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# CONSTRUCTION OF ALL WZW ORBIFOLD ACTIONS

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**Abstract**

In terms of group orbifold elements with definite monodromy, we give a construction for the action functionals of the twisted sectors of all WZW orbifolds. Surprisingly, locality of the theory dictates a form of the general twisted current algebra in which the twisted right and left mover current algebras are not a priori copies of each other. For the permutation orbifolds and the inner-automorphic orbifolds we are able to show by a mode relabelling that the situation is equivalent to copies, but we do not have an argument that this is always the case. In an extension, we also construct the actions for all the orbifolds of any nonlinear sigma model with a symmetry which acts linearly on the coordinates. Finally, implications for orbifold conformal field theory are discussed, including a conjectured set of orbifold Einstein equations for those sigma model orbifolds in this class which are also 1-loop conformal.

# 1 Introduction

Recently, a construction[1,2] was given for the chiral stress tensors of all sectors of the general current-algebraic orbifold

$$\frac{A(H)}{H}, \quad H \subset Aut(g) \tag{1.1}$$

where $A(H)$ is any current-algebraic conformal field theory on semisimple $g$ with a finite symmetry group $H$. This construction drew heavily on recent advances[3,4] in the theory of cyclic permutation orbifolds, and the construction has been worked out in further detail for the general $\mathbb{Z}_\lambda$ permutation orbifold[1], the general $S_N$ permutation orbifold[2], the general inner-automorphic orbifold[2], the general WZW orbifold[1,2], the general coset orbifold[5,2] and the cyclic coset orbifolds[5].

The basic principles of the construction are given in Refs. 1 and 2:

- The *$H$-eigenvalue problem*, whose eigendata encodes the action of $H$ on the untwisted currents (the action of $H$ in the adjoint of $g$).
- *Eigencurrents*, which are constructed from the currents using the eigendata of the $H$-eigenvalue problem.
- *Local isomorphisms* on the sphere, from the eigencurrents to twisted currents with definite monodromy.
- *Derived local isomorphisms* for the orbifold stress tensors.

Following these principles, one encounters *orbifold duality transformations* which express twisted tensors in the orbifold as discrete Fourier transforms of the corresponding tensors in the symmetric CFT $A(H)$.

In this paper, we restrict our attention to the general WZW orbifold

$$\frac{A_g(H)}{H} \subset \frac{A(H)}{H}, \quad H \subset Aut(g) \tag{1.2}$$

while extending the construction to obtain the action formulation of each sector of all $A_g(H)/H$. This development requires that we consider some natural extensions of the principles above:

- The *extended $H$-eigenvalue problem*, whose eigendata encode the action of $H$ in any representation $T$ of $g$.
- *Eigen-objects*, including the left and right mover eigencurrents and now the eigengroup elements, which are constructed from the group elements using the eigendata of the extended $H$-eigenvalue problem.
- Local isomorphisms on the cylinder $(\xi, t)$, from the eigen-objects to twisted objects, including the twisted left and right mover currents and *group orbifold elements* with definite monodromy.
- Derived local isomorphisms for the orbifold stress tensors and WZW orbifold actions.

Many new orbifold duality transformations and some surprises are encountered in this extension.

The organization of this paper is best summarized by the names of its sections:

2. The General Semisimple WZW Model
3. The Automorphism Group $H$
4. The Eigen-objects of Sector $\sigma$
5. The General WZW Orbifold $A_g(H)/H$
6. Orbifold Geometry
7. WZW Orbifold Examples
8. Implications for Orbifold Conformal Field Theory.

A central result is the WZW orbifold action on the usual solid cylinder

$$\hat{S}_\sigma(\mathcal{T}) = -\frac{1}{8\pi}\int d^2\xi\, \widehat{Tr}\big(\mathcal{M}(\mathcal{T},\sigma)\, \hat{g}^{-1}(\mathcal{T},\sigma)\partial_+\hat{g}(\mathcal{T},\sigma)\, \hat{g}^{-1}(\mathcal{T},\sigma)\partial_-\hat{g}(\mathcal{T},\sigma)\big)$$
$$-\frac{1}{12\pi}\int_\Gamma \widehat{Tr}\big(\mathcal{M}(\mathcal{T},\sigma)\, (\hat{g}^{-1}(\mathcal{T},\sigma)d\hat{g}(\mathcal{T},\sigma))^3\big), \quad \sigma = 0,...,N_c-1 \tag{1.3a}$$

$$\hat{g}(\mathcal{T},\xi+2\pi,\sigma)_{N(r)\mu}{}^{N(s)\nu} = e^{-\frac{2\pi i}{R(\sigma)}(N(r)-N(s))}\hat{g}(\mathcal{T},\xi,\sigma)_{N(r)\mu}{}^{N(s)\nu} \tag{1.3b}$$

for each twisted representation $\mathcal{T}$ in each sector $\sigma$ of each orbifold $A_g(H)/H$. Here (see Sec. 5), $N_c$ is the number of conjugacy classes of $H$ and $\mathcal{M}(\mathcal{T},\sigma)$ is called the twisted data matrix. The objects $\{\hat{g}\}$ are the appropriate set of group orbifold elements with definite monodromy (1.3b) when they are taken around the cylinder $\xi \to \xi + 2\pi$. The integers $N(r) \equiv N(\mathcal{T}, r(\sigma))$ and $R(\sigma) \equiv R(\mathcal{T},\sigma)$ depend explicitly on the symmetry group $H$, the sector $\sigma$ and the twisted representation $\mathcal{T}$. The equations of motion of the WZW orbifold action are

$$\hat{J}(\mathcal{T},\sigma) = -\frac{i}{2}\hat{g}^{-1}(\mathcal{T},\sigma)\partial_+\hat{g}(\mathcal{T},\sigma), \quad \hat{\bar{J}}(\mathcal{T},\sigma) = -\frac{i}{2}\hat{g}(\mathcal{T},\sigma)\partial_-\hat{g}^{-1}(\mathcal{T},\sigma) \tag{1.4a}$$

$$\partial_-\hat{J}(\mathcal{T},\sigma) = \partial_+\hat{\bar{J}}(\mathcal{T},\sigma) = 0 \tag{1.4b}$$

where $\hat{J}$ and $\hat{\bar{J}}$ are the matrix forms of the twisted left and right mover currents. These local relations require that the monodromies of $\hat{J}$ and $\hat{\bar{J}}$ are the same when both are taken around the cylinder.

But this simple fact dictates a form of the general twisted current algebra in which the twisted right and left mover current algebras are not a priori copies of each other (see Subsec. 5.4). For the permutation orbifolds and the inner-automorphic orbifolds we are able to show by a mode relabelling (see Sec. 7) that the situation is equivalent to copies, but we do not have a proof for the general case (see Subsec. 8.2).

In an extension, we also construct the actions

$$\hat{S}_\sigma^{NLS} = \frac{1}{8\pi\alpha'}\int d^2\xi\, \left(\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}_\sigma,\sigma) + \hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}_\sigma,\sigma)\right)\partial_+\hat{x}_\sigma^{n(r)\mu}\partial_-\hat{x}_\sigma^{n(s)\nu} \tag{1.5a}$$

$$\hat{x}_\sigma^{n(r)\mu}(\xi+2\pi) = \hat{x}_\sigma^{n(r)\mu}(\xi)e^{\frac{2\pi i}{\rho(\sigma)}n(r)}, \quad \sigma = 0,\ldots,N_c-1 \tag{1.5b}$$

$$\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}_\sigma(\xi+2\pi),\sigma) = e^{-\frac{2\pi i}{\rho(\sigma)}(n(r)+n(s))}\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}_\sigma(\xi),\sigma) \tag{1.5c}$$

$$\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}_\sigma(\xi+2\pi),\sigma) = e^{-\frac{2\pi i}{\rho(\sigma)}(n(r)+n(s))}\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}_\sigma(\xi),\sigma) \tag{1.5d}$$

for each twisted sector $\sigma$ of all the orbifolds of any nonlinear sigma model with a symmetry group $H$ which acts linearly on the coordinates (see Subsec. 6.3). Here the quantities $\hat{x}_\sigma$, $\hat{G}(\hat{x}_\sigma,\sigma)$ and $\hat{B}(\hat{x}_\sigma,\sigma)$ are the twisted Einstein coordinates, the twisted Einstein metric and the twisted $B$ field respectively, and the integers $\rho(\sigma)$ and $n(r) \equiv n(r(\sigma))$ depend explicitly on the original sigma model, its symmetry group and the sector.

The classical results above are extended to orbifold conformal field theory in Sec. 8, including a canonical operator realization of the general twisted current algebra (see Subsec. 8.1) and further discussion of the general current-algebraic orbifold $A(H)/H$ (see Subsec. 8.2). We also obtain a set of orbifold Einstein equations (see Subsec. 8.3) which are satisfied in these sigma model orbifolds when the symmetric sigma model is 1-loop conformal, and which we conjecture will control the twisted sectors of the sigma model orbifolds which are also 1-loop conformal.

A number of closely-related topics are discussed in the appendices, including a new set of braid relations (Appendix A) which are associated to the extended $H$-eigenvalue problem and an independent derivation of some of our central results using the method of twisted affine Lie groups (Appendix E).

# 2 The General Semisimple WZW Model

## 2.1 Notation

In the following subsections, we will review the general semisimple WZW model $A_g$ in the composite notation

$$(T_a)_\alpha{}^\beta, \quad g(T,\xi)_\alpha{}^\beta \tag{2.1a}$$

$$[T_a,T_b] = if_{ab}{}^c T_c, \quad M(k,T)_\alpha{}^\beta = \left(\frac{k}{y(T)}\right)_\alpha \delta_\alpha{}^\beta \tag{2.1b}$$

$$Tr(M(k,T)T_aT_b) = G_{ab}, \quad G_{ac}G^{cb} = \delta_a{}^b \quad ,[M(k,T),T_a] = 0 \tag{2.1c}$$

$$g(T,\xi) = e^{i\beta^a(\xi)T_a}, \quad g(T,\xi)g^\dagger(T,\xi) = \mathbb{1}, \quad [M(k,T),g(T,\xi)] = 0 \tag{2.1d}$$

$$a,b,c = 1,\ldots,\dim g, \quad \alpha,\beta = 1,\ldots,\dim T, \quad \xi \in [0,2\pi)\ . \tag{2.1e}$$

Here $T_a$ is any matrix representation of the compact semisimple Lie algebra $g$, with metric $G_{ab}$ and structure constants $f_{ab}{}^c$. The diagonal *data matrix* $M(k,T)$ stores information about the Dynkin indices $y$ of the matrix irreps and the affine levels $k$ of each simple component of $g$. The group elements $g(T,\xi,t)$ parameterize the corresponding semisimple group manifold, with tangent space coordinates $\beta$, on a base space which is a cylinder $(\xi,t)$. As seen in (2.1), we often suppress the time coordinate $t$.

When needed, the composite notation can be replaced by the explicit notation

$$g = \oplus_I \mathfrak{g}^I, \quad I = 0,\ldots,K-1 \tag{2.2a}$$

$$a \longrightarrow a(I), I, \quad \alpha \longrightarrow \alpha(I), I \tag{2.2b}$$

$$G_{ab} \longrightarrow G_{aI,bJ} = k_I \eta^I_{a(I),b(I)} \delta_{IJ}, \quad G^{ab} \longrightarrow G^{aI,bJ} = k_I^{-1} \eta_I^{a(I),b(I)} \delta^{IJ} \tag{2.2c}$$

$$f_{ab}{}^c \longrightarrow f_{aI,bJ}{}^{cK} = f^I_{a(I),b(I)}{}^{c(I)} \delta_{IJ} \delta_J{}^K \tag{2.2d}$$

$$(T_a)_\alpha{}^\beta \quad \longrightarrow \quad (T_{aI})_{\alpha J}{}^{\beta K} = (T^I_a)_{\alpha(I)}{}^{\beta(I)} \delta_{IJ} \delta_J{}^K \tag{2.2e}$$

$$M(k,T)_\alpha{}^\beta \longrightarrow M(k,T)_{\alpha I}{}^{\beta J} = \frac{k_I}{y_I(T^I)} \delta_{\alpha(I)}{}^{\beta(I)} \delta_I{}^J \tag{2.2f}$$

$$M^{-1}(k,T)_\alpha{}^\beta \longrightarrow M^{-1}(k,T)_{\alpha I}{}^{\beta J} = \frac{y_I(T^I)}{k_I} \delta_{\alpha(I)}{}^{\beta(I)} \delta_I{}^J \tag{2.2g}$$

$$Tr(T^I_a T^J_b) = y_I(T^I) \eta^I_{a(I),b(I)} \delta^{IJ} \tag{2.2h}$$

$$g(T)_\alpha{}^\beta \longrightarrow g(T)_{\alpha I}{}^{\beta J} = g_I(T^I)_{\alpha(I)}{}^{\beta(I)} \delta_I{}^J, \quad g_I(T^I) = e^{i \sum_a \beta^{aI} T^I_a} \ . \tag{2.2i}$$

Here $I$ is the semisimplicity index and $a(I)$, $\alpha(I)$ are indices associated to the simple component $\mathfrak{g}^I$, with structure constants, Killing metric, matrix irrep and affine level $f^I$, $\eta^I$, $T^I$ and $k_I$ respectively.

These forms simplify considerably for permutation-invariant systems,

$$\mathfrak{g}^I \simeq \mathfrak{g}, \quad a(I) = a, \quad \alpha(I) = \alpha \tag{2.3a}$$

$$\eta^I_{ab} = \eta_{ab}, \quad \eta_I^{ab} = \eta^{ab}, \quad f^{I\,c}_{ab} = f_{ab}{}^c, \quad T^I_a = T_a, \quad k_I = k \tag{2.3b}$$

$$[T_a, T_b] = i f_{ab}{}^c, \quad Tr(T_a T_b) = y(T) \eta_{ab}, \quad M(k,T) = \frac{k}{y(T)} \mathbb{1} \tag{2.3c}$$

$$g_I(T)_\alpha{}^\beta = (e^{\sum_a i \beta^{aI} T_a})_\alpha{}^\beta, \quad a,b = 1, ..., \dim \mathfrak{g}, \quad \alpha, \beta = 1, ..., \dim T \tag{2.3d}$$

where we have taken the same structure constants, Killing metric, matrix irrep $T$ and affine level for each simple copy $\mathfrak{g}^I \simeq \mathfrak{g}$. The relations in (2.3c) hold as well for simple $g$.

## 2.2 Local WZW Dynamics

We begin with a list of well-known local relations in the general semisimple WZW model $A_g$, leaving the mode form of the corresponding current algebra to the following subsection.

- Equal-time current algebra and stress tensors

$$\{J_a(\xi), J_b(\eta)\} = 2\pi i \big( f_{ab}{}^c J_c(\xi) \delta(\xi - \eta) + G_{ab} \partial_\xi \delta(\xi - \eta) \big) \tag{2.4a}$$

$$\{\bar{J}_a(\xi), \bar{J}_b(\eta)\} = 2\pi i \big( f_{ab}{}^c \bar{J}_c(\xi) \delta(\xi - \eta) - G_{ab} \partial_\xi \delta(\xi - \eta) \big) \tag{2.4b}$$

$$\{J_a(\xi), \bar{J}_b(\eta)\} = 0, \quad a,b,c = 1, ..., \dim g \tag{2.4c}$$

$$T_g(\xi) = \frac{1}{2\pi} L^{ab}_{g,\infty} J_a(\xi) J_b(\xi) = \frac{1}{4\pi} G^{ab} J_a(\xi) J_b(\xi) \tag{2.5a}$$

$$\bar{T}_g(\xi) = \frac{1}{2\pi} L^{ab}_{g,\infty} \bar{J}_a(\xi) \bar{J}_b(\xi) = \frac{1}{4\pi} G^{ab} \bar{J}_a(\xi) \bar{J}_b(\xi) \tag{2.5b}$$

$$\{T_g(\xi), T_g(\eta)\} = \ i( \ T_g(\xi) + T_g(\eta) \ ) \ \partial_\xi \delta(\xi - \eta) \tag{2.5c}$$

$$\{\bar{T}_g(\xi), \bar{T}_g(\eta)\} = -i(\ \bar{T}_g(\xi) + \bar{T}_g(\eta)\ )\ \partial_\xi \delta(\xi - \eta) \tag{2.5d}$$

$$\{T_g(\xi), \bar{T}_g(\eta)\} = 0 \tag{2.5e}$$

$$\{T_g(\xi), J_a(\eta)\} = \ iJ_a(\xi)\partial_\xi\delta(\xi-\eta), \quad \{\bar{T}_g(\xi), \bar{J}_a(\eta)\} = -i\bar{J}_a(\xi)\partial_\xi\delta(\xi-\eta) \tag{2.6a}$$

$$\{T_g(\xi), \bar{J}_a(\eta)\} = \{\bar{T}_g(\xi), J_a(\eta)\} = 0\ . \tag{2.6b}$$

Here $\{\cdot\,,\cdot\}$ are equal-time Poisson brackets multiplied by a convenient extra factor of $i$. The stress-tensor coefficient $L^{ab}_{g,\infty} = G^{ab}/2$ in (2.5a), (2.5b) is the high-level limit of the inverse inertia tensor $L^{ab}_g$ of the affine-Sugawara construction[6–11] on $g$.

- Group elements

$$\{J_a(\xi), g(T,\eta)\} = 2\pi\delta(\xi-\eta)g(T,\eta)T_a, \quad \{\bar{J}_a(\xi), g(T,\eta)\} = -2\pi\delta(\xi-\eta)T_a\, g(T,\eta) \tag{2.7}$$

$$\{T_g(\xi), g(T,\eta)\} = \ \delta(\xi-\eta)g(T,\eta)J(T,\eta) \tag{2.8a}$$

$$\{\bar{T}_g(\xi), g(T,\eta)\} = -\delta(\xi-\eta)\bar{J}(T,\eta)\, g(T,\eta) \tag{2.8b}$$

$$J(T,\xi) \equiv J_a(\xi)G^{ab}T_b, \quad \bar{J}(T,\xi) \equiv \bar{J}_a(\xi)G^{ab}T_b \tag{2.8c}$$

$$J_a(\xi) = Tr\big(M(k,T)T_aJ(T,\xi)\big), \quad \bar{J}_a(\xi) = Tr\big(M(k,T)T_a\bar{J}(T,\xi)\big)\ . \tag{2.8d}$$

Here, $J(T)$ and $\bar{J}(T)$ are the matrix currents and $M(k,T)$ is the data matrix in (2.1).

- Hamiltonian formulation

$$H_g = \int_0^{2\pi} d\xi\ (T_g(\xi) + \bar{T}_g(\xi)), \quad P_\xi = -P^\xi = \int_0^{2\pi} d\xi\ (T_g(\xi) - \bar{T}_g(\xi)) \tag{2.9a}$$

$$\partial_m A(\xi,t) = i\{P_m, A(\xi,t)\}, \quad \partial_m = (\partial_t, \partial_\xi) \tag{2.9b}$$

$$\partial_- J_a(\xi,t) = \partial_+\bar{J}_a(\xi,t) = 0, \quad \partial_- T_g(\xi,t) = \partial_+\bar{T}_g(\xi,t) = 0, \quad \partial_\pm \equiv \partial_t \pm \partial_\xi \tag{2.10a}$$

$$\partial_t g(T,\xi,t) = i\big(g(T,\xi,t)J(T,\xi,t) - \bar{J}(T,\xi,t)g(T,\xi,t)\big) \tag{2.10b}$$

$$\partial_\xi g(T,\xi,t) = i\big(g(T,\xi,t)J(T,\xi,t) + \bar{J}(T,\xi,t)g(T,\xi,t)\big) \tag{2.10c}$$

$$\partial_+ g(T,\xi,t) = 2ig(T,\xi,t)J(T,\xi,t), \quad \partial_- g(T,\xi,t) = -2i\bar{J}(T,\xi,t)g(T,\xi,t) \tag{2.10d}$$

$$\delta(\xi-\eta)\partial_+ g(T,\eta,t) = 2i\{T_g(\xi,t), g(T,\eta,t)\} \tag{2.10e}$$

$$\delta(\xi-\eta)\partial_- g(T,\eta,t) = 2i\{\bar{T}_g(\xi,t), g(T,\eta,t)\}\ . \tag{2.10f}$$

The relations in (2.10e) and (2.10f) follow from (2.10d) and (2.8a-b).

- Action formulation[12,11]

$$S_{WZW}(T) = -\frac{1}{8\pi}\int d^2\xi\; Tr\big(M(k,T)\; g^{-1}(T)\partial_+ g(T)\; g^{-1}(T)\partial_- g(T)\big)$$
$$-\frac{1}{12\pi}\int_\Gamma Tr\big(M(k,T)\; (g^{-1}(T)dg(T))^3\;\big), \quad d^2\xi \equiv dt\, d\xi \tag{2.11a}$$
$$= -\frac{1}{8\pi}\sum_I \frac{k_I}{y_I(T^I)}\int d^2\xi\; Tr\big(\; g_I^{-1}(T^I)\partial_+ g_I(T^I)\; g_I^{-1}(T^I)\partial_- g_I(T^I)\big) + \ldots \tag{2.11b}$$
$$(g^{-1}dg)^3 = dt\, d\xi\, d\rho\; \epsilon^{ABC}(g^{-1}\partial_A g)(g^{-1}\partial_B g)(g^{-1}\partial_C g), \quad \{A,B,C\} = \{t,\xi,\rho\} \tag{2.11c}$$
$$J(T,\xi,t) = -\frac{i}{2}g^{-1}(T,\xi,t)\partial_+ g(T,\xi,t) = J_a(\xi,t)G^{ab}T_b \tag{2.12a}$$
$$\bar{J}(T,\xi,t) = -\frac{i}{2}g(T,\xi,t)\partial_- g^{-1}(T,\xi,t) = \bar{J}_a(\xi,t)G^{ab}T_b \tag{2.12b}$$
$$\partial_- J(T,\xi,t) = \partial_+ \bar{J}(T,\xi,t) = 0 \tag{2.12c}$$
$$J_a(\xi,t) = -\frac{i}{2}Tr\big(\; M(k,T)T_a\, g^{-1}(T,\xi,t)\partial_+ g(T,\xi,t)\;\big) \tag{2.12d}$$
$$\bar{J}_a(\xi,t) = -\frac{i}{2}Tr\big(\; M(k,T)T_a\, g(T,\xi,t)\partial_- g^{-1}(T,\xi,t)\;\big)\;. \tag{2.12e}$$

Here $M(k,T)$ is the data matrix in (2.1) and (2.2f), $\Gamma$ is the usual solid cylinder and $\epsilon^{ABC}$ with $\epsilon^{t\xi\rho} = 1$ is the Levi-Civita density. The WZW action equations of motion (2.12a)-(2.12c) follow because the data matrix is invertible and commutes with the group elements (see Eq. (2.1)). Moreover, the action equations of motion are equivalent to the Hamiltonian equations of motion (2.10a)-(2.10c).

## 2.3 Mode Form of the Current Algebra

Using the equations of motion (2.10a) or (2.12c), we may expand the currents in modes

$$J_a(\xi,t) = \sum_m J_a(m)e^{-im(\xi+t)}, \quad \bar{J}_a(\xi,t) = \sum_m \bar{J}_a(m)e^{-im(\xi-t)} \tag{2.13}$$

and use the equal-time algebra (2.4) to obtain the mode form of the current algebra

$$\{J_a(m), J_b(n)\} = if_{ab}{}^c J_c(m+n) + mG_{ab}\delta_{m+n,0} \tag{2.14a}$$
$$\{\bar{J}_a(m), \bar{J}_b(n)\} = if_{ab}{}^c \bar{J}_c(m+n) - mG_{ab}\delta_{m+n,0} \tag{2.14b}$$
$$\{J_a(m), \bar{J}_b(n)\} = 0\;. \tag{2.14c}$$

The left mover algebra (2.14a) is the bracket form of the general affine Lie algebra[13,14,6], which we call affine $g$. The algebra (2.14) is isomorphic to affine$(g \oplus g)$. To see this, use the mode-number reversed relabelling of the right mover modes

$$\bar{\bar{J}}_a(m) \equiv \bar{J}_a(-m), \quad \bar{J}_a(\xi,t) = \sum_m \bar{\bar{J}}_a(m)e^{im(\xi-t)} \tag{2.15a}$$

$$\{\bar{\bar{J}}_a(m), \bar{\bar{J}}_b(n)\} = if_{ab}{}^c \bar{\bar{J}}_c(m+n) + mG_{ab}\delta_{m+n,0} \tag{2.15b}$$

to rectify the right mover algebra (2.14b) into a commuting copy (2.15b) of the left mover algebra.

The light-cone current algebra

$$J_a(t_+) \equiv J_a(\xi, t), \quad \bar{J}(t_-) \equiv \bar{J}(\xi, t), \quad t_\pm \equiv t \pm \xi \tag{2.16a}$$

$$\{J_a(t_+), J_b(t'_+)\} = 2\pi i\big(f_{ab}{}^c J_c(t'_+)\delta(t_+ - t'_+) + G_{ab}\partial_{t_+}\delta(t_+ - t'_+)\big) \tag{2.16b}$$

$$\{\bar{J}_a(t_-), \bar{J}_b(t'_-)\} = 2\pi i\big(f_{ab}{}^c \bar{J}_c(t'_-)\delta(t_- - t'_-) + G_{ab}\partial_{t_-}\delta(t_- - t'_-)\big) \tag{2.16c}$$

$$\{J_a(t_+), \bar{J}_b(t_-)\} = 0 \tag{2.16d}$$

also follows from (2.13) and (2.14).

# 3 The Automorphism Group $H$

## 3.1 Action in Representation $T$

Let $H \subset Aut(g)$ be a symmetry group (automorphism group) of the WZW model $A_g(H)$ and pick one representative $h_\sigma \in H$ of each conjugacy class of $H$. At the Hamiltonian level, the $H$-symmetry requires that[15,1,2]

$$J_a(\xi)' \equiv w(h_\sigma)_a{}^b J_b(\xi), \quad \bar{J}_a(\xi)' \equiv w(h_\sigma)_a{}^b \bar{J}_b(\xi), \quad w(h_\sigma)w^\dagger(h_\sigma) = 1, \quad w(h_0) = 1 \tag{3.1a}$$

$$\{J_a(\xi)', J_b(\eta)'\} = 2\pi i\big(f_{ab}{}^c J_c(\xi)'\delta(\xi - \eta) + G_{ab}\partial_\xi\delta(\xi - \eta)\big) \tag{3.1b}$$

$$\{\bar{J}_a(\xi)', \bar{J}_b(\eta)'\} = 2\pi i\big(f_{ab}{}^c \bar{J}_c(\xi)'\delta(\xi - \eta) - G_{ab}\partial_\xi\delta(\xi - \eta)\big) \tag{3.1c}$$

$$T_g(\xi)' = \frac{G^{ab}}{4\pi} J_a(\xi)' J_b(\xi)' = T_g(\xi), \quad \bar{T}_g(\xi)' = \frac{G^{ab}}{4\pi} \bar{J}_a(\xi)' \bar{J}_b(\xi)' = \bar{T}_g(\xi) \tag{3.1d}$$

$$w(h_\sigma) \in H \subset Aut(g), \quad \sigma = 0, \ldots, N_c - 1 \tag{3.1e}$$

where $w(h_\sigma)$ is the action of $h_\sigma \in H$ in the adjoint of $g$ and $N_c$ is the number of conjugacy classes of $H$. These requirements are satisfied[15,1,2] when the metric $G$ and the structure constants $f$ of $g$ satisfy

$$w(h_\sigma)_a{}^c w(h_\sigma)_b{}^d G_{cd} = G_{ab}, \quad w(h_\sigma)_a{}^d w(h_\sigma)_b{}^e f_{de}{}^f w^\dagger(h_\sigma)_f{}^c = f_{ab}{}^c, \quad \forall h_\sigma \in H \subset Aut(g). \tag{3.2}$$

In what follows, these relations are called the $H$-symmetry of $G$ and $f$.

At the action level, we require the symmetry as follows

$$g(T,\xi)' \equiv W(h_\sigma;T)g(T,\xi)W^\dagger(h_\sigma;T), \quad g^{-1}(T,\xi)' = W(h_\sigma;T)g^{-1}(T,\xi)W^\dagger(h_\sigma;T) \tag{3.3a}$$

$$W(h_\sigma;T)_\alpha{}^\beta \quad : \quad W(h_\sigma;T)W^\dagger(h_\sigma;T) = \mathbb{1}, \quad W(h_0;T) = \mathbb{1} \tag{3.3b}$$

$$S_{WZW}[M,g]' \equiv S_{WZW}[M,g'] = S_{WZW}[M,g], \tag{3.3c}$$

$$W(h_\sigma;T) \in H \subset Aut(g), \quad \sigma = 0, \ldots, N_c - 1 \tag{3.3d}$$

where $W(h_\sigma;T)$ is the action of $h_\sigma \in H$ in matrix representation $T$ of $g$. This requirement is satisfied when the data matrix $M$ has the $H$-symmetry

$$W^\dagger(h_\sigma;T)M(k,T)W(h_\sigma;T) = M(k,T), \quad \forall h_\sigma \in H \subset Aut(g) \tag{3.4}$$

because $S_{WZW}(T)$ is a trace class functional.

To find the relation between the actions $w(h_\sigma)$ and $W(h_\sigma;T)$, we also require that $H$ is an automorphism group of the mixed relations

$$\{J_a(\xi)', g(T,\eta)'\} = 2\pi\delta(\xi-\eta)g(T,\eta)'T_a, \quad \{\bar{J}_a(\xi)', g(T,\eta)'\} = -2\pi\delta(\xi-\eta)T_a\, g(T,\eta)' \tag{3.5a}$$

$$J(T,\xi)' = J_a(\xi)'G^{ab}T_b = -\frac{i}{2}g^{-1}(T,\xi)'\partial_+ g(T,\xi)' \tag{3.5b}$$

$$\bar{J}(T,\xi)' = \bar{J}_a(\xi)'G^{ab}T_b = -\frac{i}{2}g(T,\xi)'\partial_- g^{-1}(T,\xi)' \,. \tag{3.5c}$$

In conjunction with (3.2), we find that these relations are satisfied iff

$$W^\dagger(h_\sigma;T)T_aW(h_\sigma;T) = w(h_\sigma)_a{}^bT_b, \quad \forall h_\sigma \in H \subset Aut(g) \tag{3.6a}$$

$$W(h_\sigma;T)T_aW^\dagger(h_\sigma;T) = w^\dagger(h_\sigma)_a{}^bT_b \tag{3.6b}$$

$$w^\dagger(h_\sigma)_a{}^b(W^\dagger(h_\sigma;T)T_bW(h_\sigma;T)) = w(h_\sigma)_a{}^b(W(h_\sigma;T)T_bW^\dagger(h_\sigma;T)) = T_a \tag{3.6c}$$

where (3.6b) and (3.6c) are equivalent forms of (3.6a). Because these relations are unfamiliar, we outline the steps for the verification of (3.5b)

$$J_a(\xi)'G^{ab}T_b = w(h_\sigma)_a{}^bJ_b(\xi)G^{ac}T_c = J_b(\xi)G^{ca}w(h_\sigma)_a{}^bT_c = J_b(\xi)G^{ba}(\,w^\dagger(h_\sigma)_a{}^cT_c\,) \tag{3.7a}$$

$$= J_b(\xi)G^{ba}(\,W(h_\sigma;T)T_aW^\dagger(h_\sigma;T)\,) = W(h_\sigma;T)(J_b(\xi)G^{ba}T_a)W^\dagger(h_\sigma;T) \tag{3.7b}$$

$$= W(h_\sigma;T)J(T,\xi)\;W^\dagger(h_\sigma;T) = -\frac{i}{2}g^{-1}(T,\xi)'\partial_+ g(T,\xi)' \tag{3.7c}$$

where (3.2) and (3.6b) were used in (3.7a) and (3.7b) respectively.

In what follows, we call (3.6a) the *linkage relation* and we will refer to the equivalent forms in (3.6c) as the $H$-invariance of the representation matrices $T$. It is clear that the linkage relation, which constrains $W$ given $w$, lies at the heart of the $H$-symmetry of $A_g(H)$.

Here are some important consequences of the linkage relation. First, the linkage relation guarantees that an $H$-transformation of a group element is still a group element,

$$g(T,\xi) = e^{i\beta^a(\xi)T_a} \tag{3.8a}$$

$$g(T,\xi)' = W(h_\sigma;T)g(T,\xi)W^\dagger(h_\sigma;T) = e^{i\beta^a(\xi)W(h_\sigma;T)T_aW^\dagger(h_\sigma;T)} = e^{i\beta^a(\xi)'T_a} \tag{3.8b}$$

$$\beta^a(\xi)' \equiv \beta^b(\xi)w^\dagger(h_\sigma)_b{}^a \tag{3.8c}$$

where the linkage relation in the form (3.6b) was used to obtain the final form.

The linkage relation also tells us that $W$ reduces to $w$

$$W(h_\sigma;T^{adj}) = w(h_\sigma), \quad (T_a^{adj})_b{}^c \equiv -if_{ab}{}^c \tag{3.9}$$

when $T$ is taken to be the adjoint representation. In this case, the linkage relation is equivalent to the $H$-symmetry of the structure constants $f$ in (3.2). One also finds that

$$T^{adj}(s) \equiv sT^{adj}s^{-1} \longrightarrow W(h_\sigma; T^{adj}(s)) = sw(h_\sigma)s^{-1} \tag{3.10}$$

for similarity-transformed versions of the adjoint representation.

Finally, the linkage relation gives us information about the orders $\rho(\sigma)$ and $R(T,\sigma)$ of $w(h_\sigma)$ and $W(h_\sigma;T)$ respectively. These are the smallest integers which satisfy

$$w(h_\sigma)^{\rho(\sigma)} = 1, \quad W(h_\sigma;T)^{R(T,\sigma)} = \mathbb{1} \tag{3.11a}$$

$$R(T^{adj},\sigma) = \rho(\sigma)\ . \tag{3.11b}$$

Using these definitions and the linkage relation, we find in general that

$$[W(h_\sigma;T)^{\rho(\sigma)}, T_a] = 0, \quad w(h_\sigma)^{R(T,\sigma)} = 1, \quad \tfrac{R(T,\sigma)}{\rho(\sigma)} \in \mathbb{Z}_{>0} \tag{3.12}$$

so that $R(T,\sigma)$ can be a positive multiple of $\rho(\sigma)$. In what follows, we will often abbreviate $R(\sigma) \equiv R(T,\sigma)$.

## 3.2 The Extended $H$-Eigenvalue Problem

For the action $w(h_\sigma)$ of $h_\sigma \in H$ in the adjoint, the *$H$-eigenvalue problem*

$$w(h_\sigma)_a{}^b U^\dagger(\sigma)_b{}^{n(r)\mu} = U^\dagger(\sigma)_a{}^{n(r)\mu} E_{n(r)}(\sigma), \quad \sigma = 0, \ldots, N_c - 1 \tag{3.13a}$$

$$E_{n(r)}(\sigma) = e^{-2\pi i \frac{n(r)}{\rho(\sigma)}}, \quad n(r) \equiv n(r(\sigma)) \in \mathbb{Z} \tag{3.13b}$$

$$E_{n(r)\pm\rho(\sigma)}(\sigma) = E_{n(r)}(\sigma), \quad U^\dagger(\sigma)_a{}^{n(r)\pm\rho(\sigma),\mu} = U^\dagger(\sigma)_a{}^{n(r)\mu} \tag{3.13c}$$

$$U^\dagger(\sigma)U(\sigma) = 1, \quad U^\dagger(0) = 1, \quad E_{n(r)}(0) = \rho(0) = 1 \tag{3.13d}$$

$$U(\sigma)_{n(r)\mu}{}^a U^\dagger(\sigma)_a{}^{n(s)\nu} = \delta_{n(r)\mu}{}^{n(s)\nu}, \quad \delta_{n(r)\mu}{}^{n(s)\nu} \equiv \delta_{n(r)-n(s),0 \bmod \rho(\sigma)}\delta_\mu{}^\nu \tag{3.13e}$$

$$\bar{n}(r) = n(r) - \rho(\sigma)\lfloor \tfrac{n(r)}{\rho(\sigma)} \rfloor, \quad \bar{n}(r) \in \{0, ..., \rho(\sigma) - 1\} \tag{3.13f}$$

was defined and studied in Refs. 1, 5 and 2. Here $r \equiv r(\sigma)$ runs over the degenerate subspaces of the $H$-eigenvalue problem. The unitary eigenvector matrix $U^\dagger(\sigma)$ and the eigenvalues $E(\sigma)$ are periodic with period $\rho(\sigma)$ in the spectral index $n(r)$, while $\mu = \mu(r(\sigma))$ is a degeneracy index. The pull-backs $\bar{n}(r)$ are identified in Ref. 2 as the twist classes of the twisted currents of sector $\sigma$.

Correspondingly, we define now the *extended $H$-eigenvalue problem*

$$W(h_\sigma;T)_\alpha{}^\beta U^\dagger(T,\sigma)_\beta{}^{N(r)\mu} = U^\dagger(T,\sigma)_\alpha{}^{N(r)\mu} E_{N(r)}(T,\sigma), \quad \sigma = 0, \ldots, N_c - 1 \tag{3.14a}$$

$$E_{N(r)}(T,\sigma) = e^{-2\pi i \frac{N(r)}{R(\sigma)}}, \quad N(r) \equiv N(T, r(\sigma)) \in \mathbb{Z} \tag{3.14b}$$

$$E_{N(r)\pm R(\sigma)}(T,\sigma) = E_{N(r)}(T,\sigma), \quad U^\dagger(T,\sigma)_\alpha{}^{N(r)\pm R(\sigma),\mu} = U^\dagger(T,\sigma)_\alpha{}^{N(r)\mu} \tag{3.14c}$$

$$U^\dagger(T,\sigma)U(T,\sigma) = \mathbb{1}, \quad U^\dagger(T,0) = \mathbb{1}, \quad E_{N(r)}(T,0) = R(T,0) = 1 \tag{3.14d}$$

$$U(T,\sigma)_{N(r)\mu}{}^\alpha U^\dagger(T,\sigma)_\alpha{}^{N(s)\nu} = \delta_{N(r)\mu}{}^{N(s)\nu}, \ \delta_{N(r)\mu}{}^{N(s)\nu} \equiv \delta_{N(r)-N(s),0 \bmod R(\sigma)}\delta_\mu{}^\nu \tag{3.14e}$$

$$\bar{N}(r) = N(r) - R(\sigma)\lfloor \tfrac{N(r)}{R(\sigma)} \rfloor, \quad \bar{N}(r) \in \{0, ..., R(\sigma) - 1\} \tag{3.14f}$$

$$W = U^\dagger E U,\ W^\dagger = U^\dagger E^* U,\ E^* E = \mathbb{1} \tag{3.15a}$$

$$W U^\dagger = U^\dagger E,\ U W = E U,\ U^* W^* = E^* U^*,\ U W^\dagger = E^* U,\ W^\dagger U^\dagger = U^\dagger E^* \tag{3.15b}$$

$$E(T,\sigma)_{N(r)\mu}{}^{N(s)\nu} \equiv \delta_{N(r)\mu}{}^{N(s)\nu} E_{N(r)}(T,\sigma) \tag{3.15c}$$

for the action $W(h_\sigma;T)$ of $h_\sigma \in H$ in matrix representation $T$. Here, the order $R(\sigma) \equiv R(T,\sigma)$, the spectral index $N(r) \equiv N(T,r(\sigma))$ and the degeneracy index $\mu \equiv \mu(T,r(\sigma))$ can all depend on $T$. The relations in (3.15) are alternate forms of the extended eigenvalue problem.

Because of the linkage relation (3.6), the two eigenvalue problems (3.13) and (3.14) are not independent. In particular, the extended problem includes the original problem as the special case when $T = T^{adj}$:

$$W(h_\sigma;T^{adj}) = w(h_\sigma), \quad R(T^{adj},\sigma) = \rho(\sigma), \quad (T_a^{adj})_b{}^c = -i f_{ab}{}^c \tag{3.16a}$$

$$E(T^{adj},\sigma) = E(\sigma), \quad N(r) = n(r), \quad U^\dagger(T^{adj},\sigma) = U^\dagger(\sigma)\ . \tag{3.16b}$$

Moreover, Appendix A remarks on a set of braid relations, induced by the linkage relation, among the quantities of the two eigenvalue problems. Solutions of the extended $H$-eigenvalue problem are discussed for general permutation groups and general groups of inner automorphisms in Sec. 7.

It is known[1,2] that the eigenvectors $U^\dagger(\sigma)$ of the $H$-eigenvalue problem are Fourier basis elements with periodicity $\rho(\sigma)$ in the spectral index $n(r)$, and similarly the eigenvectors $U^\dagger(T,\sigma)$ of the extended problem are Fourier basis elements with periodicity $R(\sigma) = R(T,\sigma)$ in the spectral index $N(r)$. In what follows, all quantities inherit the periodicities

$$N(r) \longrightarrow N(r) \pm R(\sigma), \quad n(r) \longrightarrow n(r) \pm \rho(\sigma) \tag{3.17}$$

in the spectral indices $\{n(r)\}$, $\{N(r)\}$ of the $H$-eigenvalue problems. Since $R(T,\sigma)$ can be a multiple of $\rho(\sigma)$, this means in particular that all the orbifold duality transformations of sector $\sigma$ are discrete Fourier transforms with fundamental period $\rho(\sigma)$.

We will also need the behavior of the eigenvalue problems under conjugation in $H$:

$$w(h_\sigma) \longrightarrow v^\dagger(\sigma) w(h_\sigma) v(\sigma), \quad v^\dagger(\sigma) v(\sigma) = 1, \quad v(\sigma) \in H \tag{3.18a}$$

$$U^\dagger(\sigma) \longrightarrow v^\dagger(\sigma) U^\dagger(\sigma), \quad U(\sigma) \longrightarrow U(\sigma) v(\sigma) \tag{3.18b}$$

$$v(\sigma)_a{}^c v(\sigma)_b{}^d G_{cd} = G_{ab}, \quad v(\sigma)_a{}^d v(\sigma)_b{}^e f_{de}{}^f v^\dagger(\sigma)_f{}^c = f_{ab}{}^c \tag{3.18c}$$

$$W(h_\sigma;T) \longrightarrow v^\dagger(T,\sigma) W(h_\sigma;T) v(T,\sigma), \quad v^\dagger(T,\sigma) v(T,\sigma) = \mathbb{1}, \quad v(T,\sigma) \in H \tag{3.18d}$$

$$U^\dagger(T,\sigma) \longrightarrow v^\dagger(T,\sigma) U^\dagger(T,\sigma), \quad U(T,\sigma) \longrightarrow U(T,\sigma) v(T,\sigma) \tag{3.18e}$$

$$v^\dagger(T,\sigma) T_a v(T,\sigma) = v(\sigma)_a{}^b T_b\ . \tag{3.18f}$$

The relations in (3.18c) follow from (3.2) because $v(\sigma) \in H$, and the first three relations in (3.18) were given in Refs. 1 and 2. Taken together, the conjugation relations (3.18) preserve the linkage relation (3.6), and will allow us to check that all the orbifold duality transformations of this paper are class functions.

# 4 The Eigen-objects of Sector $\sigma$

## 4.1 Eigencurrents and Eigengroup Elements

*Eigen-objects* are linear combinations of fields in the $H$-invariant theory, chosen so that they have a diagonal response to the automorphism group.

Following Refs. 1 and 2, we define the *eigencurrents* $\mathcal{J}, \bar{\mathcal{J}}$ and the *eigengroup elements* $\mathcal{G}$ of sector $\sigma$ as follows:

$$\mathcal{J}_{n(r)\mu}(\xi,\sigma) \equiv \chi(\sigma)_{n(r)\mu} U(\sigma)_{n(r)\mu}{}^{a} J_a(\xi), \quad \bar{\mathcal{J}}_{n(r)\mu}(\xi,\sigma) \equiv \chi(\sigma)_{n(r)\mu} U(\sigma)_{n(r)\mu}{}^{a} \bar{J}_a(\xi) \tag{4.1a}$$

$$\mathcal{G}(T,\xi,\sigma) \equiv U(T,\sigma) g(T,\xi) U^{\dagger}(T,\sigma) \tag{4.1b}$$

$$\mathcal{G}(T,\xi,\sigma)_{N(r)\mu}{}^{N(s)\nu} = U(T,\sigma)_{N(r)\mu}{}^{\alpha} g(T,\xi,\sigma)_{\alpha}{}^{\beta} U^{\dagger}(T,\sigma)_{\beta}{}^{N(s)\nu} \tag{4.1c}$$

$$\mathcal{G}(T,\xi,\sigma)\mathcal{G}^{\dagger}(T,\xi,\sigma) = \mathbb{1}, \quad \mathcal{G}^{-1}(T,\xi,\sigma) = U(T,\sigma) g^{-1}(T,\xi,\sigma) U^{\dagger}(T,\sigma) \tag{4.1d}$$

$$\chi(\sigma)_{n(r)\pm\rho(\sigma),\mu} = \chi(\sigma)_{n(r)\mu}, \quad \chi(0)_{n(r)\mu} = 1 \tag{4.1e}$$

where $U^{\dagger}$ and $U^{\dagger}(T)$ are the eigenvector matrices respectively of the $H$-eigenvalue problem (3.13) and its extension (3.14). The quantities $\chi(\sigma)$, which satisfy the conventions in (4.1e), are otherwise arbitrary normalization constants.

The response of the eigen-objects to elements of the automorphism group are

$$\mathcal{J}_{n(r)\mu}(\xi,\sigma)' = E_{n(r)}(\sigma)\mathcal{J}_{n(r)\mu}(\xi,\sigma), \quad \bar{\mathcal{J}}_{n(r)\mu}(\xi,\sigma)' = E_{n(r)}(\sigma)\bar{\mathcal{J}}_{n(r)\mu}(\xi,\sigma) \tag{4.2a}$$

$$\mathcal{G}(T,\xi,\sigma)' = E(T,\sigma)\mathcal{G}(T,\xi,\sigma)E(T,\sigma)^*, \quad \mathcal{G}^{-1}(T,\xi,\sigma)' = E(T,\sigma)\mathcal{G}^{-1}(T,\xi,\sigma)E(T,\sigma)^* \tag{4.2b}$$

$$(\mathcal{G}(T,\xi,\sigma)')_{N(t)\mu}{}^{N(s)\nu} = e^{-\frac{2\pi i}{R(\sigma)}(N(r)-N(s))}\mathcal{G}(T,\xi,\sigma)_{N(r)\mu}{}^{N(s)\nu}\,. \tag{4.2c}$$

The diagonal response of the eigencurrents is well known in the theory of current-algebraic orbifolds[1,2]. To verify the two-sided diagonal response of the eigengroup elements, follow the steps

$$\begin{aligned}\mathcal{G}(T,\xi,\sigma)' &= U(T,\sigma)g(T,\xi)'U^{\dagger}(T,\sigma) = U(T,\sigma)W(h_\sigma;T)g(T,\xi)W^{\dagger}(h_\sigma,T)U^{\dagger}(T,\sigma)\\ &= E(T,\sigma)\ U(T,\sigma)g(T,\xi)U^{\dagger}(T,\sigma)\ E(T,\sigma)^* = E(T,\sigma)\mathcal{G}(T,\xi,\sigma)E(T,\sigma)^*\end{aligned} \tag{4.3}$$

where the forms (3.15) of the extended $H$-eigenvalue problem were used to obtain the final form.

The next stage in the orbifold program is to rewrite all local relations in the symmetric theory in terms of the corresponding eigen-objects. This step gives rise to *orbifold duality transformations*, which express the twisted tensors in the orbifold as discrete Fourier transforms of corresponding $H$-symmetric tensors in the symmetric theory. Moreover, the twisted tensors satisfy *selection rules* which are dual to the $H$-symmetry of the original tensors.

The case of the eigencurrent OPE's was worked out in Refs. 1 and 2 and a copy of the equal-time eigencurrent algebra is given in the following subsection. Here one encounters the orbifold duality transformations from the metric $G$ and structure constants $f$ to the *twisted metric* $\mathcal{G}(\sigma)$ and the *twisted structure constants* $\mathcal{F}(\sigma)$ of sector $\sigma$

$$G \underset{\sigma}{\longrightarrow} \mathcal{G}(G,\sigma), \quad f \underset{\sigma}{\longrightarrow} \mathcal{F}(f,\sigma)\,. \tag{4.4}$$

The explicit form of these duality transformations is[1,2]

$$\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma) = \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu}U(\sigma)_{n(r)\mu}{}^{a}U(\sigma)_{n(s)\nu}{}^{b}G_{ab} \tag{4.5a}$$

$$= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \tag{4.5b}$$

$$\mathcal{G}^{n(r)\mu;n(s)\nu}(\sigma) = \chi(\sigma)^{-1}_{n(r)\mu}\chi(\sigma)^{-1}_{n(s)\nu}G^{ab}U^{\dagger}(\sigma)_{a}{}^{n(r)\mu}U^{\dagger}(\sigma)_{b}{}^{n(s)\nu} \tag{4.5c}$$

$$= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma) \tag{4.5d}$$

$$\mathcal{G}^{n(r)\mu;n(t)\delta}(\sigma)\mathcal{G}_{n(t)\delta;n(s)\nu}(\sigma) = \delta_{n(s)\nu}{}^{n(r)\mu} \tag{4.5e}$$

$$\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\sigma) = \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu}\chi(\sigma)^{-1}_{n(t)\delta}U(\sigma)_{n(r)\mu}{}^{a}U(\sigma)_{n(s)\nu}{}^{b}f_{ab}{}^{c}U^{\dagger}(\sigma)_{c}{}^{n(t)\delta} \tag{4.5f}$$

$$= \delta_{n(r)+n(s)-n(t),0 \bmod \rho(\sigma)}\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma) \tag{4.5g}$$

$$\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\sigma) = 0 \quad \text{unless} \quad n(r)+n(s) \in \{n(r)\}\ . \tag{4.5h}$$

The forms (4.5b), (4.5d) and (4.5g) are solutions to selection rules which are dual to the $H$-invariance of $G$ and $f$ in (3.2). Twisted tensor indices can be lowered and raised with the twisted metric and its inverse, and the twisted structure constants with all indices down are totally antisymmetric. The twisted metric and twisted structure constants are also class functions[1,2] under conjugation in $H$

$$\mathcal{G}(U(\sigma)v(\sigma);\sigma) = \mathcal{G}(U(\sigma);\sigma), \quad \mathcal{F}(U(\sigma)v(\sigma);\sigma) = \mathcal{F}(U(\sigma);\sigma) \tag{4.6}$$

according to Eqs. (3.2), (3.18) and (4.5).

As an example beyond those of Refs. 1 and 2, we re-express the WZW action (2.11a) as a functional of the eigengroup elements:

$$S_\sigma[\mathcal{M},\mathcal{G}] \equiv S_{WZW}[M, g(\mathcal{G}(T,\sigma))] = S_{WZW}(T)$$

$$= -\frac{1}{8\pi}\int d^2\xi\ \widehat{Tr}\big(\mathcal{M}(\mathcal{T},\sigma)\ \mathcal{G}^{-1}(T,\sigma)\partial_+\mathcal{G}(T,\sigma)\ \mathcal{G}^{-1}(T,\sigma)\partial_-\mathcal{G}(T,\sigma)\big)$$

$$-\frac{1}{12\pi}\int_\Gamma \widehat{Tr}\big(\mathcal{M}(\mathcal{T},\sigma)(\ \mathcal{G}^{-1}(T,\sigma)d\mathcal{G}(T,\sigma)\ )^3\ \big) \tag{4.7a}$$

$$\widehat{Tr}(AB) \equiv \sum_{r,\mu,s,\nu} A_{N(r)\mu}{}^{N(s)\nu}B_{N(s)\nu}{}^{N(r)\mu} \tag{4.7b}$$

$$\mathcal{M}(T,\sigma) \equiv \mathcal{M}(k,T,\sigma) \equiv U(T,\sigma)M(k,T)U^{\dagger}(T,\sigma) \tag{4.7c}$$

$$\mathcal{M}^{-1}(T,\sigma) = U(T,\sigma)M^{-1}(k,T)U^{\dagger}(T,\sigma) \tag{4.7d}$$

$$\mathcal{M}(T,\sigma)_{N(r)\mu}{}^{N(s)\nu} = \sum_I \frac{k_I}{y_I(T^I)}\sum_\alpha U(T,\sigma)_{N(r)\mu}{}^{\alpha I}U^{\dagger}(T,\sigma)_{\alpha I}{}^{N(s)\nu}\ . \tag{4.7e}$$

Here we have encountered our first new orbifold duality transformation, namely the *twisted data matrix* $\mathcal{M}(\mathcal{T},\sigma)$ in (4.7c)

$$M \underset{\sigma}{\longrightarrow} \mathcal{M}(M,\sigma) \tag{4.8}$$

which is a duality transformation from the untwisted data matrix $M$ in (2.1).

Important properties of the twisted data matrix include

$$[\mathcal{M}(T,\sigma), \mathcal{G}(T,\xi,\sigma)] = 0 \tag{4.9}$$

$$E(T,\sigma)\mathcal{M}(T,\sigma)E(T,\sigma)^* = \mathcal{M}(T,\sigma), \quad \mathcal{M}(T,\sigma)_{N(r)\mu}{}^{N(s)\nu}\left(1 - e^{-\frac{2\pi i(N(r)-N(s))}{R(\sigma)}}\right) = 0 \tag{4.10a}$$

$$\mathcal{M}(T,\sigma)_{N(r)\mu}{}^{N(s)\nu} = \delta_{N(r)-N(s),0 \bmod \rho(\sigma)}\mathcal{M}(T,\sigma)_{N(r)\mu}{}^{N(r)\nu} \tag{4.10b}$$

where (4.9) follows from (2.1c). The $\mathcal{M}$-selection rule in (4.10a) is dual to the $H$-invariance (3.4) of the data matrix $M$, and the solution of the selection rule is given in (4.10b). The form of $\mathcal{M}$ is particularly simple

$$\mathcal{M}(T,\sigma)_{N(r)\mu}{}^{N(s)\nu} = \frac{k}{y(T)}\delta_{N(r)\mu}{}^{N(s)\nu} \tag{4.11}$$

for permutation-invariant WZW models and WZW models on simple $g$.

The twisted data matrix is also a class function

$$\mathcal{M}(U(T,\sigma)v(T,\sigma);\sigma) = \mathcal{M}(U(T,\sigma);\sigma) \tag{4.12}$$

according to Eqs. (3.4), (3.18) and (4.7c). We finally remark that the $H$-invariance of the action in the form (4.7)

$$S_\sigma[\mathcal{M}, \mathcal{G}(T,\sigma)'] = S_\sigma[\mathcal{M}, \mathcal{G}(T,\sigma)], \quad \mathcal{G}(T,\sigma)' = E(T,\sigma)\mathcal{G}(T,\sigma)E(T,\sigma)^* \tag{4.13}$$

follows immediately from the $\mathcal{M}$-selection rule in (4.10).

## 4.2 The Twisted Representation Matrices of Sector $\sigma$

We consider next the *twisted representation matrices* $\mathcal{T}$, which are duality transformations

$$T \underset{\sigma}{\longrightarrow} \mathcal{T}(T,\sigma) \tag{4.14}$$

from the untwisted representation matrices $T$. The explicit form of the twisted representation matrices is

$$\mathcal{T}_{n(r)\mu}(T,\sigma) \equiv \chi(\sigma)_{n(r)\mu}U(\sigma)_{n(r)\mu}{}^a\big(U(T,\sigma)T_aU^\dagger(T,\sigma)\big) \tag{4.15a}$$

$$\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(r)\mu}{}^{N(s)\nu} = \chi(\sigma)_{n(r)\mu}U(\sigma)_{n(r)\mu}{}^a\big(U(T,\sigma)T_aU^\dagger(T,\sigma)\big)_{N(r)\mu}{}^{N(s)\nu} \tag{4.15b}$$

$$T_a = T_a(\mathcal{T}) = U^\dagger(\sigma)_a{}^{n(r)\mu}\chi(\sigma)^{-1}_{n(r)\mu}\big(U^\dagger(T,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)U(T,\sigma)\big) \tag{4.15c}$$

where we have noted the inverse $T(\mathcal{T})$ in (4.15c). These objects appear for example in the equal-time brackets of the eigencurrents with the eigengroup elements

$$\{\mathcal{J}_{n(r)\mu}(\xi,\sigma), \mathcal{G}(\mathcal{T},\eta,\sigma)\} = 2\pi\delta(\xi-\eta)\mathcal{G}(\mathcal{T},\eta,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma) \tag{4.16a}$$

$$\{\bar{\mathcal{J}}_{n(r)\mu}(\xi,\sigma), \mathcal{G}(\mathcal{T},\eta,\sigma)\} = -2\pi\delta(\xi-\eta)\mathcal{T}_{n(r)\mu}(T,\sigma)\,\mathcal{G}(\mathcal{T},\eta,\sigma) \tag{4.16b}$$

$$\mathcal{G}(\mathcal{T},\xi,\sigma) \equiv \mathcal{G}(T(\mathcal{T}),\xi,\sigma) \tag{4.16c}$$

and will play a central role in the orbifold dynamics below. As seen in (4.16c), we will generally use the inverse relations (4.15c) to consider as a function of $\mathcal{T}$ those objects previously labelled by $T$.

The twisted representation matrices satisfy many important relations, including

$$\left[\mathcal{T}_{n(r)\mu}(T,\sigma),\mathcal{T}_{n(s)\nu}(T,\sigma)\right] = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\mathcal{T}_{n(r)+n(s),\delta}(T,\sigma) \tag{4.17a}$$

$$[\mathcal{T}_{n(r)\mu}(T,\sigma),\mathcal{T}_{n(s)\nu}(T,\sigma)\mathcal{G}^{n(s)\nu;n(t)\delta}(\sigma)\mathcal{T}_{n(t)\delta}(T,\sigma)] = 0 \tag{4.17b}$$

$$\mathcal{F}_{n(r)\mu}{}^{n(t)\delta;n(u)\epsilon}(\sigma) \equiv \mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(u)\epsilon}(\sigma)\mathcal{G}^{n(s)\nu;n(t)\delta}(\sigma) = -\mathcal{F}_{n(r)\mu}{}^{n(u)\epsilon;n(t)\delta}(\sigma) \tag{4.17c}$$

$$\begin{aligned}\widehat{Tr}(\mathcal{M}(\mathcal{T},\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\mathcal{T}_{n(s)\nu}(T,\sigma)) &= \mathcal{G}_{n(r)\mu,n(s)\nu}(\sigma)\\ &= \delta_{n(r)+n(s),0\,mod\,\rho(\sigma)}\,\mathcal{G}_{n(r)\mu,-n(r),\nu}(\sigma)\end{aligned} \tag{4.17d}$$

$$[\mathcal{M}(\mathcal{T},\sigma),\mathcal{T}_{n(r)\mu}(T,\sigma)] = 0, \quad \mathcal{M}(\mathcal{T},\sigma) \equiv \mathcal{M}(T(\mathcal{T}),\sigma) \tag{4.17e}$$

where $\mathcal{G}(\sigma)$ and $\mathcal{F}(\sigma)$ are the twisted metric and twisted structure constants of sector $\sigma$ in (4.5). The Casimir property (4.17b) follows from (4.17a) because of the antisymmetry of the twisted structure constants in (4.17c), while (4.17e) follows from (2.1c). The adjoint $\mathcal{T}^\dagger$ of the twisted representation matrices is discussed in Appendix B.

The twisted representation matrices also satisfy the selection rules

$$\mathcal{T}_{n(r)\mu}(T,\sigma) = E_{n(r)}(\sigma)^*\big(\,E(T,\sigma)^*\mathcal{T}_{n(r)\mu}(T,\sigma)E(T,\sigma)\,\big) \tag{4.18a}$$

$$\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta}(1-e^{2\pi i(\frac{n(r)}{\rho(\sigma)}+\frac{N(s)-N(t)}{R(\sigma)})}) = 0 \tag{4.18b}$$

$$\mathcal{T}_{n(r)\mu}(T,\sigma) = E_{n(r)}(\sigma)\big(\,E(T,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)E(T,\sigma)^*\,\big) \tag{4.18c}$$

$$\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta}(1-e^{-2\pi i(\frac{n(r)}{\rho(\sigma)}+\frac{N(s)-N(t)}{R(\sigma)})}) = 0\ . \tag{4.18d}$$

These selection rules are dual to the linkage relation (3.6) and in particular to the forms (3.6c) of the $H$-invariance of the representation matrices $T$. To see this, insert the forms (3.6c) into the duality transformation (4.15a) and use both eigenvalue problems (3.14) and (3.13). The useful intermediate relations

$$T_a(U) = w^\dagger(h_\sigma)_a{}^b E(T,\sigma)^* T_b(U)E(T,\sigma) = w(h_\sigma)_a{}^b E(T,\sigma)T_b(U)E(T,\sigma)^* \tag{4.19a}$$

$$T_a(U) \equiv U(T,\sigma)T_a U^\dagger(T,\sigma) \tag{4.19b}$$

follow from (3.6c) and the extended $H$-eigenvalue problem (3.14) alone.

The two sets of selection rules in (4.18) are equivalent, and the solution of the selection rules is the "Wigner-Eckart" relation

$$\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta} = \delta_{\frac{R(\sigma)}{\rho(\sigma)}n(r)+N(s)-N(t),0\bmod R(\sigma)}\,\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(s)+\frac{R(\sigma)}{\rho(\sigma)}n(r),\delta} \tag{4.20a}$$

$$\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(s)+\frac{R(\sigma)}{\rho(\sigma)}n(r),\delta} = 0 \quad \text{unless} \quad N(s)+\frac{R(\sigma)}{\rho(\sigma)}n(r) \in \{N(r)\} \tag{4.20b}$$

$$\delta_{\frac{R(\sigma)}{\rho(\sigma)}n(r)+N(s)-N(t),0\bmod R(\sigma)} = \delta_{n(r)+\frac{\rho(\sigma)}{R(\sigma)}(N(s)-N(t)),0\bmod\rho(\sigma)} = \delta_{\frac{n(r)}{\rho(\sigma)}+\frac{N(s)-N(t)}{R(\sigma)},0\bmod 1} \tag{4.20c}$$

in terms of the reduced twisted matrices on the right. The Kronecker delta in (4.20a) can be replaced by either of the equivalent forms in (4.20c). We remark that

$$[\,E(T,\sigma)^{\rho(\sigma)},\ \mathcal{T}_{n(r)\mu}(T,\sigma)\,] = 0 \tag{4.21}$$

follows from (4.20a) or (3.12) and the extended $H$-eigenvalue problem.

The twisted representation matrices are also class functions under conjugation in $H$

$$\mathcal{T}(U(\sigma)v(\sigma), U(T,\sigma)v(T,\sigma);\sigma) = \mathcal{T}(U(\sigma), U(T,\sigma);\sigma) \tag{4.22}$$

according to (3.18), (4.15) and the linkage relation (3.6). More generally, all the duality transformations of this paper are class functions, although we will not mention this explicitly below.

Finally, it will also be useful to define the rescaled twisted representation matrices

$$\begin{aligned}
\tilde{\mathcal{T}}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta} &\equiv \chi(T,\sigma)_{N(s)\nu}\,\chi(T,\sigma)^{-1}_{N(t)\delta}\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta} \\
&= \chi(\sigma)_{n(r)\mu}\,\chi(T,\sigma)_{N(s)\nu}\,\chi(T,\sigma)^{-1}_{N(t)\delta}U(\sigma)_{n(r)\mu}{}^{a}\big(\,U(T,\sigma)T_aU^\dagger(T,\sigma)\,\big)_{N(s)\nu}{}^{N(t)\delta} && (4.23a)\\
[\tilde{\mathcal{T}}_{n(r)\mu}(T,\sigma), \tilde{\mathcal{T}}_{n(s)\nu}(T,\sigma)] &= i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\sigma)\tilde{\mathcal{T}}_{n(t)\delta}(T,\sigma) && (4.23b)\\
\chi(T,\sigma)_{N(r)\pm R(\sigma),\mu} &= \chi(T,\sigma)_{N(r),\mu}, \quad \chi(T,0)_{N(r)\mu} = 1 && (4.23c)\\
\chi(T^{adj},\sigma)_{n(r)\mu} &\equiv \chi(\sigma)_{n(r)\mu}, \quad (T^{adj}_a)_b{}^c = -if_{ab}{}^c && (4.23d)\\
\tilde{\mathcal{T}}_{n(r)\mu}(T^{adj},\sigma)_{n(s)\nu}{}^{n(t)\delta} &= -i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\sigma) && (4.23e)
\end{aligned}$$

where $\{\chi(T,\sigma)\}$ are an extended set of normalization constants subject to the conventions in (4.23c) and (4.23d). Then the rescaled twisted matrices $\tilde{\mathcal{T}}(T^{adj},\sigma)$ are related by (4.23e) to the twisted structure constants $\mathcal{F}(\sigma)$. We shall encounter these rescaled matrices again in Sec. 6 and Appendix D.

## 4.3 The Tangent Space Eigencoordinates of Sector $\sigma$

The explicit form of the eigengroup elements in terms of the twisted representation matrices $\mathcal{T}$ is

$$\begin{aligned}
\mathcal{G}(\mathcal{T},\xi,\sigma) \equiv \mathcal{G}(T(\mathcal{T}),\xi,\sigma) &= U(T,\sigma)g(T,\xi)U^\dagger(T,\sigma) && (4.24a)\\
&= e^{i\beta^a(\xi)U(T,\sigma)T_aU^\dagger(T,\sigma)} = e^{i\mathfrak{b}^{n(r)\mu}(\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)} && (4.24b)\\
\mathfrak{b}^{n(r)\mu}(\xi,\sigma) &\equiv \chi(\sigma)^{-1}_{n(r)\mu}\,\beta^a(\xi)\,U^\dagger(\sigma)_a{}^{n(r)\mu} && (4.24c)\\
\beta^a(\xi) &= \chi(\sigma)_{n(r)\mu}\mathfrak{b}^{n(r)\mu}(\xi,\sigma)\,U(\sigma)_{n(r)\mu}{}^{a}\ . && (4.24d)
\end{aligned}$$

The quantities $\{\mathfrak{b}^{n(r)\mu}\}$ in (4.24) are called the *tangent space eigencoordinates* of sector $\sigma$, because their response to the automorphism group is diagonal

$$\mathfrak{b}^{n(r)\mu}(\xi,\sigma)' = \chi(\sigma)^{-1}_{n(r)\mu}\beta^a(\xi,\sigma)'U^\dagger(\sigma)_a{}^{n(r)\mu} = \mathfrak{b}^{n(r)\mu}(\xi,\sigma)E_{n(r)}(\sigma)^*\ . \tag{4.25}$$

To see this, use Eq. (3.8c) and the $H$-eigenvalue problem (3.13).

The closely-related response

$$(\mathfrak{b}^{n(r)\mu}(\xi,\sigma)'\mathcal{T}_{n(r)\mu}(T,\sigma)) = E(T,\sigma)(\mathfrak{b}^{n(r)\mu}(\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma))E(T,\sigma)^* \tag{4.26}$$

follows from (4.25) and the selection rules (4.18) for $\mathcal{T}$, so the equivalence of the automorphic responses

$$e^{i\mathfrak{b}^{n(r)\mu}(\xi,\sigma)'\mathcal{T}_{n(r)\mu}(T,\sigma)} = E(T,\sigma)e^{i\mathfrak{b}^{n(r)\mu}(\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)}E(T,\sigma)^* \tag{4.27a}$$

$$(\mathcal{G}(\mathcal{T},\xi,\sigma)')_{N(r)\mu}{}^{N(s)\nu} = e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\mathcal{G}(\mathcal{T},\xi,\sigma)_{N(r)\mu}{}^{N(s)\nu}$$

$$\longleftrightarrow \quad \mathfrak{b}^{n(r)\mu}(\xi,\sigma)' = \mathfrak{b}^{n(r)\mu}(\xi,\sigma)e^{2\pi i\frac{n(r)}{\rho(\sigma)}} \tag{4.27b}$$

is verified to all orders in $\mathfrak{b}$. In what follows, we often abbreviate $\mathfrak{b}(\xi) \equiv \mathfrak{b}(\xi,\sigma)$.

# 5 The General WZW Orbifold $A_g(H)/H$

## 5.1 Local Isomorphisms

To go to sector $\sigma$ of the WZW orbifold $A_g(H)/H$, we apply the principle of *local isomorphisms*[3,1,2]. This principle tells us to replace all eigen-objects by corresponding twisted objects, whose monodromies are taken to be the old automorphic responses. In the present case, we have

$$\mathcal{J}(\xi,\sigma), \bar{\mathcal{J}}(\xi,\sigma) \underset{\sigma}{\longrightarrow} \hat{J}(\xi,\sigma), \hat{\bar{J}}(\xi,\sigma) \tag{5.1a}$$

$$\mathcal{G}(\mathcal{T},\xi,\sigma) \underset{\sigma}{\longrightarrow} \hat{g}(\mathcal{T},\xi,\sigma), \quad \mathfrak{b}(\xi,\sigma) \underset{\sigma}{\longrightarrow} \hat{\beta}(\xi,\sigma), \quad \hat{g}(\mathcal{T},\xi,\sigma) = e^{i\hat{\beta}^{n(r)\mu}(\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)} \tag{5.1b}$$

$$\text{automorphism response } E_{n(r)}(\sigma) \underset{\sigma}{\longrightarrow} \text{ monodromy } E_{n(r)}(\sigma) \tag{5.1c}$$

$$\text{automorphism response } E_{N(r)}(\sigma) \underset{\sigma}{\longrightarrow} \text{ monodromy } E_{N(r)}(\sigma)\ . \tag{5.1d}$$

Here, $\hat{J}_{n(r)\mu}(\xi,\sigma)$,$\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma)$ are the twisted currents, and the set of *group orbifold elements*

$$\{\hat{g}\} = \{\hat{g}(\mathcal{T},\xi,\sigma)_{N(r)\mu}{}^{N(s)\nu}\} \tag{5.2}$$

parameterize the general group orbifold in sector $\sigma$ of $A_g(H)/H$. The quantities $\{\hat{\beta}^{n(r)\mu}\}$ are called the *twisted tangent space coordinates* of the group orbifold.

We also note the equivalent forms of the group orbifold elements

$$\beta^a(\xi) = \chi(\sigma)_{n(r)\mu}\mathfrak{b}^{n(r)\mu}(\xi,\sigma)U(\sigma)_{n(r)\mu}{}^a \underset{\sigma}{\longrightarrow} \hat{\mathfrak{b}}^a(\xi,\sigma) \equiv \chi(\sigma)_{n(r)\mu}\hat{\beta}^{n(r)\mu}(\xi,\sigma)U(\sigma)_{n(r)\mu}{}^a \tag{5.3a}$$

$$g(T,\beta(\xi)) = e^{i\beta^a(\xi)T_a} \tag{5.3b}$$

$$\hat{g}(\mathcal{T},\xi,\sigma) = U(T,\sigma)\ g(T,\beta(\xi)\underset{\sigma}{\rightarrow}\hat{\mathfrak{b}}(\xi,\sigma))\ U^\dagger(T,\sigma) \tag{5.3c}$$

$$= e^{i\hat{\mathfrak{b}}^a(\xi,\sigma)(U^\dagger(T,\sigma)T_aU^\dagger(T,\sigma))} \tag{5.3d}$$

$$= e^{i\hat{\beta}^{n(r)\mu}(\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)} \tag{5.3e}$$

where (5.3a) is obtained by combining (4.24d) and (5.1b). The objects $\{\hat{\mathfrak{b}}^a\}$ are identified in Appendix C as tangent space coordinates with twisted boundary conditions. The form in (5.3d) shows that the group orbifold elements are locally isomorphic to a reparameterization of the original group elements $g$.

The local isomorphisms also tell us that the group orbifold elements are still unitary

$$\hat{g}(\mathcal{T},\xi,\sigma)\hat{g}(\mathcal{T},\xi,\sigma)^\dagger = \hat{g}(\mathcal{T},\xi,\sigma)^\dagger\hat{g}(\mathcal{T},\xi,\sigma) = \mathbb{1}\,. \tag{5.4}$$

The complex conjugation of $\hat{\beta}$ and the unitarity of $\hat{g}$ are further discussed in Appendix B. The discussion of Appendix C gives an overview of the group and group orbifold elements discussed above, including their relation to group elements with twisted boundary conditions.

We will also need the *derived local isomorphisms*[1,2] for the stress tensors and the action

$$T_g(\xi),\bar{T}_g(\xi) \underset{\sigma}{\longrightarrow} \hat{T}_\sigma(\xi),\hat{\bar{T}}_\sigma(\xi), \quad S_\sigma[\mathcal{M},\mathcal{G}] \underset{\sigma}{\longrightarrow} \hat{S}_\sigma[\mathcal{M},\hat{g}] \tag{5.5}$$

which follow from the local isomorphisms (5.1) after rewriting the untwisted objects in terms of the eigen-objects (see e.g. (4.7)).

## 5.2 Monodromies

The monodromies of the twisted currents, the group orbifold elements and the twisted tangent space coordinates[a]

$$\hat{J}_{n(r)\mu}(\xi+2\pi,\sigma) = E_{n(r)}(\sigma)\hat{J}_{n(r)\mu}(\xi,\sigma), \quad \hat{\bar{J}}_{n(r)\mu}(\xi+2\pi,\sigma) = E_{n(r)}(\sigma)\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma) \tag{5.6a}$$

$$\hat{g}(\mathcal{T},\xi+2\pi,\sigma) = E(T,\sigma)\hat{g}(\mathcal{T},\xi,\sigma)E(T,\sigma)^*, \ \hat{\beta}^{n(r)\mu}(\xi+2\pi,\sigma) = \hat{\beta}^{n(r)\mu}(\xi,\sigma)E_{n(r)}(\sigma)^* \tag{5.6b}$$

$$\hat{g}^{-1}(\mathcal{T},\xi+2\pi,\sigma) = E(T,\sigma)\hat{g}^{-1}(\mathcal{T},\xi,\sigma)E(T,\sigma)^* \tag{5.6c}$$

$$E_{n(r)}(\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}, \quad E_{N(r)}(T,\sigma) = e^{-2\pi i\frac{N(r)}{R(\sigma)}} \tag{5.6d}$$

follow immediately from the local isomorphisms (5.1). The two-sided diagonal monodromy of the group orbifold elements in (5.6b) is a central result of this paper. The twisted tangent space coordinates $\hat{\beta}$ also have definite monodromy, a property not generally shared by other choices of coordinates on the twisted tangent space.

Moreover, the closely-related monodromy

$$\hat{\beta}^{n(r)\mu}(\xi+2\pi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma) = E(T,\sigma)\big(\,\hat{\beta}^{n(r)\mu}(\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\,\big)E(T,\sigma)^* \tag{5.7}$$

follows from (5.6) and the selection rules (4.18) for the twisted representation matrices. Then we may verify that the $\hat{\beta}$ and $\hat{g}$ monodromies in (5.6) are consistent

$$e^{i\hat{\beta}^{n(r)\mu}(\xi+2\pi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)} = E(T,\sigma)e^{i\hat{\beta}^{n(r)\mu}(\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)}E(T,\sigma)^* \tag{5.8a}$$

$$\begin{aligned} &\hat{g}(\mathcal{T},\xi+2\pi,\sigma)_{N(r)\mu}{}^{N(s)\nu} = e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\hat{g}(\mathcal{T},\xi,\sigma)_{N(r)\mu}{}^{N(s)\nu} \\ &\longleftrightarrow \quad \hat{\beta}^{n(r)\mu}(\xi+2\pi,\sigma) = \hat{\beta}^{n(r)\mu}(\xi,\sigma)e^{2\pi i\frac{n(r)}{\rho(\sigma)}} \end{aligned} \tag{5.8b}$$

[a]The automorphic response of $g$ and hence the $\xi$-monodromy of $\hat{g}$ is the same when the group orbifold elements are extended to the solid cylinder $\Gamma$.

to all orders in $\hat{\beta}$.

We also note that the multiplication law

$$\hat{g}_3(\mathcal{T},\xi,\sigma) \equiv \hat{g}_1(\mathcal{T},\xi,\sigma)\hat{g}_2(\mathcal{T},\xi,\sigma) \tag{5.9a}$$

$$\hat{\beta}_3(\xi,\sigma) = \hat{\beta}_1(\xi,\sigma) + \hat{\beta}_2(\xi,\sigma) + O(\hat{\beta}^2) \tag{5.9b}$$

$$\hat{g}_i(\mathcal{T},\xi+2\pi,\sigma) = E(T,\sigma)\hat{g}_i(\mathcal{T},\xi,\sigma)E(T,\sigma)^*, \quad i=1,2,3 \tag{5.9c}$$

preserves the monodromies of the group orbifold elements of sector $\sigma$.

## 5.3 Local WZW Orbifold Dynamics

Continuing with the local isomorphisms, the list of local WZW relations in Subsec. 2.2 gives the local dynamics of each sector $\sigma$ of the WZW orbifold $A_g(H)/H$.

- Twisted equal-time current algebra and orbifold stress tensors

$$\begin{aligned}\{\hat{J}_{n(r)\mu}(\xi,\sigma),\hat{J}_{n(s)\nu}(\eta,\sigma)\} =& 2\pi i\big(\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(\xi,\sigma)\delta(\xi-\eta)\\ &+\delta_{n(r)+n(s),0\bmod\rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\partial_\xi\delta(\xi-\eta)\big)\end{aligned} \tag{5.10a}$$

$$\begin{aligned}\{\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma),\hat{\bar{J}}_{n(s)\nu}(\eta,\sigma)\} =& 2\pi i\big(\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\xi,\sigma)\delta(\xi-\eta)\\ &-\delta_{n(r)+n(s),0\bmod\rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\partial_\xi\delta(\xi-\eta)\big)\end{aligned} \tag{5.10b}$$

$$\{\hat{J}_{n(r)\mu}(\xi,\sigma),\hat{\bar{J}}_{n(s)\nu}(\eta,\sigma)\} = 0, \quad \sigma=0,\ldots,N_c-1 \tag{5.10c}$$

$$\hat{J}_{n(r)\pm\rho(\sigma),\mu}(\xi,\sigma) = \hat{J}_{n(r)\mu}(\xi,\sigma), \quad \hat{\bar{J}}_{n(r)\pm\rho(\sigma),\mu}(\xi,\sigma) = \hat{\bar{J}}_{n(r)\mu}(\xi,\sigma) \tag{5.10d}$$

$$\hat{T}_\sigma(\xi) = \frac{1}{4\pi}\mathcal{G}^{n(r)\mu;n(s)\nu}(\sigma)\hat{J}_{n(r)\mu}(\xi,\sigma)\hat{J}_{n(s)\nu}(\xi,\sigma) \tag{5.11a}$$

$$= \frac{1}{4\pi}\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\hat{J}_{n(r)\mu}(\xi,\sigma)\hat{J}_{-n(r),\nu}(\xi,\sigma) \tag{5.11b}$$

$$\hat{\bar{T}}_\sigma(\xi) = \frac{1}{4\pi}\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma)\hat{\bar{J}}_{-n(r),\nu}(\xi,\sigma) \tag{5.11c}$$

$$L_{g,\infty} \underset{\sigma}{\longrightarrow} \mathcal{L}^{n(r)\mu;n(s)\nu}_{\hat{\mathfrak{g}}(\sigma),\,\infty}(\sigma) = \frac{1}{2}\mathcal{G}^{n(r)\mu;n(s)\nu}(\sigma) \tag{5.11d}$$

$$\hat{T}_\sigma(\xi+2\pi) = \hat{T}_\sigma(\xi), \quad \hat{\bar{T}}_\sigma(\xi+2\pi) = \hat{\bar{T}}_\sigma(\xi) \tag{5.11e}$$

$$\{\hat{T}_\sigma(\xi),\hat{T}_\sigma(\eta)\} = i(\,\hat{T}_\sigma(\xi)+\hat{T}_\sigma(\eta)\,)\,\partial_\xi\delta(\xi-\eta) \tag{5.11f}$$

$$\{\hat{\bar{T}}_\sigma(\xi),\hat{\bar{T}}_\sigma(\eta)\} = -i(\,\hat{\bar{T}}_\sigma(\xi)+\hat{\bar{T}}_\sigma(\eta)\,)\,\partial_\xi\delta(\xi-\eta) \tag{5.11g}$$

$$\{\hat{T}_\sigma(\xi),\hat{\bar{T}}_\sigma(\eta)\} = 0 \tag{5.11h}$$

$$\{\hat{T}_\sigma(\xi),\hat{J}_{n(r)\mu}(\eta,\sigma)\} = i\hat{J}_{n(r)\mu}(\xi,\sigma)\partial_\xi\delta(\xi-\eta) \tag{5.12a}$$

$$\{\hat{\bar{T}}_\sigma(\xi),\hat{\bar{J}}_{n(r)\mu}(\eta,\sigma)\} = -i\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma)\partial_\xi\delta(\xi-\eta) \tag{5.12b}$$

$$\{\hat{T}_\sigma(\xi),\hat{\bar{J}}_{n(r)\mu}(\eta,\sigma)\} = \{\hat{\bar{T}}_\sigma(\xi),\hat{J}_{n(r)\mu}(\eta,\sigma)\} = 0\,. \tag{5.12c}$$

The general twisted equal-time current algebra in (5.10) is isomorphic to the equal-time algebra of the eigencurrents $\mathcal{J}$, $\bar{\mathcal{J}}$ defined in (4.1). The coefficients $\mathcal{G}(\sigma)$ and $\mathcal{F}(\sigma)$ in (5.10) are the twisted metric and twisted structure constants in (4.5). The coefficient $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma),\infty}$ in (5.11d) is the high-level limit of the twisted inverse inertia tensor $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}$ of the general orbifold affine-Sugawara construction[1,2].

- Group orbifold elements

$$\hat{g}(\mathcal{T},\xi,\sigma)_{N(r)\pm R(\sigma),\mu}{}^{N(s)\nu} = \hat{g}(\mathcal{T},\xi,\sigma)_{N(r)\mu}{}^{N(s)\pm R(\sigma),\nu} = \hat{g}(\mathcal{T},\xi,\sigma)_{N(r)\mu}{}^{N(s)\nu} \tag{5.13a}$$

$$[\mathcal{M}(\mathcal{T},\sigma),\hat{g}(\mathcal{T},\xi,\sigma)] = 0 \tag{5.13b}$$

$$\{\hat{J}_{n(r)\mu}(\xi,\sigma),\hat{g}(\mathcal{T},\eta,\sigma)\} = 2\pi\delta(\xi-\eta)\hat{g}(\mathcal{T},\eta,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma) \tag{5.14a}$$

$$\{\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma),\hat{g}(\mathcal{T},\eta,\sigma)\} = -\,2\pi\delta(\xi-\eta)\mathcal{T}_{n(r)\mu}(T,\sigma)\,\hat{g}(\mathcal{T},\eta,\sigma) \tag{5.14b}$$

$$\begin{aligned}&\{\hat{J}_{n(r)\mu}(\xi,\sigma),\hat{g}(\mathcal{T},\eta,\sigma)_{N(s)\nu}{}^{N(t)\delta}\}\\ &= 2\pi\delta(\xi-\eta)\hat{g}(\mathcal{T},\eta,\sigma)_{N(s)\nu}{}^{N(t)-\frac{R(\sigma)}{\rho(\sigma)}n(r),\eta}\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(t)-\frac{R(\sigma)}{\rho(\sigma)}n(r),\eta}{}^{N(t)\delta}\end{aligned} \tag{5.14c}$$

$$\begin{aligned}&\{\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma),\hat{g}(\mathcal{T},\eta,\sigma)_{N(s)\nu}{}^{N(t)\delta}\}\\ &= -2\pi\delta(\xi-\eta)\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(s)+\frac{R(\sigma)}{\rho(\sigma)}n(r),\eta}\hat{g}(\mathcal{T},\eta,\sigma)_{N(s)+\frac{R(\sigma)}{\rho(\sigma)}n(r),\eta}{}^{N(t)\delta}\end{aligned} \tag{5.14d}$$

$$\{\hat{T}_\sigma(\xi),\hat{g}(\mathcal{T},\eta,\sigma)\} = \delta(\xi-\eta)\hat{g}(\mathcal{T},\eta,\sigma)\hat{J}(\mathcal{T},\eta,\sigma) \tag{5.15a}$$

$$\{\hat{\bar{T}}_\sigma(\xi),\hat{g}(\mathcal{T},\eta,\sigma)\} = -\delta(\xi-\eta)\hat{\bar{J}}(\mathcal{T},\eta,\sigma)\,\hat{g}(\mathcal{T},\eta,\sigma) \tag{5.15b}$$

$$\begin{aligned}\hat{J}(\mathcal{T},\xi,\sigma) &\equiv \hat{J}(\xi,\sigma)_{n(r)\mu}\mathcal{G}^{n(r)\mu;n(s)\nu}(\sigma)\mathcal{T}_{n(s)\nu}(T,\sigma)\\ &= \hat{J}(\xi,\sigma)_{n(r)\mu}\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma)\end{aligned} \tag{5.15c}$$

$$\begin{aligned}\hat{\bar{J}}(\mathcal{T},\xi,\sigma) &\equiv \hat{\bar{J}}(\xi,\sigma)_{n(r)\mu}\mathcal{G}^{n(r)\mu;n(s)\nu}(\sigma)\mathcal{T}_{n(s)\nu}(T,\sigma)\\ &= \hat{\bar{J}}(\xi,\sigma)_{n(r)\mu}\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma)\end{aligned} \tag{5.15d}$$

$$\hat{J}_{n(r)\mu}(\xi,\sigma) = \widehat{Tr}\big(\mathcal{M}(\mathcal{T},\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\hat{J}(\mathcal{T},\xi,\sigma)\big) \tag{5.15e}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma) = \widehat{Tr}\big(\mathcal{M}(\mathcal{T},\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\hat{\bar{J}}(\mathcal{T},\xi,\sigma)\big)\,. \tag{5.15f}$$

Eq. (5.13b) follows from the local isomorphisms, using Eqs. (4.9) and (4.17e). The last two relations in (5.14) are matrix element forms, using (4.18), of the first two relations. The objects $\hat{J}(\mathcal{T})$, $\hat{\bar{J}}(\mathcal{T})$ are the matrix forms of the twisted currents.

- WZW orbifold Hamiltonian

$$\hat{H}_\sigma = \int_0^{2\pi} d\xi\,(\hat{T}_\sigma(\xi)+\hat{\bar{T}}_\sigma(\xi)),\quad (\hat{P}_\xi)_\sigma = \int_0^{2\pi} d\xi\,(\hat{T}_\sigma(\xi)-\hat{\bar{T}}_\sigma(\xi)) \tag{5.16a}$$

$$\partial_m\hat{A}(\xi,t,\sigma) = i\{(\hat{P}_m)_\sigma,\,\hat{A}(\xi,t,\sigma)\} \tag{5.16b}$$

$$\partial_- \hat{J}_{n(r)\mu}(\xi,t,\sigma) = \partial_+ \hat{\bar{J}}_{n(r)\mu}(\xi,t,\sigma) = 0, \quad \partial_- \hat{T}_\sigma(\xi,t) = \partial_+ \hat{\bar{T}}_\sigma(\xi,t) = 0 \tag{5.17a}$$

$$\partial_t \hat{g}(\mathcal{T},\xi,t) = i(\hat{g}(\mathcal{T},\xi,t)\hat{J}(\mathcal{T},\xi,t) - \hat{\bar{J}}(\mathcal{T},\xi,t)\hat{g}(\mathcal{T},\xi,t)) \tag{5.17b}$$

$$\partial_\xi \hat{g}(\mathcal{T},\xi,t) = i(\hat{g}(\mathcal{T},\xi,t)\hat{J}(\mathcal{T},\xi,t) + \hat{\bar{J}}(\mathcal{T},\xi,t)\hat{g}(\mathcal{T},\xi,t)) \tag{5.17c}$$

$$\partial_+ \hat{g}(\mathcal{T},\xi,t) = 2i\hat{g}(\mathcal{T},\xi,t)\hat{J}(\mathcal{T},\xi,t), \quad \partial_- \hat{g}(\mathcal{T},\xi,t) = -2i\hat{\bar{J}}(\mathcal{T},\xi,t)\hat{g}(\mathcal{T},\xi,t) \tag{5.17d}$$

$$\delta(\xi-\eta)\partial_+ \hat{g}(\mathcal{T},\eta,t) = 2i\{\hat{T}_\sigma(\xi,t), \hat{g}(\mathcal{T},\eta,t)\} \tag{5.17e}$$

$$\delta(\xi-\eta)\partial_- \hat{g}(\mathcal{T},\eta,t) = 2i\{\hat{\bar{T}}_\sigma(\xi,t), \hat{g}(\mathcal{T},\eta,t)\} \ . \tag{5.17f}$$

As seen here, we often choose for brevity to suppress the sector label $\sigma$.

- WZW orbifold action

$$\hat{S}_\sigma(\mathcal{T}) \equiv \hat{S}_\sigma[\mathcal{M},\hat{g}] = -\frac{1}{8\pi}\int d^2\xi\ \widehat{Tr}\big(\ \mathcal{M}(\mathcal{T},\sigma)\ \hat{g}^{-1}(\mathcal{T},\sigma)\partial_+\hat{g}(\mathcal{T},\sigma)\ \hat{g}^{-1}(\mathcal{T},\sigma)\partial_-\hat{g}(\mathcal{T},\sigma)\ \big)$$
$$-\frac{1}{12\pi}\int_\Gamma \widehat{Tr}\big(\ \mathcal{M}(\mathcal{T},\sigma)\ (\ \hat{g}^{-1}(\mathcal{T},\sigma)d\hat{g}(\mathcal{T},\sigma)\ )^3\ \big) \tag{5.18a}$$

$$\widehat{Tr}(AB) = \sum_{r,\mu,s,\nu} A_{N(r)\mu}{}^{N(s)\nu} B_{N(s)\nu}{}^{N(r)\mu}, \quad \sigma = 0,\ldots,N_c-1 \tag{5.18b}$$

$$(\hat{g}^{-1}d\hat{g})^3 = dt\,d\xi\,d\rho\ \epsilon^{ABC}(\hat{g}^{-1}\partial_A\hat{g})(\hat{g}^{-1}\partial_B\hat{g})(\hat{g}^{-1}\partial_C\hat{g}), \quad \{A,B,C\} = \{t,\xi,\rho\} \tag{5.18c}$$

$$\hat{S}_\sigma[\mathcal{M}(\mathcal{T},\sigma),\hat{g}(\mathcal{T},\xi+2\pi,\sigma)] = \hat{S}_\sigma[\mathcal{M}(\mathcal{T},\sigma),\hat{g}(\mathcal{T},\xi,\sigma)] \tag{5.19}$$

$$\hat{J}(\mathcal{T},\xi,t) \equiv -\frac{i}{2}\hat{g}^{-1}(\mathcal{T},\xi,t)\partial_+\hat{g}(\mathcal{T},\xi,t) = \hat{J}_{n(r)\mu}(\xi,t)\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma) \tag{5.20a}$$

$$\hat{\bar{J}}(\mathcal{T},\xi,t) \equiv -\frac{i}{2}\hat{g}(\mathcal{T},\xi,t)\partial_-\hat{g}^{-1}(\mathcal{T},\xi,t) = \hat{\bar{J}}_{n(r)\mu}(\xi,t)\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma) \tag{5.20b}$$

$$\partial_-\hat{J}(\mathcal{T},\xi,t) = \partial_+\hat{\bar{J}}(\mathcal{T},\xi,t) = 0 \tag{5.20c}$$

$$\hat{J}_{n(r)\mu}(\xi,t) = -\frac{i}{2}\widehat{Tr}\big(\ \mathcal{M}(\mathcal{T},\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\,\hat{g}^{-1}(\mathcal{T},\xi,t)\partial_+\hat{g}(\mathcal{T},\xi,t)\ \big) \tag{5.20d}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,t) = -\frac{i}{2}\widehat{Tr}\big(\ \mathcal{M}(\mathcal{T},\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\,\hat{g}(\mathcal{T},\xi,t)\partial_-\hat{g}^{-1}(\mathcal{T},\xi,t)\ \big)\ . \tag{5.20e}$$

The general WZW orbifold action (5.18) is a central result of this paper. It is the correct action functional for each twisted representation $\mathcal{T}$ in each sector $\sigma$ of any WZW orbifold $A_g(H)/H$.

The trivial monodromy (5.19) of the general WZW orbifold action follows immediately from the $\hat{g}$ and $\hat{g}^{-1}$ monodromies in (5.6b), (5.6c) and the selection rule (4.10) of the twisted data matrix $\mathcal{M}$. The equations of motion (5.20a)-(5.20c) of the WZW orbifold action follow because $\mathcal{M}$ is invertible and commutes with the group orbifold elements (see (5.13b)). Moreover, the orbifold action equations of motion are equivalent to the orbifold Hamiltonian equations of motion (5.17a)-(5.17d). See Sec. 7 for further details in the case of the permutation orbifolds and the inner automorphic orbifolds.

The local orbifold relations (5.20a,b) also provide what is perhaps the simplest example of the consistent interrelation among the monodromies of $\hat{g}$, $\hat{J}_{n(r)\mu}$ and $\hat{\bar{J}}_{n(r)\mu}$: In the first place, the $\hat{g}$ and $\hat{g}^{-1}$monodromies in (5.6b) and (5.6c) give the same monodromies

$$\hat{J}(\mathcal{T},\xi+2\pi)_{N(r)\mu}{}^{N(s)\nu} = e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\hat{J}(\mathcal{T},\xi)_{N(r)\mu}{}^{N(s)\nu} \tag{5.21a}$$

$$\hat{\bar{J}}(\mathcal{T},\xi+2\pi)_{N(r)\mu}{}^{N(s)\nu} = e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\hat{\bar{J}}(\mathcal{T},\xi)_{N(r)\mu}{}^{N(s)\nu} \tag{5.21b}$$

for the matrix forms of the twisted currents. On the other hand, the selection rule (4.18) for $\mathcal{T}$ in the form

$$e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\mathcal{T}_{-n(r),\epsilon}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta} = \mathcal{T}_{-n(r),\epsilon}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta}e^{-2\pi i\frac{N(s)-N(t)}{R(\sigma)}} \tag{5.22}$$

and the monodromy of $\hat{J}_{n(r)\mu}(\xi)$ in (5.6a) give the same result

$$\begin{aligned}\hat{J}(\mathcal{T},\xi+2\pi)_{N(s)\nu}{}^{N(t)\delta} &= \hat{J}_{n(r)\mu}(\xi+2\pi)\mathcal{G}^{n(r)\mu;-n(r),\epsilon}(\sigma)\mathcal{T}_{-n(r),\epsilon}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta}\\ &= e^{-2\pi i\frac{N(s)-N(t)}{R(\sigma)}}\hat{J}(\xi)_{N(r)\mu}{}^{N(s)\nu}\end{aligned} \tag{5.23}$$

for the twisted left movers. The same consistency is verified for the twisted right mover currents $\hat{\bar{J}}_{n(r)\mu}(\xi)$ in (5.20b). As a consequence the local orbifold relations (5.20a,b) require, by themselves, what we already knew from the local isomorphisms: The twisted left and right mover currents $\hat{J}_{n(r)\mu}(\xi)$, $\hat{\bar{J}}_{n(r)\mu}(\xi)$ have the same monodromies when both are taken around the cylinder $\xi \to \xi + 2\pi$.

## 5.4 Mode Form of the General Twisted Current Algebra

Having completed the local tangent space formulation of WZW orbifolds, we turn now to the implied mode formulation.

The orbifold equations of motion (5.17a) or (5.20c) and the monodromies (5.6a) give the mode expansions

$$\hat{J}_{n(r)\mu}(\xi,t) = \sum_m \hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})e^{-i(m+\frac{n(r)}{\rho(\sigma)})(\xi+t)} \tag{5.24a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,t) = \sum_m \hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})e^{-i(m+\frac{n(r)}{\rho(\sigma)})(\xi-t)} \tag{5.24b}$$

$$\hat{J}_{n(r)\pm\rho(\sigma),\mu}(m\mp 1+\tfrac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}) = \hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) \tag{5.24c}$$

$$\hat{\bar{J}}_{n(r)\pm\rho(\sigma),\mu}(m\mp 1+\tfrac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}) = \hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) \tag{5.24d}$$

where the mode periodicities in (5.24c)-(5.24d) follow from (5.24a), (5.24b) and the periodicities of the currents in Eq. (5.10d).

Then the *general twisted current algebra*

$$\{\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}),\hat{J}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})\} = i\,\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\;\hat{J}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)})$$
$$+\,(m+\tfrac{n(r)}{\rho(\sigma)})\;\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\;\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \quad (5.25a)$$

$$\{\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}),\hat{\bar{J}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})\} = i\,\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\;\hat{\bar{J}}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)})$$
$$-\,(m+\tfrac{n(r)}{\rho(\sigma)})\;\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\;\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \quad (5.25b)$$

$$\{\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}),\hat{\bar{J}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})\} = 0, \quad \sigma = 0,\ldots,N_c-1 \quad (5.25c)$$

is obtained from the general twisted equal-time algebra (5.10). The operator analogue of the general twisted left mover current algebra (5.25a) was called $\hat{\mathfrak{g}}(\sigma) = \hat{\mathfrak{g}}(H \subset Aut(g);\sigma)$ in Refs. 1 and 2. The twisted right mover algebra in (5.25b) has the same form as the twisted left mover algebra, but with the opposite sign of the central term. Both algebras are consistent with the Jacobi identity[2], since the $\mathcal{F}$ and $\mathcal{G}$ terms satisfy the identity separately. We will find it convenient to refer to the general twisted current algebra (5.25) as

$$\hat{\mathfrak{g}}(h_\sigma) \oplus \hat{\bar{\mathfrak{g}}}(h_\sigma) \quad (5.26a)$$
$$\hat{\mathfrak{g}}(h_\sigma) \equiv \hat{\mathfrak{g}}(H \subset Aut(g);h_\sigma) \equiv \hat{\mathfrak{g}}(H \subset Aut(g);\sigma) \quad (5.26b)$$

and more generally to follow the convention $\sigma \to h_\sigma$ below.

In fact the twisted right mover algebra of sector $\sigma$ is not a new algebra. To understand this, consider first the mode-number reversed form of this algebra

$$\{\hat{\bar{\bar{J}}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}),\hat{\bar{\bar{J}}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})\} = i\mathcal{F}_{-n(r),\mu;-n(s),\nu}{}^{-n(r)-n(s),\delta}(h_\sigma)\hat{\bar{\bar{J}}}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)})$$
$$+(m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\;\mathcal{G}_{-n(r),\mu;n(r),\nu}(h_\sigma) \quad (5.27a)$$

$$\hat{\bar{\bar{J}}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) \equiv \hat{\bar{J}}_{-n(r),\mu}(-m-\tfrac{n(r)}{\rho(\sigma)}) \quad (5.27b)$$

which exhibits the correct sign of the central term.

We claim that the algebra (5.27) is isomorphic to the twisted left mover current algebra of the sector corresponding to $h_\sigma^{-1} \in H$

$$\hat{\bar{\mathfrak{g}}}(h_\sigma) \simeq \hat{\mathfrak{g}}(h_\sigma^{-1}) \;. \quad (5.28)$$

To see this, one needs to compare the corresponding forms of the $H$-eigenvalue problem

$$w(h_\sigma^{-1}) = w^\dagger(h_\sigma), \quad \rho(h_\sigma^{-1}) = \rho(h_\sigma) \quad (5.29a)$$
$$w(h_\sigma)U^\dagger(h_\sigma) = U^\dagger(h_\sigma)E_{n(r)}(h_\sigma) \quad (5.29b)$$
$$w(h_\sigma^{-1})U^\dagger(h_\sigma) = w^\dagger(h_\sigma)U^\dagger(h_\sigma) = U^\dagger(h_\sigma)E_{n(r)}(h_\sigma)^* \quad (5.29c)$$

where we used (3.15) to obtain (5.29c). This allows us to choose

$$U^\dagger(h_\sigma^{-1})_a{}^{n(r)\mu} = U^\dagger(h_\sigma)_a{}^{-n(r),\mu} \quad (5.30)$$

and then the duality transformations for $\mathcal{G}(\sigma)$ and $\mathcal{F}(\sigma)$ in (4.5) imply that

$$\mathcal{G}_{n(r)\mu;n(s)\nu}(h_\sigma^{-1}) = \mathcal{G}_{-n(r),\mu;-n(s),\nu}(h_\sigma), \quad \mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(h_\sigma^{-1}) = \mathcal{F}_{-n(r),\mu;-n(s),\nu}{}^{-n(t),\delta}(h_\sigma) \tag{5.31a}$$

$$\begin{aligned}\{\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{\bar{J}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})\} =& i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(h_\sigma^{-1})\hat{\bar{J}}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) \\ &+ (m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(h_\sigma^{-1})\end{aligned} \tag{5.31b}$$

which establishes the claim. This argument identifies the general twisted current algebra as

$$\text{affine}(\mathrm{g}\oplus\mathrm{g}) \underset{\sigma}{\longrightarrow} \hat{\mathfrak{g}}(\mathrm{h}_\sigma)\oplus\hat{\bar{\mathfrak{g}}}(\mathrm{h}_\sigma) \simeq \hat{\mathfrak{g}}(\mathrm{h}_\sigma)\oplus\hat{\mathfrak{g}}(\mathrm{h}_\sigma^{-1}) \tag{5.32}$$

in sector $\sigma$ of $A_g(H)/H$.

The identification (5.32) leaves open the question of when the twisted right and left mover current algebras are copies $\hat{\mathfrak{g}}(h_\sigma^{-1}) \simeq \hat{\mathfrak{g}}(h_\sigma)$, which we call *the rectification problem.* For the permutation orbifolds and the inner-automorphic orbifolds we shall see in Sec. 7 that it is possible to choose further rescalings

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) \equiv \theta_{n(r)\mu}(\sigma)\hat{\bar{J}}_{-n(r),\mu}(-m-\tfrac{n(r)}{\rho(\sigma)}) \tag{5.33a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,t) = \theta^{-1}_{-n(r),\mu}(\sigma)\sum_m \hat{\bar{J}}_{-n(r),\mu}(m-\tfrac{n(r)}{\rho(\sigma)})e^{i(m-\frac{n(r)}{\rho(\sigma)})(\xi-t)} \tag{5.33b}$$

which rectify the right mover algebra $\hat{\mathfrak{g}}(h_\sigma^{-1})$ into a copy $\hat{\mathfrak{g}}(h_\sigma)$ of the left mover algebra of sector $\sigma$. However, we do not have a proof that such rectification can be extended across all right mover algebras $\hat{\mathfrak{g}}(h_\sigma^{-1})$. Some general remarks on the rectification problem are also included in Subsec. 8.2.

Finally, we give the equivalent twisted light-cone current algebra

$$\hat{J}_{n(r)\mu}(t_+) \equiv \hat{J}_{n(r)\mu}(\xi,t,\sigma), \quad \hat{\bar{J}}_{n(r)\mu}(t_-) \equiv \hat{\bar{J}}_{n(r)\mu}(\xi,t,\sigma), \quad t_\pm \equiv t\pm\xi \tag{5.34a}$$

$$\begin{aligned}\{\hat{J}_{n(r)\mu}(t_+), \hat{J}_{n(s)\nu}(t'_+)\} =& 2\pi i\big(\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(t'_+)\delta(t_+-t'_+) \\ &+ \delta_{n(r)+n(s),0\bmod\rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\partial_{t_+}\delta(t_+-t'_+)\big)\end{aligned} \tag{5.34b}$$

$$\begin{aligned}\{\hat{\bar{J}}_{n(r)\mu}(t_-), \hat{\bar{J}}_{n(s)\nu}(t'_-)\} =& 2\pi i\big(\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(t'_-)\delta(t_--t'_-) \\ &+ \delta_{n(r)+n(s),0\bmod\rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\partial_{t_-}\delta(t_--t'_-)\big)\end{aligned} \tag{5.34c}$$

$$\{\hat{J}_{n(r)\mu}(t_+), \hat{\bar{J}}_{n(s)\nu}(t_-)\} = 0\ . \tag{5.34d}$$

$$\hat{J}_{n(r)\mu}(t_++2\pi) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(t_+), \quad \hat{\bar{J}}_{n(r)\mu}(t_--2\pi) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(t_-) \tag{5.34e}$$

where the light-cone monodromies in (5.34e) correspond to the usual path $\xi \longrightarrow \xi+2\pi$ around the cylinder. There is another (on shell) monodromy

$$\hat{J}_{n(r)\mu}(t_++2\pi) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(\xi,t), \quad \hat{\bar{J}}_{n(r)\mu}(t_-+2\pi) = e^{+2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)\mu}(\xi,t) \tag{5.35}$$

which corresponds to $t \longrightarrow t+2\pi$, but this is not a closed path on the cylinder or the sphere

$$t\to -i\tau,\ z=e^{\tau+i\xi},\ \bar{z}=e^{\tau-i\xi} \quad : \quad (\xi\to\xi+2\pi) \longleftrightarrow (z\to z\,e^{2\pi i},\ \bar{z}\to\bar{z}\,e^{-2\pi i}) \tag{5.36}$$

where $\tau$ is real Euclidean time.

# 6 Orbifold Geometry

## 6.1 Bracket Realization of the General Twisted Current Algebra

In this and the following subsection, we extend the tangent space considerations above to fill in the geometric steps omitted in going from the orbifold Hamiltonian to the orbifold action formulation. In particular, we will choose twisted Einstein coordinates, give the canonical (bracket) realization of the general twisted current algebra and use the result to obtain the twisted sigma model form of the WZW orbifold action. The results below can be obtained by either of two methods

- Eigen-objects and Local Isomorphisms
- Twisted Affine Lie Groups .

The second method is a straightforward extension of Clubok and Halpern's method of affine Lie groups[16] which was originally used to find a canonical (quantum) realization of affine(g $\oplus$ g). In fact, we have checked the results of these subsections both ways and, although the two methods are very different, we find total agreement. Here, we give only final results, referring the reader to Subsec. 6.4 and Appendix E for further details of the two methods.

Our first task is to choose twisted Einstein coordinates $\hat{x}(\xi) \equiv \hat{x}_\sigma(\xi)$ of definite monodromy, a property which can hold only in a preferred class of coordinate systems. For simplicity, we choose the coordinate system $\hat{x}(\xi) = \hat{\beta}(\xi)$ in which the twisted Einstein coordinates are equal to the twisted tangent space coordinates $\hat{\beta}$:

$$\hat{x}_\sigma^{n(r)\mu}(\xi) \equiv \hat{\beta}^{n(s)\nu}(\xi,\sigma)\hat{e}^{-1}(0)_{n(s)\nu}{}^{n(r)\mu} = \hat{\beta}^{n(r)\mu}(\xi,\sigma), \quad \hat{e}(0)_{n(r)\mu}{}^{n(s)\nu} = \delta_{n(r)\mu}{}^{n(s)\nu} \tag{6.1a}$$

$$\hat{x}_\sigma^{n(r)\mu}(\xi+2\pi) = \hat{x}_\sigma^{n(r)\mu}(\xi)E_{n(r)}(\sigma)^*, \quad \hat{g}(\mathcal{T},\xi,\sigma) = e^{i\hat{x}_\sigma^{n(r)\mu}(\xi)\mathcal{T}_{n(r)\mu}(\mathcal{T},\sigma)} . \tag{6.1b}$$

Here $\hat{e}(0) = 1$ is the twisted vielbein of sector $\sigma$ evaluated at $\hat{x} = 0$ (see below).

Then we can state the canonical (bracket) realization of the general twisted equal-time current algebra (5.10)

$$\hat{J}_{n(r)\mu}(\xi,\sigma) = 2\pi\hat{e}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{p}_{n(t)\delta}(\hat{B}) + \frac{1}{2}\partial_\xi\hat{x}^{n(t)\delta}\hat{e}(\hat{x})_{n(t)\delta}{}^{n(s)\nu}\mathcal{G}_{n(s)\nu;n(r)\mu}(\sigma) \tag{6.2a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma) = 2\pi\hat{\bar{e}}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{p}_{n(t)\delta}(\hat{B}) - \frac{1}{2}\partial_\xi\hat{x}^{n(t)\delta}\hat{\bar{e}}(\hat{x})_{n(t)\delta}{}^{n(s)\nu}\mathcal{G}_{n(s)\nu;n(r)\mu}(\sigma) \tag{6.2b}$$

$$\hat{p}_{n(r)\mu}(\hat{B}) = \hat{p}_{n(r)\mu} + \frac{1}{4\pi}\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x})\partial_\xi\hat{x}^{n(s)\nu} \tag{6.2c}$$

$$\{\hat{x}^{n(r)\mu}(\xi),\hat{p}_{n(s)\nu}(\eta)\} = i\delta(\xi-\eta)\delta_{n(s)\nu}{}^{n(r)\mu}, \tag{6.2d}$$

$$\hat{x}^{n(r)\mu}(\xi+2\pi) = \hat{x}^{n(r)\mu}(\xi)e^{2\pi i\frac{n(r)}{\rho(\sigma)}}, \quad \hat{p}_{n(r)\mu}(\xi+2\pi) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{p}_{n(r)\mu}(\xi) \tag{6.2e}$$

in sector $\sigma$ of $A_g(H)/H$. Here, $\mathcal{G}(\sigma)$ is the twisted tangent space metric in (4.5) and $\hat{p}(\xi) \equiv \hat{p}^\sigma(\xi)$ are the twisted Einstein momenta. Beyond $\hat{x}$ and $\hat{p}$, the result (6.2) involves the twisted vielbeins $\hat{e}$ and $\hat{\bar{e}}$ and the twisted $B$ field $\hat{B}$, defined below.

The twisted vielbeins $\hat{e}$, $\hat{\bar{e}}$ are defined as follows

$$\hat{\partial}_{n(r)\mu}(\xi) \equiv \frac{\partial}{\partial \hat{x}^{n(r)\mu}(\xi)}, \quad \hat{\partial}_{n(r)\mu}(\xi + 2\pi) = e^{-2\pi i \frac{n(r)}{\rho(\sigma)}} \hat{\partial}_{n(r)\mu}(\xi) \tag{6.3a}$$

$$\hat{e}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi)) \equiv -i\hat{g}^{-1}(\mathcal{T}, \xi)\hat{\partial}_{n(r)\mu}\hat{g}(\mathcal{T}, \xi) \equiv \hat{e}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu}\mathcal{T}_{n(s)\nu}(T, \sigma) \tag{6.3b}$$

$$\hat{e}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu} = \widehat{Tr}\big(\mathcal{M}(\mathcal{T}, \sigma)\hat{e}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi))\mathcal{T}(T, \sigma)_{n(t)\delta}\big)\mathcal{G}^{n(t)\delta;n(s)\nu}(\sigma) \tag{6.3c}$$

$$\hat{e}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi + 2\pi))_{N(s)\nu}{}^{N(t)\delta} = e^{-2\pi i(\frac{n(r)}{\rho(\sigma)} + \frac{N(s)-N(t)}{R(\sigma)})}\hat{e}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi))_{N(s)\nu}{}^{N(t)\delta} \tag{6.3d}$$

$$\hat{e}(\hat{x}(\xi + 2\pi))_{n(r)\mu}{}^{n(s)\nu} = e^{-2\pi i \frac{n(r)-n(s)}{\rho(\sigma)}}\hat{e}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu} \tag{6.3e}$$

$$\hat{e}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\,\hat{e}(\hat{x})_{n(t)\delta}{}^{n(s)\nu} = \delta_{n(r)\mu}{}^{n(s)\nu}, \quad \hat{e}^{-1}(\hat{x})_{n(r)\mu}{}^{n(s)\nu}\,\hat{e}(\mathcal{T}, \hat{x})_{n(s)\nu} = \mathcal{T}_{n(r)\mu}(T, \sigma) \tag{6.3f}$$

$$\hat{e}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi)) \equiv \hat{e}_{n(r)\mu}(\mathcal{T}, \hat{x}_\sigma(\xi), \sigma), \quad \hat{e}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu} \equiv \hat{e}(\hat{x}_\sigma(\xi), \sigma)_{n(r)\mu}{}^{n(s)\nu} \tag{6.3g}$$

$$\hat{\bar{e}}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi)) \equiv -i\hat{g}(\mathcal{T}, \xi)\hat{\partial}_{n(r)\mu}\hat{g}^{-1}(\mathcal{T}, \xi) \equiv \hat{\bar{e}}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu}\mathcal{T}_{n(s)\nu}(T, \sigma) \tag{6.4a}$$

$$\hat{\bar{e}}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu} = \widehat{Tr}\big(\mathcal{M}(\mathcal{T}, \sigma)\hat{\bar{e}}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi))\mathcal{T}(T, \sigma)_{n(t)\delta}\big)\mathcal{G}^{n(t)\delta;n(s)\nu}(\sigma) \tag{6.4b}$$

$$\hat{\bar{e}}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi + 2\pi))_{N(s)\nu}{}^{N(t)\delta} = e^{-2\pi i(\frac{n(r)}{\rho(\sigma)} + \frac{N(s)-N(t)}{R(\sigma)})}\hat{\bar{e}}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi))_{N(s)\nu}{}^{N(t)\delta} \tag{6.4c}$$

$$\hat{\bar{e}}(\hat{x}(\xi + 2\pi))_{n(r)\mu}{}^{n(s)\nu} = e^{-2\pi i \frac{n(r)-n(s)}{\rho(\sigma)}}\hat{\bar{e}}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu} \tag{6.4d}$$

$$\hat{\bar{e}}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\,\hat{\bar{e}}(\hat{x})_{n(t)\delta}{}^{n(s)\nu} = \delta_{n(r)\mu}{}^{n(s)\nu}, \quad \hat{\bar{e}}^{-1}(\hat{x})_{n(r)\mu}{}^{n(s)\nu}\,\hat{\bar{e}}(\mathcal{T}, \hat{x})_{n(s)\nu} = \mathcal{T}_{n(r)\mu}(T, \sigma) \tag{6.4e}$$

$$\hat{\bar{e}}_{n(r)\mu}(\mathcal{T}, \hat{x}(\xi)) \equiv \hat{\bar{e}}_{n(r)\mu}(\mathcal{T}, \hat{x}_\sigma(\xi), \sigma), \quad \hat{\bar{e}}(\hat{x}(\xi))_{n(r)\mu}{}^{n(s)\nu} \equiv \hat{\bar{e}}(\hat{x}_\sigma(\xi), \sigma)_{n(r)\mu}{}^{n(s)\nu} \tag{6.4f}$$

and the inverse twisted vielbeins $\hat{e}^{-1}$, $\hat{\bar{e}}^{-1}$ have the same monodromies as the twisted vielbeins. According to the selection rules (4.18) for $\mathcal{T}$, the monodromies (6.3d) and (6.4c) of $\hat{e}(\mathcal{T})$, $\hat{\bar{e}}(\mathcal{T})$ are consistent with the monodromies (6.3e) and (6.4d) of $\hat{e}$ and $\hat{\bar{e}}$. The twisted vielbeins $\hat{e}(\mathcal{T})$, $\hat{e}$ and $\hat{\bar{e}}(\mathcal{T})$, $\hat{\bar{e}}$ are also invariant respectively under left and right translations $\hat{g} \to \hat{g}_0\hat{g}$ and $\hat{g} \to \hat{g}\hat{g}_0$ on the group orbifold.

The definitions of the twisted $B$ field $\hat{B}$ and the twisted torsion field $\hat{H}$ are

$$\begin{aligned}\hat{H}_{n(r)\mu;n(s)\nu;n(t)\delta}(\hat{x}) &\equiv \hat{\partial}_{n(r)\mu}\hat{B}_{n(s)\nu;n(t)\delta}(\hat{x}) + \hat{\partial}_{n(s)\nu}\hat{B}_{n(t)\delta;n(r)\mu}(\hat{x}) + \hat{\partial}_{n(t)\delta}\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}) && (6.5a)\\ &= -i\,\widehat{Tr}(\mathcal{M}(\mathcal{T}, \sigma)\hat{e}_{n(r)\mu}(\mathcal{T}, \hat{x})[\hat{e}_{n(s)\nu}(\mathcal{T}, \hat{x}), \hat{e}_{n(t)\delta}(\mathcal{T}, \hat{x})]) && (6.5b)\\ &= \hat{e}(\hat{x})_{n(r)\mu}{}^{n(r')\mu'}\,\hat{e}(\hat{x})_{n(s)\nu}{}^{n(s')\nu'}\,\hat{e}(\hat{x})_{n(t)\delta}{}^{n(t'),\delta'}\,\mathcal{F}_{n(r')\mu';n(s')\nu';n(t')\delta'}(\sigma) && (6.5c)\end{aligned}$$

$$\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i \frac{n(r)+n(s)}{\rho(\sigma)}}\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi)), \quad \hat{B}_{n(r)\mu;n(s)\nu} = -\hat{B}_{n(s)\nu;n(r)\mu} \tag{6.6a}$$

$$\hat{H}_{n(r)\mu;n(s)\nu;n(t)\delta}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i \frac{n(r)+n(s)+n(t)}{\rho(\sigma)}}\hat{H}_{n(r)\mu;n(s)\nu;n(t)\delta}(\hat{x}(\xi)) \tag{6.6b}$$

$$\hat{H}(\hat{x}(\xi)) \equiv \hat{H}(\hat{x}_\sigma(\xi), \sigma), \quad \hat{B}(\hat{x}(\xi)) \equiv \hat{B}(\hat{x}_\sigma(\xi), \sigma) \tag{6.6c}$$

where $\hat{H}$ and the twisted structure constants $\mathcal{F}(\sigma)$ in (6.5c) are totally antisymmetric.

We will also need the definition of the twisted adjoint action $\hat{\Omega}$

$$\hat{g}(\mathcal{T},\xi,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\hat{g}^{-1}(\mathcal{T},\xi,\sigma)=\hat{\Omega}(\hat{x})_{n(r)\mu}{}^{n(s)\nu}\mathcal{T}_{n(s)\nu}(T,\sigma) \tag{6.7a}$$

$$\hat{\Omega}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{\Omega}(\hat{x})_{n(s)\nu}{}^{n(u)\epsilon}\mathcal{G}_{n(t)\delta;n(u)\epsilon}(\sigma)=\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma) \tag{6.7b}$$

$$\hat{\bar{e}}(\hat{x})_{n(r)\mu}{}^{n(s)\nu}=-\hat{e}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{\Omega}(\hat{x})_{n(t)\delta}{}^{n(s)\nu},\quad \hat{\bar{e}}^{-1}(\hat{x})_{n(r)\mu}{}^{n(s)\nu}=-\hat{\Omega}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{e}^{-1}(\hat{x})_{n(t)\delta}{}^{n(s)\nu} \tag{6.7c}$$

$$\hat{\Omega}(\hat{x})\equiv\hat{\Omega}(\hat{x}_\sigma,\sigma)\equiv e^{-i\hat{H}(\hat{x})},\quad \hat{H}(\hat{x}(\xi))\equiv\hat{H}(\hat{x}_\sigma(\xi),\sigma)\equiv\hat{x}_\sigma^{n(r)\mu}(\xi)\tilde{\mathcal{T}}_{n(r)\mu}(T^{adj},\sigma) \tag{6.7d}$$

$$\tilde{\mathcal{T}}_{n(r)\mu}(T^{adj},\sigma)_{n(s)\nu}{}^{n(t)\delta}=\chi(\sigma)_{n(s)\nu}\chi(\sigma)^{-1}_{n(t)\delta}\mathcal{T}_{n(r)\mu}(T^{adj},\sigma)_{n(s)\nu}{}^{n(t)\delta}=-i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\sigma) \tag{6.7e}$$

which relates the twisted vielbeins $\hat{e}$, $\hat{\bar{e}}$. The rescaled twisted matrices $\tilde{\mathcal{T}}(T,\sigma)$ were defined in (4.23) and the relation between the twisted adjoint action $\hat{\Omega}$ and the group orbifold element $\hat{g}$ is discussed in Appendix D.

To verify that the twisted currents (6.2) satisfy the general twisted equal-time current algebra (5.10), one uses (6.5a), (6.5b), (6.7c) and the twisted Cartan-Maurer and inverse twisted Cartan-Maurer relations

$$\hat{\partial}_{n(r)\mu}\hat{e}(\hat{x})_{n(s)\nu}{}^{n(t)\delta}-\hat{\partial}_{n(s)\nu}\hat{e}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}=\hat{e}(\hat{x})_{n(r)\mu}{}^{n(u)\epsilon}\hat{e}(\hat{x})_{n(s)\nu}{}^{n(v)\zeta}\mathcal{F}_{n(u)\epsilon;n(v)\zeta}{}^{n(t)\delta}(\sigma) \tag{6.8a}$$

$$\begin{aligned}\hat{e}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{\partial}_{n(t)\delta}\hat{e}^{-1}(\hat{x})_{n(s)\nu}{}^{n(u)\epsilon}-\hat{e}^{-1}(\hat{x})_{n(s)\nu}{}^{n(t)\delta}\hat{\partial}_{n(t)\delta}\hat{e}^{-1}(\hat{x})_{n(r)\mu}{}^{n(u)\epsilon}\\ =\mathcal{F}_{n(s)\nu;n(r)\mu}{}^{n(t)\delta}(\sigma)\hat{e}^{-1}(\hat{x})_{n(t)\delta}{}^{n(u)\epsilon}\end{aligned} \tag{6.8b}$$

which also have the same form for $\hat{e}\longrightarrow\hat{\bar{e}}$.

Finally, we give the explicit forms of the twisted vielbein $\hat{e}$ and the twisted $B$ field $\hat{B}$

$$\hat{e}(\hat{x})_{n(r)\mu}{}^{n(s)\nu}=\left(\tfrac{e^{i\hat{H}}-1}{i\hat{H}}\right)_{n(r)\mu}{}^{n(s)\nu},\quad \hat{e}(0)_{n(r)\mu}{}^{n(s)\nu}=\delta_{n(r)\mu}{}^{n(s)\nu} \tag{6.9a}$$

$$\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x})=\left(\tfrac{e^{i\hat{H}}-e^{-i\hat{H}}-2i\hat{H}}{(i\hat{H})^2}\right)_{n(r)\mu}{}^{n(t)\delta}\mathcal{G}_{n(t)\delta;n(s)\nu}(\sigma) \tag{6.9b}$$

which follow from the definitions above in the preferred coordinate system (6.1).

## 6.2 Twisted Sigma Model Form of the WZW Orbifold Action

Starting from the orbifold Hamiltonian (5.16a) and the canonical representation (6.2) of the twisted currents, we find the canonical form of the orbifold Hamiltonian

$$\hat{H}_\sigma=\int d\xi\left(2\pi\hat{G}^{n(r)\mu;n(s)\nu}(\hat{x})\hat{p}_{n(r)\mu}(\hat{B})\hat{p}_{n(s)\nu}(\hat{B})+\frac{1}{8\pi}\partial_\xi\hat{x}^{n(r)\mu}\partial_\xi\hat{x}^{n(s)\nu}\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x})\right) \tag{6.10a}$$

$$\begin{aligned}\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi))&\equiv\hat{e}(\hat{x}(\xi))_{n(r)\mu}{}^{n(t)\delta}\hat{e}(\hat{x}(\xi))_{n(s)\nu}{}^{n(u)\epsilon}\mathcal{G}_{n(t)\delta;n(u)\epsilon}(\sigma)\\ &=\hat{e}(\hat{x}(\xi))_{n(r)\mu}{}^{n(t)\delta}\hat{e}(\hat{x}(\xi))_{n(s)\nu}{}^{-n(t)\epsilon}\mathcal{G}_{n(t)\delta;-n(t),\epsilon}(\sigma)\end{aligned} \tag{6.10b}$$

$$\hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}(\xi))=\mathcal{G}^{n(t)\delta;n(u)\epsilon}(\sigma)\hat{e}^{-1}(\hat{x}(\xi))_{n(t)\delta}{}^{n(r)\mu}\hat{e}^{-1}(\hat{x}(\xi))_{n(u)\epsilon}{}^{n(s)\nu} \tag{6.10c}$$

$$\hat{G}_{n(r)\mu;n(t)\delta}(\hat{x}(\xi))\hat{G}^{n(t)\delta;n(s)\nu}(\hat{x}(\xi))=\delta_{n(r)\mu}{}^{n(s)\nu} \tag{6.10d}$$

$$\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi+2\pi))=e^{-2\pi i\frac{n(r)+n(s)}{\rho(\sigma)}}\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi)) \tag{6.10e}$$

$$\hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}(\xi+2\pi))=\hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}(\xi))e^{+2\pi i\frac{n(r)+n(s)}{\rho(\sigma)}} \tag{6.10f}$$

$$\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x})\equiv\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}_\sigma,\sigma),\quad \hat{G}^{n(r)\mu;n(s)\nu}(\hat{x})\equiv\hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}_\sigma,\sigma) \tag{6.10g}$$

where $\mathcal{G}_\bullet(\sigma)$ is the twisted tangent space metric in (4.5), $\hat{G}_\bullet(\hat{x})$ is the twisted Einstein metric and $\hat{G}^\bullet$ is the inverse twisted Einstein metric. The further results

$$\partial_t \hat{x}^{n(r)\mu} = 4\pi \hat{G}^{n(r)\mu;n(s)\nu}(\hat{x})\hat{p}_{n(s)\nu}(\hat{B}), \quad \hat{p}_{n(r)\mu}(\hat{B}) = \frac{1}{4\pi}\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x})\partial_t \hat{x}^{n(s)\nu} \tag{6.11a}$$

$$\hat{J}_{n(r)\mu}(\xi) = \frac{1}{2}\partial_+ \hat{x}^{n(t)\delta}(\xi)\hat{e}(\hat{x}(\xi))_{n(t)\delta}{}^{n(s)\nu}\mathcal{G}_{n(s)\nu;n(r)\mu}(\sigma) \tag{6.11b}$$

$$\hat{\hat{J}}_{n(r)\mu}(\xi) = \frac{1}{2}\partial_- \hat{x}^{n(t)\delta}(\xi)\hat{\hat{e}}(\hat{x}(\xi))_{n(t)\delta}{}^{n(s)\nu}\mathcal{G}_{n(s)\nu;n(r)\mu}(\sigma) \tag{6.11c}$$

then follow from the orbifold Hamiltonian equations of motion.

Finally, we may eliminate the twisted momenta $\hat{p}$ in favor of the twisted time derivatives $\partial_t \hat{x}$ to obtain the twisted sigma model form of the WZW orbifold action

$$\hat{S}_\sigma = \hat{S}_\sigma(\mathcal{T}) = \int d^2\xi\, \hat{\mathcal{L}}_\sigma(\hat{x}(\xi)), \quad \sigma = 0, \ldots, N_c - 1 \tag{6.12a}$$

$$\hat{\mathcal{L}}_\sigma(\hat{x}(\xi)) = \frac{1}{8\pi}\big(\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}) + \hat{B}_{n(r)\mu;n(s)\nu}(\hat{x})\big)\partial_+ \hat{x}^{n(r)\mu}\partial_- \hat{x}^{n(s)\nu} \tag{6.12b}$$

$$\hat{\mathcal{L}}_\sigma(\hat{x}(\xi + 2\pi)) = \hat{\mathcal{L}}_\sigma(\hat{x}(\xi)) \ . \tag{6.12c}$$

The trivial monodromy (6.12c) of the Lagrange density $\hat{\mathcal{L}}_\sigma$ follows from the monodromies of $\hat{x}$, $\hat{G}$ and $\hat{B}$ in (6.2), (6.6) and (6.10).

The twisted sigma model form (6.12) of the WZW orbifold action is equal to the form (5.18) in terms of the group orbifold elements. Equality of the kinetic terms follow from (6.3b) and (4.17d), and one needs the orbifold "Gauss' law"

$$\widehat{Tr}(\mathcal{M}\,(\hat{g}^{-1}d\hat{g})^3) = d^2\xi d\rho\, \partial_A j^A, \quad j^A \equiv \tfrac{3}{2}\epsilon^{ABC}\partial_B \hat{x}^{n(r)\mu}\partial_C \hat{x}^{n(s)\nu}\hat{B}_{n(r)\mu;n(s)\nu} \tag{6.13a}$$

$$\int_\Gamma \widehat{Tr}(\mathcal{M}\,(\hat{g}^{-1}d\hat{g})^3) = -\frac{3}{2}\int d^2\xi\, \hat{B}_{n(r)\mu;n(s)\nu}\partial_+ \hat{x}^{n(r)\mu}\partial_- \hat{x}^{n(s)\nu} \tag{6.13b}$$

to see that the $\hat{B}$ term of (6.12) is equal to the twisted WZW term in (5.18).

## 6.3 Orbifolds of the Sigma Model with a Linear Symmetry

As we shall see, the geometric results of Subsecs. 6.1 and 6.2 can also be derived by local isomorphisms. We begin this discussion more generally however by studying the orbifolds of nonlinear sigma models with a certain type of symmetry which we call linear.

Suppose we are given a nonlinear sigma model on a manifold $M$ with a set of local[b] coordinates $\{x^i\}$, and that the sigma model has a symmetry group $H$ which acts linearly on the coordinates as follows:

$$S_{NLS} = \int d^2\xi\, \mathcal{L}_{NLS}(x), \quad \mathcal{L}_{NLS}(x) = \frac{1}{8\pi\alpha'}(G_{ij}(x) + B_{ij}(x))\partial_+ x^i \partial_- x^j \tag{6.14a}$$

$$x^{i'} = x^i (w^\dagger)_i{}^j, \quad \partial_i (w^\dagger)_j{}^k = 0 \tag{6.14b}$$

[b] A complete treatment of the nonlinear sigma model should include discussion of global issues, but we will not do so here.

$$G_{ij}(x)' \equiv G_{ij}(x') = w_i{}^l w_j{}^k G_{lk}(x), \quad B_{ij}(x)' \equiv B_{ij}(x') = w_i{}^l w_j{}^k B_{lk}(x) \tag{6.14c}$$

$$\mathcal{L}_{NLS}(x)' = \mathcal{L}_{NLS}(x') = \mathcal{L}_{NLS}(x) \tag{6.14d}$$

$$w \equiv w(h_\sigma),\ \forall h_\sigma \in H \quad ; \quad i,j = 1,...,\dim M\ . \tag{6.14e}$$

Linearity of the symmetry transformations can be maintained only in certain preferred coordinate systems: In conventional language, we are considering the special case of a symmetry group $H$ with one fixed point of $M$ in each sector $\sigma$, and the preferred coordinate systems are those in which the fixed points are at the origin for all $\sigma$. This restriction defines a broad class of symmetric sigma models including (see Subsec. 6.4) the WZW models $A_g(H)$, the $H$-invariant principal chiral models and presumably the general $H$-invariant coset construction[5,2].

The $H$-eigenvalue problem for this setup has the form

$$w(h_\sigma)_i{}^j U^\dagger(\sigma)_j{}^{n(r)\mu} = U^\dagger(\sigma)_i{}^{n(r)\mu} E_{n(r)}(\sigma),\ E_{n(r)}(\sigma) = e^{-2\pi i \frac{n(r)}{\rho(\sigma)}},\ \sigma = 0,\ldots,N_c-1 \tag{6.15}$$

in analogy with the tangent space $H$-eigenvalue problem in (3.13). The relevant eigen-objects and their responses to the symmetry group are:

$$\mathcal{X}^{n(r)\mu}(x) \equiv \mathcal{X}^{n(r)\mu}_\sigma(x) \equiv \chi(\sigma)^{-1}_{n(r)\mu} x^i U^\dagger(\sigma)_i{}^{n(r)\mu}, \quad x^i(\mathcal{X}) = \chi(\sigma)_{n(r)\mu} \mathcal{X}^{n(r)\mu}_\sigma(x) U(\sigma)_{n(r)\mu}{}^i \tag{6.16a}$$

$$\mathcal{G}_{n(r)\mu;n(s)\nu}(\mathcal{X}) \equiv \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu} U(\sigma)_{n(r)\mu}{}^i U(\sigma)_{n(s)\nu}{}^j G_{ij}(x(\mathcal{X})) \tag{6.16b}$$

$$\mathcal{B}_{n(r)\mu;n(s)\nu}(\mathcal{X}) \equiv \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu} U(\sigma)_{n(r)\mu}{}^i U(\sigma)_{n(s)\nu}{}^j B_{ij}(x(\mathcal{X})) \tag{6.16c}$$

$$\mathcal{L}_{NLS}(\mathcal{X}) \equiv \mathcal{L}_{NLS}(\mathcal{X}(x)) = \frac{1}{8\pi\alpha'}(\mathcal{G}_{n(r)\mu;n(s)\nu}(\mathcal{X}) + \mathcal{B}_{n(r)\mu;n(s)\nu}(\mathcal{X}))\partial_+ \mathcal{X}^{n(r)\mu}\partial_- \mathcal{X}^{n(s)\nu} \tag{6.16d}$$

$$\mathcal{X}^{n(r)\mu}(x)' = \mathcal{X}^{n(r)\mu}(x') = \mathcal{X}^{n(r)\mu}(x) E_{n(r)}(\sigma)^* \tag{6.16e}$$

$$\mathcal{G}_{n(r)\mu;n(s)\nu}(\mathcal{X})' = \mathcal{G}_{n(r)\mu;n(s)\nu}(\mathcal{X}(x')) = E_{n(r)}(\sigma) E_{n(s)}(\sigma) \mathcal{G}_{n(r)\mu;n(s)\nu}(\mathcal{X}) \tag{6.16f}$$

$$\mathcal{B}_{n(r)\mu;n(s)\nu}(\mathcal{X})' = \mathcal{B}_{n(r)\mu;n(s)\nu}(\mathcal{X}(x')) = E_{n(r)}(\sigma) E_{n(s)}(\sigma) \mathcal{B}_{n(r)\mu;n(s)\nu}(\mathcal{X}) \tag{6.16g}$$

$$\mathcal{L}_{NLS}(\mathcal{X})' = \mathcal{L}_{NLS}(\mathcal{X}(x')) = \mathcal{L}_{NLS}(\mathcal{X})\ . \tag{6.16h}$$

Finally, we use the local isomorphisms

$$\mathcal{X} \underset{\sigma}{\longrightarrow} \hat{x}, \quad \mathcal{G}(\mathcal{X}) \underset{\sigma}{\longrightarrow} \hat{G}(\hat{x}), \quad \mathcal{B}(\mathcal{X}) \underset{\sigma}{\longrightarrow} \hat{B}(\hat{x}), \quad \mathcal{L}_{NLS}(\mathcal{X}) \underset{\sigma}{\longrightarrow} \hat{\mathcal{L}}^{NLS}_\sigma(\hat{x}) \tag{6.17a}$$

$$\text{automorphism response } E_{n(r)}(\sigma) \underset{\sigma}{\longrightarrow} \text{ monodromy } E_{n(r)}(\sigma) \tag{6.17b}$$

to obtain the action for twisted sector $\sigma$ of the sigma model orbifolds:

$$\hat{S}^{NLS}_\sigma = \int d^2\xi\, \hat{\mathcal{L}}^{NLS}_\sigma(\hat{x}), \quad \sigma = 0,\ldots,N_c-1 \tag{6.18a}$$

$$\hat{\mathcal{L}}^{NLS}_\sigma(\hat{x}) = \frac{1}{8\pi\alpha'}(\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}) + \hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}))\partial_+ \hat{x}^{n(r)\mu}\partial_- \hat{x}^{n(s)\nu} \tag{6.18b}$$

$$\hat{x}^{n(r)\mu}(\xi + 2\pi) = \hat{x}^{n(r)\mu}(\xi) e^{2\pi i \frac{n(r)}{\rho(\sigma)}} \tag{6.18c}$$

$$\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i \frac{n(r)+n(s)}{\rho(\sigma)}} \hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi)) \tag{6.18d}$$

$$\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i \frac{n(r)+n(s)}{\rho(\sigma)}} \hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi)) \tag{6.18e}$$

$$\hat{\mathcal{L}}_\sigma^{NLS}(\hat{x}(\xi+2\pi)) = \hat{\mathcal{L}}_\sigma^{NLS}(\hat{x}(\xi)) \tag{6.18f}$$

$$\hat{x} \equiv \hat{x}_\sigma(\xi), \quad \hat{G}(\hat{x}) \equiv \hat{G}(\hat{x}_\sigma(\xi),\sigma), \quad \hat{B}(\hat{x}) \equiv \hat{B}(\hat{x}_\sigma(\xi),\sigma)\ . \tag{6.18g}$$

Here $\hat{x}$, $\hat{G}$ and $\hat{B}$ are respectively the twisted Einstein coordinates, the twisted Einstein metric and the twisted $B$ field of sector $\sigma$.

The local isomorphisms also give the explicit forms of the twisted Einstein tensors,

$$x^i = \chi(\sigma)_{n(r)\mu}\mathcal{X}_\sigma^{n(r)\mu}U(\sigma)_{n(r)\mu}{}^i \underset{\sigma}{\longrightarrow} \hat{\mathcal{X}}^i(\hat{x}) \equiv \hat{\mathcal{X}}_\sigma^i(\hat{x}_\sigma) \equiv \chi(\sigma)_{n(r)\mu}\hat{x}_\sigma^{n(r)\mu}U(\sigma)_{n(r)\mu}{}^i \tag{6.19a}$$

$$\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}) = \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu}U(\sigma)_{n(r)\mu}{}^iU(\sigma)_{n(s)\nu}{}^jG_{ij}(x\underset{\sigma}{\rightarrow}\hat{\mathcal{X}}(\hat{x})) \tag{6.19b}$$

$$\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}) = \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu}U(\sigma)_{n(r)\mu}{}^iU(\sigma)_{n(s)\nu}{}^jB_{ij}(x\underset{\sigma}{\rightarrow}\hat{\mathcal{X}}(\hat{x})) \tag{6.19c}$$

in terms of the untwisted Einstein tensors $G_{ij}(x)$ and $B_{ij}(x)$. The objects $\hat{\mathcal{X}}(\hat{x})$ are identified in Appendix C as Einstein coordinates with twisted boundary conditions. Note that we do not find any selection rules for $\hat{G}$ or $\hat{B}$ at this stage because the relations (6.14c), which involve both $x$ and $x'$, are not homogeneous in $G_{ij}(x)$ or $B_{ij}(x)$.

In fact, however, selection rules do exist for the moments of the twisted tensors. To show this, we begin with the moment expansions of the tensors and the twisted tensors

$$G_{ij}(x) = \sum_{m=0}^{\infty} g_{ij,i_1\ldots i_m}x^{i_1}\cdots x^{i_m}, \quad B_{ij}(x) = \sum_{m=0}^{\infty} b_{ij,i_1\ldots i_m}x^{i_1}\cdots x^{i_m} \tag{6.20a}$$

$$\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}) = \sum_{m=0}^{\infty} \mathcal{G}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1\ldots n(r_m)\mu_m}\hat{x}^{n(r_1)\mu_1}\cdots\hat{x}^{n(r_m)\mu_m} \tag{6.20b}$$

$$\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x}) = \sum_{m=0}^{\infty} \mathcal{B}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1\ldots n(r_m)\mu_m}\hat{x}^{n(r_1)\mu_1}\cdots\hat{x}^{n(r_m)\mu_m}\ . \tag{6.20c}$$

Then substitution of the moment expansions into (6.19b) and (6.19c) give an arbitrarily large set of duality transformations

$$g_{\ldots} \underset{\sigma}{\longrightarrow} \mathcal{G}_{\ldots}(g_{\ldots},\sigma), \quad b_{\ldots} \underset{\sigma}{\longrightarrow} \mathcal{B}_{\ldots}(b_{\ldots},\sigma) \tag{6.21a}$$

$$\mathcal{G}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1\ldots n(r_m)\mu_m} \equiv \chi_{n(r)\mu}\chi_{n(s)\nu}U_{n(r)\mu}{}^iU_{n(s)\nu}{}^j\left(\prod_{k=1}^{m}\chi_{n(r_k)\mu_k}U_{n(r_k)\mu_k}{}^{i_k}\right)g_{ij,i_1\ldots i_m} \tag{6.21b}$$

$$\mathcal{B}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1\ldots n(r_m)\mu_m} \equiv \chi_{n(r)\mu}\chi_{n(s)\nu}U_{n(r)\mu}{}^iU_{n(s)\nu}{}^j\left(\prod_{k=1}^{m}\chi_{n(r_k)\mu_k}U_{n(r_k)\mu_k}{}^{i_k}\right)b_{ij,i_1\ldots i_m} \tag{6.21c}$$

$$\chi \equiv \chi(\sigma), \quad U \equiv U(\sigma) \tag{6.21d}$$

which express the twisted moments $\mathcal{G}_{\ldots}$ and $\mathcal{B}_{\ldots}$ of $\hat{G}$ and $\hat{B}$ in terms of the untwisted moments $g_{\ldots}$ and $b_{\ldots}$.

Next, we need the symmetry transformations of the untwisted moments

$$(w^\dagger)_i{}^k(w^\dagger)_j{}^l(w^\dagger)_{i_1}{}^{j_1}\cdots(w^\dagger)_{i_m}{}^{j_m}g_{kl,j_1\ldots j_m} = g_{ij,i_1\ldots i_m} \tag{6.22a}$$

$$(w^\dagger)_i{}^k(w^\dagger)_j{}^l(w^\dagger)_{i_1}{}^{j_1}\cdots(w^\dagger)_{i_m}{}^{j_m}b_{kl,j_1\ldots j_m} = b_{ij,i_1\ldots i_m} \tag{6.22b}$$

which follow from the transformations (6.14c) of the untwisted tensors. It follows from the duality transformations in (6.21) that the selection rules for the twisted moments

$$\mathcal{G}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1,\ldots,n(r_m)\mu_m}(1 - E^*_{n(r)}E^*_{n(s)}E^*_{n(r_1)}\cdots E^*_{n(r_m)}) = 0 \tag{6.23a}$$

$$\mathcal{B}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1,\ldots,n(r_m)\mu_m}(1 - E^*_{n(r)}E^*_{n(s)}E^*_{n(r_1)}\cdots E^*_{n(r_m)}) = 0 \tag{6.23b}$$

$$\mathcal{G}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1,\ldots,n(r_m)\mu_m} \propto \delta_{n(r)+n(s)+\sum_{j=1}^m n(r_j),0 \bmod \rho(\sigma)} \tag{6.23c}$$

$$\mathcal{B}_{n(r)\mu,n(s)\nu,n(r_1)\mu_1,\ldots,n(r_m)\mu_m} \propto \delta_{n(r)+n(s)+\sum_{j=1}^m n(r_j),0 \bmod \rho(\sigma)} \tag{6.23d}$$

are dual to (6.22). These selection rules guarantee[c] the correct monodromies (6.10e) and (6.6a) of $\hat{G}(\hat{x}(\xi))$ and $\hat{B}(\hat{x}(\xi))$.

The twisted canonical brackets of the sigma model orbifolds

$$\{x^i(\xi),\ p_j(\eta)\} = i\delta(\xi-\eta)\delta_j{}^i \tag{6.24a}$$

$$x^i(\xi)' = x^j(\xi)(w^\dagger)_j{}^i, \quad p_i(\xi)' \equiv w_i{}^j p_j(\xi) \tag{6.24b}$$

$$\mathcal{P}_{n(r)\mu}(\xi) \equiv \chi(\sigma)_{n(r)\mu}U_{n(r)\mu}{}^i p_i(\xi), \quad \mathcal{P}_{n(r)\mu}(\xi)' = E_{n(r)}\mathcal{P}_{n(r)\mu}(\xi) \tag{6.24c}$$

$$\{\mathcal{X}^{n(r)\mu}(\xi),\ \mathcal{P}_{n(s)\nu}(\eta))\} = i\delta(\xi-\eta)\delta_{n(s)\nu}{}^{n(r)\mu} \tag{6.24d}$$

$$\mathcal{X} \underset{\sigma}{\longrightarrow} \hat{x}, \quad \mathcal{P} \underset{\sigma}{\longrightarrow} \hat{p} \tag{6.24e}$$

$$\{\hat{x}^{n(r)\mu}(\xi),\ \hat{p}_{n(s)\nu}(\eta)\} = i\delta(\xi-\eta)\delta_{n(s)\nu}{}^{n(r)\mu} \tag{6.24f}$$

are also obtained by the addition of the local isomorphism for the Einstein momenta in (6.24e). One may also consider the twisted tangent space formulation of the sigma model orbifolds, starting from the untwisted tangent space formulation in Ref. 17.

The sigma model orbifolds described here are in general not conformal at the quantum level. To select those orbifolds which are 1-loop conformal, one will presumably need to consider the orbifold Einstein equations which we present in Subsec. 8.3.

## 6.4 Local Isomorphisms and WZW Orbifold Geometry

In this subsection, we continue the sigma model discussion above in the special case of WZW. Our emphasis here is that all the geometric results of Subsecs. 6.1 and 6.2 can be obtained by local isomorphisms.

We begin by recalling that the WZW tangent space coordinates $\beta$ have the automorphic response $\beta' = \beta w^\dagger$ (see Eq. (3.8c)). So for the special case of the WZW model in sigma model form, it is natural to choose the preferred coordinate system $x = \beta$

$$x^i = \beta^a e(0)_a{}^i, \quad e(0)_i{}^a = \delta_i{}^a, \quad x^{i\prime} = x^j w^\dagger(h_\sigma)_j{}^i, \quad i,a = 1,\ldots,\dim g \tag{6.25a}$$

$$w(h_\sigma)_i{}^j = e(0)_i{}^a w(h_\sigma)_a{}^b e(0)_b{}^j \tag{6.25b}$$

$$U^\dagger(\sigma)_i{}^{n(r)\mu} = e(0)_i{}^a U^\dagger(\sigma)_a{}^{n(r)\mu}, \quad U(\sigma)_{n(r)\mu}{}^i = U(\sigma)_{n(r)\mu}{}^a e(0)_a{}^i \tag{6.25c}$$

[c]Looking back, we see that this structure is analogous to that seen in the all-order equivalence (5.8) for the group orbifold elements $\hat{g}$. Indeed one finds that the twisted moments of $\hat{g}$ in powers of $\hat{\beta}$ satisfy selection rules which are powers of the selection rules (4.18) for $\mathcal{T}$, and moreover that these twisted moment selection rules (plus the monodromy of $\hat{\beta}$) guarantee the monodromy of $\hat{g}$. The bottom line is that selection rules are satisfied only by constant (coordinate-independent) twisted tensors.

in which $e(0) = 1$ is the left-invariant vielbein at the origin (see below), the $H$-symmetry acts linearly on the Einstein coordinates and the (Einstein) $H$-eigenvalue problem (6.15) is isomorphic to the (tangent space) $H$-eigenvalue problem (3.13). Then the analysis of the previous subsection gives the twisted sigma model form (6.12) of the WZW orbifold action.

As another example, we begin with the untwisted left and right invariant WZW vielbeins

$$e_i(x,T) = -ig^{-1}(T)\partial_i g(T) = e(x)_i{}^a T_a, \quad \bar{e}_i(x,T) = -ig(T)\partial_i g^{-1}(T) = \bar{e}(x)_i{}^a T_a \tag{6.26a}$$

$$g(T) \equiv g(T,x) = e^{iH(T,x)}, \quad H(T,x) \equiv x^i e_i{}^a(0) T_a \tag{6.26b}$$

$$e(x)_i{}^a = \left(\tfrac{e^{iH(x)}-1}{iH(x)}\right)_i{}^a, \quad H(x) \equiv x^i e_i{}^a(0) T_a^{adj}, \quad e_i{}^a(0) = \delta_i{}^a \tag{6.26c}$$

in the preferred coordinate system. The symmetry transformations of the vielbeins $e$, $\bar{e}$ and the eigenvielbeins $\mathcal{E}$, $\bar{\mathcal{E}}$ are

$$e_i(x,T)' = e_i(x',T) = -\,ig^{-1}(T)'\partial_i' g(T)' = w_i{}^j W(h_\sigma;T) e_j(x,T) W^\dagger(h_\sigma;T) \tag{6.27a}$$

$$\bar{e}_i(x,T)' = \bar{e}_i(x',T) = -\,ig(T)'\partial_i' g^{-1}(T)' = w_i{}^j W(h_\sigma;T) \bar{e}_j(x,T) W^\dagger(h_\sigma;T) \tag{6.27b}$$

$$(e(x)')_i{}^a = e(x')_i{}^a = w_i{}^j e_j{}^b (w^\dagger)_b{}^a, \quad (\bar{e}(x)')_i{}^a = \bar{e}(x')_i{}^a = w_i{}^j \bar{e}_j{}^b (w^\dagger)_b{}^a \tag{6.27c}$$

$$\mathcal{E}(\mathcal{X})_{n(r)\mu}{}^{n(s)\nu} \equiv \chi_{n(r)\mu}\chi^{-1}_{n(s)\nu} U_{n(r)\mu}{}^i e(x(\mathcal{X}))_i{}^a (U^\dagger)_a{}^{n(s)\nu}, \quad \mathcal{E}(\mathcal{X}) = \chi U e(x(\mathcal{X})) U^\dagger \chi^{-1} \tag{6.27d}$$

$$\bar{\mathcal{E}}(\mathcal{X})_{n(r)\mu}{}^{n(s)\nu} \equiv \chi_{n(r)\mu}\chi^{-1}_{n(s)\nu} U_{n(r)\mu}{}^i \bar{e}(x(\mathcal{X}))_i{}^a (U^\dagger)_a{}^{n(s)\nu}, \quad \bar{\mathcal{E}}(\mathcal{X}) = \chi U \bar{e}(x(\mathcal{X})) U^\dagger \chi^{-1} \tag{6.27e}$$

$$(\mathcal{E}(\mathcal{X})')_{n(r)\mu}{}^{n(s)\nu} = \mathcal{E}(\mathcal{X}(x'))_{n(r)\mu}{}^{n(s)\nu} = E_{n(r)} \mathcal{E}(\mathcal{X})_{n(r)\mu}{}^{n(s)\nu} E^*_{n(s)} \tag{6.27f}$$

$$(\bar{\mathcal{E}}(\mathcal{X})')_{n(r)\mu}{}^{n(s)\nu} = \bar{\mathcal{E}}(\mathcal{X}(x'))_{n(r)\mu}{}^{n(s)\nu} = E_{n(r)} \bar{\mathcal{E}}(\mathcal{X})_{n(r)\mu}{}^{n(s)\nu} E^*_{n(s)} \tag{6.27g}$$

$$-i\mathcal{G}^{-1}(\mathcal{T},\xi)\partial_{n(r)\mu}\mathcal{G}(\mathcal{T},\xi) = \mathcal{E}(\mathcal{X})_{n(r)\mu}{}^{n(s)\nu} \mathcal{T}_{n(s)\nu}, \quad \partial_{n(r)\mu} \equiv \frac{\partial}{\partial \mathcal{X}^{n(r)\mu}} \tag{6.27h}$$

$$-i\mathcal{G}(\mathcal{T},\xi)\partial_{n(r)\mu}\mathcal{G}^{-1}(\mathcal{T},\xi) = \bar{\mathcal{E}}(\mathcal{X})_{n(r)\mu}{}^{n(s)\nu} \mathcal{T}_{n(s)\nu}\,. \tag{6.27i}$$

The eigencoordinate $\mathcal{X}$ is defined in (6.16) and we have suppressed all sector dependence in these relations. Then the local isomorphisms

$$\mathcal{X} \underset{\sigma}{\longrightarrow} \hat{x}, \quad \mathcal{E}(\mathcal{X}) \underset{\sigma}{\longrightarrow} \hat{e}(\hat{x}), \quad \bar{\mathcal{E}}(\mathcal{X}) \underset{\sigma}{\longrightarrow} \hat{\bar{e}}(\hat{x}) \tag{6.28a}$$

$$\text{automorphism response } E_{n(r)}(\sigma) \underset{\sigma}{\longrightarrow} \text{ monodromy } E_{n(r)}(\sigma) \tag{6.28b}$$

give the relations (6.3b), (6.4a), the twisted vielbein monodromies (6.3e) and (6.4d) and the explicit form of the twisted vielbein

$$\hat{e}(\hat{x})_{n(r)\mu}{}^{n(s)\nu} = \chi_{n(r)\mu}\chi^{-1}_{n(s)\nu} U_{n(r)\mu}{}^i\, e(x \underset{\sigma}{\rightarrow} \hat{\mathcal{X}}(\hat{x}))_i{}^a (U^\dagger)_a{}^{n(s)\nu}, \quad \hat{e}(\hat{x}) = \chi U e(x \underset{\sigma}{\rightarrow} \hat{\mathcal{X}}(\hat{x})) U^\dagger \chi^{-1} \tag{6.29}$$

in terms of the untwisted vielbein $e_i{}^a(x)$. The objects $\hat{\mathcal{X}}(\hat{x})$ are defined in (6.19a). The same result holds for $\hat{\bar{e}}(\hat{x})$ with $e \longrightarrow \bar{e}$.

Following the previous subsection, we introduce the moment expansions of the untwisted vielbein $e$ and the twisted vielbein $\hat{e}$ to obtain

$$e(x)_i{}^a = \sum_{m=0}^{\infty} e_{i,i_1\ldots i_m}{}^a x^{i_1} \cdots x^{i_m} \tag{6.30a}$$

$$(w^\dagger)_i{}^j(w^\dagger)_{i_1}{}^{j_1}\cdots(w^\dagger)_{i_m}{}^{j_m}e_{j,j_1\ldots j_m}{}^b w_b{}^a = e_{i,i_1\ldots i_m}{}^a \tag{6.30b}$$

$$\hat{e}(\hat{x})_{n(r)\mu}{}^{n(s)\nu} = \sum_{m=0}^{\infty}\mathcal{E}_{n(r)\mu,n(r_1)\mu_1\ldots n(r_m)\mu_m}{}^{n(s)\nu}\hat{x}^{n(r_1)\mu_1}\ldots\hat{x}^{n(r_m)\mu_m} \tag{6.30c}$$

$$\mathcal{E}_{n(r)\mu,n(r_1)\mu_1\ldots n(r_m)\mu_m}{}^{n(s)\nu}=\chi_{n(r)\mu}\chi^{-1}_{n(s)\nu}U_{n(r)\mu}{}^i\left(\prod_{j=1}^{m}\chi_{n(r_j)\mu_j}U_{n(r_j)\mu_j}{}^{i_j}\right)e_{i,i_1\ldots i_m}{}^a(U^\dagger)_a{}^{n(s)\nu} \tag{6.30d}$$

$$\mathcal{E}_{n(r)\mu,n(r_1)\mu_1\ldots n(r_m)\mu_m}{}^{n(s)\nu}(1-E^*_{n(r)}E^*_{n(r_1)}\cdots E^*_{n(r_m)}E_{n(s)}) = 0 \tag{6.30e}$$

$$\mathcal{E}_{n(r)\mu,n(r_1)\mu_1\ldots n(r_m)\mu_m}{}^{n(s)\nu}=\delta_{n(r)-n(s)+\sum_{i=1}^m n(r_i),0\bmod\rho(\sigma)}\mathcal{E}_{n(r)\mu,n(r_1)\mu_1\ldots n(r_m)\mu_m}{}^{n(r)+\sum_{i=1}^m n(r_i)}. \tag{6.30f}$$

Here (6.30d) is an infinite set of duality transformations

$$e_{\ldots}\underset{\sigma}{\longrightarrow}\mathcal{E}_{\ldots}(e_{\ldots},\sigma) \tag{6.31}$$

which express the twisted vielbein moments in terms of the untwisted vielbein moments. The selection rules (6.30e) for the twisted vielbein moments (and their solution in (6.30f)) follow from the duality transformations and the symmetry transformations (6.30b) of the untwisted moments. These selection rules guarantee the monodromy (6.3e) of the twisted vielbein. Similar moment expansions lead to duality transformations and selection rules which guarantee the monodromy of $\hat{\bar{e}}(\hat{x})$.

In particular, the duality transformations (6.30d) imply

$$e(0)_i{}^a = e_i{}^a = \delta_i{}^a \longrightarrow \tag{6.32a}$$

$$\hat{e}(0)_{n(r)\mu}{}^{n(s)\nu}=\mathcal{E}_{n(r)\mu}{}^{n(s)\nu}=\chi_{n(r)\mu}\chi^{-1}_{n(s)\nu}U_{n(r)\mu}{}^i e(0)_i{}^a(U^\dagger)_a{}^{n(s)\nu}=\delta_{n(r)\mu}{}^{n(s)\nu} \tag{6.32b}$$

so that the preferred twisted coordinate system $\hat{x}=\hat{\beta}$ chosen in Subsec. 6.1 is dual to the preferred untwisted coordinate system $x=\beta$ chosen in (6.25). Moreover, Eq.(6.19a) is now recognized as Eq. (5.3a).

Using (6.26c), (6.27d) and (6.29), the eigenvielbein and the twisted vielbein can be evaluated in closed form

$$\mathcal{E}(\mathcal{X}) = \frac{e^{i\mathcal{H}(\mathcal{X})}-1}{i\mathcal{H}(\mathcal{X})},\quad \mathcal{H}(\mathcal{X}) = \chi U H(x(\mathcal{X}))U^\dagger\chi^{-1} = \mathcal{X}^{n(r)\mu}\tilde{\mathcal{T}}_{n(r)\mu}(T^{adj}),\quad \mathcal{E}(0)=1 \tag{6.33a}$$

$$\hat{e}(\hat{x}) = \frac{e^{i\hat{H}(\hat{x})}-1}{i\hat{H}(\hat{x})},\quad \hat{H}(\hat{x}) = \hat{x}^{n(r)\mu}\tilde{\mathcal{T}}_{n(r)\mu}(T^{adj}),\quad \hat{e}(0)=1 \tag{6.33b}$$

in agreement with (6.9).

The bracket realization (6.2) of the general current algebra also follows by the local isomorphisms above from Bowcock's bracket realization[18]

$$J_a(\xi) = 2\pi e(x(\xi))_a{}^i p_i(B,\xi) + \frac{1}{2}\partial_\xi x^i(\xi)e(x(\xi))_i{}^b G_{ba} \tag{6.34a}$$

$$\bar{J}_a(\xi) = 2\pi\bar{e}(x(\xi))_a{}^i p_i(B,\xi) - \frac{1}{2}\partial_\xi x^i(\xi)\bar{e}(x(\xi))_i{}^b G_{ba} \tag{6.34b}$$

$$\{x^i(\xi),p_j(\eta)\} = i\delta(\xi-\eta)\delta_j{}^i,\quad p_i(B,\xi) = p_i(\xi) + \frac{1}{4\pi}B_{ij}(x(\xi))\partial_\xi x^j(\xi) \tag{6.34c}$$

$$\partial_i B_{jk}(x) + \partial_j B_{ki}(x) + \partial_k B_{ij}(x) = -i\,Tr\big(M(k,T)e_i(x,T)[e_j(x,T),e_k(x,T)]\big) \tag{6.34d}$$

in the untwisted theory. To see this, use (6.34) to check first that the eigencurrents $\mathcal{J}_{n(r)\mu}$, $\bar{\mathcal{J}}_{n(r)\mu}$ have the same form when rewritten in terms of $\mathcal{G}(\sigma)$ and the eigen-objects, and then use the local isomorphisms. Similarly, the untwisted WZW relations

$$H_g = \int d\xi (2\pi G^{ab} e(x)_a{}^i e(x)_b{}^j p_i(B) p_j(B) + \frac{1}{8\pi} \partial_\xi x^i \partial_\xi x^j e(x)_i{}^a e(x)_j{}^b G_{ab}) \tag{6.35a}$$

$$J_a(\xi) = \frac{1}{2} \partial_+ x^i(\xi) e(x(\xi))_i{}^b G_{ba}, \quad \bar{J}_a(\xi) = \frac{1}{2} \partial_- x^i(\xi) \bar{e}(x(\xi))_i{}^b G_{ba} \tag{6.35b}$$

give the WZW orbifold results (6.10a) and (6.11).

## 6.5 The Duality Algorithm

All the local orbifold results of this paper have been derived by the method of eigen-objects and local isomorphisms. Looking back however we see that, just as observed in Refs. 1 and 2, all of these results can also be obtained by the *duality algorithm*

$$a \underset{\sigma}{\longrightarrow} n(r)\mu, \quad \alpha \underset{\sigma}{\longrightarrow} N(r)\mu \tag{6.36a}$$

$$G \underset{\sigma}{\longrightarrow} \mathcal{G}(\sigma), \quad f \underset{\sigma}{\longrightarrow} \mathcal{F}(\sigma), \quad T \underset{\sigma}{\longrightarrow} \mathcal{T}(\sigma) \tag{6.36b}$$

$$J, \bar{J} \underset{\sigma}{\longrightarrow} \hat{J}, \hat{\bar{J}}, \quad T, \bar{T} \underset{\sigma}{\longrightarrow} T_\sigma, \bar{T}_\sigma \tag{6.36c}$$

$$g \underset{\sigma}{\longrightarrow} \hat{g}, \quad M \underset{\sigma}{\longrightarrow} \mathcal{M}(\mathcal{T}, \sigma), \quad S_{WZW}(T) \underset{\sigma}{\longrightarrow} \hat{S}_\sigma(\mathcal{T}) \tag{6.36d}$$

$$\beta \underset{\sigma}{\longrightarrow} \hat{\beta}, \quad p \underset{\sigma}{\longrightarrow} \hat{p} \tag{6.36e}$$

$$i \underset{\sigma}{\longrightarrow} n(r)\mu, \quad x \underset{\sigma}{\longrightarrow} \hat{x}, \quad G(x) \underset{\sigma}{\longrightarrow} \hat{G}(\hat{x}) \tag{6.36f}$$

$$B(x) \underset{\sigma}{\longrightarrow} \hat{B}(\hat{x}), \quad H(x) \underset{\sigma}{\longrightarrow} \hat{H}(\hat{x}), \quad S_{NLS} \underset{\sigma}{\longrightarrow} \hat{S}_\sigma^{NLS} \tag{6.36g}$$

$$e(x) \underset{\sigma}{\longrightarrow} \hat{e}(\hat{x}), \quad \bar{e}(x) \underset{\sigma}{\longrightarrow} \hat{\bar{e}}(\hat{x}) \tag{6.36h}$$

from the corresponding local relations in the symmetric theory. The only proviso is that the duality algorithm must be used *before* the selection rules are applied. For example, Eq. (2.5a) and the duality algorithm give Eq. (5.11a) for the left mover stress tensor of the orbifold, and then the selection rules give (5.11b).

The duality algorithm is more than a useful mnemonic: One finds that each step needed to connect any two orbifold results follows by the duality algorithm from the corresponding step needed to connect the corresponding relations in the $H$-symmetric theory. As an example, the reader is invited to compare the steps in the verification of the Virasoro algebra (5.11f) of the orbifold from Eqs. (5.10a) and (5.11a) with the corresponding steps in the derivation of the original Virasoro algebra (2.5c) from Eqs. (2.4a) and (2.5a).

The duality algorithm also holds for the orbifold Einstein equations of Subsec. 8.3.

# 7 WZW Orbifold Examples

For brevity, we concentrate in these examples on the central features of our construction:

- the explicit forms of the action $W(h_\sigma;T)$ in matrix representation $T$
- solution of the extended $H$-eigenvalue problems
- rectification of the twisted right mover current algebras
- the explicit forms of the twisted representation matrices $\mathcal{T}$
- the explicit forms of the WZW orbifold actions, and the separability of the WZW orbifold actions when the twisted current algebras are semisimple.

## 7.1 About WZW Permutation Orbifolds

To study WZW permutation orbifolds $A_g(H)$, $H(\text{permutation}) \subset S_N$ we begin with the current-algebraic formulation of the untwisted permutation-invariant theory,

$$g = \oplus_{I=0}^{K-1}\mathfrak{g}^I, \quad K \leq N, \quad \mathfrak{g}^I \simeq \mathfrak{g}, \quad a \longrightarrow aI, \quad \alpha \longrightarrow \alpha I, \tag{7.1a}$$

$$G_{aI,bJ} = k\eta_{ab}\delta_{IJ}, \quad f_{aI,bJ}{}^{cK} = f_{ab}{}^c\delta_{IJ}\delta_J{}^K, \quad (T_{aI})_{\alpha J}{}^{\beta K} = (T_a)_\alpha{}^\beta\delta_{IJ}\delta_J{}^K \tag{7.1b}$$

$$T(\xi) = \frac{\eta^{ab}}{4\pi k}J_{aI}(\xi)J_{bI}(\xi), \quad \bar{T}(\xi) = \frac{\eta^{ab}}{4\pi k}\bar{J}_{aI}(\xi)\bar{J}_{bI}(\xi) \tag{7.1c}$$

$$Tr(T_aT_b) = y(T)\eta_{ab}, \quad M(k,T)_{\alpha I}{}^{\beta J} = \frac{k}{y(T)}\delta_\alpha{}^\beta\delta_I{}^J \tag{7.1d}$$

$$I = 0,...,K-1, \quad K \leq N, \quad a = 1,...,\dim\mathfrak{g}, \quad \alpha = 1,...,\dim T \tag{7.1e}$$

where each copy $\mathfrak{g}$ is taken at level $k$ with Killing metric $\eta_{ab}$, structure constant $f_{ab}{}^c$ and irrep $T_a$.

At this level, the theory is invariant under the symmetry transformations[2]

$$w(h_\sigma)_{aI}{}^{bJ} = \delta_a{}^b w(h_\sigma)_I{}^J, \quad \forall h_\sigma \in H(permutation) \subset Aut(g) \tag{7.2}$$

where $w(h_\sigma)$ is the action of $H$ in the adjoint. Then we must find $W(h_\sigma;T)$, the corresponding action of $h_\sigma$ in matrix representation $T$. It is straightforward to check that

$$W(h_\sigma;T)_{\alpha I}{}^{\beta J} = \delta_\alpha{}^\beta w(h_\sigma)_I{}^J \tag{7.3}$$

solves the linkage relation (3.6a) so long as

$$w^\dagger(h_\sigma)_I{}^J w(h_\sigma)_J{}^K = w(h_\sigma)_J{}^I\delta_I{}^K, \quad \forall_{I,J,K} \tag{7.4}$$

and these relations indeed hold for all $h_\sigma \in H$ because each $w(h_\sigma)_I{}^J$ is a permutation matrix.

The simple forms (7.2) and (7.3) tell us that, for permutation orbifolds, the extended $H$-eigenvalue problem (3.14) and the $H$-eigenvalue problem (3.13) are both simply related to the solution of the reduced eigenvalue problem for $w(h_\sigma)_I{}^J$

$$n(r)\mu \longrightarrow n(r)aj, \quad N(r)\mu \longrightarrow N(r)\alpha j \tag{7.5a}$$

$$R(T,\sigma)=\rho(\sigma),\quad E(T,\sigma)=E(\sigma),\quad N(r)=n(r) \tag{7.5b}$$

$$U^\dagger(T,\sigma)_{\alpha I}{}^{n(r)\beta j}=\delta_\alpha{}^\beta U^\dagger(\sigma)_I{}^{n(r)j},\quad U^\dagger(\sigma)_{aI}{}^{n(r)bj}=\delta_a{}^b U^\dagger(\sigma)_I^{n(r)j} \tag{7.5c}$$

$$w(h_\sigma)_I{}^J U^\dagger(\sigma)_J{}^{n(r)j}=U^\dagger(\sigma)_I{}^{n(r)j}E_{n(r)}(\sigma),\quad E_{n(r)}(\sigma)=e^{-2\pi i\frac{n(r)}{\rho(\sigma)}} \tag{7.5d}$$

given in (7.5d). The degeneracy indices of the permutation orbifolds are conventionally called $(aj)$. This gives the simplified formulae

$$\begin{aligned}\mathcal{G}_{n(r)aj;n(s)bl}(\sigma)&=k\eta_{ab}\chi(\sigma)_{n(r)aj}\chi(\sigma)_{n(s)bl}\sum_I U(\sigma)_{n(r)j}{}^I U(\sigma)_{n(s)l}{}^I\\&=\delta_{n(r)+n(s),0\bmod\rho(\sigma)}\mathcal{G}_{n(r)aj;-n(r),bl}(\sigma)\end{aligned} \tag{7.6a}$$

$$\begin{aligned}\mathcal{G}^{n(r)aj;n(s)bl}(\sigma)&=\frac{\eta^{ab}}{k}\chi(\sigma)^{-1}_{n(r)aj}\chi(\sigma)^{-1}_{n(s)bl}\sum_I U^\dagger(\sigma)_I{}^{n(r)j}U^\dagger(\sigma)_I{}^{n(s)l}\\&=\delta_{n(r)+n(s),0\bmod\rho(\sigma)}\mathcal{G}^{n(r)aj;-n(r),bl}(\sigma)\end{aligned} \tag{7.6b}$$

$$\begin{aligned}\mathcal{F}_{n(r)aj;n(s)bl}{}^{n(t)cm}(\sigma)&=f_{ab}{}^c\chi(\sigma)_{n(r)aj}\chi(\sigma)_{n(s)bl}\chi(\sigma)^{-1}_{n(t)cm}\sum_I U(\sigma)_{n(r)j}{}^I U(\sigma)_{n(s)l}{}^I U^\dagger(\sigma)_I{}^{n(t)m}\\&=\delta_{n(r)+n(s)-n(t),0\bmod\rho(\sigma)}\mathcal{F}_{n(r)aj;n(s)bl}{}^{n(r)+n(s),cm}(\sigma)\end{aligned} \tag{7.6c}$$

$$\mathcal{T}_{n(r)aj}(T,\sigma)_{n(s)\alpha l}{}^{n(t)\beta m}=(T_a)_\alpha{}^\beta\chi(\sigma)_{n(r)aj}\sum_I U(\sigma)_{n(r)j}{}^I U(\sigma)_{n(s)l}{}^I U^\dagger(\sigma)_I{}^{n(t)m} \tag{7.6d}$$

$$=\delta_{n(r)+n(s)-n(t),0\bmod\rho(\sigma)}\mathcal{T}_{n(r)aj}(T,\sigma)_{n(s)\alpha l}{}^{n(r)+n(s),\beta m} \tag{7.6e}$$

$$\hat{T}_\sigma(\xi)=\frac{1}{4\pi}\mathcal{G}^{n(r)aj;-n(r),bl}(\sigma)\hat{J}_{n(r)aj}(\xi,\sigma)\hat{J}_{-n(r),bl}(\xi,\sigma) \tag{7.6f}$$

$$\hat{\bar{T}}_\sigma(\xi)=\frac{1}{4\pi}\mathcal{G}^{n(r)aj;-n(r),bl}(\sigma)\hat{\bar{J}}_{n(r)aj}(\xi,\sigma)\hat{\bar{J}}_{-n(r),bl}(\xi,\sigma) \tag{7.6g}$$

$$\hat{J}_{n(r)aj}(\xi+2\pi,\sigma)=e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)aj}(\xi,\sigma),\quad \hat{\bar{J}}_{n(r)aj}(\xi+2\pi,\sigma)=e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)aj}(\xi,\sigma) \tag{7.6h}$$

$$\hat{g}(\mathcal{T},\hat{x}_\sigma)=e^{i\hat{x}_\sigma^{n(r)aj}\mathcal{T}_{n(r)aj}(T,\sigma)},\quad \hat{x}_\sigma^{n(r)aj}(\xi+2\pi)=\hat{x}_\sigma^{n(r)aj}(\xi)e^{2\pi i\frac{n(r)}{\rho(\sigma)}}\qquad(\hat{\beta}=\hat{x}) \tag{7.7a}$$

$$\hat{g}(\mathcal{T},\hat{x}_\sigma(\xi+2\pi))_{n(r)\alpha j}{}^{n(s)\beta l}=e^{-2\pi i\frac{n(r)-n(s)}{\rho(\sigma)}}\hat{g}(\mathcal{T},\hat{x}_\sigma(\xi))_{n(r)\alpha j}{}^{n(s)\beta l} \tag{7.7b}$$

$$\mathcal{M}(\mathcal{T},\sigma)_{n(r)aj}{}^{n(s)bl}=\frac{k}{y(T)}\delta_{n(r)aj}{}^{n(s)bl} \tag{7.7c}$$

$$\begin{aligned}\hat{S}_\sigma(\mathcal{T})=&-\frac{k}{y(T)}\Bigg(\frac{1}{8\pi}\int d^2\xi\,\widehat{Tr}\big(\,\hat{g}^{-1}(\mathcal{T},\sigma)\,\partial_+\hat{g}(\mathcal{T},\sigma)\,\hat{g}(\mathcal{T},\sigma)^{-1}\partial_-\hat{g}(\mathcal{T},\sigma)\,\big)\\&-\frac{1}{12\pi}\int_\Gamma\widehat{Tr}\big((\hat{g}(\mathcal{T},\sigma)^{-1}d\hat{g}(\mathcal{T},\sigma))^3\big)\Bigg),\quad \widehat{Tr}(AB)\equiv\sum_{r,s,a,b,j,l}A_{n(r)aj}{}^{n(s)bl}A_{n(s)bl}{}^{n(r)aj}\end{aligned} \tag{7.8a}$$

$$\hat{J}_{n(r)aj}(\xi)=-\frac{i}{2}\frac{k}{y(T)}(\mathcal{T}_{n(r)aj})_{n(s)\alpha l}{}^{n(r)+n(s),\beta m}\left(\hat{g}^{-1}(\mathcal{T},\xi)\partial_+\hat{g}(\mathcal{T},\xi)\right)_{n(r)+n(s),\beta m}{}^{n(s)\alpha l} \tag{7.8b}$$

$$\hat{\bar{J}}_{n(r)aj}(\xi)=-\frac{i}{2}\frac{k}{y(T)}(\mathcal{T}_{n(r)aj})_{n(s)\alpha l}{}^{n(r)+n(s),\beta m}\left(\hat{g}(\mathcal{T},\xi)\partial_-\hat{g}^{-1}(\mathcal{T},\xi)\right)_{n(r)+n(s),\beta m}{}^{n(s)\alpha l} \tag{7.8c}$$

which will be further evaluated in the examples below.

## 7.2 The WZW Cyclic Permutation Orbifolds $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$

For the cyclic permutation group $H = \mathbb{Z}_\lambda$, it is known[1] that

$$n(r) = r, \quad \bar{n}(r) = \bar{r}, \quad E_r(\sigma) = e^{-2\pi i \frac{r}{\rho(\sigma)}} \tag{7.9a}$$

$$U^\dagger(\sigma)_I{}^{rj} = \frac{1}{\sqrt{\rho(\sigma)}} e^{2\pi i \frac{N(\sigma) r (j-I)}{\lambda}} \delta_{j, I \bmod \frac{\lambda}{\rho(\sigma)}}, \quad \chi(\sigma)_{raj} = \sqrt{\rho(\sigma)} \tag{7.9b}$$

$$\mathcal{G}_{raj;sbl}(\sigma) = \rho(\sigma) k \eta_{ab} \delta_{jl} \delta_{r+s, 0 \bmod \rho(\sigma)}, \quad \mathcal{G}^{raj;sbl}(\sigma) = \frac{1}{\rho(\sigma) k} \eta^{ab} \delta^{jl} \delta_{r+s, 0 \bmod \rho(\sigma)}, \tag{7.9c}$$

$$\mathcal{F}_{raj;sbl}{}^{tcm}(\sigma) = f_{ab}{}^c \delta_{jl} \delta_l{}^m \delta_{r+s-t, 0 \bmod \rho(\sigma)} \tag{7.9d}$$

$$a = 1, \ldots, \dim \mathfrak{g}, \quad \bar{r} = 0, \ldots, \rho(\sigma) - 1, \quad j = 0, \ldots, \tfrac{\lambda}{\rho(\sigma)} - 1, \quad \sigma = 0, \ldots, \lambda - 1 \tag{7.9e}$$

where $I = 0, \ldots, \lambda - 1$ and the integers $N(\sigma)$ are defined in Ref. 1. In the standard notation $\hat{J}^{(r)}_{aj} \equiv \hat{J}_{raj}$, $\hat{\bar{J}}^{(r)}_{aj} \equiv \hat{\bar{J}}_{raj}$, this gives the algebraic results for the WZW cyclic permutation orbifolds $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$:

$$\hat{T}_\sigma(\xi) = \frac{1}{4\pi} \frac{\eta^{ab}}{k\rho(\sigma)} \sum_{j=0}^{\frac{\lambda}{\rho(\sigma)}-1} \sum_{r=0}^{\rho(\sigma)-1} \hat{J}^{(r)}_{aj}(\xi, \sigma) \hat{J}^{(-r)}_{bj}(\xi, \sigma) \tag{7.10a}$$

$$\hat{\bar{T}}_\sigma(\xi) = \frac{1}{4\pi} \frac{\eta^{ab}}{k\rho(\sigma)} \sum_{j=0}^{\frac{\lambda}{\rho(\sigma)}-1} \sum_{r=0}^{\rho(\sigma)-1} \hat{\bar{J}}^{(r)}_{aj}(\xi, \sigma) \hat{\bar{J}}^{(-r)}_{bj}(\xi, \sigma) \tag{7.10b}$$

$$\hat{J}^{(r)}_{aj}(\xi + 2\pi, \sigma) = e^{-2\pi i \frac{r}{\rho(\sigma)}} \hat{J}^{(r)}_{aj}(\xi, \sigma), \quad \hat{\bar{J}}^{(r)}_{aj}(\xi + 2\pi, \sigma) = e^{-2\pi i \frac{r}{\rho(\sigma)}} \hat{\bar{J}}^{(r)}_{aj}(\xi, \sigma) \tag{7.10c}$$

$$\hat{J}^{(r)}_{aj}(\xi, \sigma) = \sum_m \hat{J}^{(r)}_{aj}(m + \tfrac{r}{\rho(\sigma)}) e^{-i(m + \frac{r}{\rho(\sigma)})(\xi + t)} \tag{7.10d}$$

$$\hat{\bar{J}}^{(r)}_{aj}(\xi, \sigma) = \sum_m \hat{\bar{J}}^{(r)}_{aj}(m + \tfrac{r}{\rho(\sigma)}) e^{-i(m + \frac{r}{\rho(\sigma)})(\xi - t)} \tag{7.10e}$$

$$\hat{J}^{(r \pm \rho(\sigma))}_{aj}(m \mp 1 + \tfrac{r \pm \rho(\sigma)}{\rho(\sigma)}) = \hat{J}^{(r)}_{aj}(m + \tfrac{n(r)}{\rho(\sigma)}) \tag{7.10f}$$

$$\hat{\bar{J}}^{(r \pm \rho(\sigma))}_{aj}(m \mp 1 + \tfrac{r \pm \rho(\sigma)}{\rho(\sigma)}) = \hat{\bar{J}}^{(r)}_{aj}(m + \tfrac{n(r)}{\rho(\sigma)}) \tag{7.10g}$$

$$\begin{aligned} &\{\hat{J}^{(r)}_{aj}(m + \tfrac{r}{\rho(\sigma)}), \hat{J}^{(s)}_{bl}(n + \tfrac{s}{\rho(\sigma)})\} \\ &= \delta_{jl}\big( i f_{ab}{}^c \hat{J}^{(r+s)}_{cj}(m + n + \tfrac{r+s}{\rho(\sigma)}) + (\rho(\sigma)k)\eta_{ab}(m + \tfrac{r}{\rho(\sigma)}) \delta_{m+n+\frac{r+s}{\rho(\sigma)}, 0} \big) \end{aligned} \tag{7.11a}$$

$$\begin{aligned} &\{\hat{\bar{J}}^{(r)}_{aj}(m + \tfrac{r}{\rho(\sigma)}), \hat{\bar{J}}^{(s)}_{bl}(n + \tfrac{s}{\rho(\sigma)})\} \\ &= \delta_{jl}\big( i f_{ab}{}^c \hat{\bar{J}}^{(r+s)}_{cj}(m + n + \tfrac{r+s}{\rho(\sigma)}) + \rho(\sigma) k \eta_{ab}(m + \tfrac{r}{\rho(\sigma)}) \delta_{m+n+\frac{r+s}{\rho(\sigma)}, 0} \big) \end{aligned} \tag{7.11b}$$

$$\{\hat{J}^{(r)}_{aj}(m+\tfrac{r}{\rho(\sigma)}),\hat{\hat{J}}^{(s)}_{bl}(n+\tfrac{s}{\rho(\sigma)})\}=0 \tag{7.11c}$$

$$\hat{\hat{J}}^{(r)}_{aj}(m+\tfrac{r}{\rho(\sigma)})\equiv\hat{\hat{J}}^{(-r)}_{aj}(-m-\tfrac{r}{\rho(\sigma)})\ . \tag{7.11d}$$

Here the twisted left mover algebra[1,2] is a semisimple orbifold affine algebra[3], with commuting orbifold affine subalgebras labelled by $j$, at orbifold affine level $\hat{k}=\rho(\sigma)k$. Moreover, we have used the mode-number reversed relabelling (7.11d) to rectify the twisted right mover algebra into a copy (7.11b) of the twisted left mover algebra.

For the group orbifold elements of $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$, we find from Subsec. 7.1

$$\mathcal{T}^{(r)}_{aj}(T,\sigma)\equiv\mathcal{T}_{raj}(T,\sigma),\quad \hat{x}^{aj}_{(r)}(\xi,\sigma)\equiv\hat{x}^{raj}(\xi,\sigma) \tag{7.12a}$$

$$\mathcal{T}^{(r)}_{aj}(T,\sigma)_{s\alpha l}{}^{t\beta m}=(T_a)_\alpha{}^\beta\delta_{jl}\delta_l{}^m\delta_{r+s-t,0\ mod\,\rho(\sigma)} \tag{7.12b}$$

$$[\mathcal{T}^{(r)}_{aj}(T,\sigma),\mathcal{T}^{(s)}_{bl}(T,\sigma)]=\delta_{jl}if_{ab}{}^c\mathcal{T}^{(r+s)}_{cj}(T,\sigma) \tag{7.12c}$$

$$\hat{g}(\mathcal{T},\xi,\sigma)=e^{i\hat{x}^{aj}_{(r)}(\xi,\sigma)\mathcal{T}^{(r)}_{aj}(T,\sigma)} \tag{7.12d}$$

$$\hat{g}(\mathcal{T},\xi+2\pi,\sigma)_{r\alpha j}{}^{s\beta l}=e^{-\frac{2\pi i}{\rho(\sigma)}(r-s)}\hat{g}(\mathcal{T},\xi,\sigma)_{r\alpha j}{}^{s\beta l},\ \hat{x}^{aj}_{(r)}(\xi+2\pi,\sigma)=\hat{x}^{aj}_{(r)}(\xi,\sigma)e^{2\pi i\frac{r}{\rho(\sigma)}} \tag{7.12e}$$

where $\mathcal{T}^{(r)}_{aj}$ in (7.12b) are the twisted representation matrices in this case. The algebra of the twisted representation matrices in (7.12c) shows the same semisimplicity (labelled by $j$) as the twisted left and right mover current algebras in (7.11), and this semisimplicity is reflected as well in the group orbifold elements and the WZW orbifold action:

$$\mathcal{T}^{(r)}_{aj}(T,\sigma)_{s\alpha l}{}^{t\beta m}=\delta_{jl}\delta_l{}^m\mathcal{T}^{(r)}_a(T,\sigma)_{s\alpha}{}^{t\beta},\quad \mathcal{T}^{(r)}_a(T,\sigma)_{s\alpha}{}^{t\beta}\equiv(T_a)_\alpha{}^\beta\delta_{r+s-t,0\,\mathrm{mod}\,\rho(\sigma)} \tag{7.13a}$$

$$\hat{g}(\mathcal{T},\xi,\sigma)_{r\alpha j}{}^{s\beta l}=\delta_j{}^l\hat{g}_j(\mathcal{T},\xi,\sigma)_{r\alpha}{}^{s\beta},\quad \hat{g}_j(\mathcal{T},\xi,\sigma)\equiv e^{i\sum_{r\alpha}\hat{x}^{aj}_{(r)}(\xi,\sigma)\mathcal{T}^{(r)}_a(T,\sigma)} \tag{7.13b}$$

$$\hat{g}_j(\mathcal{T},\xi+2\pi,\sigma)_{r\alpha}{}^{s\beta}=e^{-\frac{2\pi i}{\rho(\sigma)}(r-s)}\hat{g}_j(\mathcal{T},\xi,\sigma)_{r\alpha}{}^{s\beta},\quad \hat{x}^{aj}_{(r)}(\xi+2\pi,\sigma)=\hat{x}^{aj}_{(r)}(\xi,\sigma)e^{2\pi i\frac{r}{\rho(\sigma)}} \tag{7.13c}$$

$$\hat{S}_\sigma(\mathcal{T})\equiv-\frac{k}{y(T)}\sum_{j=0}^{\frac{\lambda}{\rho(\sigma)}-1}\Big(\frac{1}{8\pi}\int d^2\xi\,\widehat{Tr}\big(\,\hat{g}_j^{-1}(\mathcal{T},\sigma)\partial_+\hat{g}_j(\mathcal{T},\sigma)\ \hat{g}_j^{-1}(\mathcal{T},\sigma)\partial_-\hat{g}_j(\mathcal{T},\sigma)\,\big)$$
$$+\frac{1}{12\pi}\int_\Gamma\widehat{Tr}\big(\,(\,\hat{g}_j^{-1}(\mathcal{T},\sigma)d\hat{g}_j(\mathcal{T},\sigma)\,)^3\,\big)\Big) \tag{7.13d}$$

$$\widehat{Tr}(AB)\equiv\sum_{r,s=0}^{\rho(\sigma)-1}\sum_{\alpha,\beta}A_{r\alpha}{}^{s\beta}B_{s\beta}{}^{r\alpha}\ . \tag{7.13e}$$

In particular, the group orbifold element $\hat{g}$ in (7.13b) is block diagonal with unitary square blocks $\hat{g}_j$ of size $\rho(\sigma)\times\rho(\sigma)$, and the WZW orbifold action (7.13d) is separable according to these blocks. The further relations

$$\hat{J}^{(r)}_{aj}(\xi,\sigma)=-\frac{i}{2}\frac{k}{y(T)}\widehat{Tr}\left(\mathcal{T}^{(r)}_a(T,\sigma)\hat{g}_j^{-1}(\mathcal{T},\xi,\sigma)\partial_+\hat{g}_j(\mathcal{T},\xi,\sigma)\right) \tag{7.14a}$$

$$\hat{\hat{J}}^{(r)}_{aj}(\xi,\sigma)=-\frac{i}{2}\frac{k}{y(T)}\widehat{Tr}\left(\mathcal{T}^{(r)}_a(T,\sigma)\hat{g}_j(\mathcal{T},\xi,\sigma)\partial_-\hat{g}_j^{-1}(\mathcal{T},\xi,\sigma)\right) \tag{7.14b}$$

$$\{\hat{J}^{(r)}_{aj}(\xi,\sigma),\hat{g}(\mathcal{T},\eta,\sigma)_{s\alpha l}{}^{t\beta m}\} = 2\pi\delta(\xi-\eta)\hat{g}(\mathcal{T},\eta,\sigma)_{s\alpha l}{}^{t-r,\gamma m}(T_a)_\gamma{}^\beta\delta_j{}^m \tag{7.15a}$$

$$\{\hat{J}^{(r)}_{aj}(\xi,\sigma),\hat{g}_l(\mathcal{T},\eta,\sigma)_{s\alpha}{}^{t\beta}\} = 2\pi\delta(\xi-\eta)\hat{g}_l(\mathcal{T},\eta,\sigma)_{s\alpha}{}^{t-r,\gamma}(T_a)_\gamma{}^\beta\delta_{jl} \tag{7.15b}$$

$$\{\hat{\bar{J}}^{(r)}_{aj}(\xi,\sigma),\hat{g}(\mathcal{T},\eta,\sigma)_{s\alpha l}{}^{t\beta m}\} = -2\pi\delta(\xi-\eta)\delta_{jl}(T_a)_\alpha{}^\gamma\hat{g}(\mathcal{T},\eta,\sigma)_{s+r,\gamma l}{}^{t\beta m} \tag{7.15c}$$

$$\{\hat{\bar{J}}^{(r)}_{aj}(\xi,\sigma),\hat{g}_l(\mathcal{T},\eta,\sigma)_{s\alpha}{}^{t\beta}\} = -2\pi\delta(\xi-\eta)\delta_{jl}(T_a)_\alpha{}^\gamma\hat{g}_l(\mathcal{T},\eta,\sigma)_{s+r,\gamma}{}^{t\beta} \tag{7.15d}$$

show that the orbifold affine subalgebra labelled by $j$ acts only on the block $\hat{g}_j$.

## 7.3 The WZW Permutation Orbifolds $A_g(S_N)/S_N$

For the permutation group $H = S_N$, it is known[2] that

$$I \longrightarrow \hat{j}j, \quad n(r)l \longrightarrow \hat{l}l \tag{7.16a}$$

$$U^\dagger(\vec{\sigma})_{\hat{j}j}{}^{\hat{l}l} = \frac{\delta_j{}^l}{\sqrt{\sigma_j}}e^{-2\pi i\frac{\hat{j}\hat{l}}{\sigma_j}}, \quad E^j_{\hat{j}}(\vec{\sigma}) = e^{-2\pi i\frac{\hat{j}}{\sigma_j}}, \quad \frac{N(r)}{R(\sigma)} = \frac{n(r)}{\rho(\sigma)} = \frac{\hat{j}}{\sigma_j} \tag{7.16b}$$

$$\sum_{j=0}^{n(\vec{\sigma})-1}\sigma_j = N, \qquad \sigma_{j+1} \leqslant \sigma_j \tag{7.16c}$$

$$\hat{J}_{n(r)aj} \longrightarrow J^{(\hat{j})}_{aj} \equiv \hat{J}_{\hat{j}aj}, \quad \hat{\bar{J}}_{n(r)aj} \longrightarrow \bar{J}^{(\hat{j})}_{aj} \equiv \hat{\bar{J}}_{\hat{j}aj}, \quad \chi_{\hat{j}aj}(\vec{\sigma}) = \sqrt{\sigma_j} \tag{7.16d}$$

$$\mathcal{G}_{\hat{j}aj;\hat{l}bl}(\vec{\sigma}) = \delta_{jl}\sigma_j k\eta_{ab}\delta_{\hat{j}+\hat{l},0\bmod\sigma_j}, \quad \mathcal{G}^{\hat{j}aj;\hat{l}bl}(\vec{\sigma}) = \frac{\delta^{jl}}{\sigma_j}\frac{\eta^{ab}}{k}\delta_{\hat{j}+\hat{l},0\bmod\sigma_j} \tag{7.16e}$$

$$\mathcal{F}_{\hat{j}aj;\hat{l}bl}{}^{\hat{m}\,cm}(\vec{\sigma}) = f_{ab}{}^c\delta_{jl}\delta_l{}^m\delta_{\hat{j}+\hat{l},\hat{m}\bmod\sigma_j} \tag{7.16f}$$

$$j,l = 0,\ldots,n(\vec{\sigma})-1, \quad \hat{\bar{j}} = 0,\ldots,\sigma_j-1, \quad \hat{\bar{l}} = 0,\ldots,\sigma_l-1\,. \tag{7.16g}$$

Here each sector $\vec{\sigma} = \{\sigma_j\}$ is a partition of $N$, which also labels a representative element $h_{\vec{\sigma}}$ of each conjugacy class of $S_N$. (The integer $\hat{j}$ runs within each disjoint cycle $\sigma_j$ of $h_{\vec{\sigma}}$, while the integer $j$ runs over the set of all disjoint cycles.)

This leads to the algebraic results for the WZW permutation orbifolds $A_g(S_N)/S_N$

$$\hat{T}_{\vec{\sigma}}(\xi) = \frac{\eta^{ab}}{4\pi k}\sum_{j=0}^{n(\vec{\sigma})-1}\frac{1}{\sigma_j}\sum_{\hat{j}=0}^{\sigma_j-1}\hat{J}^{(\hat{j})}_{aj}(\xi,\vec{\sigma})\hat{J}^{(-\hat{j})}_{bj}(\xi,\vec{\sigma}) \tag{7.17a}$$

$$\hat{\bar{T}}_{\vec{\sigma}}(\xi) = \frac{\eta^{ab}}{4\pi k}\sum_{j=0}^{n(\vec{\sigma})-1}\frac{1}{\sigma_j}\sum_{\hat{j}=0}^{\sigma_j-1}\hat{\bar{J}}^{(\hat{j})}_{aj}(\xi,\vec{\sigma})\hat{\bar{J}}^{(-\hat{j})}_{bj}(\xi,\vec{\sigma}) \tag{7.17b}$$

$$\hat{J}^{(\hat{j})}_{aj}(\xi+2\pi,\vec{\sigma}) = e^{-2\pi i\frac{\hat{j}}{\sigma_j}}\hat{J}^{(\hat{j})}_{aj}(\xi,\vec{\sigma}), \quad \hat{\bar{J}}^{(\hat{j})}_{aj}(\xi+2\pi,\vec{\sigma}) = e^{-2\pi i\frac{\hat{j}}{\sigma_j}}\hat{\bar{J}}^{(\hat{j})}_{aj}(\xi,\vec{\sigma}) \tag{7.18a}$$

$$\hat{J}^{(\hat{j})}_{aj}(\xi,t,\vec{\sigma}) = \sum_m \hat{J}^{(\hat{j})}_{aj}(m+\tfrac{\hat{j}}{\sigma_j})e^{-i(m+\frac{\hat{j}}{\sigma_j})(\xi+t)} \tag{7.18b}$$

$$\hat{\bar{J}}^{(\hat{j})}_{aj}(\xi,t,\vec{\sigma}) = \sum_m \hat{\bar{J}}^{(\hat{j})}_{aj}(m+\tfrac{\hat{j}}{\sigma_j})e^{-i(m+\frac{\hat{j}}{\sigma_j})(\xi-t)} \tag{7.18c}$$

$$\hat{J}_{aj}^{(\hat{j}\pm\sigma_j)}(m\mp 1+\tfrac{\hat{j}\pm\sigma_j}{\sigma_j}) = \hat{J}_{aj}^{(\hat{j})}(m+\tfrac{\hat{j}}{\sigma_j}), \quad \hat{\bar{J}}_{aj}^{(\hat{j}\pm\sigma_j)}(m\mp 1+\tfrac{\hat{j}\pm\sigma_j}{\sigma_j}) = \hat{\bar{J}}_{aj}^{(\hat{j})}(m+\tfrac{\hat{j}}{\sigma_j}) \tag{7.18d}$$

$$\{\hat{J}_{aj}^{(\hat{j})}(m+\tfrac{\hat{j}}{\sigma_j}), \hat{J}_{bl}^{(\hat{l})}(n+\tfrac{\hat{l}}{\sigma_l})\}=\delta_{jl}\big(if_{ab}{}^{c}\hat{J}_{cj}^{(\hat{j}+\hat{l})}(m+n+\tfrac{\hat{j}+\hat{l}}{\sigma_j})+(\sigma_j k)\eta_{ab}(m+\tfrac{\hat{j}}{\sigma_j})\delta_{m+n+\frac{\hat{j}+\hat{l}}{\sigma_j},0}\big) \tag{7.19a}$$

$$\{\hat{\bar{J}}_{aj}^{(\hat{j})}(m+\tfrac{\hat{j}}{\sigma_j}), \hat{\bar{J}}_{bl}^{(\hat{l})}(n+\tfrac{\hat{l}}{\sigma_l})\}=\delta_{jl}\big(if_{ab}{}^{c}\hat{\bar{J}}_{cj}^{(\hat{j}+\hat{l})}(m+n+\tfrac{\hat{j}+\hat{l}}{\sigma_j})+(\sigma_j k)\eta_{ab}(m+\tfrac{\hat{j}}{\sigma_j})\delta_{m+n+\frac{\hat{j}+\hat{l}}{\sigma_j},0}\big) \tag{7.19b}$$

$$\{\hat{J}_{aj}^{(\hat{j})}(m+\tfrac{\hat{j}}{\sigma_j}), \hat{\bar{J}}_{bl}^{(\hat{l})}(n+\tfrac{\hat{l}}{\sigma_l})\}=0 \tag{7.19c}$$

$$\hat{\bar{J}}_{aj}^{(\hat{j})}(m+\tfrac{\hat{j}}{\sigma_j}) \equiv \hat{\bar{J}}_{aj}^{(-\hat{j})}(-m-\tfrac{\hat{j}}{\sigma_j}) \ . \tag{7.19d}$$

Here the twisted left mover algebra[2] in (7.19a) is a semisimple orbifold affine algebra[3], with commuting subalgebras labelled by $j$, at orbifold affine level $\hat{k}_j = \sigma_j k$. Moreover, we have used the mode-number reversed relabelling (7.19d) to rectify the twisted right mover algebra into a copy (7.19b) of the twisted left mover algebra.

It is known[19,1,2] that the twisted left mover current algebra for all permutation orbifolds is a collection of commuting orbifold affine algebras at various orbifold affine levels, and it is clear that the mode-number reversed relabellings (7.11d) and (7.19d) rectify each commuting right mover subalgebra separately. It follows that the mode-number reversed relabelling

$$\hat{\bar{J}}_{n(r)aj}(m+\tfrac{n(r)}{\rho(\sigma)}) \equiv \hat{\bar{J}}_{-n(r),aj}(-m-\tfrac{n(r)}{\rho(\sigma)}) \tag{7.20}$$

will rectify the twisted right mover algebra ($\hat{\mathfrak{g}}(h_\sigma^{-1}) \simeq \mathfrak{g}(h_\sigma)$) for all sectors of all permutation orbifolds.

Continuing with the group orbifold elements of $A_g(S_N)/S_N$, we find

$$\mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma}) \equiv \mathcal{T}_{\hat{j}aj}(T,\vec{\sigma}), \quad \hat{x}_{(\hat{j})}^{aj}(\xi,\vec{\sigma}) \equiv \hat{x}^{\hat{j}aj}(\xi,\vec{\sigma}) \tag{7.21a}$$

$$\mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma})_{\hat{l}\alpha l}{}^{\hat{m}\,\beta m} = (T_a)_\alpha{}^\beta \delta_{jl}\delta_l{}^m \delta_{\hat{j}+\hat{l}-\hat{m},0\bmod\sigma_j} \tag{7.21b}$$

$$[\mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma}), \mathcal{T}_{bl}^{(\hat{l})}(T,\vec{\sigma})] = \delta_{jl} i f_{ab}{}^c \mathcal{T}_{cj}^{(\hat{j}+\hat{l})}(T,\vec{\sigma}) \tag{7.21c}$$

$$\hat{g}(T,\xi,\vec{\sigma}) = e^{i\hat{x}_{(\hat{j})}^{aj}(\xi,\vec{\sigma})\mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma})}, \quad \hat{x}_{(\hat{j})}^{aj}(\xi+2\pi,\vec{\sigma}) = \hat{x}_{(\hat{j})}^{aj}(\xi,\vec{\sigma})e^{+2\pi i\frac{\hat{j}}{\sigma_j}} \tag{7.21d}$$

$$\hat{g}(\mathcal{T},\xi+2\pi,\vec{\sigma})_{\hat{j}\alpha j}{}^{\hat{l}\beta l} = e^{-2\pi i(\frac{\hat{j}}{\sigma_j}-\frac{\hat{l}}{\sigma_l})}\hat{g}(\mathcal{T},\xi,\vec{\sigma})_{\hat{j}\alpha j}{}^{\hat{l}\beta l} \quad \alpha = 1,...,\dim T \tag{7.21e}$$

where $\mathcal{T}_{aj}^{(\hat{j})}$ in (7.21b) are the twisted representation matrices in this case.

The decomposition corresponding to the semisimplicity of the twisted left and right mover current algebras in (7.19) is

$$\hat{g}(\mathcal{T},\xi,\vec{\sigma})_{\hat{j}\alpha j}{}^{\hat{j}'\beta j'} = \delta_j{}^{j'}\hat{g}_j(\mathcal{T},\xi,\vec{\sigma})_{\hat{j}\alpha}{}^{\hat{j}'\beta}, \quad \hat{g}_j(\mathcal{T},\xi,\vec{\sigma}) \equiv e^{i\sum_{a\hat{j}}\hat{x}_{(\hat{j})}^{aj}(\xi,\vec{\sigma})\mathcal{T}_a^{(\hat{j})}(T,\vec{\sigma})} \tag{7.22a}$$

$$(\mathcal{T}_a^{(\hat{j})}(T,\vec{\sigma}))_{\hat{j}'\alpha}{}^{\hat{j}''\beta} \equiv \delta_{\hat{j}+\hat{j}'-\hat{j}'',0\bmod\sigma_j}(T_a)_\alpha{}^\beta, \quad \bar{\hat{j}},\bar{\hat{j}}',\bar{\hat{j}}'' = 0,...,\sigma_j-1 \tag{7.22b}$$

$$\hat{g}_j(\mathcal{T},\xi+2\pi,\vec{\sigma})_{\hat{j}\alpha}{}^{\hat{j}'\beta} = e^{-\frac{2\pi i}{\sigma_j}(\hat{j}-\hat{j}')}\hat{g}_j(\mathcal{T},\xi,\vec{\sigma})_{\hat{j}\alpha}{}^{\hat{j}'\beta} \tag{7.22c}$$

$$\hat{S}_{\vec{\sigma}}(\mathcal{T}) = -\frac{k}{y(T)} \sum_{j=0}^{n(\vec{\sigma})-1} \left( \frac{1}{8\pi} \int d^2\xi \, \widehat{Tr}_j\big( \, \hat{g}_j^{-1}(\mathcal{T},\vec{\sigma}) \, \partial_+ \hat{g}_j(\mathcal{T},\vec{\sigma}) \, \hat{g}_j^{-1}(\mathcal{T},\vec{\sigma}) \partial_- \hat{g}_j(\mathcal{T},\vec{\sigma}) \, \big) \right.$$

$$\left. + \frac{1}{12\pi} \int_\Gamma \widehat{Tr}_j\big( \, (\, \hat{g}_j^{-1}(\mathcal{T},\vec{\sigma}) d\hat{g}_j(\mathcal{T},\vec{\sigma}) \,)^3 \, \big) \right) \tag{7.22d}$$

$$\widehat{Tr}_j(AB) \equiv \sum_{\hat{j},\hat{j}'=0}^{\sigma_j-1} \sum_{\alpha,\beta} A_{\hat{j}\alpha}{}^{\hat{j}'\beta} B_{\hat{j}'\beta}{}^{\hat{j}\alpha} \, . \tag{7.22e}$$

In particular, we see that the group orbifold element $\hat{g}$ is block diagonal with unitary square blocks $\hat{g}_j$ of size $\sigma_j \times \sigma_j$, and the WZW orbifold action (7.22d) is separable according to these blocks. The same structure is seen in the relations between the group orbifold elements and the twisted currents

$$\hat{J}_{aj}^{(\hat{j})}(\xi,\vec{\sigma}) = -\frac{i}{2}\frac{k}{y(T)} \widehat{Tr}_j \left( \mathcal{T}_a^{(\hat{j})}(T,\vec{\sigma}) \hat{g}_j^{-1}(\mathcal{T},\xi,\vec{\sigma}) \partial_+ \hat{g}_j(\mathcal{T},\xi,\vec{\sigma}) \right) \tag{7.23a}$$

$$\hat{\bar{J}}_{aj}^{(\hat{j})}(\xi,\vec{\sigma}) = -\frac{i}{2}\frac{k}{y(T)} \widehat{Tr})_j \left( \mathcal{T}_a^{(\hat{j})}(T,\vec{\sigma}) \hat{g}_j(\mathcal{T},\xi,\vec{\sigma}) \partial_- \hat{g}_j^{-1}(\mathcal{T},\xi,\vec{\sigma}) \right) \tag{7.23b}$$

$$\{\hat{J}_{aj}^{(\hat{j})}(\xi,\vec{\sigma}), \hat{g}(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}\alpha l}{}^{\hat{m}\,\beta m}\} = 2\pi\delta(\xi-\eta)\hat{g}(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}\alpha l}{}^{\hat{m}-\hat{j},\gamma m}(T_a)_\gamma{}^\beta \delta_j{}^m \tag{7.24a}$$

$$\{\hat{J}_{aj}^{(\hat{j})}(\xi,\vec{\sigma}), \hat{g}_l(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}\alpha}{}^{\hat{l}'\beta}\} = 2\pi\delta(\xi-\eta)\hat{g}_l(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}\alpha}{}^{\hat{l}'-\hat{j},\gamma}(T_a)_\gamma{}^\beta \delta_{jl} \tag{7.24b}$$

$$\{\hat{\bar{J}}_{aj}^{(\hat{j})}(\xi,\vec{\sigma}), \hat{g}(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}\alpha l}{}^{\hat{m}\,\beta m}\} = -\,2\pi\delta(\xi-\eta)\delta_{jl}(T_a)_\alpha{}^\gamma \hat{g}(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}+\hat{j},\gamma l}{}^{\hat{m}\,\beta m} \tag{7.24c}$$

$$\{\hat{\bar{J}}_{aj}^{(\hat{j})}(\xi,\vec{\sigma}), \hat{g}_l(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}\alpha}{}^{\hat{l}'\beta}\} = -\,2\pi\delta(\xi-\eta)\delta_{jl}(T_a)_\alpha{}^\gamma \hat{g}_l(\mathcal{T},\eta,\vec{\sigma})_{\hat{l}+\hat{j},\gamma}{}^{\hat{l}'\beta} \tag{7.24d}$$

so that the orbifold affine subalgebra labelled by $j$ acts only on the block $\hat{g}_j$.

## 7.4 The Inner-automorphic WZW Orbifolds $A_g(H(d))/H(d)$

Let $h_\sigma \in H(d) \subset Aut(g)$ be an element of any group $H(d)$ of inner automorphisms of simple $g$. Following Ref. 2, we know the matrix action $w(h_\sigma)$

$$w(h_\sigma)_\alpha{}^\beta = \delta_\alpha{}^\beta e^{2\pi i\sigma\alpha\cdot d}, \quad w(h_\sigma)_A{}^B = \delta_A{}^B, \quad w(h_\sigma)_\alpha{}^A = w(h_\sigma)_A{}^\alpha = 0 \tag{7.25a}$$

$$\alpha \in \Delta(g), \quad A = 1...\text{rank } g, \quad \sigma = 0,\ldots,\rho(1)-1, \quad N_c = \rho(1) \tag{7.25b}$$

of $h_\sigma$ on the untwisted currents in the Cartan-Weyl basis. This is the action of $h_\sigma$ in the adjoint $(T_a^{adj})_b{}^c = -if_{ab}{}^c$. Refs. 20 and 2 discuss the conditions on the vector $d$ such that $H(d)$ is a group of finite order.

Then the matrix action $W(h_\sigma;T)$ of $h_\sigma$ in representation $T$ must solve the linkage relation (3.6a) in the form

$$W^\dagger(h_\sigma;T)T_\alpha W(h_\sigma;T) = e^{2\pi i\sigma\alpha\cdot d}T_\alpha, \quad W^\dagger(h_\sigma;T)T_A W(h_\sigma;T) = T_A \, . \tag{7.26}$$

To solve these equations, it is convenient to introduce the weight basis of matrix representation $T$

$$(T_A)_\lambda{}^{\lambda'} \equiv \langle T,\lambda|H_A(0)|T,\lambda'\rangle = \lambda_A\delta_{\lambda,\lambda'}, \quad (T_\alpha)_\lambda{}^{\lambda'} \equiv \langle T,\lambda|E_\alpha(0)|T,\lambda'\rangle \propto \delta_{\lambda,\lambda'+\alpha} \tag{7.27}$$

where $\{\lambda\}$ are the weights of the representation. The solution of the linkage relation

$$W(h_\sigma;T)_\lambda{}^{\lambda'} = \delta_\lambda{}^{\lambda'}e^{-2\pi i\sigma\lambda\cdot d}, \quad W^\dagger(h_\sigma;T)_\lambda{}^{\lambda'} = \delta_\lambda{}^{\lambda'}e^{+2\pi i\sigma\lambda\cdot d} \tag{7.28}$$

is obtained immediately in this basis. Note that the adjoint matrices of the weight basis are similarity transformations of $(T_a^{adj})_b{}^c = -if_{ab}{}^c$, so that $W$ does not reduce to $w$ in this case.

We discuss the orders $\rho(\sigma)$ and $R(T,\sigma)$ of $w(h_\sigma)$ and $W(h_\sigma;T)$, which are the smallest integers satisfying

$$e^{2\pi i\sigma\alpha\cdot d\rho(\sigma)} = 1, \quad e^{-2\pi i\sigma\lambda\cdot dR(T,\sigma)} = 1\ . \tag{7.29}$$

These conditions imply that

$$\begin{cases} R(T,\sigma) = \rho(\sigma) & \text{when } T \text{ is in the conjugacy class of the root lattice} \\ e^{-2\pi i\sigma\lambda_{min}\cdot dR(T,\sigma)} = 1 & \text{when } T \text{ is in the conjugacy class of } \lambda_{min} \text{ plus the root lattice} \end{cases} \tag{7.30}$$

where $\{\lambda_{min}\}$ are the minimal weights of simple $g$, given in Ref. 21. As the simplest example, we find

$$R(T,\sigma) = \begin{cases} \rho(\sigma) & \text{for integral spin } j \\ 2\rho(\sigma) & \text{for half-integral spin } j \end{cases} \tag{7.31}$$

for $g = SU(2)$. In what follows, we revert to our abbreviation $R(\sigma) \equiv R(T,\sigma)$.

In this case the actions $w(h_\sigma)$ and $W(h_\sigma;T)$ are diagonal, so the solution of the $H$-eigenvalue problem and the extended $H$-eigenvalue problem can be taken as

$$U^\dagger(h_\sigma) = \chi(\sigma) = 1, \quad U^\dagger(T,h_\sigma) = \mathbb{1}, \quad \mathcal{G} = G, \quad \mathcal{F} = f, \quad \mathcal{T} = T \tag{7.32a}$$

$$\tfrac{n_\alpha(r)}{\rho(\sigma)} = -\sigma\alpha\cdot d, \quad n_A = 0, \quad \tfrac{N(r)}{R(\sigma)} = \sigma\lambda\cdot d \tag{7.32b}$$

$$\mathcal{M}(\mathcal{T},\sigma) = M(k,T) = \frac{k}{y(T)}\mathbb{1}\ . \tag{7.32c}$$

This tells us that all the selection rules discussed above are satisfied naturally by the untwisted objects in the Cartan-Weyl basis and weight basis. For example, the selection rule (4.18) for $\mathcal{T}$ takes the form

$$e^{-2\pi i\sigma(\lambda-\lambda')\cdot d}(T_A)_\lambda{}^{\lambda'} = (T_A)_\lambda{}^{\lambda'}, \quad e^{-2\pi i\sigma(\lambda-\lambda')\cdot d}(T_\alpha)_\lambda{}^{\lambda'} = e^{-2\pi i\sigma\alpha\cdot d}(T_\alpha)_\lambda{}^{\lambda'} \tag{7.33}$$

which is easily checked from (7.27).

This gives the stress tensors and twisted currents of the inner-automorphic WZW orbifolds $A_g(H_d)/H_d$

$$\hat{T}_\sigma(\xi) = \frac{1}{4\pi k}(\hat{H}_A(\xi,\sigma)\hat{H}^A(\xi,\sigma) + \hat{E}_\alpha(\xi,\sigma)\hat{E}_{-\alpha}(\xi,\sigma)) \tag{7.34a}$$

$$\hat{\bar{T}}_\sigma(\xi) = \frac{1}{4\pi k}(\hat{\bar{H}}_A(\xi,\sigma)\hat{\bar{H}}^A(\xi,\sigma) + \hat{\bar{E}}_\alpha(\xi,\sigma)\hat{\bar{E}}_{-\alpha}(\xi,\sigma)) \tag{7.34b}$$

$$\hat{H}_A(\xi+2\pi,\sigma) = \hat{H}_A(\xi,\sigma), \quad \hat{E}_\alpha(\xi+2\pi,\sigma) = e^{2\pi i\alpha\cdot d}\hat{E}_\alpha(\xi,\sigma) \tag{7.35a}$$

$$\hat{\bar{H}}_A(\xi+2\pi,\sigma) = \hat{\bar{H}}_A(\xi,\sigma), \quad \hat{\bar{E}}_\alpha(\xi+2\pi,\sigma) = e^{2\pi i\alpha\cdot d}\hat{\bar{E}}_\alpha(\xi,\sigma) \tag{7.35b}$$

$$\hat{H}_A(\xi,t,\sigma) = \sum_m \hat{H}_A(m)e^{-im(\xi+t)}, \quad \hat{E}_\alpha(\xi,t,\sigma) = \sum_m \hat{E}_\alpha(m-\sigma\alpha\cdot d)e^{-i(m-\sigma\alpha\cdot d)(\xi+t)} \tag{7.35c}$$

$$\hat{\bar{H}}_A(\xi,t,\sigma) = \sum_m \hat{\bar{H}}_A(m)e^{-im(\xi-t)}, \quad \hat{\bar{E}}_\alpha(\xi,t,\sigma) = \sum_m \hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d)e^{-i(m-\sigma\alpha\cdot d)(\xi-t)} \tag{7.35d}$$

$$\begin{aligned} \{\hat{H}_A(m),\hat{H}_B(n)\} &= km\delta_{AB}\delta_{m+n,0}, \\ \{\hat{H}_A(m),\hat{E}_\alpha(n-\sigma\alpha\cdot d)\} &= \alpha_A\hat{E}_\alpha(m+n-\sigma\alpha\cdot d) \end{aligned} \tag{7.36a}$$

$$\begin{aligned} &\{\hat{E}_\alpha(m-\sigma\alpha\cdot d),\hat{E}_\beta(n-\sigma\beta\cdot d)\} \\ &= \begin{cases} N_\gamma(\alpha,\beta)\hat{E}_\gamma(m+n-\sigma\gamma\cdot d) & \text{if } \alpha+\beta=\gamma \\ \alpha\cdot\hat{H}(m+n) + k(m-\sigma\alpha\cdot d)\delta_{m+n,0} & \text{if } \alpha+\beta=0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{7.36b}$$

$$N_{-\gamma}(-\alpha,-\beta) = -N_\gamma(\alpha,\beta), \quad A=1,...,\operatorname{rank} g, \quad \alpha\in\Delta(g), \quad \sigma=0,...,\rho(1)-1\ . \tag{7.36c}$$

The twisted left mover algebra in (7.36) is the bracket form of the usual set[2]

$$\hat{\mathfrak{g}}(h_\sigma) = \hat{\mathfrak{g}}(H(d)\subset Aut(g);h_\sigma), \quad \sigma=0,...,\rho(1)-1 \tag{7.37}$$

of inner-automorphically twisted affine Lie algebras, and the commuting twisted right mover algebra $\bar{\hat{\mathfrak{g}}}(h_\sigma)\simeq\hat{\mathfrak{g}}(h_\sigma^{-1})$ has the same form except for the sign change of the central terms.

In this case also we have been able to solve the rectification problem for the twisted right movers. The simplest rectification is

$$\begin{aligned} \{\hat{\bar{\bar{H}}}_A(m),\hat{\bar{\bar{H}}}_B(n)\} &= km\delta_{AB}\delta_{m+n,0}, \\ \{\hat{\bar{\bar{H}}}_A(m),\hat{\bar{\bar{E}}}_\alpha(n-\sigma\alpha\cdot d)\} &= \alpha_A\hat{\bar{\bar{E}}}_\alpha(m+n-\sigma\alpha\cdot d) \end{aligned} \tag{7.38a}$$

$$\begin{aligned} &\{\hat{\bar{\bar{E}}}_\alpha(m-\sigma\alpha\cdot d),\hat{\bar{\bar{E}}}_\beta(n-\sigma\beta\cdot d)\} \\ &= \begin{cases} N_\gamma(\alpha,\beta)\hat{\bar{\bar{E}}}_\gamma(m+n-\sigma\gamma\cdot d) & \text{if } \alpha+\beta=\gamma \\ \alpha\cdot\hat{\bar{\bar{H}}}(m+n) + k(m-\sigma\alpha\cdot d)\delta_{m+n,0} & \text{if } \alpha+\beta=0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{7.38b}$$

$$\hat{\bar{\bar{H}}}_A(m) \equiv -\hat{\bar{H}}_A(-m), \quad \hat{\bar{\bar{E}}}_\alpha(m-\sigma\alpha\cdot d) = -\hat{\bar{E}}_{-\alpha}(-m+\sigma\alpha\cdot d) \tag{7.38c}$$

so that $\hat{\mathfrak{g}}(h_\sigma^{-1})\simeq\hat{\mathfrak{g}}(h_\sigma)$ for all sectors $\sigma$ of the orbifolds $A_g(H(d))/H(d)$.

We close this discussion with a collection of results involving the group orbifold elements $\hat{g}$ of $A_g(H(d))/H(d)$:

$$\hat{g}(T,\xi,\sigma) = e^{i(\hat{x}^A(\xi,\sigma)T_A + \hat{x}^\alpha(\xi,\sigma)T_\alpha)} \qquad (\hat{\beta} = \hat{x}) \tag{7.39a}$$

$$\hat{x}^A(\xi+2\pi,\sigma) = \hat{x}^A(\xi,\sigma), \quad \hat{x}^\alpha(\xi+2\pi,\sigma) = \hat{x}^\alpha(\xi,\sigma)e^{-2\pi i\sigma\alpha\cdot d} \tag{7.39b}$$

$$\hat{g}(T,\xi+2\pi,\sigma)_\lambda{}^{\lambda'} = e^{-2\pi i\sigma(\lambda-\lambda')\cdot d}\hat{g}(T,\xi,\sigma)_\lambda{}^{\lambda'} \tag{7.39c}$$

$$\{\hat{H}_A(\xi,\sigma),\hat{g}(T,\eta,\sigma)\} = 2\pi\delta(\xi-\eta)\hat{g}(T,\eta,\sigma)T_A \tag{7.40a}$$

$$\{\hat{E}_\alpha(\xi,\sigma),\hat{g}(T,\eta,\sigma)\} = 2\pi\delta(\xi-\eta)\hat{g}(T,\eta,\sigma)T_\alpha \tag{7.40b}$$

$$\{\hat{\bar{H}}_A(\xi,\sigma),\hat{g}(T,\eta,\sigma)\} = -\,2\pi\delta(\xi-\eta)T_A\hat{g}(T,\eta,\sigma) \tag{7.40c}$$

$$\{\hat{\bar{E}}_\alpha(\xi,\sigma),\hat{g}(T,\eta,\sigma)\} = -\,2\pi\delta(\xi-\eta)T_\alpha\hat{g}(T,\eta,\sigma) \tag{7.40d}$$

$$\begin{aligned}\hat{S}_\sigma(T) = &-\frac{k}{y(T)}\Big(\frac{1}{8\pi}\int d^2\xi\,\widehat{Tr}\big(\,\hat{g}^{-1}(T,\sigma)\partial_+\hat{g}(T,\sigma)\;\hat{g}^{-1}(T,\sigma)\partial_-\hat{g}(T,\sigma)\big)\\ &+\frac{1}{12\pi}\int_\Gamma \widehat{Tr}\big((\,\hat{g}^{-1}(T,\sigma)d\hat{g}(T,\sigma)\,)^3\,\big)\Big), \quad \widehat{Tr}(AB) \equiv \sum_{\lambda,\lambda'} A_\lambda{}^{\lambda'}B_{\lambda'}{}^\lambda\,.\end{aligned} \tag{7.41}$$

Using the $T$ selection rules (7.33), the reader is invited to verify the all-order equivalence of the monodromies (7.39c) and (7.39b).

## 7.5 A $U(1)/\mathbb{Z}_2$ Example

Orbifolds $A_g(H)/H$ on abelian $g$ are included in our construction. As an example, we consider here the simple case

$$g = U(1) = \text{Cartan }(SU(2)), \quad H = \mathbb{Z}_2 \tag{7.42}$$

and we will follow the details through explicitly only for the spin $j = \frac{1}{2}$ representation. The untwisted theory is defined by

$$g(T,\xi) = e^{ix(\xi)T_3}, \quad T_a = \frac{1}{2}\sigma_a, \quad M(k,T) = 2, \quad S = \frac{1}{8\pi}\int d^2\xi\,\partial_+x\,\partial_-x \tag{7.43a}$$

$$J(\xi)\equiv J_3(\xi)=\frac{1}{2}\partial_+x(\xi), \quad \bar{J}(\xi)\equiv\bar{J}_3(\xi)=-\frac{1}{2}\partial_-x(\xi), \quad G=G^{-1}=1, \quad L_\infty=L=\frac{1}{2} \tag{7.43b}$$

where we have chosen $\beta = x$ and $\{\vec{\sigma}\}$ are the Pauli matrices. The action of the $\mathbb{Z}_2$ symmetry group is

$$x(\xi) \longrightarrow -x(\xi): \quad J(\xi) \longrightarrow -J(\xi), \quad \bar{J}(\xi) \longrightarrow -\bar{J}(\xi) \tag{7.44}$$

in sector $\sigma = 1$.

Then the $H$-eigenvalue problem gives the $\frac{1}{2}$-integral moded scalar field[22,2]

$$\rho(1) = 2, \quad \bar{n}(r) = 1, \quad \chi = 1, \quad \mathcal{G} = \mathcal{G}^{-1} = 1 \tag{7.45a}$$

$$\hat{T}_{\sigma=1}(\xi) = \frac{1}{4\pi}\hat{J}(\xi)\hat{J}(\xi), \quad \hat{\bar{T}}_{\sigma=1}(\xi) = \frac{1}{4\pi}\hat{\bar{J}}(\xi)\hat{\bar{J}}(\xi) \tag{7.45b}$$

$$\hat{J}(\xi+2\pi) = -\hat{J}(\xi), \quad \hat{\bar{J}}(\xi+2\pi) = -\hat{\bar{J}}(\xi) \tag{7.45c}$$

$$\hat{J}(\xi) = \sum_m \hat{J}(m+\tfrac{1}{2})e^{-i(m+\frac{1}{2})(\xi+t)}, \quad \hat{\bar{J}}(\xi) = \sum_m \hat{\bar{J}}(m+\tfrac{1}{2})e^{-i(m+\frac{1}{2})(\xi-t)} \tag{7.45d}$$

$$\{\hat{J}(m+\tfrac{1}{2}), \hat{J}(n+\tfrac{1}{2})\} = -\{\hat{\bar{J}}(m+\tfrac{1}{2}), \hat{\bar{J}}(n+\tfrac{1}{2})\} = (m+\tfrac{1}{2})\delta_{m+n+1,0} \tag{7.45e}$$

$$\{\hat{J}(m+\tfrac{1}{2}), \hat{\bar{J}}(n+\tfrac{1}{2})\} = 0\ . \tag{7.45f}$$

in twisted sector $\sigma = 1$ of the $U(1)/\mathbb{Z}_2$ orbifold. The mode-number reversed relabelling of the twisted right movers

$$\hat{\bar{\bar{J}}}(m+\tfrac{1}{2}) \equiv \hat{\bar{J}}(-m-\tfrac{1}{2}) \tag{7.46}$$

gives a copy of the twisted left mover algebra, and it is clear that mode-number reversed relabelling will rectify any multiple component orbifold of this type ($x^i \to -x^i$).

For the extended $H$-eigenvalue problem, we find

$$W(T) = e^{i\pi T_2} = i\sigma_2, \quad W^\dagger(T)T_3W(T) = -T_3, \quad R(1) = 4 \tag{7.47a}$$

$$U^\dagger(T) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad E(T) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp\left(-\tfrac{2\pi i}{4}\begin{pmatrix} N(1) & 0 \\ 0 & N(2) \end{pmatrix}\right) \tag{7.47b}$$

$$N(1) = -1, \quad N(2) = +1 \tag{7.47c}$$

and hence the further orbifold results

$$\mathcal{T}(T) = U(T)T_3U^\dagger(T) = T_1, \quad \mathcal{G}(T,\xi) = U(T)g(T)U^\dagger(T) = e^{ix(\xi)T_1} \tag{7.48a}$$

$$\hat{g}(\mathcal{T},\xi) = e^{i\hat{x}(\xi)T_1} = \cos\frac{\hat{x}(\xi)}{2} + i\sigma_1 \sin\frac{\hat{x}(\xi)}{2}, \quad \hat{x}(\xi+2\pi) = -\hat{x}(\xi) \tag{7.48b}$$

$$\hat{g}(\mathcal{T},\xi+2\pi)_{N(r)}{}^{N(s)} = e^{-\frac{2\pi i}{4}(N(r)-N(s))}\hat{g}(\mathcal{T},\xi)_{N(r)}{}^{N(s)} \tag{7.48c}$$

$$\hat{S} = \frac{1}{8\pi}\int d^2\xi\, \partial_+\hat{x}\partial_-\hat{x}, \quad \mathcal{M} = 2 \tag{7.48d}$$

in the single twisted sector. The reader is invited to check that (7.48b) is consistent with (7.48c), so that one finds trivial monodromy for the diagonal group orbifold elements and monodromy -1 for the off-diagonal group orbifold elements.

The exponential form of $W(T)$ in (7.47a) allows us to treat all spins $j$, but we shall not do so here.

# 8 Implications for Orbifold Conformal Field Theory

## 8.1 Operator Realization of the General Twisted Current Algebra

The canonical operator realization of the general twisted currents

$$\hat{J}_{n(r)\mu}(\xi,\sigma) = 2\pi\hat{e}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{p}_{n(t)\delta}(\hat{B}) + \frac{1}{2}\partial_\xi\hat{x}^{n(t)\delta}\hat{e}(\hat{x})_{n(t)\delta}{}^{n(s)\nu}\mathcal{G}_{n(s)\nu;n(r)\mu}(\sigma) \tag{8.1a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma) = 2\pi\hat{\bar{e}}^{-1}(\hat{x})_{n(r)\mu}{}^{n(t)\delta}\hat{p}_{n(t)\delta}(\hat{B}) - \frac{1}{2}\partial_\xi\hat{x}^{n(t)\delta}\hat{\bar{e}}(\hat{x})_{n(t)\delta}{}^{n(s)\nu}\mathcal{G}_{n(s)\nu;n(r)\mu}(\sigma) \tag{8.1b}$$

$$\hat{p}_{n(r)\mu}(\hat{B}) = \hat{p}_{n(r)\mu} + \frac{1}{4\pi}\hat{B}_{n(r)\mu;n(s)\nu}(\hat{x})\partial_\xi\hat{x}^{n(s)\nu} \tag{8.1c}$$

$$[\hat{x}^{n(r)\mu}(\xi), \hat{p}_{n(s)\nu}(\eta)] = i\delta(\xi-\eta)\delta_{n(s)\nu}{}^{n(r)\mu} \tag{8.1d}$$

$$\hat{x}^{n(r)\mu}(\xi+2\pi) = \hat{x}^{n(r)\mu}(\xi)e^{+2\pi i\frac{n(r)}{\rho(\sigma)}}, \quad \hat{p}_{n(r)\mu}(\xi+2\pi) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{p}_{n(r)\mu}(\xi) \tag{8.2a}$$

$$\hat{J}_{n(r)\mu}(\xi+2\pi) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(\xi), \quad \hat{\bar{J}}_{n(r)\mu}(\xi+2\pi) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)\mu}(\xi) \tag{8.2b}$$

$$\begin{aligned}[\hat{J}_{n(r)\mu}(\xi), \hat{J}_{n(s)\nu}(\eta)] = 2\pi i\Big(&\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(\xi)\delta(\xi-\eta)\\ &+\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\partial_\xi\delta(\xi-\eta)\Big)\end{aligned} \tag{8.3a}$$

$$\begin{aligned}[\hat{\bar{J}}_{n(r)\mu}(\xi), \hat{\bar{J}}_{n(s)\nu}(\eta)] = 2\pi i\Big(&\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\xi)\delta(\xi-\eta)\\ &-\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\partial_\xi\delta(\xi-\eta)\Big)\end{aligned} \tag{8.3b}$$

$$[\hat{J}_{n(r)\mu}(\xi), \hat{\bar{J}}_{n(s)\nu}(\eta)] = 0 \tag{8.3c}$$

is the quantum analogue of the canonical bracket realization in Eq. (6.2). The twisted vielbeins $\hat{e}$, $\hat{\bar{e}}$ and the twisted $B$ field $\hat{B}$ are the same as those defined in Subsec. 6.1. In the untwisted sector $\sigma = 0$, this result reduces to the semisimple generalization of Clubok and Halpern's canonical operator realization[16] of affine(g $\oplus$ g).

Following Subsec. 6.4, the result (8.1) can be derived by local isomorphisms from the Clubok-Halpern realization. An independent derivation by the method of twisted affine Lie groups is given in Appendix E.

## 8.2 More about All Current-Algebraic Orbifolds

The twisted left mover currents and stress tensors of the general current-algebraic orbifold $A(H)/H$, $H \subset Aut(g)$ were given in Refs. 1 and 2. The implicit assumption in these and prior references was that twisted right mover currents and stress tensors could be taken as copies of the left movers. However, locality of the WZW orbifold actions has led us to question that assumption and, in what follows, we translate the classical lessons above into the operator formulation of the general current-algebraic orbifold.

In the symmetric CFT $A(H)$, the OPE's and automorphic responses of the currents and eigencurrents are left-right copies:

$$J_a(z)J_b(w) = \frac{G_{ab}}{(z-w)^2} + \frac{if_{ab}{}^c}{z-w}J_c(w) + O((z-w)^0), \quad \bar{J}_a(\bar{z})\bar{J}_b(\bar{w}) = \frac{G_{ab}}{(\bar{z}-\bar{w})^2} + \frac{if_{ab}{}^c}{\bar{z}-\bar{w}}\bar{J}_c(\bar{w}) + O((\bar{z}-\bar{w})^0) \tag{8.4a}$$

$$J_a(z)' = w(h_\sigma)_a{}^b J_b(z), \quad \bar{J}_a(\bar{z})' = w(h_\sigma)_a{}^b \bar{J}_b(\bar{z}) \tag{8.4b}$$

$$\mathcal{J}_{n(r)\mu}(z) = \chi_{n(r)\mu}U_{n(r)\mu}{}^a J_a(z), \quad \bar{\mathcal{J}}_{n(r)\mu}(z) = \chi_{n(r)\mu}U_{n(r)\mu}{}^a \bar{J}_a(\bar{z}) \tag{8.4c}$$

$$\mathcal{J}_{n(r)\mu}(z)' = E_{n(r)}\mathcal{J}_{n(r)\mu}(z), \quad \bar{\mathcal{J}}_{n(r)\mu}(\bar{z})' = E_{n(r)}\bar{\mathcal{J}}_{n(r)\mu}(\bar{z}) \,. \tag{8.4d}$$

To preserve the locality of the theory, however, we have learned that the statement of the local current isomorphisms must be refined to

$$\mathcal{J}, \bar{\mathcal{J}} \underset{\sigma}{\longrightarrow} \hat{J}, \hat{\bar{J}} \tag{8.5a}$$

$$\text{automorphism response } E_{n(r)}(\sigma) \underset{\sigma}{\longrightarrow} \text{ same-path monodromy } E_{n(r)}(\sigma)\ . \tag{8.5b}$$

This gives the twisted current system of $A(H)/H$

$$\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(w) = \frac{\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}}{(z-w)^2} + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(w)}{z-w} + O((z-w)^0) \tag{8.6a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{w}) = \frac{\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}}{(\bar{z}-\bar{w})^2} + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\bar{w})}{\bar{z}-\bar{w}} + O((\bar{z}-\bar{w})^0) \tag{8.6b}$$

$$\hat{J}_{n(r)\mu}(ze^{2\pi i}) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(z), \quad \hat{\bar{J}}_{n(r)\mu}(\bar{z}e^{-2\pi i}) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)\mu}(\bar{z}) \tag{8.6c}$$

in which the OPE's are left-right copies and the monodromies, although not left-right copies, are the same when the same path is followed. Using Eq. (5.36), we see that this system is the Euclidean quantum analogue of the twisted light-cone system (5.34). For comparison, the twisted current system for the situation with left-right copies is given in Appendix F.

The monodromies (8.6c) give the mode expansions

$$\hat{J}_{n(r)\mu}(z) = \sum_{m\in\mathbb{Z}} \hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})z^{-(m+\frac{n(r)}{\rho(\sigma)})-1} \tag{8.7a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z}) = \sum_{m\in\mathbb{Z}} \hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})\bar{z}^{+(m+\frac{n(r)}{\rho(\sigma)})-1} \tag{8.7b}$$

and then the OPE's in (8.6) give the general twisted current algebra $\hat{\mathfrak{g}}(h_\sigma)\oplus\hat{\mathfrak{g}}(h_\sigma^{-1})$

$$[\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{J}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) + (m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \tag{8.8a}$$

$$[\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{\bar{J}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) - (m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \tag{8.8b}$$

$$[\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{\bar{J}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] = 0, \quad m,n\in\mathbb{Z}, \qquad \sigma = 0,\ldots,N_c-1 \tag{8.8c}$$

in sector $\sigma$ of $A(H)/H$. This result is the operator analogue of the bracket algebra (5.25). We have also checked that the orbifold adjoint of the currents

$$\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})^\dagger = \sum_\nu \mathcal{R}_{n(r)\mu}{}^{-n(r),\nu}(\sigma)\hat{J}_{-n(r),\nu}(-m-\tfrac{n(r)}{\rho(\sigma)}) \tag{8.9a}$$

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})^\dagger = \sum_\nu \mathcal{R}_{n(r)\mu}{}^{-n(r),\nu}(\sigma)\hat{\bar{J}}_{-n(r),\nu}(-m-\tfrac{n(r)}{\rho(\sigma)}) \tag{8.9b}$$

defines a real form of the general twisted current algebra (8.8). Here, $\mathcal{R}$ is the orbifold conjugation matrix of Ref. 2.

The integral affine subalgebra of (8.8) is

$$[\hat{J}_{0\mu}(m), \hat{J}_{0\nu}(n)] = i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{J}_{0\delta}(m+n) + m\delta_{m+n,0}\mathcal{G}_{0\mu,0\nu}(\sigma) \tag{8.10a}$$

$$[\hat{\bar{J}}_{0\mu}(m), \hat{\bar{J}}_{0\nu}(n)] = i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{\bar{J}}_{0\delta}(m+n) + m\delta_{m+n,0}\mathcal{G}_{0\mu,0\nu}(\sigma) \tag{8.10b}$$

$$[\hat{J}_{0\mu}(m), \hat{\bar{J}}_{0\nu}(0)] = 0, \quad \hat{\bar{J}}_{0\mu}(m) \equiv \hat{\bar{J}}_{0\mu}(-m) \tag{8.10c}$$

where we have rectified the twisted right mover algebra by the relabelling in (8.10c). We also note the Lie subalgebra

$$[\hat{J}_{0\mu}(0), \hat{J}_{0\nu}(0)] = i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{J}_{0\delta}(0), \quad [\hat{\bar{J}}_{0\mu}(0), \hat{\bar{J}}_{0\nu}(0)] = i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{\bar{J}}_{0\delta}(0) \tag{8.11a}$$

$$[\hat{J}_{0\mu}(0), \hat{\bar{J}}_{0\nu}(0)] = 0, \quad \hat{\bar{J}}_{0\mu}(0) \equiv \hat{J}_{0\mu}(0) \tag{8.11b}$$

generated by the zero modes of the integral affine subalgebra.

Using derived isomorphisms, we also obtain the stress tensors of $A(H)/H$ and the OPE's of the stress tensors with the twisted currents

$$\hat{T}_\sigma(z) = \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{J}_{n(r)\mu}(z)\hat{J}_{-n(r),\nu}(z) :, \quad \hat{T}_\sigma(ze^{2\pi i}) = \hat{T}_\sigma(z) \tag{8.12a}$$

$$\hat{\bar{T}}_\sigma(\bar{z}) = \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{-n(r),\nu}(\bar{z}) :, \quad \hat{\bar{T}}_\sigma(\bar{z}e^{2\pi i}) = \hat{\bar{T}}_\sigma(\bar{z}) \tag{8.12b}$$

$$\mathcal{L}^{n(r)\mu;n(s)\nu}(\sigma) = \chi(\sigma)^{-1}_{n(r)\mu}\chi(\sigma)^{-1}_{n(s)\nu}L^{ab}U^\dagger(\sigma)_a{}^{n(r)\mu}U^\dagger(\sigma)_b{}^{n(s)\nu}, \quad \sigma = 0, \ldots, N_c - 1 \tag{8.12c}$$

$$\hat{c}(\sigma) = \hat{\bar{c}}(\sigma) = 2\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma) = 2G_{ab}L^{ab} = c = \bar{c} \tag{8.12d}$$

$$\begin{aligned}\hat{T}_\sigma(z)\hat{J}_{n(r)\mu}(w) =& \mathcal{M}_{n(r)\mu}{}^{n(r)\mu}(\sigma)[\frac{1}{(z-w)^2} + \frac{\partial_w}{(z-w)}]\hat{J}_{n(r)\mu}(w) \\ &+ \frac{\mathcal{N}_{n(r)\mu}{}^{n(s)\nu;n(r)-n(s),\delta} : \hat{J}_{n(s)\nu}(w)\hat{J}_{n(r)-n(s),\delta}(w) :}{z-w} + O((z-w)^0)\end{aligned} \tag{8.13a}$$

$$\hat{\bar{T}}_\sigma(\bar{z})\hat{\bar{J}}_{n(r)\mu}(\bar{w}) = \mathcal{M}_{n(r)\mu}{}^{n(r)\mu}(\sigma)[\frac{1}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}}}{(\bar{z}-\bar{w})}]\hat{\bar{J}}_{n(r)\mu}(\bar{w}) \tag{8.13b}$$

$$+ \frac{\mathcal{N}_{n(r)\mu}{}^{n(s)\nu;n(r)-n(s),\delta} : \hat{\bar{J}}_{n(s)\nu}(w)\hat{\bar{J}}_{n(r)-n(s),\delta}(\bar{w}) :}{\bar{z}-\bar{w}} + O((\bar{z}-\bar{w})^0) \tag{8.13c}$$

where $\mathcal{L}(\sigma)$ in (8.12c) is the twisted inverse inertia tensor of sector $\sigma$ and $:(\cdot):$ means OPE normal ordering[4,1,2]. Both stress tensors are Virasoro when the $H$-invariant[15] inverse inertia tensor $L^{ab}$ satisfies the Virasoro master equation[23–25]. The twisted tensors $\mathcal{M}$ and $\mathcal{N}$ in (8.13) are given as duality transformations in Ref. 2. Together with the general twisted current system in (8.6), we see that the basic OPE's of current-algebraic orbifold theory are left-right copies.

For permutation orbifolds it is clear from our discussion and Ref. 2 that the ground state (twist field state) of sector $\sigma$ satisfies

$$\hat{J}_{n(r)\mu}(m + \tfrac{n(r)}{\rho(\sigma)} \geq 0)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{J}_{n(r)\mu}(m + \tfrac{n(r)}{\rho(\sigma)} \leq 0) = 0 \tag{8.14a}$$

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}\leq 0)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}\geq 0) = 0 \tag{8.14b}$$

$$\hat{J}_{0\mu}(0)|0\rangle_\sigma = \hat{\bar{J}}_{0\mu}(0)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{J}_{0\mu}(0) = {}_\sigma\langle 0|\hat{\bar{J}}_{0\mu}(0) = 0 \tag{8.14c}$$

so that the ground state of each sector is a singlet under the zero modes (the conserved quantities) of the integral affine subalgebra. It is not difficult to check that these ground states have the same conformal weights under the right and left mover stress tensors

$$L(m\geq 0)|0\rangle_\sigma = \bar{L}(m\geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\Delta}_0(\sigma)|0\rangle_\sigma \tag{8.15}$$

where $\hat{\Delta}_0(\sigma)$ is given in Ref. 2.

For the special case of the WZW Orbifolds $A_g(H)/H$, take[1,2]

$$\mathcal{L}_{\hat{g}(\sigma)}^{n(r)\mu;n(s)\nu}(\sigma) = \chi(\sigma)^{-1}_{n(r)\mu}\chi(\sigma)^{-1}_{n(s)\nu}L_g^{ab}U^\dagger(\sigma)_a{}^{n(r)\mu}U^\dagger(\sigma)_b{}^{n(s)\nu} \tag{8.16a}$$

$$\mathcal{M}_{n(r)\mu}{}^{n(s)\nu}(\sigma) = \delta_{n(r)\mu}{}^{n(s)\nu}, \quad \mathcal{N}_{n(r)\mu}{}^{n(s)\nu;n(t)\delta}(\sigma) = 0 \tag{8.16b}$$

where $L_g^{ab}$ is the inverse inertia tensor of the affine-Sugawara construction[6-11] on $g$. In this case our group orbifold elements $\hat{g}$ are closely related to the (Virasoro and) twisted affine primary fields of the WZW orbifolds, a subject which deserves further study.

We also include two general remarks on the rectification problem (see Subsec. 5.4 and Sec. 7). First, in order to succeed in a rectification, it seems that we need to be able to find $g$ automorphisms $A(\sigma)$ which relate the action $w(h_\sigma^{-1}) = w^\dagger(h_\sigma)$ to the action $w(h_\sigma)$:

$$A^\dagger(\sigma)w^\dagger(h_\sigma)A(\sigma) = w(h_\sigma), \quad A^\dagger(\sigma)A(\sigma) = 1, \quad A(\sigma)\in Aut(g)\ . \tag{8.17}$$

Here is a quick argument that the existence of such automorphisms is sufficient to rectify the twisted right mover algebra. The currents $\hat{\mathcal{J}}$, $\hat{\bar{\mathcal{J}}}$ with twisted boundary conditions (see Ref. 1 and App. C) satisfy

$$\hat{\mathcal{J}}_a(z)\hat{\mathcal{J}}_b(w) = \tfrac{G_{ab}}{(z-w)^2} + \tfrac{if_{ab}{}^c}{z-w}\hat{\mathcal{J}}_c(w) + O((z-w)^0),\ \hat{\bar{\mathcal{J}}}_a(\bar{z})\hat{\bar{\mathcal{J}}}_b(\bar{w}) = \tfrac{G_{ab}}{(\bar{z}-\bar{w})^2} + \tfrac{if_{ab}{}^c}{\bar{z}-\bar{w}}\hat{\bar{\mathcal{J}}}_c(\bar{w}) + O((\bar{z}-\bar{w})^0) \tag{8.18a}$$

$$\hat{\mathcal{J}}_a(ze^{2\pi i}) = w(h_\sigma)_a{}^b\hat{\mathcal{J}}_b(z),\ \hat{\bar{\mathcal{J}}}_a(\bar{z}e^{-2\pi i}) = w(h_\sigma)_a{}^b\hat{\bar{\mathcal{J}}}_b(\bar{z})\ . \tag{8.18b}$$

But then we find from (8.17) that the linear combinations $\hat{\bar{\bar{\mathcal{J}}}}$ of the right movers

$$\hat{\bar{\bar{\mathcal{J}}}}_a(\bar{z}) \equiv A(\sigma)_a{}^b\hat{\bar{\mathcal{J}}}(\bar{z})_b \tag{8.19a}$$

$$\hat{\bar{\bar{\mathcal{J}}}}_a(\bar{z})\hat{\bar{\bar{\mathcal{J}}}}_b(\bar{w}) = \tfrac{G_{ab}}{(\bar{z}-\bar{w})^2} + \tfrac{if_{ab}{}^c}{\bar{z}-\bar{w}}\hat{\bar{\bar{\mathcal{J}}}}_c(\bar{w}) + O((\bar{z}-\bar{w})^0), \quad \hat{\bar{\bar{\mathcal{J}}}}_a(\bar{z}e^{2\pi i}) = w(h_\sigma)_a{}^b\hat{\bar{\bar{\mathcal{J}}}}_b(\bar{z}) \tag{8.19b}$$

satisfy the same OPE's and twisted boundary conditions as the left movers $\hat{\mathcal{J}}_a(z)$. It follows that the monodromy decompositions and mode algebras of both are identical.

Finally, although we have so far found no examples, suppose that there are some twisted current algebras whose twisted right movers cannot be rectified. The possibility that the twisted right and left mover current algebras might not be copies of each other has not

been mentioned explicitly in the literature, but in fact there is no contradiction between this possibility and what is known about partition functions and characters of general orbifolds[26]. The point is that when one sees

$$Z = \sum_{\sigma,i} |\chi_i(q,\sigma)|^2 \tag{8.20}$$

in the literature, it is supposed to be read as

$$Z = \sum_{\sigma,i} \chi_i(q,\sigma)\chi_{i^\vee}(\bar{q},\sigma) \tag{8.21}$$

where $i^\vee$ denotes the charge conjugate field of $i$. Moreover, it is known that when $i$ is a representation in the sectors twisted by $h_\sigma$ then $i^\vee$ is a representation in the sector twisted by $h_\sigma^{-1}$ (so that the two sectors can combine to the identity). Thus, in our notation, what is required by (8.21)

$$Z = \sum_{\sigma,i} \chi_i(q,h_\sigma)\chi_{i^\vee}(\bar{q},h_\sigma^{-1}) \tag{8.22}$$

is in precise agreement with the form $\hat{\mathfrak{g}}(h_\sigma) \oplus \hat{\mathfrak{g}}(h_\sigma^{-1})$ of the general twisted current algebra (8.8).

## 8.3 Orbifold Einstein Equations

The nonlinear sigma model is 1-loop conformal when the sigma model Einstein equations[27–32]

$$R_{ij} + \frac{1}{4}H_{ki}{}^{l}H^{k}{}_{lj} - 2\nabla_i\nabla_j\phi = 0, \quad (\nabla^k - 2\nabla^k\phi)H_{kij} = 0 \tag{8.23}$$

are satisfied, where $\phi$ is the dilaton field.

Using eigen-objects and local isomorphisms (see below), we have been able to find a set of *orbifold Einstein equations*

$$\hat{R}_{n(r)\mu;n(s)\nu}(\hat{x}) + \frac{1}{4}\hat{H}_{n(t)\delta;n(r)\mu}{}^{n(u)\epsilon}(\hat{x})\,\hat{H}^{n(t)\delta}{}_{n(u)\epsilon;n(s)\nu}(\hat{x}) - 2\hat{\nabla}_{n(r)\mu}\hat{\nabla}_{n(s)\nu}\hat{\phi}(\hat{x}) = 0 \tag{8.24a}$$

$$\left(\hat{\nabla}^{n(t)\delta} - 2\hat{\nabla}^{n(t)\delta}\hat{\phi}(\hat{x})\right)\hat{H}_{n(t)\delta;n(r)\mu;n(s)\nu}(\hat{x}) = 0, \quad \sigma = 0,\ldots,N_c-1 \tag{8.24b}$$

$$\hat{x} \equiv \hat{x}_\sigma, \quad \hat{R}(\hat{x}) \equiv \hat{R}(\hat{x}_\sigma,\sigma), \quad \hat{H}(\hat{x}) \equiv \hat{H}(\hat{x}_\sigma,\sigma), \quad \hat{\phi}(\hat{x}) \equiv \hat{\phi}(\hat{x}_\sigma,\sigma) \tag{8.24c}$$

which hold in sector $\sigma$ of the sigma model orbifolds of Subsec. 6.3 when the Einstein equations (8.23) hold in the symmetric sigma model. Here $\{\hat{x}\}$ are the twisted Einstein coordinates, $\hat{\phi}(\hat{x})$ is the orbifold dilaton and it turns out that all the twisted tensors are constructed in analogy to the usual untwisted Einstein tensors, following the duality algorithm

$$i \underset{\sigma}{\longrightarrow} n(r)\mu, \quad x^i \underset{\sigma}{\longrightarrow} \hat{x}^{n(r)\mu}, \quad \partial_i \underset{\sigma}{\longrightarrow} \hat{\partial}_{n(r)\mu}, \quad \nabla_i \underset{\sigma}{\longrightarrow} \hat{\nabla}_{n(r)\mu} \tag{8.25a}$$

$$G_{ij}(x) \underset{\sigma}{\longrightarrow} \hat{G}_{n(r)\mu;n(s)\nu}(\hat{x}), \quad G^{ij}(x) \underset{\sigma}{\longrightarrow} \hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}) \tag{8.25b}$$

$$\Gamma_{ij}{}^{k}(x) \underset{\sigma}{\longrightarrow} \hat{\Gamma}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\hat{x}), \quad R_{ijkl}(x) \underset{\sigma}{\longrightarrow} \hat{R}_{n(r)\mu;n(s)\nu;n(t)\delta;n(u)\epsilon}(\hat{x}) \tag{8.25c}$$

(see also Subsec. 6.5). For example, the twisted torsion field $\hat{H}$ satisfies Eq. (6.5a) in terms of the twisted $B$ field $\hat{B}$, while the twisted Christoffel connections $\hat{\Gamma}$ have the form

$$\hat{\Gamma}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\hat{x}) = \frac{1}{2}\big(\,\hat{\partial}_{(n(r)\mu}\hat{G}_{n(s)\nu);n(u)\epsilon}(\hat{x}) - \hat{\partial}_{n(u)\epsilon}\hat{G}_{n(r)\mu;n(s)\nu}(\hat{x})\,\big)\,\hat{G}^{n(u)\epsilon;n(t)\delta}(\hat{x}) \tag{8.26}$$

in terms of the twisted Einstein metric $\hat{G}_\bullet$ and the inverse twisted Einstein metric $\hat{G}^\bullet$.

We also obtain the explicit forms and monodromies of all the twisted tensors, including the results for $\hat{G}_\bullet$ and $\hat{B}$ in Subsec. 6.3, and for example,

$$\hat{x}^{n(r)\mu}(\xi + 2\pi) = \hat{x}^{n(r)\mu}(\xi)e^{2\pi i\frac{n(r)}{\rho(\sigma)}} \tag{8.27}$$

$$\hat{\phi}(\hat{x}) = \phi(x \underset{\sigma}{\rightarrow} \hat{\mathfrak{X}}(\hat{x})), \quad \hat{\phi}(\hat{x}(\xi + 2\pi)) = \hat{\phi}(\hat{x}(\xi)) \tag{8.28}$$

$$\hat{\phi}_{n(r)\mu}(\hat{x}) \equiv \hat{\nabla}_{n(r)\mu}\hat{\phi}(\hat{x}) = \hat{\partial}_{n(r)\mu}\hat{\phi}(\hat{x}), \quad \hat{\phi}_{n(r)\mu}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\phi}_{n(r)\mu}(\hat{x}(\xi)) \tag{8.29a}$$

$$\hat{\nabla}_{n(r)\mu}\hat{\phi}_{n(s)\nu}(\hat{x}) = \hat{\partial}_{n(r)\mu}\hat{\phi}_{n(s)\nu}(\hat{x}) - \hat{\Gamma}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\hat{x})\hat{\phi}_{n(t)\delta}(\hat{x}) \tag{8.29b}$$

$$\hat{\nabla}_{n(r)\mu}\hat{\phi}_{n(s)\nu}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i\frac{n(r)+n(s)}{\rho(\sigma)}}\hat{\nabla}_{n(r)\mu}\hat{\phi}_{n(s)\nu}(\hat{x}(\xi)) \tag{8.29c}$$

$$\hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}) = \chi^{-1}_{n(r)\mu}\chi^{-1}_{n(s)\nu}G^{ij}(x \underset{\sigma}{\rightarrow} \hat{\mathfrak{X}}(\hat{x}))(U^\dagger)_i{}^{n(r)\mu}(U^\dagger)_j{}^{n(s)\nu} \tag{8.30a}$$

$$\hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}(\xi + 2\pi)) = \hat{G}^{n(r)\mu;n(s)\nu}(\hat{x}(\xi))e^{2\pi i\frac{n(r)+n(s)}{\rho(\sigma)}} \tag{8.30b}$$

$$\hat{\Gamma}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\hat{x}) = \chi_{n(r)\mu}\chi_{n(s)\nu}\chi^{-1}_{n(t)\delta}U_{n(r)\mu}{}^iU_{n(s)\nu}{}^j\Gamma_{ij}{}^k(x \underset{\sigma}{\rightarrow} \hat{\mathfrak{X}}(\hat{x}))(U^\dagger)_k{}^{n(t)\delta} \tag{8.31a}$$

$$\hat{\Gamma}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i\frac{n(r)+n(s)-n(t)}{\rho(\sigma)}}\hat{\Gamma}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\hat{x}(\xi)) \tag{8.31b}$$

$$\hat{R}_{n(r)\mu;n(s)\nu}(\hat{x}) = \hat{R}_{n(r)\mu;n(s)\nu;n(t)\delta;n(u)\epsilon}(\hat{x})\hat{G}^{n(t)\delta;n(u)\epsilon}(\hat{x}) \tag{8.32a}$$

$$= \chi_{n(r)\mu}\chi_{n(s)\nu}U_{n(r)\mu}{}^iU_{n(s)\nu}{}^j(R_{ijkl}G^{kl})(x \underset{\sigma}{\rightarrow} \hat{\mathfrak{X}}(\hat{x})) \tag{8.32b}$$

$$\hat{R}_{n(r)\mu;n(s)\nu;n(t)\delta;n(u)\epsilon}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i\frac{n(r)+n(s)+n(t)+n(u)}{\rho(\sigma)}}\hat{R}_{n(r)\mu;n(s)\nu;n(t)\delta;n(u)\epsilon}(\hat{x}(\xi)) \tag{8.32c}$$

$$\hat{R}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi + 2\pi)) = e^{-2\pi i\frac{n(r)+n(s)}{\rho(\sigma)}}\hat{R}_{n(r)\mu;n(s)\nu}(\hat{x}(\xi)) \tag{8.32d}$$

$$\hat{H}_{n(r)\mu;n(s)\nu;n(t)\delta}(\hat{x}) = \chi_{n(r)\mu}\chi_{n(s)\nu}\chi_{n(t)\delta}U_{n(r)\mu}{}^iU_{n(s)\nu}{}^jU_{n(t)\delta}{}^kH_{ijk}(x \underset{\sigma}{\rightarrow} \hat{\mathfrak{X}}(\hat{x})) \tag{8.33}$$

$$U_{n(r)\mu}{}^l \equiv U(\sigma)_{n(r)\mu}{}^l, \quad (U^\dagger)_{n(r)\mu}{}^l \equiv U^\dagger(\sigma)_{n(r)\mu}{}^l, \quad \chi_{n(r)\mu} \equiv \chi(\sigma)_{n(r)\mu}\,. \tag{8.34}$$

The quantities $\hat{\mathfrak{X}}(\hat{x})$ are defined in (6.19a), so that each twisted tensor is determined as a function of the twisted coordinates $\hat{x}$, given the coordinate dependence of the untwisted tensors. As in Subsec. 6.3, selection rules which guarantee the indicated monodromies can be obtained for the moments of each twisted tensor.

The derivation of this result goes as follows: All Lorentz tensors in the symmetric sigma model (6.14) transform under the symmetry group $H$ with a factor $w$ on the left for a down

index and a factor of $w^\dagger$ on the right for an up index. For example, we have (6.14b), (6.14c) and

$$G^{ij}(x)' = G^{ij}(x') = G^{kl}(x)(w^\dagger)_k{}^i(w^\dagger)_l{}^j, \quad \partial_i' = w_i{}^j\partial_j, \quad \nabla_i' = w_i{}^j\nabla_j \tag{8.35a}$$

$$\Gamma_{ij}{}^l(x)' = \Gamma_{ij}{}^l(x') = w_i{}^p w_j{}^q \Gamma_{pq}{}^s(x)(w^\dagger)_s{}^l, \quad \phi(x)' = \phi(x') = \phi(x)\ . \tag{8.35b}$$

This tells us that the corresponding eigentensors should be defined with a factor $\chi U$ or $\chi^{-1}U^\dagger$ for each up or down index respectively. The Einstein equations have the same form in terms of the eigentensors as they do in terms of the original tensors. Then the orbifold Einstein equations (8.24) and the explicit forms (8.27)-(8.34) follow by local isomorphisms. In the end, each twisted down index carries a monodromy $e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}$ and each twisted up index carries a monodromy $e^{+2\pi i\frac{n(r)}{\rho(\sigma)}}$, so that objects which are Lorentz scalars, such as the orbifold curvature scalar $\hat{R}$ and the orbifold dilaton $\hat{\phi}$, have trivial monodromy.

The orbifold Einstein equations (8.24) are very likely to be the condition that twisted sector $\sigma$ of a sigma model orbifold is conformal, but a direct check (a one-loop computation using the orbifold sigma model action $\hat{S}_\sigma^{NLS}$ in (6.18)) is necessary to be sure. This situation is analogous to that of the general orbifold Virasoro master equation[1,2] which follows from local isomorphisms, but which also needed a direct check using orbifold OPE's. It is expected that this check will go through for the sigma model orbifolds because (as seen for the general orbifold Virasoro master equation) the short-distance singularities of the orbifold computation are expected to follow the duality algorithm (8.25).

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## Appendix A. New Braid Relations

The braid relations

$$W^\dagger(h_\sigma;T)T_{n(r)\mu}(\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}T_{n(r)\mu}(\sigma)W^\dagger(h_\sigma;T),\ T_{n(r)\mu}(\sigma) \equiv \chi(\sigma)_{n(r)\mu}U(\sigma)_{n(r)\mu}{}^a T_a \tag{A.1}$$

follow easily (start with $W^\dagger T_{n(r)\mu}W$) from the linkage relation (3.6) and the $H$-eigenvalue problem (3.13). It is expected that the linkage relation leads to other connections of this type between the $H$-eigenvalue problem and its extended form. We will encounter the matrices $T_{n(r)\mu}$ again in Appendix C.

## Appendix B. Unitarity and the Orbifold Matrix Adjoint Operation

For the untwisted representation matrices $T$ and the untwisted tangent space coordinates $\beta$, we have

$$T_a^\dagger = \rho_a{}^b T_b, \quad \beta^{a*} = \beta^b \rho_b{}^{a*}, \tag{B.1a}$$
$$\rho^* \rho = 1, \quad w(h_\sigma)^* \, \rho \, w^\dagger(h_\sigma) = \rho \tag{B.1b}$$

where dagger is matrix adjoint, star is complex conjugation and $\rho$ is the complex conjugation matrix of Refs. 25 and 2. It follows from (B.1) that the linkage relation (3.6a) is preserved under the matrix adjoint operation, and that the untwisted group elements are unitary

$$g(T,\xi) = e^{i\beta^a(\xi)T_a}, \quad g^\dagger(T,\xi) = e^{-i\beta^{a*}T_a^\dagger} = g^{-1}(T,\xi) \tag{B.2}$$

as stated in the text.

Using Eqs. (4.24d), (5.1b) and following Ref. 2, we find for the corresponding eigenobjects and twisted structures:

$$\mathcal{T}_{n(r)\mu}(T,\sigma)^\dagger = \mathcal{R}_{n(r)\mu}{}^{n(s)\nu}(\sigma)\mathcal{T}_{n(s)\nu}(T,\sigma) = \mathcal{R}_{n(r)\mu}{}^{-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma) \tag{B.3a}$$
$$\mathfrak{b}^{n(r)\mu}(\xi)^* = \mathfrak{b}^{n(s)\nu}(\xi)\mathcal{R}_{n(s)\nu}{}^{n(r)\mu}(\sigma)^* = \mathfrak{b}^{-n(r),\nu}(\xi)\mathcal{R}_{-n(r),\nu}{}^{n(r)\mu}(\sigma)^* \tag{B.3b}$$
$$\hat{\beta}^{n(r)\mu}(\xi)^* = \hat{\beta}^{n(s)\nu}(\xi)\mathcal{R}_{n(s)\nu}{}^{n(r)\mu}(\sigma)^* = \hat{\beta}^{-n(r),\nu}(\xi)\mathcal{R}_{-n(r),\nu}{}^{n(r)\mu}(\sigma)^* \tag{B.3c}$$

$$\begin{aligned}\mathcal{R}_{n(r)\mu}{}^{n(s)\nu}(\sigma) =& \chi(\sigma)_{n(r)\mu}{}^* \, \chi(\sigma)^{-1}_{n(s)\nu} \, U(\sigma)_{n(r)\mu}{}^{a*} \rho_a{}^b U^\dagger(\sigma)_b{}^{n(s)\nu} \\ =& \delta_{n(r)+n(s),\,0\,mod\,\rho(\sigma)} \mathcal{R}_{n(r)\mu}{}^{-n(r),\nu}(\sigma), \quad \mathcal{R}(\sigma)^*\mathcal{R}(\sigma) = 1.\end{aligned} \tag{B.3d}$$

Eq.(B.3a) defines the *orbifold matrix adjoint operation*, where $\mathcal{R}(\sigma)$ in (B.3d) is the orbifold conjugation matrix of Ref. 2. The orbifold conjugation matrix is dual to the conjugation matrix $\rho$, and also controls the orbifold adjoint operation of the twisted current modes (see Eq. (8.9)). Unitarity of the eigengroup elements and the group orbifold elements

$$\mathcal{G}(\mathcal{T},\xi,\sigma) = e^{i\mathfrak{b}^{n(r)\mu}(\xi)\mathcal{T}_{n(r)\mu}(T,\sigma)} \longrightarrow \mathcal{G}^\dagger(\mathcal{T},\xi,\sigma) = \mathcal{G}^{-1}(\mathcal{T},\xi,\sigma). \tag{B.4a}$$
$$\hat{g}(\mathcal{T},\xi,\sigma) = e^{i\hat{\beta}^{n(r)\mu}(\xi)\mathcal{T}_{n(r)\mu}(T,\sigma)} \longrightarrow \hat{g}^\dagger(\mathcal{T},\xi,\sigma) = \hat{g}^{-1}(\mathcal{T},\xi,\sigma) \tag{B.4b}$$

follows from the relations in (B.3).

## Appendix C. Twisted Boundary Conditions

Other objects of interest in WZW orbifolds are the group orbifold elements $\hat{\mathcal{G}}$ with twisted boundary conditions

$$\hat{\mathcal{G}}(T,\xi,\sigma)_\alpha{}^\beta : \qquad \hat{\mathcal{G}}(T,\xi+2\pi,\sigma) = W(h_\sigma;T)\hat{\mathcal{G}}(T,\xi,\sigma)W^\dagger(h_\sigma;T) \tag{C.1}$$

where $W(h_\sigma;T)$ is the action of $h_\sigma \in H \subset Aut(g)$ in matrix representation $T$ (see Subsec. 3.1). These are objects with mixed monodromy which are locally isomorphic to the untwisted group elements $g$:

$$\hat{\mathcal{G}}(T,\xi,\sigma) = U^\dagger(T,\sigma)\hat{g}(\mathcal{T},\xi,\sigma)U(T,\sigma) = e^{i\hat{\beta}^{n(r)\mu}(\xi,\sigma)T_{n(r)\mu}(\sigma)} = e^{i\hat{\mathfrak{b}}^a(\xi,\sigma)T_a} \tag{C.2a}$$
$$T_{n(r)\mu}(\sigma) = \chi(\sigma)_{n(r)\mu}U(\sigma)_{n(r)\mu}{}^a T_a, \quad \hat{\mathfrak{b}}^a(\xi,\sigma) \equiv \chi(\sigma)_{n(r)\mu}\hat{\beta}^{n(r)\mu}(\xi,\sigma)U(\sigma)_{n(r)\mu}{}^a \tag{C.2b}$$

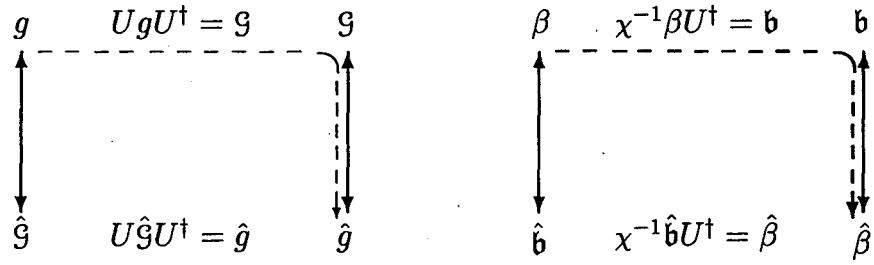
$$\hat{\mathfrak{b}}^a(\xi+2\pi,\sigma) = \hat{\mathfrak{b}}^b(\xi,\sigma)w^\dagger(h_\sigma)_b{}^a \tag{C.2c}$$

$$\hat{g}(\mathcal{T},\xi,\sigma)=U(\mathcal{T},\sigma)\hat{\mathcal{G}}(\mathcal{T},\xi,\sigma)U^\dagger(\mathcal{T},\sigma),\ \hat{\beta}^{n(r)\mu}(\xi,\sigma)=\hat{\mathfrak{b}}^a(\xi,\sigma)U^\dagger(\sigma)_a{}^{n(r)\mu}\chi(\sigma)^{-1}_{n(r)\mu}\,. \tag{C.2d}$$

The matrices $T_{n(r)\mu}$ were defined in Appendix A and the quantities $\{\hat{\mathfrak{b}}^a\}$, which appeared in (5.3), are identified in (C.2c) as tangent space coordinates with twisted boundary conditions. According to (C.2d), the monodromy decompositions of $\hat{\mathcal{G}}$ and $\hat{\mathfrak{b}}$ give the group orbifold elements $\hat{g}$ and the twisted tangent space coordinates $\hat{\beta}$. In summary, we have the relations

$$g = e^{i\beta\cdot\mathcal{T}},\quad \mathcal{G} = e^{i\mathfrak{b}\cdot\mathcal{T}},\quad \hat{\mathcal{G}} = e^{i\hat{\mathfrak{b}}\cdot\mathcal{T}},\quad \hat{g} = e^{i\hat{\beta}\cdot\mathcal{T}}\,. \tag{C.3}$$

The main discussion of the text follows the dotted line in the commuting diagram of Fig. 1, but the same results can be obtained by following the path $g \to \hat{\mathcal{G}} \to \hat{g}$.




Each vertical double arrow is a local isomorphism
$g$ = Lie group elements: trivial monodromy, mixed under automorphisms
$\mathcal{G}$ = eigengroup elements: trivial monodromy, diagonal under automorphisms
$\hat{g}$ = group orbifold elements with definite monodromy
$\hat{\mathcal{G}}$ = group orbifold elements with twisted boundary conditions

Fig. 1: Group and group orbifold elements

A closely-related commuting diagram (see Fig. 2) was given for currents and orbifold currents in Ref. 1. In particular, the currents $\hat{\mathcal{J}}$, $\hat{\bar{\mathcal{J}}}$ with twisted boundary conditions

$$\hat{\mathcal{J}}_a(\xi+2\pi,\sigma) = w(h_\sigma)_a{}^b\hat{\mathcal{J}}_b(\xi,\sigma),\quad \hat{\bar{\mathcal{J}}}_a(\xi+2\pi,\sigma) = w(h_\sigma)_a{}^b\hat{\bar{\mathcal{J}}}_b(\xi,\sigma) \tag{C.4a}$$

$$\hat{\mathcal{J}}_a(\xi,\sigma)=U^\dagger(\sigma)_a{}^{n(r)\mu}\chi(\sigma)^{-1}_{n(r)\mu}\hat{J}_{n(r)\mu}(\xi,\sigma),\ \hat{\bar{\mathcal{J}}}_a(\xi,\sigma)=U^\dagger(\sigma)_a{}^{n(r)\mu}\chi(\sigma)^{-1}_{n(r)\mu}\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma) \tag{C.4b}$$

$$\hat{J}_{n(r)\mu}(\xi,\sigma)=\chi(\sigma)_{n(r)\mu}U(\sigma)_{n(r)\mu}{}^a\hat{\mathcal{J}}_a(\xi,\sigma),\ \hat{\bar{J}}_{n(r)\mu}(\xi,\sigma)=\chi(\sigma)_{n(r)\mu}U(\sigma)_{n(r)\mu}{}^a\hat{\bar{\mathcal{J}}}_a(\xi,\sigma) \tag{C.4c}$$

are objects with mixed monodromy which are locally isomorphic to the untwisted currents $J$, $\bar{J}$. According to (C.4c), the monodromy decompositions of $\hat{\mathcal{J}}$, $\hat{\bar{\mathcal{J}}}$ give the twisted currents $\hat{J}$, $\hat{\bar{J}}$ of the text.

$$
\begin{tikzcd}
(J,\bar{J}) \arrow[r, dashed, bend right, "\chi U(J,\bar{J}) = (\mathcal{J},\bar{\mathcal{J}})"] \arrow[d, leftrightarrow] & (\mathcal{J},\bar{\mathcal{J}}) \arrow[d, leftrightarrow] \\
(\hat{\mathcal{J}},\hat{\bar{\mathcal{J}}}) \arrow[r, dashed, bend right, "\chi U(\hat{\mathcal{J}},\hat{\bar{\mathcal{J}}}) = (\hat{J},\hat{\bar{J}})"] & (\hat{J},\hat{\bar{J}})
\end{tikzcd}
$$

Each vertical double arrow is a local isomorphism
$J, \bar{J}$ = currents: trivial monodromy, mixed under automorphisms
$\mathcal{J}, \bar{\mathcal{J}}$ = eigencurrents: trivial monodromy, diagonal under automorphisms
$\hat{J}, \hat{\bar{J}}$ = twisted currents with definite monodromy
$\hat{\mathcal{J}}, \hat{\bar{\mathcal{J}}}$ = currents with twisted boundary conditions

Fig. 2: Currents and orbifold currents

Finally, one may consider Einstein coordinates $\hat{\mathfrak{X}}$ with twisted boundary conditions

$$\hat{\mathfrak{X}}^i_\sigma(\xi + 2\pi) = \hat{\mathfrak{X}}^j_\sigma(\xi) w^\dagger (h_\sigma)_j{}^i \tag{C.5}$$

in the geometric description of WZW orbifolds or the sigma model orbifolds of Subsec. 6.3. These coordinates are also objects with mixed monodromy

$$\hat{x}^{n(r)\mu}_\sigma(\xi) = \chi(\sigma)^{-1}_{n(r)\mu} \hat{\mathfrak{X}}^i_\sigma(\xi) U^\dagger(\sigma)_i{}^{n(r)\mu}, \quad \hat{\mathfrak{X}}^i_\sigma(\xi) = \chi(\sigma)_{n(r)\mu} \hat{x}^{n(r)\mu}_\sigma(\xi) U(\sigma)_{n(r)\mu}{}^i \tag{C.6}$$

which are locally isomorphic to the untwisted Einstein coordinates $x^i$. We have encountered the objects $\hat{\mathfrak{X}} \equiv \hat{\mathfrak{X}}(\hat{x})$ and the local isomorphism $x \underset{\sigma}{\longrightarrow} \hat{\mathfrak{X}}$ in Eqs. (6.19), (6.29) and (8.27)-(8.34). According to (C.6), the monodromy decomposition of $\hat{\mathfrak{X}}$ gives our twisted Einstein coordinates $\hat{x}$. These results and the discussion of the text give the commuting diagram shown in Fig. 3. For the special case of the WZW orbifolds, the preferred coordinate system chosen in the text is $x = \beta$, $\mathfrak{X} = \mathfrak{b}$, $\hat{\mathfrak{X}} = \hat{\mathfrak{b}}$ and $\hat{x} = \hat{\beta}$ where $\beta$, $\mathfrak{b}$, $\hat{\mathfrak{b}}$ and $\beta$ are the tangent space coordinates and orbifold tangent space coordinates in Fig. 1.

$$
\begin{tikzcd}
x \arrow[r, dashed, bend right, "\chi^{-1} x U^\dagger = \mathfrak{X}"] \arrow[d, leftrightarrow] & \mathfrak{X} \arrow[d, leftrightarrow] \\
\hat{\mathfrak{X}} \arrow[r, dashed, bend right, "\chi^{-1} \hat{\mathfrak{X}} U^\dagger = \hat{x}"] & \hat{x}
\end{tikzcd}
$$

Each vertical double arrow is a local isomorphism
$x$ = coordinates: trivial monodromy, mixed under automorphism
$\mathfrak{X}$ = eigencoordinates: trivial monodromy, diagonal under automorphisms
$\hat{x}$ = twisted coordinates with definite monodromy
$\hat{\mathfrak{X}}$ = coordinates with twisted boundary conditions

Fig. 3: Coordinates and orbifold coordinates

## Appendix D. Rescaled Group Orbifold Elements

Another interesting class of objects are the rescaled group orbifold elements $\tilde{g}$

$$\tilde{g}(\tilde{\mathcal{T}},\xi,\sigma) \equiv \chi(T,\sigma)\hat{g}(\mathcal{T},\xi,\sigma)\chi(T,\sigma)^{-1} = e^{i\hat{x}_\sigma^{n(r)\mu}(\xi)\tilde{\mathcal{T}}_{n(r)\mu}(T,\sigma)} \quad (\hat{\beta}=\hat{x}) \tag{D.1a}$$

$$\chi(T,\sigma)_{N(r)\mu}{}^{N(s)\nu} = \delta_{N(r)\mu}{}^{N(s)\nu}\chi_{N(r)\mu}(T,\sigma) \tag{D.1b}$$

$$\tilde{g}(\tilde{\mathcal{T}},\xi+2\pi,\sigma))_{N(r)\mu}{}^{N(s)\nu} = e^{-\frac{2\pi i}{R(\sigma)}(N(r)-N(s))}\tilde{g}(\tilde{\mathcal{T}},\xi,\sigma))_{N(r)\mu}{}^{N(s)\nu} \tag{D.1c}$$

$$\hat{g}(\mathcal{T},\xi,\sigma) = \chi(T,\sigma)^{-1}\tilde{g}(\tilde{\mathcal{T}},\xi,\sigma)\chi(T,\sigma) \tag{D.1d}$$

where $\hat{g}$ is the group orbifold element and $\tilde{\mathcal{T}}_{n(r)\mu}$ are the rescaled twisted representation matrices defined in Eq. (4.23). We have already encountered an example of the rescaled group orbifold elements, namely the twisted adjoint action $\hat{\Omega}$ in Subsec. 6.1

$$\hat{\Omega}(\hat{x}(\xi)) = e^{-i\hat{H}(\hat{x}(\xi))} = e^{-i\hat{x}_\sigma^{n(r)\mu}(\xi)\tilde{\mathcal{T}}_{n(r)\mu}(T^{adj},\sigma)} = \tilde{g}^{-1}(\tilde{\mathcal{T}}(T^{adj}),\xi,\sigma) \tag{D.2}$$

which also occurs in the twisted vielbeins and twisted $B$ fields.

We have refrained from using the rescaled group orbifold elements because their unitarity depends on the choice of the normalization constants $\chi(T,\sigma)$. In particular, using the unitarity of the group orbifold element $\hat{g}$, one finds that

$$\tilde{g}^\dagger(\tilde{\mathcal{T}},\xi,\sigma) = (\chi(T,\sigma)\chi(T,\sigma)^*)^{-1}\tilde{g}^{-1}(\tilde{\mathcal{T}},\xi,\sigma)(\chi(T,\sigma)\chi(T,\sigma)^*) \tag{D.3}$$

so that the rescaled group orbifold elements $\tilde{g}$ are unitary only when $\chi\chi^* \propto \mathbb{1}$ .

Nevertheless, the results of this paper can be rewritten in terms of the rescaled group orbifold elements, for example,

$$\begin{aligned}\hat{S}_\sigma(\tilde{\mathcal{T}}) \equiv \hat{S}_\sigma(\mathcal{T}(\tilde{\mathcal{T}})) = &-\frac{1}{8\pi}\int d^2\xi\, \widehat{Tr}\big(\tilde{\mathcal{M}}(\tilde{\mathcal{T}},\sigma)\,\tilde{g}^{-1}(\tilde{\mathcal{T}},\sigma)\partial_+\tilde{g}(\tilde{\mathcal{T}},\sigma)\,\tilde{g}^{-1}(\tilde{\mathcal{T}},\sigma)\partial_-\tilde{g}(\tilde{\mathcal{T}},\sigma)\big) \\ &-\frac{1}{12\pi}\int_\Gamma \widehat{Tr}\big(\tilde{\mathcal{M}}(\tilde{\mathcal{T}},\sigma)\,(\,\tilde{g}^{-1}(\tilde{\mathcal{T}},\sigma)d\tilde{g}(\tilde{\mathcal{T}},\sigma)\,)^3\big)\end{aligned} \tag{D.4a}$$

$$\tilde{\mathcal{M}}(\tilde{\mathcal{T}},\sigma) \equiv \chi(T,\sigma)\mathcal{M}(\mathcal{T},\sigma)\chi(T,\sigma)^{-1} \tag{D.4b}$$

$$\{\hat{J}_{n(r)\mu}(\xi,\sigma),\tilde{g}(\tilde{\mathcal{T}},\eta,\sigma)\} = 2\pi\delta(\xi-\eta)\tilde{g}(\tilde{\mathcal{T}},\eta,\sigma)\tilde{\mathcal{T}}_{n(r)\mu}(T,\sigma) \tag{D.4c}$$

$$\{\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma),\tilde{g}(\tilde{\mathcal{T}},\eta,\sigma)\} = -2\pi\delta(\xi-\eta)\tilde{\mathcal{T}}_{n(r)\mu}(T,\sigma)\tilde{g}(\tilde{\mathcal{T}},\eta,\sigma) \tag{D.4d}$$

$$\hat{J}_{n(r)\mu}(\xi,\sigma)\mathcal{G}^{n(r)\mu,-n(r),\nu}(\sigma)\tilde{\mathcal{T}}_{-n(r),\nu}(T,\sigma) = -\frac{i}{2}\tilde{g}^{-1}(\tilde{\mathcal{T}},\xi,\sigma)\partial_+\tilde{g}(\tilde{\mathcal{T}},\xi,\sigma) \tag{D.4e}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,\sigma)\mathcal{G}^{n(r)\mu,-n(r),\nu}(\sigma)\tilde{\mathcal{T}}_{-n(r),\nu}(T,\sigma) = -\frac{i}{2}\tilde{g}(\tilde{\mathcal{T}},\xi,\sigma)\partial_-\tilde{g}^{-1}(\tilde{\mathcal{T}},\xi,\sigma) \tag{D.4f}$$

by using the rescaled twisted matrices $\tilde{\mathcal{T}}$ and the inverse relation (D.1d).

## Appendix E. Twisted Affine Lie Groups

The method of affine Lie groups[16] is easily extended to give quantum or classical left and right mover canonical realizations of the currents associated to any infinite dimensional Lie algebra. In what follows we will work out the quantum and classical extension to *twisted affine Lie groups*, which gives the realizations of the general twisted current algebra $\hat{\mathfrak{g}}(h_\sigma) \oplus \hat{\mathfrak{g}}(h_\sigma^{-1})$ discussed in the text.

We begin with a set of abstract modes $\hat{E}$ which satisfy the general twisted current algebra $\hat{\mathfrak{g}}(h_\sigma)$ in Ref. 2

$$[\hat{E}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{E}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] = i\,\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\;\hat{E}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) + (m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\,. \tag{E.1}$$

This algebra can be recast as an infinite dimensional Lie algebra

$$[\hat{\mathcal{E}}_{\hat{L}}, \hat{\mathcal{E}}_{\hat{M}}] = i\tilde{\mathcal{F}}_{\hat{L}\hat{M}}{}^{\hat{N}}\hat{\mathcal{E}}_{\hat{N}} \tag{E.2a}$$

$$\hat{\mathcal{E}}_{\hat{L}} = (\hat{E}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), 1), \quad \hat{L} = ((n(r)\mu, m+\tfrac{n(r)}{\rho(\sigma)}), \hat{y}_*), \quad \hat{\mathcal{E}}_{\hat{y}_*} = 1 \tag{E.2b}$$

$$\tilde{\mathcal{F}}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)};n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}{}^{n(t)\delta,p+\frac{n(t)}{\rho(\sigma)}} = \delta_{m+n-p+\frac{n(r)+n(s)-n(t)}{\rho(\sigma)},0}\,\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\sigma) \tag{E.2c}$$

$$\tilde{\mathcal{F}}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)};n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}{}^{\hat{y}_*} = -i(m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\,\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma) \tag{E.2d}$$

whose non-zero structure constants are given in (E.2c) and (E.2d). The elements $\hat{\gamma}$ of the corresponding twisted affine Lie group are defined as

$$\hat{\gamma} = e^{iy}\hat{\psi}(\hat{x}(t)), \quad \hat{\psi}(\hat{x}(t)) \equiv exp\big(i\hat{x}^{n(r)\mu;m+\frac{n(r)}{\rho(\sigma)}}(t)\hat{E}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})\big) \tag{E.3a}$$

$$\hat{x}^{n(r)\pm\rho(\sigma),\mu;m\mp1+\frac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}}(t) = \hat{x}^{n(r)\mu;m+\frac{n(r)}{\rho(\sigma)}}(t) \tag{E.3b}$$

$$\hat{e}(0)_{\hat{L}}{}^{\hat{M}} = -\hat{\bar{e}}(0)_{\hat{L}}{}^{\hat{M}} = \delta_{\hat{L}}{}^{\hat{M}} \tag{E.3c}$$

where $\{\hat{x}(t)\}$ are the ordinary (mode) coordinates at time $t$, the coordinate $y$ corresponds to the central term and $\hat{\psi}(\hat{x}(t))$ is the reduced twisted affine Lie group element. Here we have chosen the preferred coordinate system (E.3c) on the affine Lie group, where $\hat{e}(0) = 1$ is the twisted mode vielbein at the origin (see below). This choice ultimately guarantees that the local operators of the construction have definite monodromy. More generally, the twisted mode vielbeins are defined as

$$\hat{e}_{\hat{L}} \equiv -i\hat{\gamma}^{-1}\partial_{\hat{L}}\hat{\gamma} = \hat{e}_{\hat{L}}{}^{\hat{M}}\hat{\mathcal{E}}_{\hat{M}},, \quad \hat{\bar{e}}_{\hat{L}} \equiv -i\hat{\gamma}\partial_{\hat{L}}\hat{\gamma}^{-1} = \hat{\bar{e}}_{\hat{L}}{}^{\hat{M}}\hat{\mathcal{E}}_{\hat{M}}, \quad \partial_{\hat{L}} \equiv \frac{\partial}{\partial\hat{x}^{\hat{L}}} \tag{E.4a}$$

$$\hat{e} \equiv \hat{e}(t), \quad \hat{\bar{e}} \equiv \hat{\bar{e}}(t), \quad \partial_{\hat{L}} \equiv \partial_{\hat{L}}(t) \tag{E.4b}$$

in terms of the elements $\hat{\gamma}$ of the twisted affine Lie group. As seen in (E.4b), we often suppress the fixed time label $t$.

The method of Ref. 16 then gives the general twisted current modes as the reduced twisted affine Lie derivatives

$$\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) = -i(\hat{e}^{-1})_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}\hat{\mathcal{D}}_{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}} \tag{E.5a}$$

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) = -i(\hat{\bar{e}}^{-1})_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}\hat{\bar{\mathcal{D}}}_{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}} \tag{E.5b}$$

$$\hat{\mathcal{D}}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}} \equiv \partial_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}} - i\,\hat{e}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{\hat{y}_*} \tag{E.5c}$$

$$\hat{\bar{\mathcal{D}}}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}} \equiv \partial_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}} + i\,\hat{\bar{e}}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{\hat{y}_*} \tag{E.5d}$$

$$\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t)\hat{g}(\hat{x}(t)) = \hat{g}(\hat{x}(t))\hat{E}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) \tag{E.5e}$$

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t)\hat{g}(\hat{x}(t)) = -\hat{E}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})\hat{g}(\hat{x}(t)) \tag{E.5f}$$

which satisfy (at fixed $t$) the general twisted current algebra $\hat{\mathfrak{g}}(h_\sigma)\oplus\hat{\mathfrak{g}}(h_\sigma^{-1})$ in (8.8). The sign reversal of the right mover central term in (8.8) is in fact completely natural from the point of view of affine Lie groups (see Eq. (2.12) and (2.13) of Ref. 16) and twisted affine Lie groups.

Following Ref. 16, we also find the equivalent $\hat{B}$ form of the twisted currents

$$\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) = -i(\hat{e}^{-1})_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}\hat{\mathcal{D}}_{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}(\hat{B}) + \frac{1}{2}\hat{\Omega}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{\hat{y}_*} \tag{E.6a}$$

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) = -i(\hat{\bar{e}}^{-1})_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}\hat{\mathcal{D}}_{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}(\hat{B}) - \frac{1}{2}(\hat{\Omega}^{-1})_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{\hat{y}_*} \tag{E.6b}$$

$$\hat{\mathcal{D}}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}(\hat{B}) \equiv \partial_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}} \tag{E.6c}$$

$$+\frac{i}{2}\hat{B}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)};n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}(\hat{e}^{-1})_{n(t)\delta,p+\frac{n(t)}{\rho(\sigma)}}{}^{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}\mathcal{G}^{n(t)\delta,p+\frac{n(t)}{\rho(\sigma)};n(u)\epsilon,q+\frac{n(u)}{\rho(\sigma)}}(\sigma)\hat{\Omega}_{n(u)\epsilon,q+\frac{n(u)}{\rho(\sigma)}}{}^{\hat{y}_*}$$

$$\hat{\Omega}_{\hat{L}}{}^{\hat{M}} \equiv (e^{-i\hat{h}})_{\hat{L}}{}^{\hat{M}}, \quad \hat{h} \equiv \sum_{r,\mu,m}\hat{x}^{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}\tilde{\mathcal{T}}^{adj}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}, \quad (\tilde{\mathcal{T}}^{adj}_{\hat{L}})_{\hat{M}}{}^{\hat{N}} \equiv -i\tilde{\mathcal{F}}_{\hat{L};\hat{M}}{}^{\hat{N}} \tag{E.6d}$$

$$\hat{B}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)};n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}} \equiv \left(\frac{e^{i\hat{h}}-e^{-i\hat{h}}-2i\hat{h}}{(i\hat{h})^2}\right)_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{n(t)\delta,p+\frac{n(t)}{\rho(\sigma)}}\mathcal{G}_{n(t)\delta,p+\frac{n(t)}{\rho(\sigma)};n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}(\sigma) \tag{E.6e}$$

$$\mathcal{G}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)};n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}(\sigma) \equiv \delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma)\ . \tag{E.6f}$$

Evaluating this result in the untwisted sector[d] we find some typos in Ref. 16: The second (barred) equations in (2.30a), (2.33a), (3.6), (4.6a) and (4.31a) should read $\hat{\Omega}\longrightarrow\hat{\Omega}^{-1}$. Also, the right hand sides of Eqs. (4.22a,b) should have an extra factor -1.

The local twisted fields of the text are sums of the twisted modes above:

[d]For $\sigma=0$ and simple $g$, our results can be compared to those of Ref. 16 in the special coordinate system $e_{i\mu}{}^{am}(0)=\delta_{i\mu}{}^{am}$ where $am\simeq i\mu$. We find that our $B$ field is identified with $-kB$ in that reference, and moreover our quantities $y$, $e_{i\mu}{}^{y^*}$, $(e^{-1})_{i\mu}{}^{y^*}$, $\Omega_{am}{}^{y^*}$ and $(\Omega^{-1})_{am}{}^{y^*}$ also contain an extra factor of $k$.

$$\hat{J}_{n(r)\mu}(\xi,t) = \sum_m e^{-i(m+\frac{n(r)}{\rho(\sigma)})\xi}\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) \tag{E.7a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\xi,t) = \sum_m e^{-i(m+\frac{n(r)}{\rho(\sigma)})\xi}\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) \tag{E.7b}$$

$$\hat{x}^{n(r)\mu}(\xi,t) = \sum_m e^{i(m+\frac{n(r)}{\rho(\sigma)})\xi}\hat{x}^{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}(t) \tag{E.8a}$$

$$\hat{p}_{n(r)\mu}(\xi,t) = -\frac{i}{2\pi}\sum_m e^{-i(m+\frac{n(r)}{\rho(\sigma)})\xi}\partial_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}(t) \tag{E.8b}$$

$$[\hat{x}^{n(r)\mu}(\xi,t),\hat{p}_{n(s)\nu}(\eta,t)] = i\delta(\xi-\eta)\delta_{n(s)\nu}{}^{n(r)\mu} \tag{E.8c}$$

$$\hat{e}(\xi,t)_{n(r)\mu}{}^{n(s)\nu} = \sum_m e^{i(n-m+\frac{n(s)-n(r)}{\rho(\sigma)})\xi}\hat{e}(t)_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}{}^{n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}, \quad \forall n\in\mathbb{Z} \tag{E.8d}$$

$$\hat{B}(\xi,t)_{n(r)\mu;n(s)\nu} = \sum_m e^{-i(n+m+\frac{n(s)+n(r)}{\rho(\sigma)})\xi}\hat{B}(t)_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)};n(s)\nu,n+\frac{n(s)}{\rho(\sigma)}}, \quad \forall n\in\mathbb{Z} \tag{E.8e}$$

$$\hat{g}(\mathcal{T},\xi,t,\sigma)_{N(s)\nu}{}^{N(t)\delta} = \sum_p e^{i(q-p+\frac{N(t)-N(s)}{R(\sigma)})\xi}(e^{i\hat{\mathcal{H}}(\mathcal{T},t,\sigma)})_{N(s)\nu,p+\frac{N(s)}{R(\sigma)}}{}^{N(t)\delta,q+\frac{N(t)}{R(\sigma)}}, \ \forall q\in\mathbb{Z} \tag{E.9a}$$

$$\hat{\mathcal{H}}(\mathcal{T},t,\sigma) \equiv \hat{x}^{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}(t)\mathcal{T}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}(T,\sigma) \tag{E.9b}$$

$$\mathcal{T}_{n(r)\mu,m+\frac{n(r)}{\rho(\sigma)}}(T,\sigma)_{N(s)\nu,p+\frac{N(s)}{R(\sigma)}}{}^{N(t)\delta,q+\frac{N(t)}{R(\sigma)}} \equiv \delta_{m+p-q+\frac{n(r)}{\rho(\sigma)}+\frac{N(s)-N(t)}{R(\sigma)},0}\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta}. \tag{E.9c}$$

The mode expansions of $\hat{e}^{-1}$, $\hat{\bar{e}}$ and $\hat{\bar{e}}^{-1}$ have the same forms as the twisted vielbein in (E.8d). The objects $\mathcal{T}_{n(r)\mu}$ in (E.9) are the twisted representation matrices of the text.

The time dependence of the twisted current modes

$$\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) = e^{-i(m+\frac{n(r)}{\rho(\sigma)})t}\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) \tag{E.10a}$$

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)},t) = e^{i(m+\frac{n(r)}{\rho(\sigma)})t}\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) \tag{E.10b}$$

follows from the operator WZW orbifold Hamiltonian $\hat{H}_\sigma = L_\sigma(0)+\bar{L}_\sigma(0)$, where $L_\sigma(0)$ and $\bar{L}_\sigma(0)$ are the zero modes of the left and right mover orbifold affine-Sugawara constructions discussed in Subsec. 8.2. The time independent modes in (E.10) are the twisted current modes of the text. To obtain the classical realization of the twisted currents in (6.2), one needs only replace operator $\hat{p}$ above by classical $\hat{p}$, so that the bracket (6.2d) is satisfied.

## Appendix F. Twisted Left-Right Current Copies

We have seen that locality of the WZW orbifold action dictates a form of the general twisted current algebra (see Subsecs. 5.4 and 8.2) in which the twisted right and left mover current algebras are not a priori copies of each other. For comparison, we give here the twisted current system for left-right current copies:

$$\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(w) = \frac{\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}}{(z-w)^2} + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(w)}{z-w} + O((z-w)^0) \tag{F.1a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{w}) = \frac{\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}}{(\bar{z}-\bar{w})^2} + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\bar{w})}{\bar{z}-\bar{w}} + O((\bar{z}-\bar{w})^0) \tag{F.1b}$$

$$\hat{J}_{n(r)\mu}(ze^{2\pi i}) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(z),\ \hat{\bar{J}}_{n(r)\mu}(\bar{z}e^{2\pi i}) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)\mu}(\bar{z}),\ \sigma=0,\dots,N_c-1. \tag{F.1c}$$

In this case, the monodromies are the same but the paths followed are not. This gives the mode expansions

$$\hat{J}_{n(r)\mu}(z) = \sum_{m\in\mathbb{Z}} \hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})z^{-(m+\frac{n(r)}{\rho(\sigma)})-1}, \quad \hat{\bar{J}}_{n(r)\mu}(\bar{z}) = \sum_{m\in\mathbb{Z}} \hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})\bar{z}^{-(m+\frac{n(r)}{\rho(\sigma)})-1} \tag{F.2}$$

and the twisted current algebra $\hat{\mathfrak{g}}(h_\sigma)\oplus\hat{\mathfrak{g}}(h_\sigma)$

$$[\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{J}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) + (m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \tag{F.3a}$$

$$[\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{\bar{J}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) + (m+\tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \tag{F.3b}$$

$$[\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{\bar{J}}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] = 0, \quad m,n\in\mathbb{Z}, \quad \sigma=0,\dots,N_c-1\,. \tag{F.3c}$$

The stress tensors and OPE's of the stress tensors with the twisted currents are still given by (8.12) and (8.13).

We also note that the copy system (F.1) can be obtained from the eigencurrent OPE's and automorphic responses in (8.4) by the *non-local* current isomorphisms

$$\hat{\mathcal{J}}\,,\hat{\bar{\mathcal{J}}} \underset{\sigma}{\longrightarrow} \hat{J}\,,\hat{\bar{J}} \tag{F.4a}$$

$$\text{automorphism response } E_{n(r)}(\sigma) \underset{\sigma}{\longrightarrow} \text{ opposite-path monodromy } E_{n(r)}(\sigma) \tag{F.4b}$$

which should be compared with the local current isomorphisms in Eq. (8.5).

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