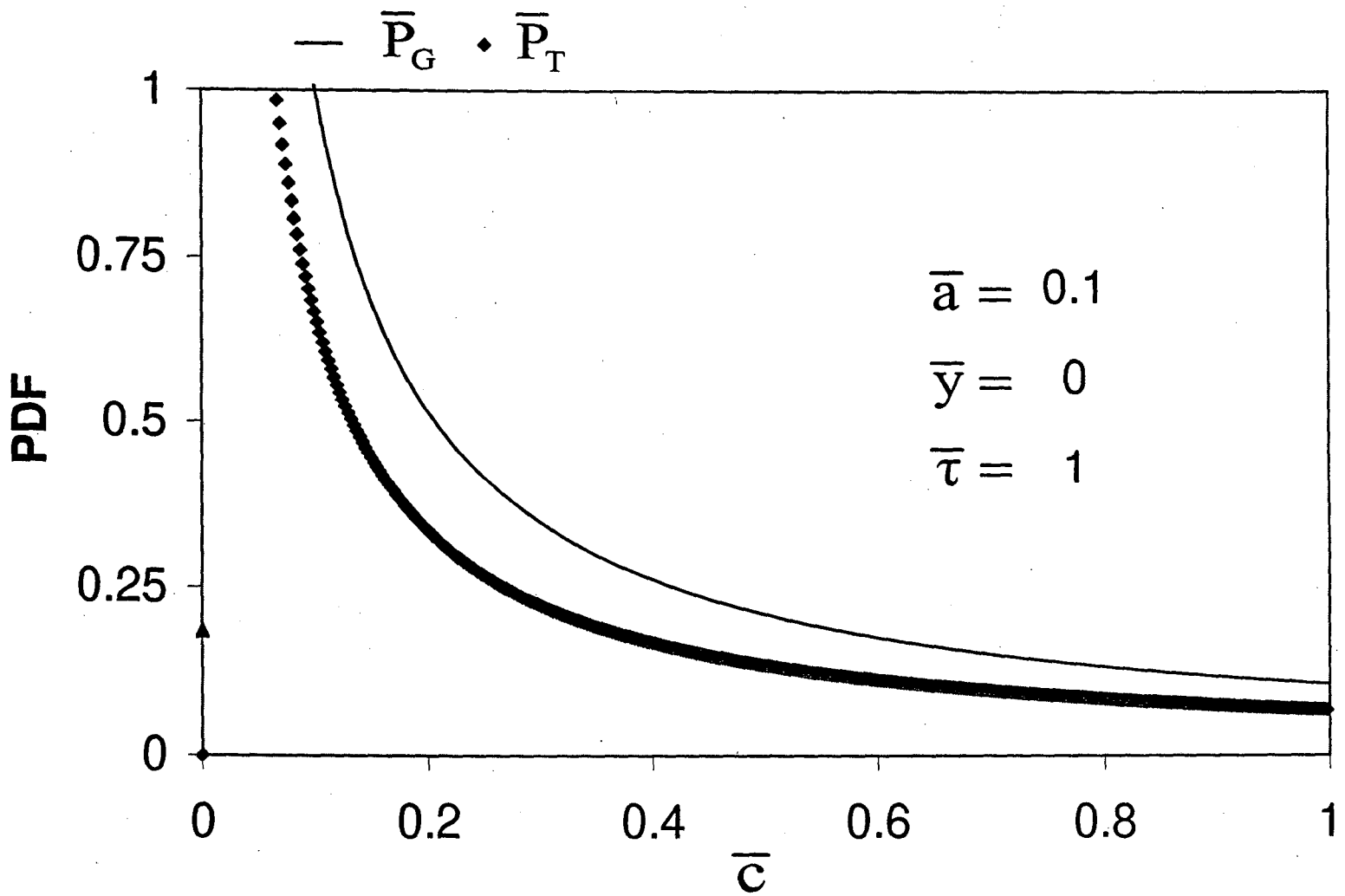

Figure 12. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0$ and $\overline{\tau}=1$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=\overline{c}_2=4\times10^{-5}$.

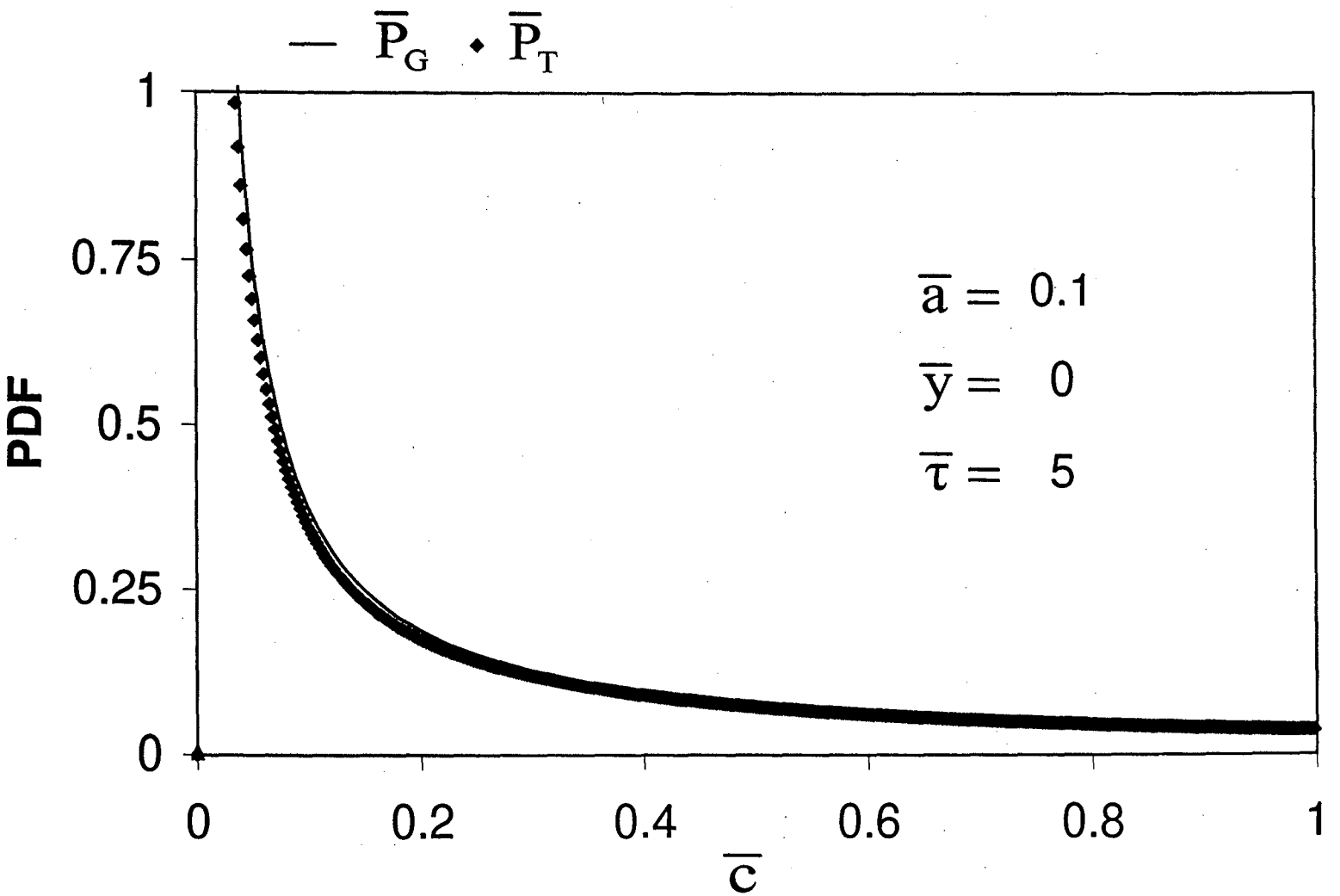

Figure 13. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0$ and $\overline{\tau}=5$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=\overline{c}_2=2\times10^{-22}$.

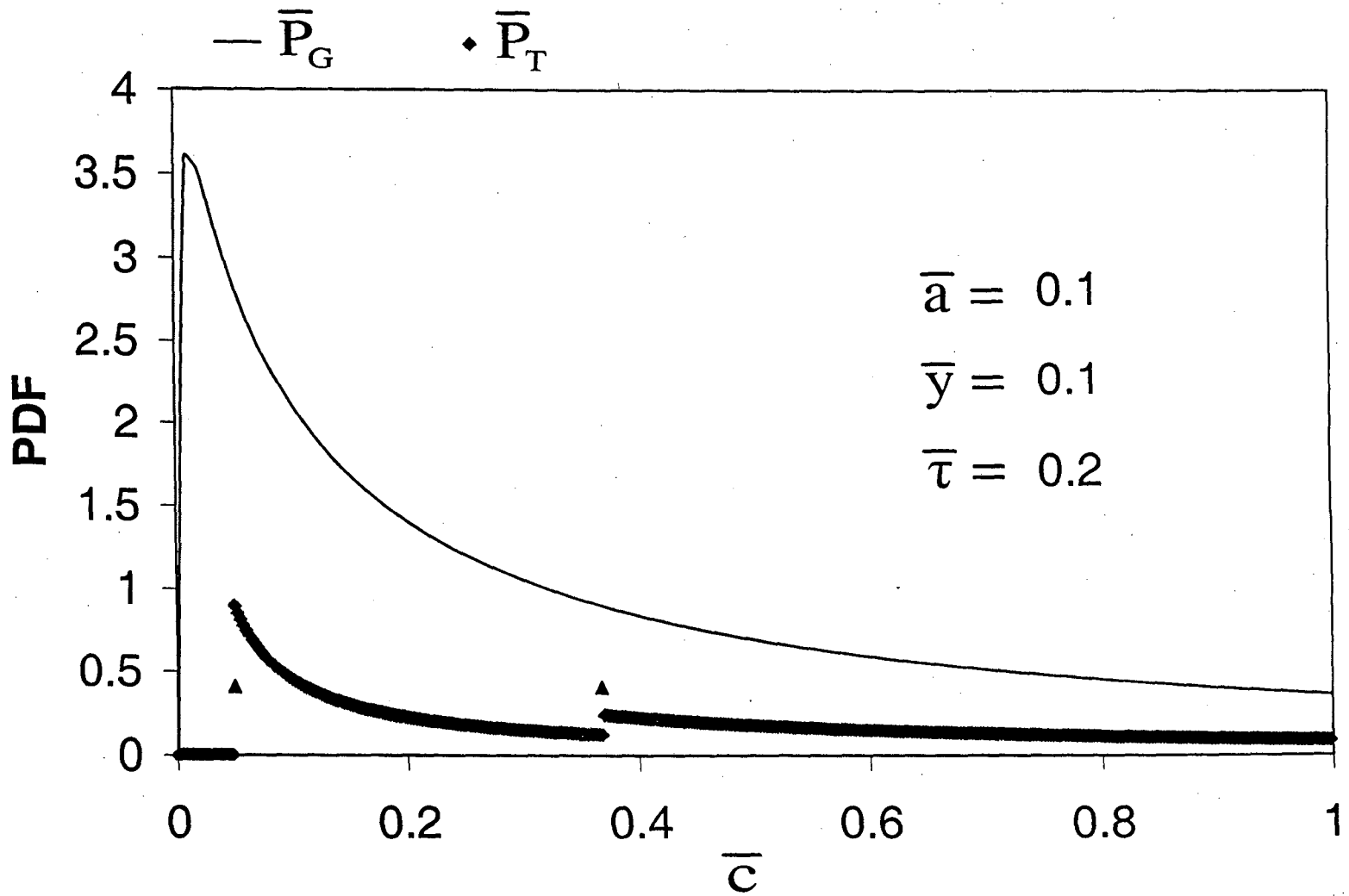

Figure 14. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0.1$ and $\overline{\tau}=0.2$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.04978$ *and* $\overline{c}_3=0.3679$.

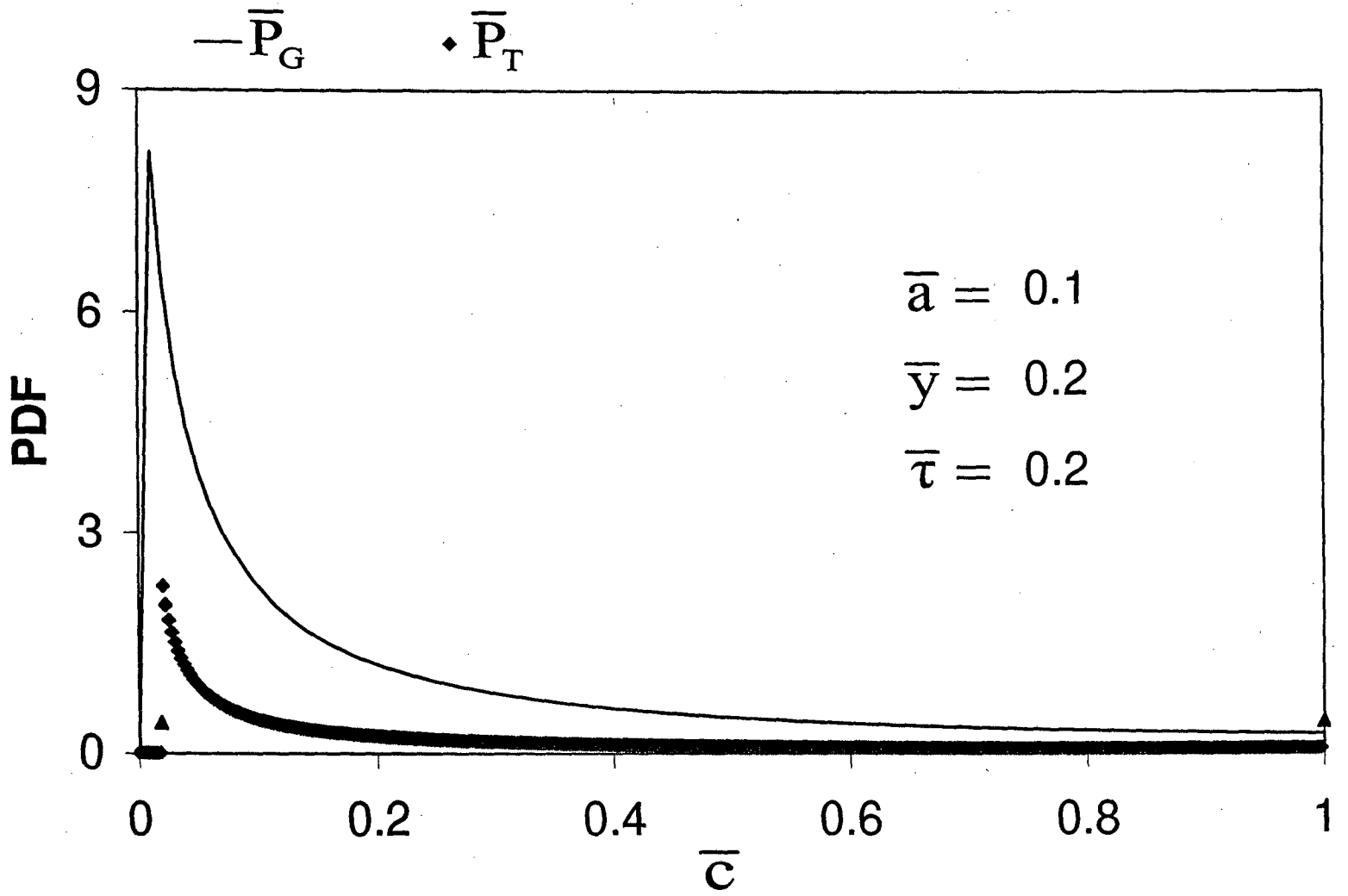

Figure 15. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0.2$ and $\overline{\tau}=0.2$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.01831$ *and* $\overline{c}_3=1$.

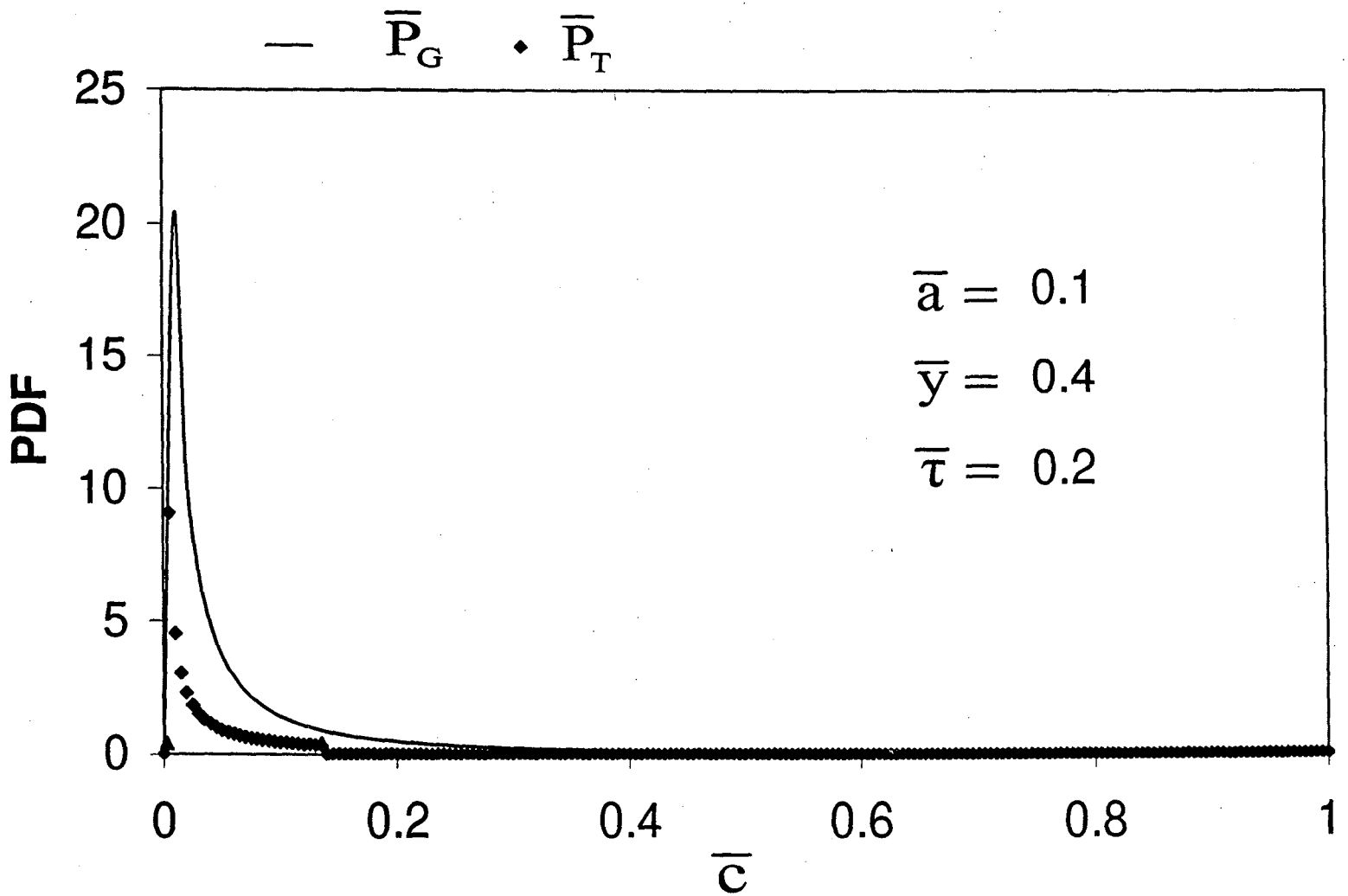

Figure 16. $\overline{P}_G(\overline{y},\overline{\tau};\overline{c})$ for Gaussian velocity process and $\overline{P}_T(\overline{y},\overline{\tau};\overline{c})$ for telegraph velocity process at $\overline{y}=0.4$ and $\overline{\tau}=0.2$ for $\overline{a}=0.1$. The symbol ▲ shows the value of the coefficient of the δ-function at $\overline{c}_1=0.006738$ *and* $\overline{c}_3=0.1353$.

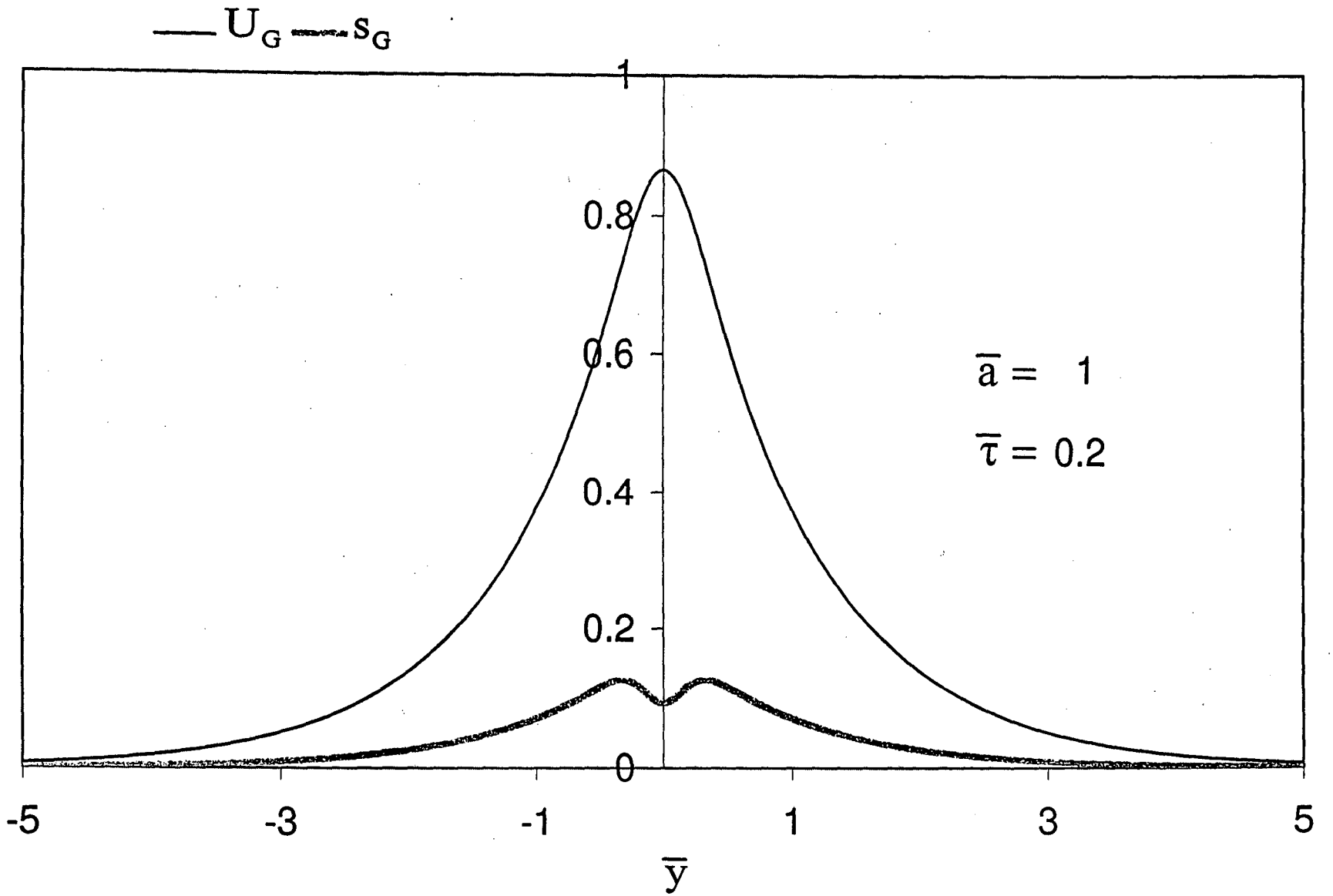

Figure 17. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for Gaussian random velocity at $\bar{\tau} = 0.2$ and $\bar{a} = 1$.

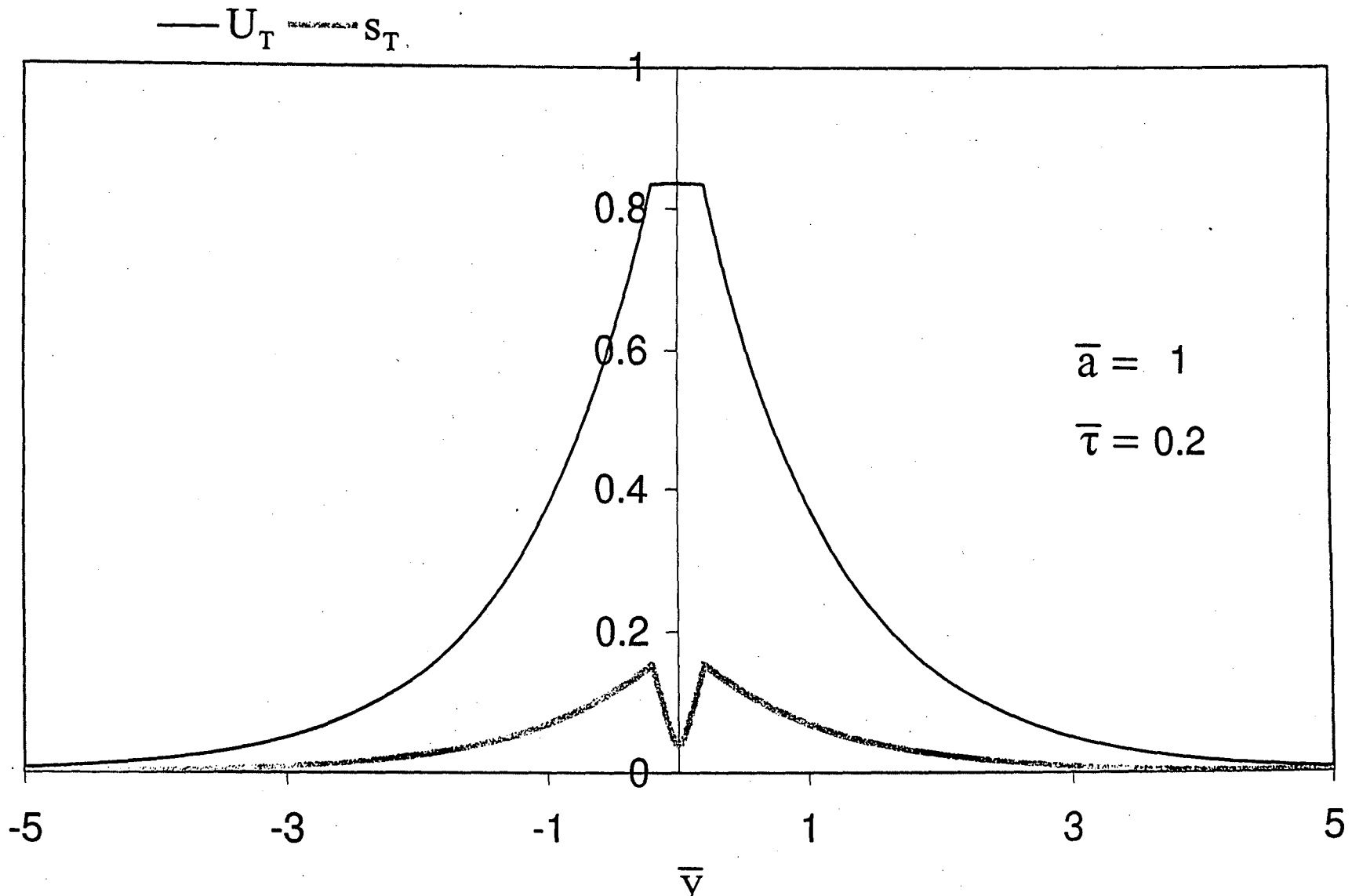

Figure 18. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for telegraph random velocity at $\bar{\tau} = 0.2$ and $\bar{a} = 1$.

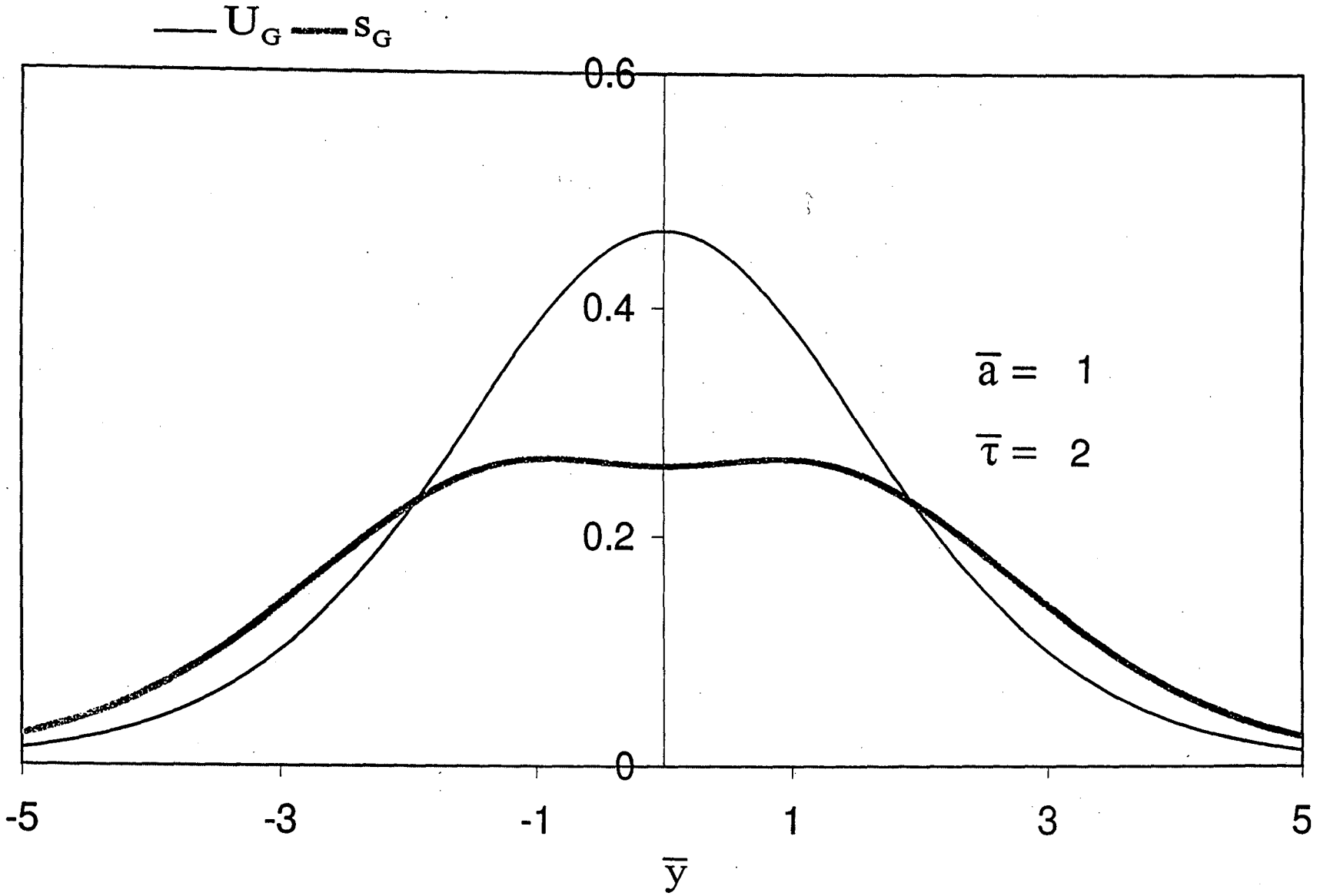

Figure 19. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for Gaussian random velocity at $\bar{\tau} = 2$ and $\bar{a} = 1$.

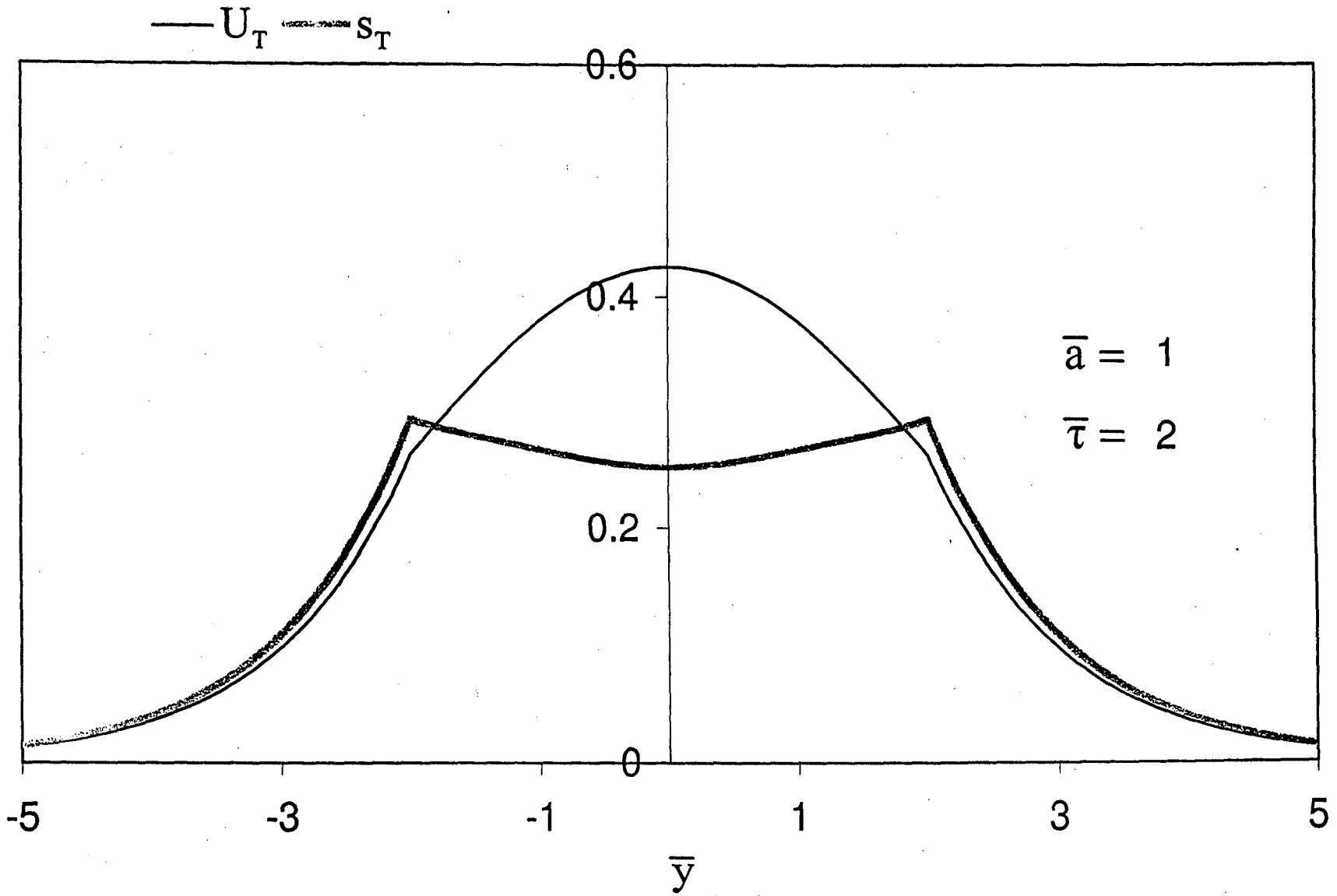

Figure 20. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for telegraph random velocity at $\bar{\tau} = 2$ and $\bar{a} = 1$.

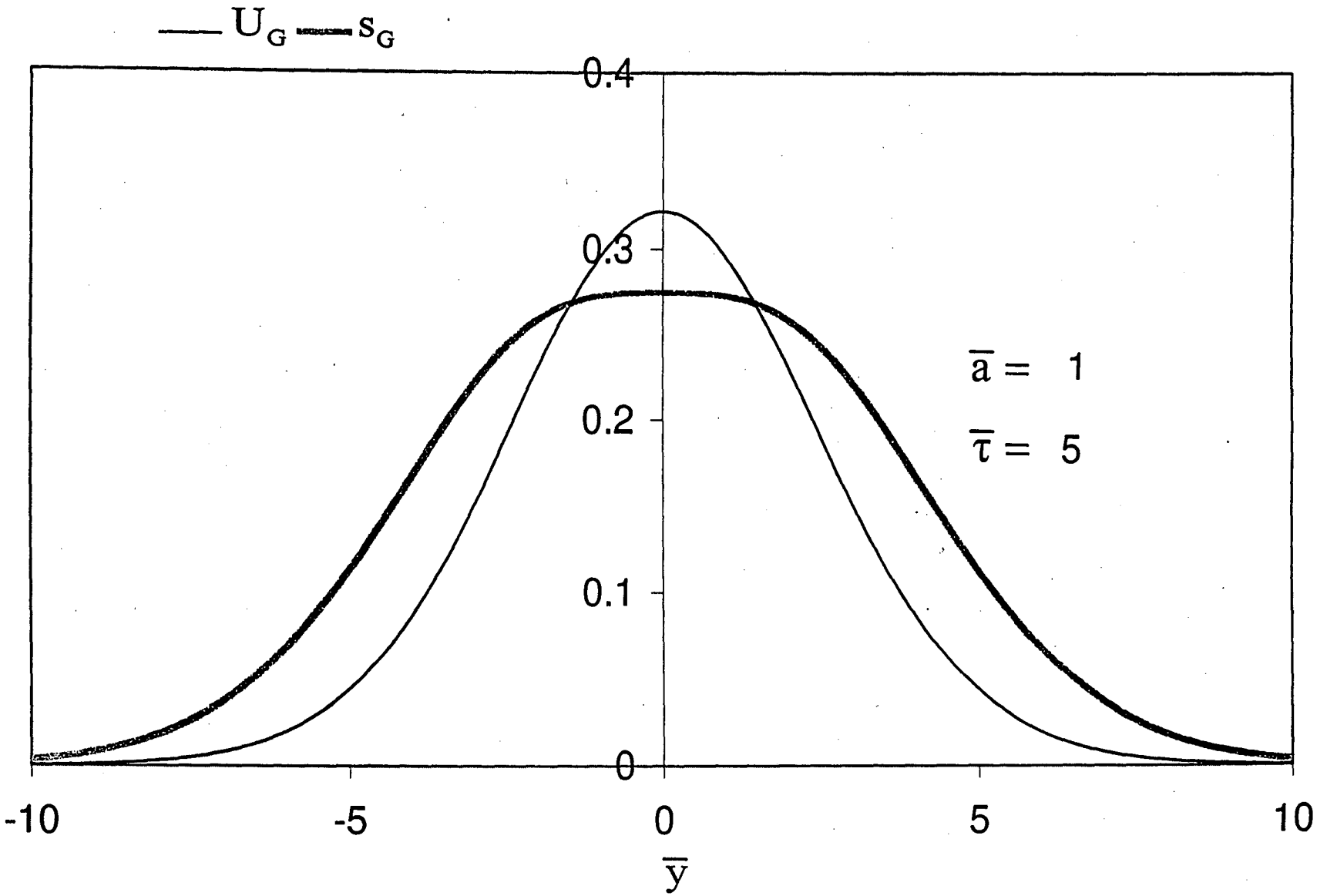

Figure 21. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for Gaussian random velocity at $\bar{\tau} = 5$ and $\bar{a} = 1$.

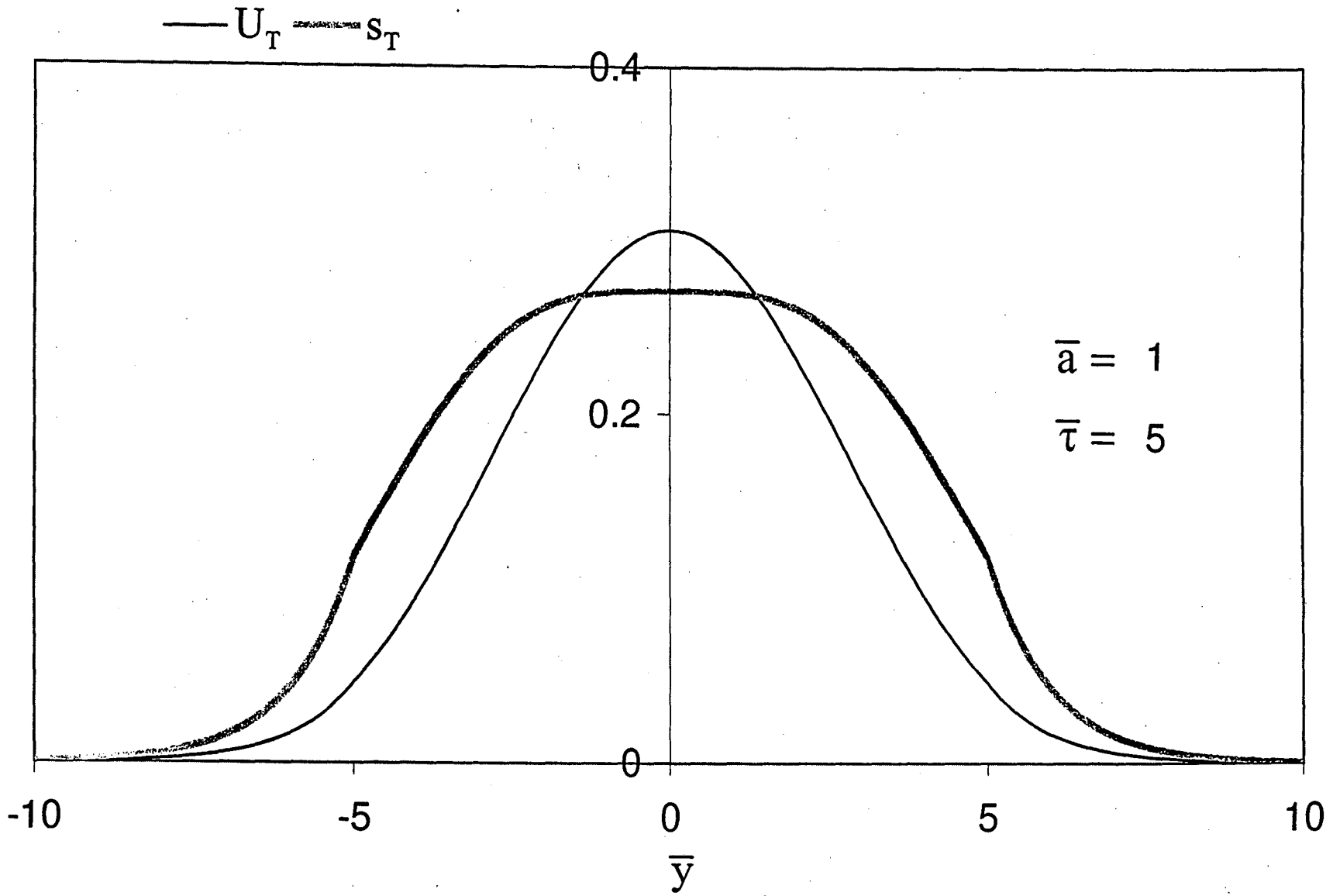

Figure 22. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for telegraph random velocity at $\bar{\tau} = 5$ and $\bar{a} = 1$.

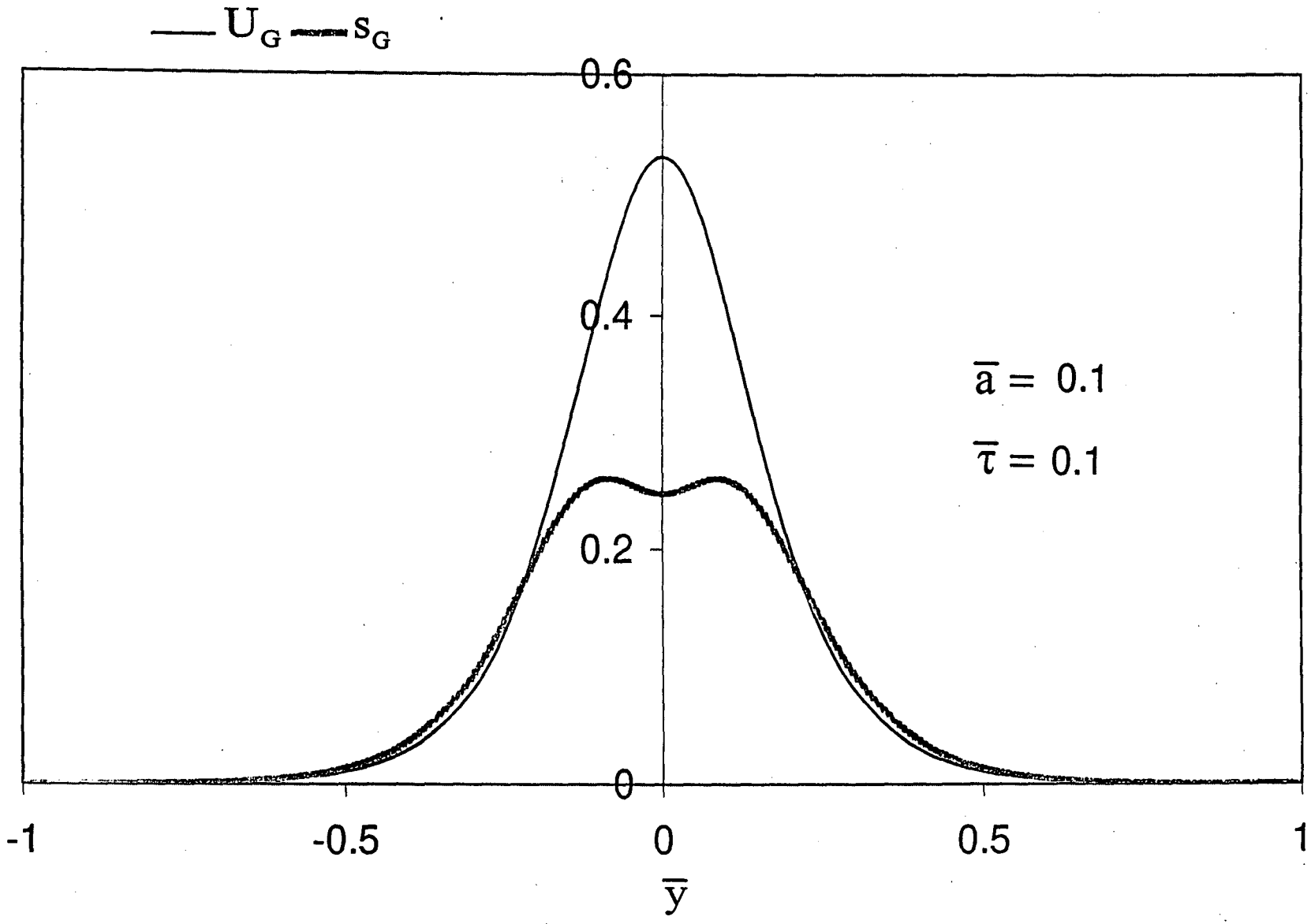

Figure 23. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for Gaussian random velocity at $\bar{\tau} = 0.1$ and $\bar{a} = 0.1$.

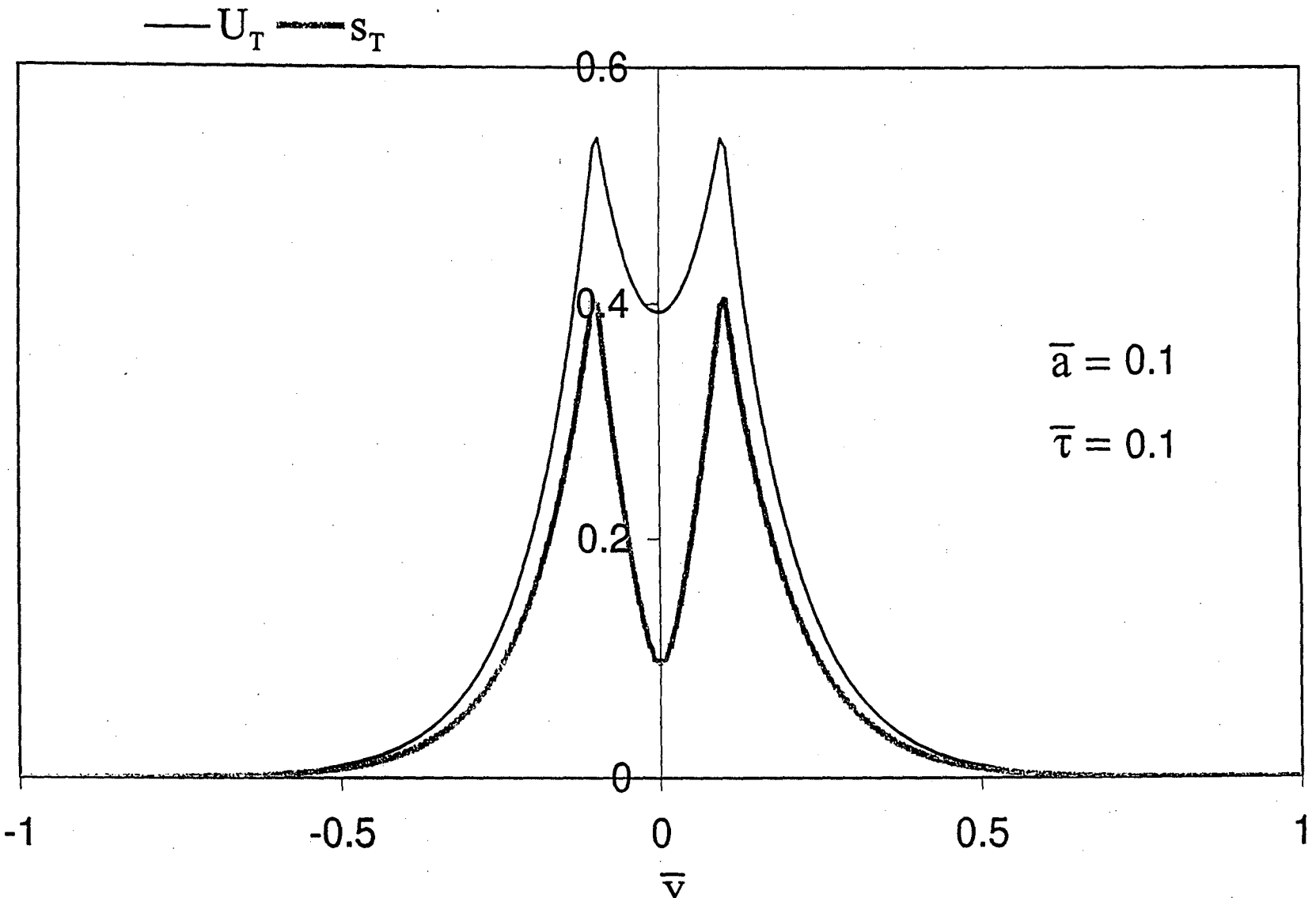

Figure 24. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for telegraph random velocity at $\bar{\tau} = 0.1$ and $\bar{a} = 0.1$.

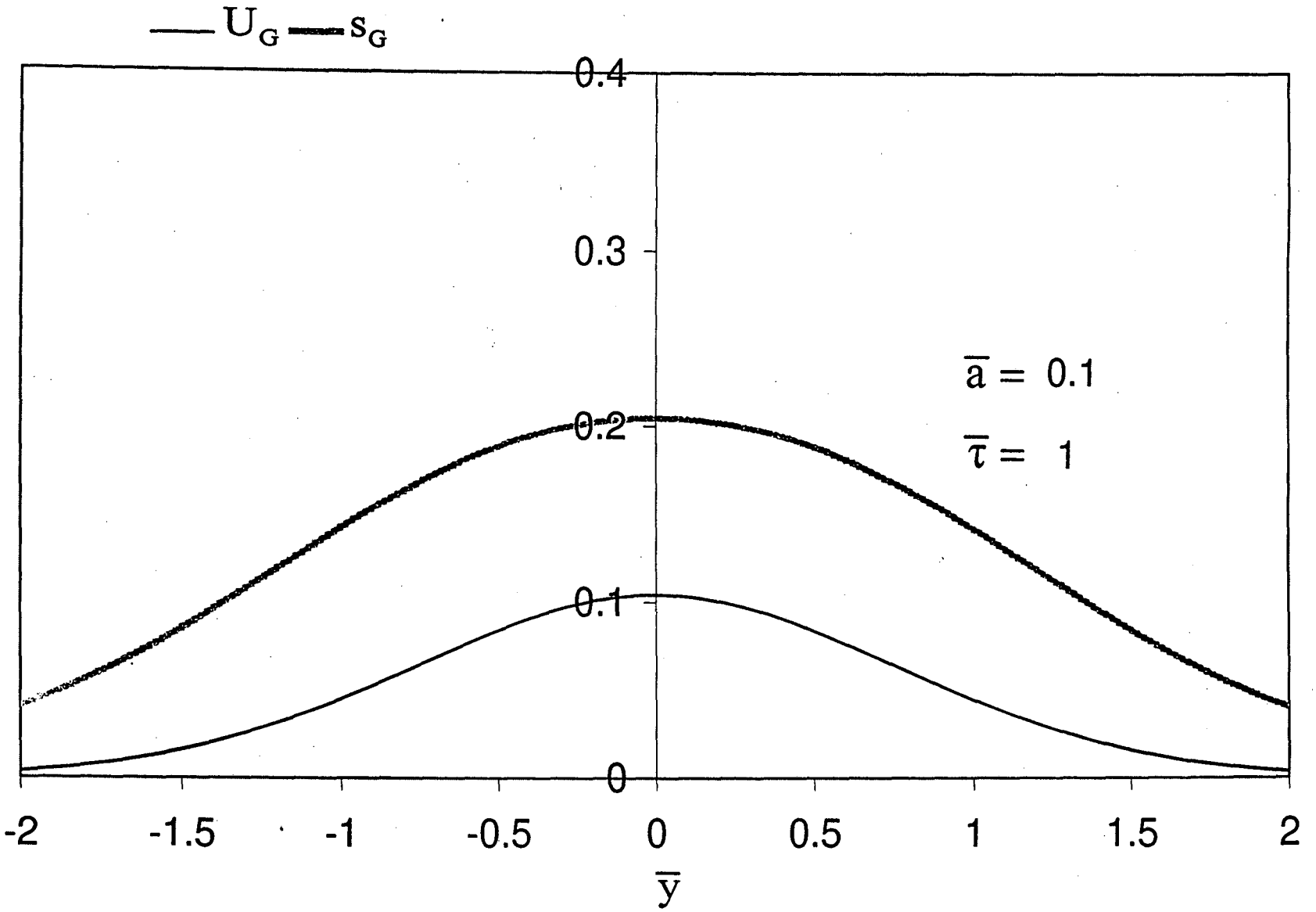

Figure 25. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for Gaussian random velocity at $\bar{\tau}=1$ and $\bar{a}=0.1$.

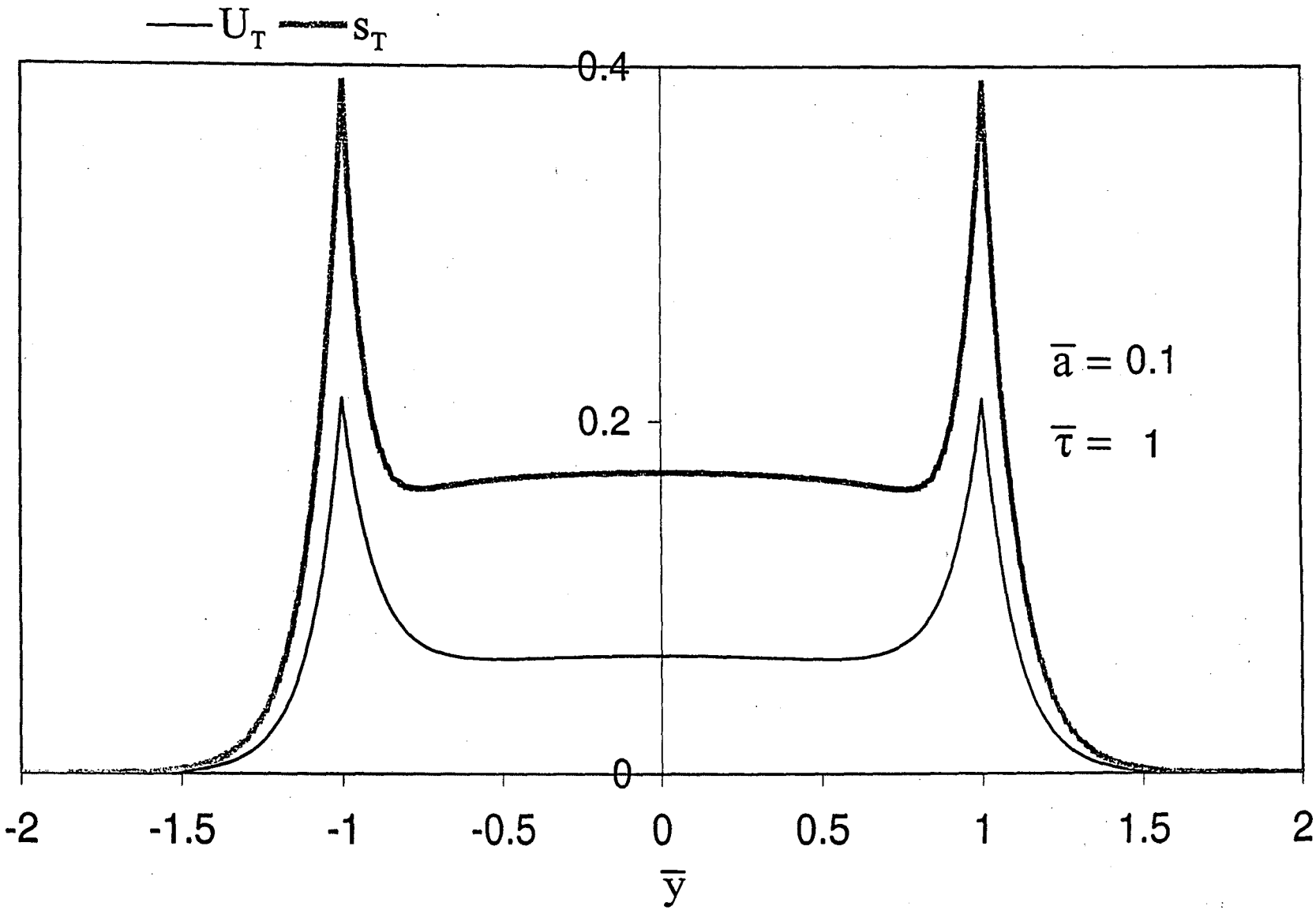

Figure 26. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for telegraph random velocity at $\bar{\tau}=1$ and $\bar{a}=0.1$.

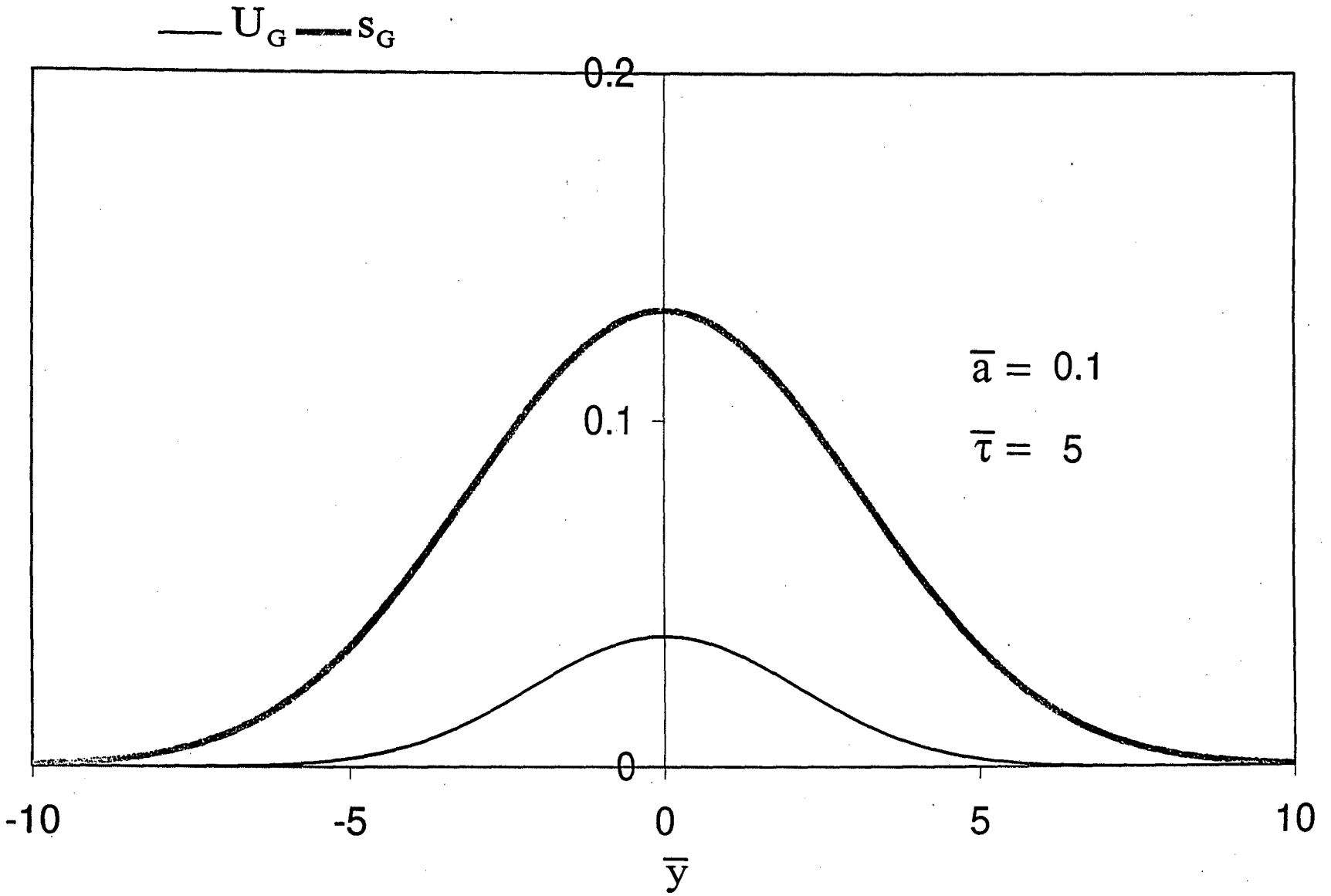

Figure 27. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for Gaussian random velocity at $\bar{\tau} = 5$ and $\bar{a} = 0.1$.

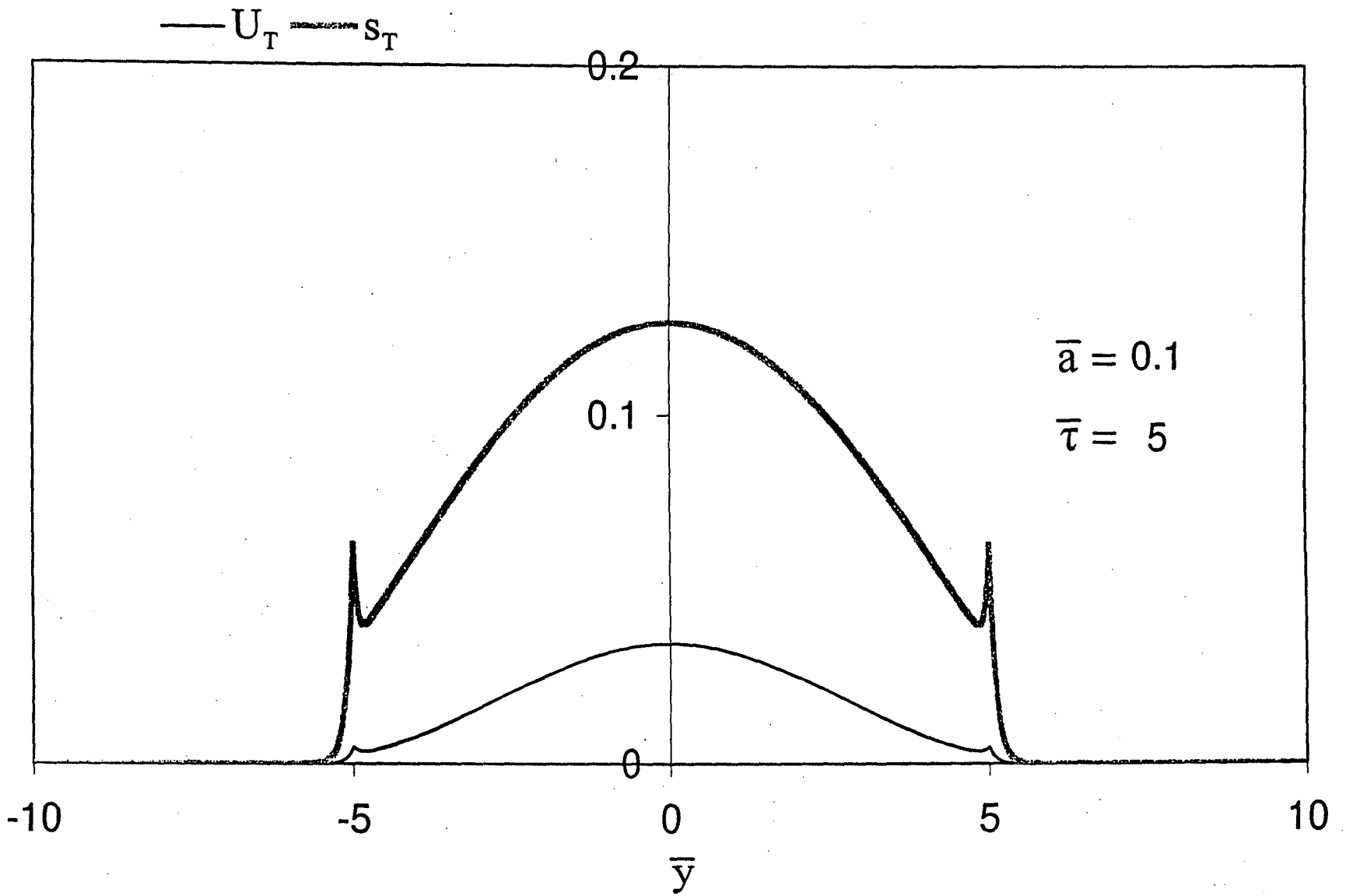

Figure 28. The dimensionless mean concentration $U(\bar{y},\bar{\tau})$ and the dimensionless mean square deviation $s(\bar{y},\bar{\tau})$ for telegraph random velocity at $\bar{\tau} = 5$ and $\bar{a} = 0.1$.

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