LBNL-48165
Preprint

ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY

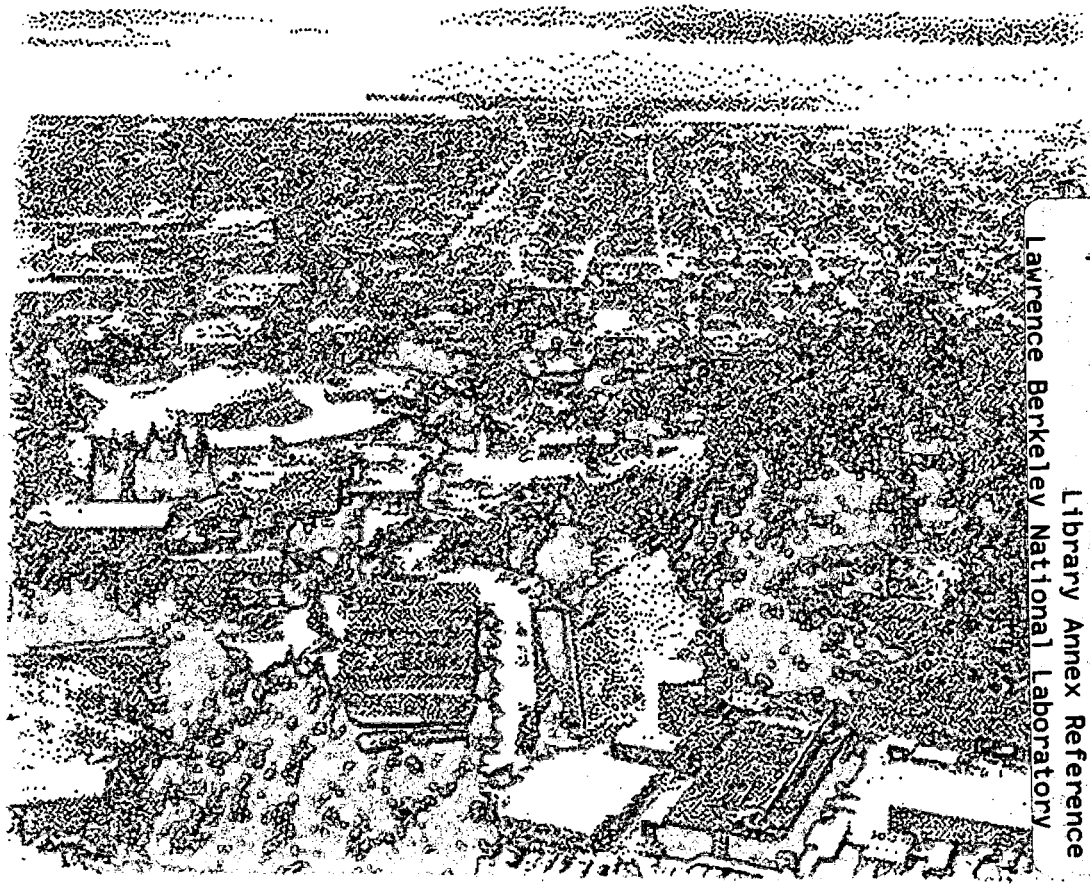
One Loop Partition Function in the Presence of Tachyon Background and Corrections to Tachyon Condensation

Korkut Bardakci and Anatoly Konechny

Physics Division

July 2001

Submitted to
Nuclear Physics B



REFERENCE COPY
Does Not
Circulate
Library Annex Reference
Lawrence Berkeley National Laboratory
Copy 1
LBNL-48165

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

LBNL-48165

One Loop Partition Function in the Presence of Tachyon Background and Corrections to Tachyon Condensation

Korkut Bardakci and Anatoly Konechny

Department of Physics
University of California, Berkeley

and

Physics Division
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

July 2001

This work was supported by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, and in part by National Science Foundation Grant No. PHY-95-14797.

July 11, 2001

LBNL-48165
UCB-PTH-01/16
hep-th/0105098

One loop partition function in the presence of tachyon background and corrections to tachyon condensation *

Korkut Bardakci† and Anatoly Konechny‡

Department of Physics
University of California at Berkeley
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

Abstract

We compute the one-loop partition function for quadratic tachyon background in open string theory. Both closed and open string representations are developed. Using these representations we study the one-loop divergences in the partition function in the presence of the tachyon background. The divergences due to the open and closed string tachyons are treated by analytic continuation in the tachyon mass squared. We pay particular attention to the imaginary part of the analytically continued expressions. The last one gives the decay rate of the unstable vacuum. The dilaton tadpole is also given some partial consideration. The partition function is further used to study corrections to tachyon condensation processes describing brane descent relations. Assuming the boundary string field theory prescription for construction of the string field action via partition function holds at one loop level we study the one-loop corrections to the tachyon potential and to the tensions of lower-dimensional branes.

*This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation grant PHY-95-14797.

†email address: kbardakci@lbl.gov
‡email address: konechny@thsrv.lbl.gov

1 Introduction

First attempts to find a stable nonperturbative vacuum in bosonic string theory were made way back in the 70's [1], [2], [3]. Part of the complexity of the problem comes from the fact that tachyon condensation is an off-shell problem. With the advent of string field theory new attempts were made [4], [5] that studied tachyon potential and provided more evidence that a new stable vacuum indeed exists. Later an important insight into the problem was provided by A. Sen [6] who among other things put forward a conjecture that the hight of the tachyon potential in open bosonic string theory is equal to the tension of the space-filling D25-brane. This point of view as well as string field theory methods were further developed in a series of papers [7], [8], [9], [10], [11] (and references therein). The tachyon condensation is believed to yield a complete decoupling of the open string states. The D25-brane desintegration into the closed string vacuum may go through various metastable phases described by lower-dimensional branes [8]. These descent relations are in general easier to study than the complete condensation.

In the papers cited above the cubic string field theory was one of the primary tools of investigation. Recently it was realized that another version of open string theory nowadays christened as Boundary String Field Theory (BSFT) can be very useful in studying the question. The BSFT was put forward by E. Witten in [12] and further developed in the papers of E. Witten [13], K. Li and E. Witten [14], and S. Shatashvili [15], [16]. Using BSFT methods the exact tree level tachyon potential was derived in [34], [35] and the Sen's conjecture regarding the hight of the potential was shown to be true. In particular BSFT was shown to describe most elegantly brane descent relations. We would like to note that the BSFT in its spirit is very similar to the old sigma model approach (see [21], [22] for a review and [23] for a recent discussion). The picture of tachyon condensation in bosonic BSFT (as well as the BSFT itself) was further developed in [36], [39], [37], [40], [41], [38].

In this paper, we investigate one loop corrections to the effective action of the tachyon field, probed by the mixed boundary conditions

$$\frac{\partial X}{\partial \tau} = uX, \tag{1}$$

first studied in [13] and later used in many subsequent papers. There are several motivations for studying this problem; for example, one would like to see whether the system stays weakly coupled as the tachyon rolls down the potential and also one would like to test Sen's conjectures. Our aim in this paper is more modest; we wish to carry out a divergence free and internally consistent one loop correction to the tachyon potential and the D-brane tension. In our analysis we take the approach of BSFT. BSFT gives a (background independent) prescription of how to compute a space-time action in the presence of an open string (off-shell) background. The prescription was only developed at tree level string theory. In view of the lack of a general theoretical foundation of a quantum BSFT, we proceed with a speculative procedure for computing the effective space-time action that extends the tree level prescription in the most direct manner. The key ingredient in the computation is a one-loop partition function in the presence of tachyon background (see section 2 for a discussion). This

amplitude can be considered as a closed string propagating at tree level for a (Euclidean) time T between initial and final states representing the boundary conditions (boundary states). This is represented by a cylinder graph. We choose the boundary conditions to be independent of the time T in this picture. In section 3, the cylinder is mapped into an annulus by a conformal tranformation, and the resulting boundary conditions are shown to depend on T. We should point out that the imposition of the simple (T independent) boundary conditions in the cylinder picture, as opposed to, say, in the annulus picture, is somewhat arbitrary, although in our opinion, a natural choice. In section 4, we compute the cylinder amplitude up to an overall normalization constant using BSFT, and in section 5, we perform an independent check on our result by computing the same amplitude in the annular region.

We should stress that the calculations described so far were carried out at a fixed modulus, which is the time T for the cylinder or the ratio of the two radii for the annulus. To complete the calculation of the one loop partition function, we have to add the contribution of the ghost sector and integrate over the modulus with a suitable measure. In the case of the usual boundary conditions (Neumann or Dirichlet), conformal invariance uniquely determines both the ghost contribution and the modular measure. Since in our problem the boundary conditions violate conformal invariance, we know no convincing way of uniquely fixing these contributions. In the absence of a guiding principle, we have decided to keep the ghost contribution the same as in the conformal case, and use the same conformal measure in the integration over the modulus T in the cylinder picture (see eq. (14)). Although this is an ad hoc recipe, it has the virtue of being simple and having the correct limit as $u \to 0$ (Neumann boundary). We should point out that in principle there is an ambiguity even in the calculation of the tree level open string amplitude with non-conformal boundary conditions [13]. The result of the calculation would in general depend on the region of the world sheet chosen; for example, a calculation done using the upper half plane would give a different result than the one done using a circle. This is a consequence of the lack of invariance under the conformal tranformation connecting the two regions. Of course, this does not mean that the ambiguity in the calculation of the amplitude corresponds to an ambiguity in the resulting physics. Since the calculation of the tree amplitude in [13] rests on firm foundation, namely BRST invariance [12], it is generally believed that amplitudes calculated in different regions must be related by field redefinitions, which do not change the underlying physics. However, in the case of the one loop amplitude, there is no such well founded starting point, and our naive prescription may need to be modified in the future[§]. In spite of these reservations, we believe that it is of some interest to carry the calculation to the end to find the correction to the tachyon potential. As we shall argue later, the final results appear to be reasonable and self consistent.

An alternative way of looking at the cylinder amplitude is to view it as the calculation of the partition function of an open string with mixed boundary conditions corresponding to (1). This calculation, which is technically more involved then the calculation of the

[§]After this work was completed preprints [49], [50] appeared that propose a different scheme for computing loop corrections in BSFT.

cylinder amplitude, is carried out in section 8. The final answer is in a partially implicit form difficult to compare with the earlier result in detail. However, the open string picture has its advantages. For example, some of the divergences of the amplitude are easier to handle, and the undetermined overall constant of the previous calculation is easily fixed.

We would like to remark that the computation of the boundary state and the one-loop partition function in two channels is essentially independent of its further use in the construction of space-time effective action and we believe it to be of interest by itself.

Finally, we would like to discuss briefly the divergences encountered in the integration over the modulus. These divergences are caused by the tachyons present in both the open and closed string channels, and by the dilaton in the closed string channel. We think of the divergences due to the tachyons as being similar to the superficial divergences encountered in the integral representation of the tree level string amplitudes. These latter divergences are easily circumvented by appropriate analytic continuations in external momenta. The same idea of analytic continuation can be used for the tachyonic divergences [27], with a resulting complex tachyon potential. This is, of course, due to the instability of the vacuum in the presence of the tachyon. An alternative approach, which we will not use, is to cancel the tachyon divergence by a tree level counter term (Fischler- Susskind mechanism [31]). The divergence due to the dilaton, however, has to be canceled by the Fischler-Susskind mechanism when it is present. However, in this paper we will restrict ourselves to the situation when there is no dilatonic divergence and no need for tree level counter terms. This happens in the process describing the descent relation of D25 brane to a D25-p brane with $p > 2$.

The paper is organized as follows. In section 2 we give a general discussion of BSFT and the loop corrections in it. Section 3 contains a further discussion of the boundary conditions at one loop corresponding to the quadratic tachyon perturbation. In section 4 we compute the corresponding boundary state and find the expression for partition function in the closed string channel. In section 5 we give an alternative computation via Green's function on the annulus and discuss renormalization conditions. In section 6 we remind the reader about the situation with one-loop divergences in the conformal case and about the analytic continuation treatment of the tachyonic divergences. The divergences in the closed string channel in the presence of the tachyon background are considered in section 7. In section 8 we develop the open string channel description of the partition function. We approximately compute the modified Casimir energy for the boundary conditions describing the open string channel. The open string tachyonic divergence comes from the contributions of the ground state (whose energy is given by the Casimir energy) and the first excited state. We derive a general integral representation for the Casimir energy as well as an approximate expression valid for small coupling constant. These results are used then to study the general form of the tachyon divergence. In section 9 we discuss the dilaton divergence and compute the one-loop correction to the tachyon potential. In section 10 we derive the tensions of the lower-dimensional branes by finding the asymptotic value of the action in the limit $a, u \to \infty$. We conclude with a discussion and a list of unsolved questions in section 11.

2 Bosonic boundary string field theory

The starting point of the original paper [12] in which BSFT were introduced was a Batalin-Vilkovisky (BV) formalism on the space of sigma model boundary perturbations. Consider a world sheet action defined on a unit disc on the complex plane with standard metric that has the form

$$S = S_0 + \int_{\partial\Sigma} d\phi\, \mathcal{V} \tag{2}$$

that is S is equal to a sum of the standard free action S_0 in the bulk corresponding to a fixed closed string background and a boundary perturbation specified by some local operator of ghost number zero $\mathcal{V}$ constructed from the fields X_μ and ghosts b, c. The space of such operators is considered to be a phase space of the BSFT. Note that the lack of precise definition of the admissible class of operators $\mathcal{V}$ (or equivalently a space of boundary conditions) is still a major problem of BSFT.

A further assumption is that $\mathcal{V}$ can be represented as $\mathcal{V} = b_{-1}\mathcal{O}$. In the situation when ghosts and matter are decoupled one has $\mathcal{O} = c\mathcal{V}$. (Strictly speaking in Witten's formulation $\mathcal{O}$ is the main object specifying a point in the phase space. However we will soon assume that matter and ghosts decouple and work only with $\mathcal{V}$'s.) Then the Witten's BV antibracket is defined as

$$\omega(\delta\mathcal{O}_1, \delta\mathcal{O}_2) = \int d\phi_1 \int d\phi_2 \langle \delta\mathcal{O}_1(\phi_1)\delta\mathcal{O}_2(\phi_2)\rangle$$

where $\langle ... \rangle$ stands for a correlator in the presence of the background $\mathcal{V}$ $(\mathcal{O})$, i.e. the point in phase space at which we evaluate ω.

The string field action $\mathcal{S}$ is defined as a Hamiltonian for the vector field specified on the phase space by the BRST operator Q that is assumed to be determined by the standard bulk part S_0. Let us expand the boundary perturbation $\mathcal{O}$ in terms of some basis $\mathcal{O}_i$:

$$\mathcal{O} = \sum_i \lambda^i \mathcal{O}_i$$

where λ^i are coupling constants (coordinates on the phase space). We can write then the following equation for $\mathcal{S}$

$$\frac{\partial \mathcal{S}}{\partial \lambda^i} = \frac{1}{2}\int d\phi_1 \int d\phi_2 \langle \mathcal{O}_i(\phi_1)\{Q, \mathcal{O}(\phi_2)\}\rangle \tag{3}$$

Since $Q^2 = 0$ the string field action $\mathcal{S}$ should satisfy the (classical) master equation

$$\{\mathcal{S}, \mathcal{S}\}_{BV} = 0$$

where $\{.,.\}_{BV}$ is the BV bracket specified by ω.

The action $\mathcal{S}$ is a string field tree-level action. A natural way to extend Witten's formulation to the full quantum theory would be to consider a quantum master equation

$$\hbar\Delta_\rho \mathcal{S} + \frac{1}{2}\{\mathcal{S}, \mathcal{S}\}_{BV} = 0 .$$

To specify the operator Δ_ρ one needs to supply the phase space with a measure ρ [17]. Then Δ_ρ is an operator that acting on a function A gives the divergence

$$\mathrm{div}_\rho A = \frac{1}{\rho}\partial_{\lambda_i}(\rho A^i)$$

where A^i is a vector field corresponding to A. The loop expansion then would correspond to the expansion of $\mathcal{S}$ in powers of $\hbar$. The density ρ is essentially an independent ingredient in the formalism (see [18] for a thorough discussion). Lacking a rigorous definition of the phase space itself the idea of finding some natural measure on it does not look very promising at the moment. Thus we will have to proceed in some other way to define the loop corrections.

From now on we will talk only about the situation when matter and ghosts are decoupled and $\mathcal{O} = c\mathcal{V}$. In paper [13] (see also [15]) it was shown on general grounds that a solution to (3) (for the case when matter and ghosts are decoupled) must be of the form

$$\mathcal{S} = (1 + V^i \frac{\partial}{\partial \lambda^i})Z \tag{4}$$

where Z is the disc partition function corresponding to (2) and V^i is some vector field on the space of open string fields $\mathcal{V}$. The equation of motions following from this action are always linear.

It was shown later by Shatashvili [16] that a more careful treatment of total derivatives in correlation functions leads to a natural modification of (4) allowing nonlinearities in the equations of motion and proposed the following relation

$$\mathcal{S} = (1 + \beta^i \frac{\partial}{\partial \lambda^i})Z \tag{5}$$

where β^i is the beta function corresponding to coupling constant λ^i. This relation was shown to be true in the first order in conformal perturbation theory [16]. Note that in order to account for the nonlinear contributions in β_i (contributions of contact terms) in the framework of the original BV formalism one has to modify the BRST operator Q that now has to be dependent on λ^i. As long as the beta function is linear (and one can always choose locally such set of coordinates λ^i when this is true) the equations (4) and (5) seem to be equivalent. However that describes only one coordinate patch in the whole phase space manifold. In particular the set of coordinates in which the beta function is linear is singular when perturbations $\mathcal{V}$ approach the mass shell. But as long as we stay far off shell that is a situation of primary interest in the case with tachyon condensation this coordinate system works well. (See [35], [38] for a discussion of the on shell behavior of BSFT.)

We will take formula (5) as a starting point for constructing the BSFT action. Written in that form the BSFT action can be easily linked with the sigma model approach (see [21] for a general review and [23] for a recent discussion on the relation of the sigma model approach and BSFT). Indeed it was noticed a long time ago [20] that being a generating functional for scattering amplitudes the renormalized sigma model partition function is a natural candidate for string theory effective action. And this identification works quite well in the vicinity of the

mass shell of massless particles. However if one wants to include tachyons in the sigma model approach (that is natural because perturbations corresponding to tachyons are relevant and do not change the renormalizability unlike the massive string fields) than the identification $\mathcal{S} = Z^{ren}$ does not work. It does not give the correct equations of motion because

$$\frac{\partial Z}{\partial \lambda} = \int d\phi \langle \mathcal{V}(\phi) \rangle$$

does not vanish in general at the conformal point $\lambda = 0$ if $\mathcal{V}$ has conformal dimension 1 that is for example the case for the constant tachyon mode. The second term in the expression (5) corrects that problem. Thus if we substitute $\beta(\lambda) = -\lambda + \mathcal{O}(\lambda^2)$ we will get

$$\frac{\partial \mathcal{S}}{\partial \lambda} = \left(\frac{\partial}{\partial \lambda} \mathcal{O}(\lambda^2) \right) Z$$

that evidently vanishes at the original fixed point $\lambda = 0$. This means that in a coordinate patch in which the beta function is linear (5) gives the correct equations of motion. It is believed that in general there exists a nonsingular metric G_{ij} defined on the whole manifold of string fields such that

$$\frac{\partial \mathcal{S}}{\partial \lambda^i} = G_{ij}(\lambda)\beta^j .$$

In the sigma model approach a generating functional for scattering amplitudes that includes all string loop corrections is given by the total renormalized sigma model partition function. For the open string theory it has the form

$$Z = \sum_b \frac{g^{-1+b}}{b!} Z_b$$

where the sum is over world sheets with b-boundaries. Moreover the beta functions of massless fields are known to receive loop corrections coming from modular infinities (see [22] for a review). We see then that from the sigma model point of view formula (5) has a natural generalization that includes loop corrections.

It should be noted that an off shell extension of the sigma model approach to string theory involves a great deal of arbitrariness having to do with gauge fixing and field redefinitions. When doing loop corrections the world sheet metrics and the sigma model backgrounds need to be chosen consistently at each order of perturbation theory. Since in the problem at hand the bulk part of the sigma model action is fixed (that corresponds to a fixed closed string background) it seems to be natural to integrate over the moduli using a closed string picture of the amplitude at each order. In this picture we consider a $b-1$-loop open string vacuum amplitude as a b-point tree level scattering of closed string states $|\mathcal{V}\rangle$ specified by the open string background functional $\mathcal{V}$. In the case when the perturbation $\mathcal{V}$ is conformal this correspondence is well established and the corresponding closed string state $|\mathcal{V}\rangle$ is called a boundary state [24]. The normalization of $|\mathcal{V}\rangle$ is fixed by the equality of open and closed string channel representations for the one-loop partition function

$$\mathrm{Tr} e^{-H_{\mathcal{V}}^{open} T} = \langle \mathcal{V} | e^{-H_0^{cl} \pi / T} | \mathcal{V} \rangle . \tag{6}$$

In the left hand side of this equation we have an open string partition function on a strip of length T with periodic boundary conditions in the time direction and the perturbed open string Hamiltonian $H_{\mathcal{V}}^{open}$. In the right hand side we have a closed string amplitude evaluated on a cylinder of length π/T using the free closed string Hamiltonian H_0^{cl}. The overlap of $|\mathcal{V}\rangle$ with the closed string Fock space vacuum $|0\rangle$ is called a boundary entropy [33] and is proportional to the disc partition function

$$\langle 0|\mathcal{V}\rangle \sim Z_{disc}(\mathcal{V}) . \tag{7}$$

In this paper we construct a boundary state satisfying (6), (7) for the case of quadratic tachyon perturbation on the boundary. Lacking a general quantum BSFT theory we explore an ad hoc prescription for a one loop corrected BSFT effective space-time action that uses formula (5) and the one loop partition function. Note that this construction is guaranteed to give the correct one loop corrections to the on-shell amplitudes. However apriory it is not clear whether this prescription has all of the desired properties for the truly off-shell quantities. We believe that whatever the correct one loop prescription may be many of the features present in our speculative construction, such as the imaginary part of the tachyon potential and loop corrections to the brane tensions, should remain. In the next section we will discuss in more detail the 1-loop boundary conditions corresponding to the quadratic tachyon background.

3 Boundary condition

As it was discussed in [35] a boundary perturbation corresponding to quadratic tachyon profile is particularly useful for describing descent relation between unstable D-branes. The quadratic profile has a unique property of preserving its shape along the RG flow, i.e. the corresponding modes (coupling constants) can be consistently decoupled from all other string modes.

In the conventions of Witten's paper [13] the particular background we are interested in is specified (at tree level) by the following action on a unit disc

$$S = \frac{1}{8\pi}\Big(\int d\phi \int r dr \partial_\alpha X \partial^\alpha X + u \int_0^{2\pi} d\phi X^2(\phi)\Big) + a \tag{8}$$

which is written for a single string coordinate field X. Here r, ϕ are polar coordinates, u and a are (nonnegative) constants.

Due to the lack of conformal invariance it is not immediately obvious what boundary conditions describe the same background at the loop level. The disc representation (8) has to do with some particular off-shell gauge fixing. Since the boundary term in (8) is rotationally invariant it is naturally to expect that at the one-loop level the same background must be represented by boundary conditions on an annulus $r_0 \leq |z| \leq 1$, $z = re^{i\phi}$ in such a way that on each circle $|z| = 1$ and $|z| = r_0$ the boundary term is equivalent to the one in (8). A conformal transformation that interchanges the two circles is the inversion $z \mapsto r_0^2/\bar{z}$. So one

can take for the circle $|z| = 1$ exactly the same boundary term as at the tree level and on the circle $|z| = r_0$ the one obtained from it by the aforementioned conformal transformation. The boundary conditions then read

$$\frac{\partial}{\partial r}X + uX = 0\,, \quad r = 1\,, \qquad -r_0\frac{\partial}{\partial r}X + uX = 0\,, \quad r = r_0\,. \tag{9}$$

The one-loop partition function for the quadratic tachyon potential was considered in a number of papers [43], [44], [45]. The boundary conditions on the annulus considered in papers [45], [46] (section 4) match with ours. The other papers consider boundary conditions that either do not have a relative sign in (9) that we think is quite unnatural from the point of view that tachyon does not carry any charge, or do not have a factor of r_0 in the second condition in which case we think the two boundaries are not treated on equal footing.

These boundary conditions take the most transparent form on a cylinder with coordinates $\sigma = \phi$, $\tau = \ln r$ with ranges $0 \leq \sigma < 2\pi$, $-t \leq \tau \leq 0$, $t = -\ln r_0$. They can be represented by means of a boundary state $|u\rangle$ in the Fock space of the first quantized closed string theory. It is defined up to an overall constant by the equation

$$\frac{\partial X}{\partial \tau}|u\rangle_\tau = uX|u\rangle_\tau\,. \tag{10}$$

Then the boundary condition (9) is represented on the cylinder by the boundary state $|u\rangle_{\tau=0}$ being the initial state at the right end of the cylinder and by the conjugated state $\langle u|_{\tau=-t}$ being the final state at the left end. The partition function for the cylinder of length t is then given by the expression

$$Z_1(u, a, t) = e^{-2a}\langle u|e^{(L_0+\tilde{L}_0)t}|u\rangle \tag{11}$$

where $|u\rangle \equiv |u\rangle_{\tau=0}$.

Note that the tree level partition function $Z_0(u, a)$ corresponding to the action (8) is given up to an overall numerical constant C to be discussed later by the overlap of $|u\rangle$ with the closed string vacuum

$$Z_0(a, u) = C \cdot e^{-a}\langle 0|u\rangle \tag{12}$$

where we prefer not to include the factor e^{-a} in the definition of $|u\rangle$. The overlap itself in the conformal situation gives the value of the boundary entropy [33]. Equation (12) along with equation (10) allows one to compute $|u\rangle$ up to an overall numerical constant. The inversion used on the annulus corresponds to the reflection about the middle of the cylinder that interchanges the two ends of the cylinder. Evidently the boundary conditions are symmetric with respect to this reflection.

It is instructive to note that the tree level representation as it is fixed on the unit disc favors the cylindrical quantization. For instance the same boundary conditions on an infinite strip will be time dependent. This suggests that quite generally in BSFT one may think of the open string configuration space as some suitable space of boundary states.

Consider now a conformally equivalent cylinder with coordinates $\sigma' = -\tau \cdot \frac{\pi}{t}$, $\tau' = \sigma \cdot \frac{\pi}{t}$ ranging as $0 \leq \sigma' \leq \pi$, $0 \leq \tau' \leq \frac{2\pi^2}{t} \equiv 2\pi T$ with the boundaries $\tau' = 0$ and $\tau' = 2\pi T$ identified.

The boundary conditions (10) being mapped on this cylinder take the form

$$-\frac{\partial X}{\partial \sigma'} + \frac{u}{T} X = 0\,, \quad \sigma' = 0\,, \qquad \frac{\partial X}{\partial \sigma'} + \frac{u}{T} X = 0\,, \quad \sigma' = \pi \tag{13}$$

where $T = \frac{\pi}{t}$.

This picture corresponds to an open string with boundary conditions (13) propagating in a loop of length $2\pi T$ in Euclidean time. The nonlocality in time of the boundary conditions (13) stems from the fact that the boundary perturbation at hand is not conformally invariant. Still it is relatively easy to quantize the theory in this representation that is quite useful in studying the partition function in the appropriate region of the moduli space.

The full one-loop partition function is obtained from (11) by integration over the moduli space. We assume the measure of integration to be the same as in the conformal case, given by the appropriate ghost determinant. In general we will consider all 26 coordinates of the string X_μ with the boundary condition being either the ones specified by the background (8) or the usual Neumann ones. In that case the sigma model action on the annulus has the form

$$S = \frac{1}{8\pi}\Big(\int d\phi \int r dr \sum_{\mu=1}^{26} \partial_\alpha X_\mu \partial^\alpha X^\mu + \sum_{i=1}^{D} u_i \Big(\int_{|z|=1} d\phi X_i^2(\phi) + \int_{|z|=r_0} d\phi X_i^2(\phi)\Big)\Big) + 2a\,.$$

The background is thus specified by the coupling constants u_i, $i = 1, \ldots, D$, a. The full boundary state corresponding to these boundary conditions is then a tensor product

$$|B\rangle = \prod_{\mu=1}^{26} |B_i\rangle\,, \quad |B_i\rangle = |u_i\rangle\,, i = 1, \ldots, D\,, \quad |B_\mu\rangle = |N\rangle\,, \quad \mu > D\,.$$

With this notation in mind we can write now the full one-loop partition function as

$$Z_1(u_i, a) = e^{-2a} \int_{-\infty}^{0} \frac{dt}{2\pi} e^{2t} f^2(-t) \langle B| e^{(L_0 + \tilde{L}_0)t} |B\rangle \tag{14}$$

where

$$f^2(t) = (f(t))^2 = \prod_{n=1}^{\infty} (1 - e^{-2tn})^2$$

comes from the ghost determinant.

4 The partition function in the closed string channel

We now proceed to calculate the boundary state (10) and the partition function in the closed string channel. The mode expansion for a single coordinate field X has the form

$$X = \hat{q} - 2i\hat{p}\tau + \sum_{1}^{\infty} \frac{1}{\sqrt{m}} \Big[a_m^\dagger e^{m\tau + im\sigma} + \tilde{a}_m^\dagger e^{m\tau - im\sigma} + a_m e^{-m\tau - im\sigma} + \tilde{a}_m e^{-m\tau + im\sigma}\Big] \tag{15}$$

and the commutation relations are

$$[\hat{q},\hat{p}] = i\,, \qquad [a_m, a_n^\dagger] = \delta_{mn}\,.$$

Our conventions here coincide with the ones in [24] that correspond to the choice $\alpha' = 2$ (in accord with the factor $1/8\pi$ in front of the action (8)). Substituting this mode expansion into (10) we obtain

$$\begin{aligned} i\hat{p}|u\rangle &= \frac{-u}{2(1-u\tau)}\hat{q}|u\rangle\,, \\ \tilde{a}_m|u\rangle &= \left(\frac{m-u}{m+u}\right)e^{2m\tau}a_m^\dagger|u\rangle\,, \\ a_m|u\rangle &= \left(\frac{m-u}{m+u}\right)e^{2m\tau}\tilde{a}_m^\dagger|u\rangle\,. \end{aligned}$$

It is easy now to find a solution to these equations

$$|u\rangle_\tau = N(u)\cdot\frac{\sqrt{u}}{\sqrt{1-u\tau}}\cdot exp\Big(\sum_{m=1}^{\infty}\left(\frac{m-u}{m+u}\right)a_m^\dagger\tilde{a}_m^\dagger e^{2m\tau}\Big)exp\Big(-\frac{u\hat{q}^2}{4(1-u\tau)}\Big)|0\rangle\,. \tag{16}$$

Here we inserted the factor $\frac{\sqrt{u}}{\sqrt{1-u\tau}}$ corresponding to the normalization of the spreading Gaussian wave packet $exp(-\frac{uq^2}{4(1-u\tau)})$ that gives the correct time evolution. The time independent constant $N(u)$ can be obtained up to a numerical factor from equation (12). The tree level partition factor for the boundary interaction (8) was found in [13] to be equal to

$$Z_0(u) = \frac{1}{\sqrt{u}}\prod_{k=1}^{\infty}\frac{1}{1+u/k}e^{u/k} = \sqrt{u}e^{\gamma u}\Gamma(u) \tag{17}$$

where γ is the Euler constant. We keep the overall normalization of $Z_0(u)$ as it appeared in [13]. Thus by (12) we have $N(u) = N_0\cdot Z_0(u)$ where N_0 is a numerical constant. An analogous boundary state in the supersymmetric case was computed in [42].

Plugging the boundary state (16) into formula (14) we obtain

$$Z_1(u,a) = e^{-2a}\langle u|_{\tau=-t}u\rangle_{\tau=0} = (N_0)^2\cdot(e^{-a}Z_0(u))^2\cdot\frac{\sqrt{u/\pi}}{\sqrt{2+ut}}\prod_{m=1}^{\infty}\frac{1}{1-e^{-2mt}\left(\frac{m-u}{m+u}\right)^2}\,. \tag{18}$$

We can fix the value of N_0 by comparing to the open string channel (see section 8): $N_0 = 1/(\sqrt{2})$. It comes out that with this normalization the proportionality constant C in (12) is equal to $1/\sqrt{2\pi}$. In addition to that we fix the normalization of Neumann boundary state for a coordinate X compactified as $X \sim X + R$ by matching contributions of the zero modes as

$$\lim_{u\to\infty}\frac{1}{\sqrt{u}} \sim \frac{R}{\sqrt{2\pi}}\,.$$

With this normalization the full one-loop partition function (14) for D coordinates satisfying the boundary conditions (10) and $26-D$ satisfying the Neumann ones reads

$$\begin{aligned} Z_1(u_i,a) &= \frac{V_{26-D}}{(\sqrt{16\pi^2})^{26-D}} e^{-2a} \prod_{i=1}^{D} \Big[(Z_0(u_i))^2 \cdot \\ &\int_0^\infty \frac{dt}{2\pi} e^{2t} f^{D-24}(t) \sqrt{\frac{u_i}{8\pi(1+tu_i/2)}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-2mt}\left(\frac{m-u_i}{m+u_i}\right)^2} \Big] . \end{aligned} \tag{19}$$

5 Green's function on the annulus, partition function and renormalization

In the previous section we computed the one-loop partition function using computationally the shortest path. It is instructive to do an independent computation via Green's function on the annulus. In particular in this approach the renormalization implicitly hidden in the factor $(Z_0(u))^2$ becomes transparent.

Consider an annulus $r_0 \leq |z| \leq 1$. A Green's function $G(z,z')$ satisfying

$$-\frac{1}{4\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} G(z,z') = \delta^2(z,z')$$

and boundary conditions (9) can be computed by exploiting an ansatz corresponding to a decomposition into a particular solution and a general solution of the homogeneous equation represented in a form of Lorant expansion plus an additional $\ln|z|^2$ term:

$$G(z,z') = -\ln|z-z'|^2 + \sum_{k=-\infty}^{+\infty} (z^k f_k(z',\bar{z}') + \bar{z}^k \bar{f}_k(z',\bar{z}')) + C(z',\bar{z}')\ln|z|^2$$

where $f_k(z',\bar{z}')$ and $C(z',\bar{z}')$ are some unknown functions.

Plugging in the ansatz into (9) and solving it we obtain

$$\begin{aligned} G(z,z') &= -\ln|z-z'|^2 + \frac{2}{u} - \frac{2\left(1-\frac{u}{2}|z'|^2\right)\left(1-\frac{u}{2}|z|^2\right)}{u(2-u\ln r_0)} + \\ &\sum_{k\neq 0}[(z\bar{z}')^k + (\bar{z}z')^k]\frac{k^2-u^2}{k((k+u)^2-r_0^{2k}(k-u)^2)} + \\ &\sum_{k=1}^{+\infty}\Big[\left(\frac{zr_0^2}{z'}\right)^k + \left(\frac{z'r_0^2}{z}\right)^k + \left(\frac{\bar{z}r_0^2}{\bar{z}'}\right)^k + \left(\frac{\bar{z}'r_0^2}{\bar{z}}\right)^k\Big]\frac{(k-u)^2}{k((k+u)^2-r_0^{2k}(k-u)^2)} . \end{aligned} \tag{20}$$

This Green's function can be explicitly checked to satisfy $G(z,z') = G(\frac{r_0^2}{\bar{z}}, z')$.

To determine the partition function $Z_1(u,r_0)$ one can write out two kinds of equation corresponding to the variation of $\ln Z_1(u,r_0)$ with respect to u and the modulus r_0:

$$\frac{\partial \ln Z_1(u,r_0)}{\partial u} = -\frac{1}{8\pi}\left(\int_{|z|=r_0} d\phi \langle X^2(z)\rangle + \int_{|z|=1} d\phi \langle X^2(z)\rangle\right) , \tag{21}$$

$$\frac{\partial \ln Z_1(u,r_0)}{\partial r_0} = -\frac{r_0}{\pi(1-r_0^2)}\int_{r_0\leq|z|\leq 1} d^2z\left(\frac{T_{zz}}{\bar{z}^2}+\frac{T_{\bar{z}\bar{z}}}{z^2}\right) \tag{22}$$

where T_{zz} and $T_{\bar{z}\bar{z}}$ are the components of the stress-energy tensor. (See [25] for the derivation of the second equation. Note a factor of $-1/4\pi$ missed in their formula (4.11).)

The correlator $\langle X^2(\phi)\rangle$ entering the first equation can be obtained by renormalizing the values of Green's function $G(z,z')$ on each of the boundaries in the limit when z approaches z'. One finds that the divergences come from the bulk part $-\ln|z-z'|^2$ and from the terms in the second line of equation (20). The last divergence is exactly of the same form as the first one and has a natural interpretation as a divergence coming from the image charge. Thus one can show that on each component of the boundary the divergent part of the two-point correlator is

$$lim_{\phi\to\phi'}G(\phi,\phi') = -2\ln|1-e^{i(\phi-\phi')}|^2 + \text{finite part}. \tag{23}$$

Subtracting the divergent part we obtain

$$\langle X^2(z)\rangle_{|z|=r_0} + \langle X^2(z)\rangle_{|z|=1} = \frac{4}{u}\Big[\frac{1-u\ln r_0}{2-u\ln r_0}\Big] + 8\sum_{k=1}^{\infty}\frac{r_0^{2k}(k-u)(2k-u)-u(k+u)}{k((k+u)^2-r_0^{2k}(k-u)^2)}.$$

This expression can be now plugged into equation (21) and integrated. This gives the partition function up to a factor that may depend on r_0. To fix this factor one may utilize the second equation (22). Before explaining how this is done let us make some remarks on the subtraction we performed to make the two-point correlator finite. First let us make an obvious remark that we employed the point splitting regularization. It is important to stay consistently with the same regularization when analyzing the system in various coordinates (see the sections below that deal with the open string channel). The corresponding counterterms are constants on each boundary that are equal in value. Thus this subtraction is a renormalization of the coupling constant a. Of course this subtraction is the same as the one made in the tree level calculation [13]. In the calculation via the boundary state described in the previous section this regularization is implicitly present in the normalization factor $(Z_0(u))$ of the boundary state that enters squared (via two boundaries) in the partition function. As we will see below the above subtraction is the only subtraction logarithmic in scale that is needed to render the partition function finite. Therefore the beta functions (including the parts corresponding to the classical dimensions) for the coupling constants a and u are

$$\beta_u = -u\,, \qquad \beta_a = -a-u\,. \tag{24}$$

In a more general case when one has boundary conditions (9) in D directions one has

$$\beta_{u_i} = -u_i\,,\ \ i=1,\dots D\,, \qquad \beta_a = -a-\sum_{i=1}^{D}u_i\,. \tag{25}$$

When using equation (22) one first uses the Green's function (20) to compute the components of the stress-energy tensor

$$T_{zz}(z) = lim_{z\to z'}\frac{1}{2}\Big[\frac{\partial^2 G(z,z')}{\partial z\partial z'} - \frac{1}{(z-z')^2}\Big].$$

A straightforward computation yields

$$T_{zz} = \frac{1}{z^2}\Big[-\frac{u}{4(2-u\ln r_0)} - \sum_{k=1}^{\infty}\frac{r_0^{2k}k(k-u)^2}{(k+u)^2 - r_0^{2k}(k-u)^2}\Big].$$

Now we can plug this and the complex conjugated expression for $T_{\bar{z}\bar{z}}$ into (22) and integrate it. After checking the result against the same kind of computation made using equation (21) we obtain that up to an arbitrary overall numerical factor the partition function coincides with the one computed in the previous section (equation (18)).

6 One-loop bosonic string divergences

In this section we remind the reader of the situation with one-loop divergences in the conformal case. Consider an open bosonic string satisfying Dirichlet boundary condition in D directions and the Neumann ones the remaining $26-D$ dimensions. Its one-loop partition function written in the open string sector has the form

$$Z_1 = \int_0^\infty \frac{dT}{2T}\mathrm{Tr}e^{-2\pi T L_0} = \frac{V_{26-D}}{(\sqrt{8\pi^2\alpha'})^{26-D}}\int_0^\infty \frac{dT}{2T^2}T^{-(24-D)/2}e^{2\pi T}f(\pi T)^{-24} \tag{26}$$

This integral diverges at $T \to 0$. The divergent part in the trace comes from the open string tachyon states and is equal to

$$I = \frac{V_{26-D}}{(\sqrt{8\pi^2\alpha'})^{26-D}}\int_R^\infty \frac{dT}{2T}T^{-\alpha}e^{2\pi T} \tag{27}$$

where we put in some cut-off R and $\alpha = (26-D)/2$. The physical origin of this divergence is in the wrong sign of the tachyon mass squared. The divergent part can be represented as a point particle path-integral (see Polchinski's book [28] for a detailed discussion)

$$I \sim \int \frac{dl}{2l}\int dp^{26-D}e^{-(p^2-1)l/2}$$

where l is a proper time along the particle world line. This amplitude gives the (connected part of) one-loop vacuum amplitude for an open string tachyon vibrating in $26-D$ dimensions. Equivalently in field theory the above formula can be rewritten as a contribution of the tachyon to the vacuum energy density (in space-time)

$$I \sim -\frac{1}{2}\int \frac{dp^{26-D}}{(2\pi)^{26-D}}\ln(p^2-1)$$

that clearly indicates (by the presence of negative values under the logarithm) that the correctly defined amplitude may develop an imaginary part.

The divergent part (27) can be defined by means of analytic continuation. Here our discussion closely follows paper [27]. One treats the exponential in (27) as a parameter b,

evaluating the integral for $b = -2\pi$ and then rotating in complex plane as $b \mapsto be^{-\pi i}$. The direction of rotation can be fixed by carefully inserting the $i\epsilon$ in the propagator (see [27]) for details). It is easy to obtain the imaginary part of the analytically continued integral using the formula

$$\mathrm{Im} \int_R^\infty \frac{dx}{x} x^{-\alpha} e^{(b+i\epsilon)x} = \frac{\pi}{\Gamma(1+\alpha)} b^\alpha . \tag{28}$$

This way we get for (27)

$$\mathrm{Im} I = V_{26-D} \frac{\pi}{2\Gamma\left(14 - \frac{D}{2}\right)} \left(\frac{|m^2_{\text{o. s. tach.}}|}{4\pi} \right)^{(26-D)/2} \tag{29}$$

where $m^2_{\text{o. s. tach.}} = -1/\alpha'$ is the open string tachyon mass squared.

The appearance of imaginary part for the partition function and thus for the one-loop space-time effective tachyon action of course signifies the instability of the system. It is known in field theory [29], [30] that an imaginary part of the vacuum energy gives a decay rate of unstable vacuum. More precisely the decay rate per unit volume is $\Gamma = 2\mathrm{Im}E$.

One can also compute the real part of the analytically continued expression. However the real part will depend on the cutoff and needs to be pasted together with the other part of the partition function. The resulting number can be obtained numerically (see [27] for an example of this type of computation).

The divergences due to closed string states can be studied by performing a closed/open string channel duality transformation $t = \pi/T$. The partition function (26) takes the form

$$Z_1 = \frac{V_{26-D}}{2\pi(\sqrt{8\pi^2\alpha'})^{26-D}} \int_0^\infty dt t^{-D/2} e^{2t} f^{-24}(t) . \tag{30}$$

When $D > 2$ the divergent part comes only from the closed string tachyon states

$$I' = \frac{V_{26-D}}{2\pi(\sqrt{8\pi^2\alpha'})^{26-D}} \int_R^\infty dt t^{-D/2} e^{2t}$$

This expression corresponds to a vacuum-vacuum tree-level diagram for the closed string tachyon

$$I' \sim \int \frac{dp^{26-D}}{(2\pi)^{26-D}} \frac{1}{p^2 - 2} .$$

Using formula (28) we obtain for the imaginary part

$$\mathrm{Im} I' = V_{26-D} \frac{2^{D/2-2}}{\Gamma(D/2)} \left(\frac{|m^2_{\text{c. s. tach.}}|}{32\pi^2} \right)^{(26-D)/2} \tag{31}$$

where $m^2_{\text{c. s. tach.}} = -4/\alpha'$ is the closed string tachyon mass squared.

When $D \leq 2$ one also has a divergence corresponding to a dilaton tadpole. This is a true physical divergence that cannot be eliminated by analytic continuation and is due to

the propagation of massless particles between the vacuum states. The momentum space propagator for a massless particle is $1/p^2$ that diverges on the mass shell. The momentum conservation on the other hand requires that the massless particle emerges from the vacuum with the zero momentum, i.e. precisely on the mass shell. The problem with this divergence can be resolved by a shift of the dilaton background [31] that results in adding a source term to the space-time effective action.

7 Divergences in the closed string channel in the presence of tachyon background

In this section we consider divergences of the one-loop partition function (19) coming from the closed string sector. By looking at the boundary state (16) and the expression for the partition function (14) we see that divergences at $t \to 0$ come from the vacuum part of the boundary state

$$|\text{tach.}\rangle \equiv N(u) \cdot \sqrt{u} exp\Big(-\frac{u\hat{q}^2}{4}\Big)|0\rangle \tag{32}$$

that describes a tachyon with a Gaussian wave function and in the case when $D \leq 2$ from the part

$$|\text{dil.}\rangle \equiv N(u) \cdot \Big(\frac{1-u}{1+u}\Big) \cdot \sqrt{u} a_1^\dagger \tilde{a}_1^\dagger exp\Big(-\frac{u\hat{q}^2}{4}\Big)|0\rangle \tag{33}$$

corresponding to a dilaton with the same Gaussian wave function.

For $D \geq 3$ there is no divergence due to the dilaton (c.f. the discussion in the previous section). Physically one may think about this fact as follows. By plugging the state (33) into expression (14) for the partition function we find that the contribution of this state up to an overall constant is

$$I \sim \langle \text{dil.}| \int_0^\infty dt\, e^{-\hat{p}^2 t} |\text{dil.}\rangle = \langle \text{dil.}| \frac{1}{\hat{p}^2} |\text{dil.}\rangle = \int d^{26}p \frac{1}{p^2} e^{-\sum_{i=1}^D \frac{p_i^2}{2u_i}}.$$

Fourier transforming this expression we obtain

$$I \sim \int d^{26}q \frac{1}{q^{24}} e^{-\frac{1}{2}\sum_{i=1}^D u_i q_i^2}$$

that can be interpreted as a Coulomb potential self-energy of a matter that has a Gaussian density distribution in D-directions. For $D > 3$ it is finite. The Gaussian factor plays a role of effective infrared regulator. However in the limit $u \to 0$ the divergence will show up as a more singular behavior of the partition function.

Let us postpone a further investigation of the dilaton divergences until section 9 and consider now divergences due to the tachyon state. The divergent contribution of state (32) to the partition function has the form

$$I_{\text{cl. tach.}}(u) = \frac{V_{26-D}}{(\sqrt{16\pi^2})^{26-D}} e^{-2a} \prod_{i=1}^D \Big[(Z_0(u_i))^2 \int_0^\infty \frac{dt}{2\pi} e^{2t} \sqrt{\frac{u_i}{8\pi(1+tu_i/2)}}\Big].$$

For the sake of simplifying the computation we will assume that all u_i's are equal to a single parameter u. Then changing the exponent 2 to a complex parameter $-b$ and shifting the integration variable we can rewrite the above integral as

$$I_{\text{cl. tach.}}(u,b) = \frac{V_{26-D}}{(\sqrt{16\pi^2})^{26-D}}(e^{-a}(Z_0(u))^D)^2 \frac{1}{2\pi(4\pi)^{D/2}} e^{2b/u} \int_{2/u}^{\infty} dt' \frac{e^{-bt'}}{(t')^{D/2}} .$$

Applying formula (28) we obtain

$$\text{Im} I_{\text{cl. tach.}}(u) = \frac{V_{26-D}}{(\sqrt{16\pi^2})^{26-D}}(e^{-a}(Z_0(u))^D)^2 \frac{1}{(4\pi)^{D/2}} e^{-4/u} \frac{2^{D/2-2}}{\Gamma(D/2)} . \tag{34}$$

As we will show in section 10 minimizing the space-time effective action with respect to a the limit of the quantity $e^{-a}(Z_0(u))^D$ along the RG trajectory $u \to \infty$, $a \to \infty$ is a number $(\sqrt{2\pi})^D$ just as in the tree level computation [35]. Thus the imaginary part (34) as $u =\to \infty$ approaches the finite value

$$\text{Im} I_{\text{cl. tach.}}(\infty) = \frac{V_{26-D}(2\pi)^{D/2}}{4(\sqrt{16\pi^2})^{26-D}\Gamma(D/2)} . \tag{35}$$

Up to a numerical factor (that essentially comes from $(e^{-a}Z_0(u))^{2D}$) ((35) agrees with (31) and we see that for this divergence the naive prescription of first taking the $u \to \infty$ limit that yields the Dirichlet boundary conditions and then performing the analytic continuation gives the same result as the accurate procedure described above. We see however that along the RG flow the imaginary part due to the closed string tachyon grows (in absolute value) from zero at $u = 0$ to the constant value (35).

The closed string tachyon one-loop divergence in the presence of the quadratic open string tachyon background was also considered in [48] where a different treatment than analytic continuation is proposed.

8 Open string channel

The divergences of the partition function (19) in the limit $t \to 0$ are best studied in the open string channel that describes an open string propagating in a (Euclidean) time loop of length $2\pi T = 2\pi^2/t$ and satisfying the boundary conditions (13) that for the sake of readers convenience we write here once more:

$$-\frac{\partial X}{\partial \sigma} + \frac{u}{T}X = 0\,, \quad \sigma = 0\,, \qquad \frac{\partial X}{\partial \sigma} + \frac{u}{T}X = 0\,, \quad \sigma = \pi\,. \tag{36}$$

(To simplify the notations in this section we will drop the primes at the open string world sheet coordinates σ', τ'.) Although these boundary conditions have a nonlocal time-dependence via the factors of $\frac{1}{T}$ in order to compute the partition function

$$Z_1 = \int_0^{\infty} \frac{dT}{2T} \text{Tr} e^{-2\pi T H(u/T)} \tag{37}$$

we may first canonically quantize the theory with the boundary conditions

$$-\frac{\partial X}{\partial \sigma} + vX = 0\,, \quad \sigma = 0\,, \qquad \frac{\partial X}{\partial \sigma} + vX = 0\,, \quad \sigma = \pi \tag{38}$$

where v is a constant, write down the partition function and then replace v with u/T. The Hamiltonian $H(v)$ will have a spectrum of the form

$$h = E_{Cas}(v) + \sum_{n=0}^{\infty} N_n \lambda_n(v)$$

where $E_{Cas}(v)$ is the Casimir energy in the background (38), $\lambda_n(v)$ are oscillator frequencies and N_n - the occupation numbers. The partition function (37) is thus of the form

$$Z_1 = \int_0^{\infty} \frac{dT}{2T} e^{-2\pi T E_{Cas}(u/T)} \prod_{n=0}^{\infty} \frac{1}{1 - e^{-2\pi T \lambda_n(u/T)}} \tag{39}$$

If the spectrum in the presence of u- background is not significantly modified the divergences in (39) will come only from a finite number of the first excited states, i.e. the ground state with energy $E_{Cas}(u/T)$ and possibly some number of the low lying excited states.

We start by deriving the spectrum of frequencies. A general solution to the Laplace equation

$$\left(\frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \sigma^2}\right) X = 0$$

with the boundary conditions (38) can be expanded into eigenfunctions as

$$X(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} e^{\lambda_n \tau} (\alpha_n e^{-i\sigma\lambda_n} + \beta_n e^{i\sigma\lambda_n})\,.$$

By plugging in this expansion into the boundary conditions (38) we obtain the system of two linear homogeneous equations. The corresponding determinant factorizes as

$$\Delta = \left(\tan\left(\frac{\pi\lambda_n}{2}\right) - \frac{v}{\lambda_n}\right)\left(\cot\left(\frac{\pi\lambda_n}{2}\right) + \frac{v}{\lambda_n}\right).$$

Thus we obtain two spectral equations

$$\begin{aligned} \tan\left(\frac{\pi\lambda_n}{2}\right) &= \frac{v}{\lambda_n}\,, \\ \cot\left(\frac{\pi\lambda_n}{2}\right) &= -\frac{v}{\lambda_n} \end{aligned} \tag{40}$$

defining the even and odd parity eigenvalues respectively. We would like to remark here that the spectral problem we are looking at here is equivalent to the following quantum mechanical model. Consider a one-dimensional system on a circle with coordinate $0 \leq x < 2\pi$ subject

to $\mathbf{Z}_2$ orbifold identification $x \sim 2\pi - x$. If we consider a quantum mechanical particle in this space with a potential

$$V(x) = 2u\delta(x) + 2u\delta(x - \pi)$$

then for the corresponding Schroedinger operator one obtains exactly the same spectrum as (40) provided one restricts himself to the solutions that are even under the reflection $x \to x - 2\pi$.

Alternatively both parity branches can be combined in a single equation

$$\tan(\pi\lambda_n) = \frac{2\lambda_n v}{\lambda_n^2 - v^2}.$$

It is clear from this equation or equations (40) that for sufficiently large n one recovers the regular Regge spectrum with the first correction of the form

$$\lambda_n = n + \frac{2v}{\pi n} + \mathcal{O}((v/n)^2). \tag{41}$$

For $v < 1$ equation (41) is a fairly good approximation to all eigenvalues with the exception of the lowest eigenvalue. The last one is approximately

$$\lambda_0 \approx \sqrt{\frac{2v}{\pi}} \tag{42}$$

with the correction being of the order of $v^{3/2}$.

Despite the fact that we cannot solve the transcendental equations (40) exactly (and thus cannot use the direct mode summation formula) we still will be able to derive an integral formula for $E_{Cas}(v)$ valid for all (nonnegative) values of v. Before we go into that let us first consider as a warmup the small v case when the expressions (41), (42) provide a good approximation for the spectrum. Using those expressions and the mode summation method we can compute an approximate expression for the Casimir energy that is valid up to the terms depending on v as v^p, $p > 1$. Note that after we substitute $v = u/T$ and plug these terms into the Hamiltonian we obtain factors of the form $e^{-2\pi u^p T^{1-p}}$ that tend to 1 as $T \to \infty$. Thus for small u our approximation will allow us to derive the leading divergence of the partition function as $T \to \infty$. Moreover, despite its seemingly limited range of use, this asymptotics essentially sets the imaginary part of the analytically continued $T \to \infty$ divergence. This is due to the following observation. The imaginary part should not depend on the lower cutoff in T-integration. Thus for any value of u we can place the cutoff high enough so that $v = u/T$ is small and thus the approximation we are talking about is useful.

To compute the Casimir energy in such a way that the resulting partition function will match the one computed in the closed string channel we must employ the equivalent regularization scheme. Thus we should proceed by using the point splitting regularization. Note that there is a subtlety here. The point splitting parameter $\phi' - \phi = \epsilon$ used on the annulus (23) corresponds to a point splitting $\epsilon' = \epsilon \cdot T$ on the strip in the open channel. The factor T comes from the coordinate mapping

$$z = e^{-\frac{\sigma + i\tau}{T}}$$

relating the two pictures. Here z is the complex coordinate on the annulus and σ, τ are coordinates on the strip.

With this identification in mind we can write the point slitting regulated Casimir energy for small values of v as

$$E_{Cas}(v) = \frac{1}{2}\lambda_0 + \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n e^{-\epsilon'\lambda_n} = \sqrt{\frac{v}{2\pi}} + \frac{1}{2}\sum_{n=1}^{\infty}\left(n + \frac{2v}{\pi n}\right)e^{-\epsilon'(n+2v/(\pi n))} + o(v) =$$
$$\sqrt{\frac{v}{2\pi}} - \frac{1}{24} + \frac{1}{2T^2\epsilon^2} - \frac{v}{\pi} - \frac{v}{\pi}\ln\epsilon - \frac{v}{\pi}\ln T + o(\epsilon) + o(v)$$

Subtracting the divergent quadratic and logarithmic parts we obtain after sending $\epsilon \to 0$

$$E_{Cas}(v) = -\frac{1}{24} + \sqrt{\frac{v}{2\pi}} - \frac{v}{\pi} - \frac{v}{\pi}\ln T + o(v) \tag{43}$$

Therefore it follows that

$$lim_{T\to\infty}\mathrm{Tr}e^{-2\pi TH(u/T)} = T^{2u}e^{2u}exp\left(\frac{\pi T}{12} - 2\sqrt{2\pi uT} - 2\pi T\cdot f(u/T)\right) \tag{44}$$

where $f(u)$ is some unknown function that has the property

$$lim_{T\to\infty}T\cdot f(u/T) = 0\,.$$

We independently obtained the same asymptotics (up to an overall exponent e^{Cu} for which we were unable to pin down the value of C) by analyzing the expression for the partition function in the closed string sector by means of the Euler-Maclaurin summation formula.

We would like to derive now a general integral formula for the Casimir energy $E_{Cas}(v)$ that is valid for all values of v. This can be done as follows. It is easy to show that the spectrum λ_n coincides (except for the zero poit) with the set of zeroes of an entire analytic function

$$\phi(z) = e^{i\pi z}(v+iz)^2 - e^{-i\pi z}(v-iz)^2\,.$$

The regulated Casimir energy then can be represented as a contour integral

$$E_{Cas}(v,\epsilon') = \frac{1}{4\pi i}\oint ze^{-\epsilon' z}d\ln\phi(z) \tag{45}$$

where the contour should encircle the positive eigenvalues and can be conveniently chosen to consist of two slanted rays: $z = (i+\delta)x$, $z = (-i+\delta)x$ where $\delta > 0$ and x runs from zero to infinity. One should also keep in mind that we do not include the zero into the spectrum. Although formally it does not contribute to the regulated infinite sum of the eigenvalues one may still wish to avoid potential troubles by modifying the contour by cutting out small initial segmets of the rays and connecting their endpoints by a small half-arc.

To simplify the manipulations below we note that both δ and the point splitting parameter ϵ' act as regulators, and one can achieve the same result by keeping only ϵ' and assuming that it has an appropriate imaginary part that provides a damping exponential factor. With this

in mind and taking also into account that the function $\phi(z)$ is odd we can write $E_{Cas}(v,\epsilon')$ as the following integral

$$E_{Cas}(v,\epsilon') = -\frac{1}{2\pi}\int_0^\infty xcos(\epsilon' x) d\ln\left(e^{\pi x}(v+x)^2 - e^{-\pi x}(v-x)^2\right).$$

Factoring out the term $e^{\pi x}(u+x)^2$ we obtain after one partial integration

$$E_{Cas}(v,\epsilon') = \frac{1}{2\pi}\int_0^\infty \ln\left(1 - e^{-2\pi x}\left(\frac{x-v}{x+v}\right)^2\right) d(xcos(\epsilon' x))$$
$$-\frac{1}{2}\int_0^\infty xcos(\epsilon' x)dx - \frac{1}{\pi}\int_0^\infty \frac{xcos(\epsilon' x)}{x+v}dx.$$

Here the first integral is finite as $\epsilon' \to 0$, the second term is quadratically divergent and the third one is logarithmically divergent. The divergences are regulated by imaginary part of ϵ' that is chosen with an appropriate sign. Let us show how to treat the logarithmic divergence. We can rewrite the logarithmically divergent part I_{log} as

$$I_{log} = \frac{v}{\pi}e^{v\epsilon'}(-E_{\rm i}(-v\epsilon'))$$

where $E_{\rm i}(x)$ denotes the exponential integral function. Its asymptotics near zero is such that (see for example [47]) $-E_{\rm i}(-x) = -\gamma - \ln(x) + o(x)$, $|argx| < \pi$ where γ is the Euler constant. Using this asumptotics we can go ahead subtract the infinities end send $\epsilon = \epsilon'/T$ to zero. We obtain the following expression

$$E_{Cas}(v) = \frac{1}{2\pi}\int_0^\infty \ln\left(1 - e^{-2\pi x}\left(\frac{x-v}{x+v}\right)^2\right)dx - \frac{v}{\pi}\ln(vT) - \frac{v}{\pi}\gamma. \tag{46}$$

Note that the first term in this expression we could have easily gotten starting from the closed string expression (19) and using the Euler-Maclaurin summation formula. Thus we see that in terms of that formula the open string channel Casimir energy is essentially the integral approximation to the infinite series while the corrections (that can be written for example as integrals involving saw-tooth function and its integrals) correspond to the excited states. Note that the first term in (46) gives the standard conformal Casimir energy $-\frac{1}{24}$ in the limits $v \to 0$ and $v \to \infty$.

Let us discuss now the divergence $T \to \infty$ ($t \to 0$) of the partition function. The general form of asymptotics is given by formula (44) in which the function $f(x)$ is in principle extractable from our general expression (46). This function cannot be dropped because the corresponding contributions are still divergent. Also note that the states coming from the excited levels of the first oscillator with frequency λ_0 also have negative energy and thus contribute to the divergence (44) an overall factor

$$\frac{1}{1-e^{-2\pi T\lambda_0(u/T)}} \approx \frac{1}{1-e^{-2\sqrt{2\pi Tu}}}.$$

Note that in the limit $u \to 0$ this factor restores the contribution of zero modes

$$\frac{1}{2\sqrt{2\pi T u}}.$$

By comparing the $u \to 0$ behavior of the partition function in the closed string channel (18) to the expression above we find the normalization of the boundary state (16).

Thus we can write down the following general expression for the tachyon divergent part of the partition function for 26 scalars with D scalars satisfying the boundary condition (36) has the form

$$\begin{aligned} I_{\text{o. tach.}}(u) &= e^{2u} \frac{V_{26-D}}{(16\pi^2)^{(26-D)/2}} \int_R^\infty \frac{dT}{2T} T^{2u} T^{-(26-D)/2} \cdot \\ & exp\Big(2\pi T + 2D\sqrt{2\pi u T} - 2\pi D T \cdot f(u/T)\Big) \frac{1}{1 - e^{-2D\sqrt{2\pi T u} - 2D\pi T \phi(u/T)}} \end{aligned} \tag{47}$$

where $\phi(u)$ stands for a correction to $\lambda_0(u)$, $b = 2\pi$ to be rotated to -2π, and R is a cutoff. Further analysis of this formula is obstructed by two facts: our integrable representation (46) is not very useful in analytic manipulations and second, we do not have any explicit analytic expression for $\phi(u)$.

The terms contained in $f(u)$ and $\phi(u)$ are responsible for the flow from the Neumann to the Dirichlet spectrum. In the course of that flow the even integer eigenvalues get shifted by one unit and become the odd ones and vice versa. Since the limit $u \to \infty$ sends the variable u/T to infinity and the limit $T \to \infty$ sends it to zero for a fixed cutoff R the contributions of f and ϕ become more and more important in that limit. However as we argued above for the purposes of computing the imaginary part of the analytic continuation one may always adjust the lower cutoff so that the terms in f in ϕ are all subdominant. One can expand in this terms so that the typical expression u-dependent term in the integral is of the form

$$O(u) \cdot \int_R^\infty e^{-bT + 2D\sqrt{2\pi u T}} T^{\alpha(u)} \tag{48}$$

where $\alpha(u)$ and $O(u)$ are such that $\alpha(u) \to constant$, $O(u) \to 0$ as $u \to 0$. Each term of this form can be analytically continued in b.

One may be interested in two kinds of questions about the analytic continuation and the imaginary part. First one may worry whether taking the limits $u \to 0$ $u \to \infty$ commutes with analytic continuation, or in other words do we recover the standard analytically continued Dirichlet and Neumann partition functions. If this were not true it would clearly signal some inconsistency of the analytic continuation procedure applied to the off shell situation at hand. In view of expansion (48) above it seems to us that this is not the case.

A second, less formal question one may be interested in has to do with the physical interpretation of the imaginary part. Since it gives the decay rate of the unstable vacuum, it could be used as a measure of the vacuum stability. It would be very interesting then to see how it behaves along the RG trajectory. One may expect that the flow monotonically decreases to the value set by (29) for the appropriate D. Unfortunately we did not gain enough analytical control over the open string channel to see that.

9 Correction to the tachyon potential

Consider now a boundary perturbation specified by coupling constants u_i switched on in D directions with the other $26-D$ Neumann directions being compactified as $X_i \sim X_i + R_i$, $i = D+1, \ldots, 26$. We will restrict ourselves to the case $D \geq 3$ so we will not have to deal with the dilaton divergence directly from the start, although inevitably it will show up in the $u \to 0$ limit.

We assume that the space-time one-loop effective action is given by the expression

$$S = (1 + \beta_a \frac{\partial}{\partial a} + \sum_{i=1}^{D} \beta_{u_i} \frac{\partial}{\partial u_i})(\frac{1}{g} Z_0(u,a) + Z_1^{a.c.}(u,a))$$

where $Z_0(u,a) = e^{-a} \prod_{i=1}^{D} Z_0(u_i) \prod_{i=D+1}^{26} \frac{R_i}{\sqrt{4\pi}}$ is the tree-level partition function, g is a string coupling constant and

$$Z_1^{a.c.}(u,a) = e^{-2a} Z_1^{a.c.}(u_1, \ldots, u_D)$$

stands for the one-loop partition function (19) defined by means of analytic continuation. For $D \geq 3$ it takes finite values and contains an imaginary part due to both closed and open string tachyons.

At this point we would like to restore the factors of α'. This can be easily done by the substitutions: $u \mapsto u\alpha'/2$, $R_i \mapsto R_i\sqrt{2/\alpha'}$. Substituting the beta functions (25) we obtain

$$\begin{aligned} S &= \frac{1}{g} e^{-a} (a + \sum_i \frac{\alpha'}{2} u_i - u_i \frac{\partial}{\partial u_i} + 1) \prod_i Z_0(u_i) \prod_{i=D+1}^{26} \frac{R_i}{\sqrt{2\pi\alpha'}} + \\ & e^{-2a} (2a + \sum_i \alpha' u_i - u_i \frac{\partial}{\partial u_i} + 1) Z_1^{a.c.}(u_1, \ldots, u_D) . \end{aligned} \tag{49}$$

On the other hand this expression should coincide with the space-time action

$$S = T_{25} \int d^{26}X \, [f(T) \partial_i T \partial^i T + V(T) + \text{higher derivative terms}] \tag{50}$$

evaluated on the quadratic tachyon profile

$$T(X) = a + \frac{1}{4} \sum_i u_i X_i^2 . \tag{51}$$

The tree level computation [34], [35] gives

$$f(T) = e^{-T} , \qquad V(T) = e^{-T}(1+T)$$

and the value of $T_{25}^{tree} = \frac{1}{g}(2\pi\alpha')^{-13}$ where we inserted the $1/g$ factor to match with our conventions. Our considerations below follow more closely paper [35]. Note that our normalizations are slightly different from those in [35]. Their coupling constant u is 4 times our u.

Let us now first study the expression (49) in the limit $u \to 0$. We can represent $Z_1^{a.c.}$ as a sum of the "bulk part" that comes from integration over a region of moduli space with cutoffs on both ends, and the analytically continued "tails" coming from the two boundaries of the moduli space. As u goes to zero the bulk part tends to the bulk part of the Neumann partition function. The contributions to the tails due to open and closed string tachyons were discussed in the previous sections. We argued that both analytically continued tails due to tachyons tend to the corresponding tachyon contributions of the string with Neumann boundary conditions. The only problem in the $u \to 0$ comes from the dilaton states.

For simplicity let us assume $u_1 = \ldots = u_D = u$. Then the dilaton contribution to the partition function that hides in it the usual dilaton tadpole has the form

$$\begin{aligned} I_{\text{dilaton}}(u,a) &= (e^{-a}(Z_0(u))^D)^2 \frac{V_{26-D}}{(\sqrt{16\pi^2})^{26-D}} \cdot ((24-D) + D\left(\frac{1-u}{1+u}\right)^2) \cdot \\ & \int_0^\infty \frac{dt}{2\pi} \frac{1}{((2/u+t)4\pi)^{D/2}} . \end{aligned} \tag{52}$$

Evaluating the integral we obtain an asymptotics of the form

$$I_{\text{dilaton}}(u) \sim Const \cdot \frac{1}{u} \cdot \left(\frac{1}{\sqrt{u}}\right)^D , \quad u \to 0$$

that contains an extra $1/u$ factor standing at the usual volume element.

In the space-time effective action this term can be represented by a nonlocal term of the form

$$\frac{1}{\partial^2} e^{-2T}(1+2T) .$$

Thus as $u \to 0$ we have an explicit infrared problem in our action. It must be resolved by introducing a new massless degree of freedom about which we of course were aware from the very beginning - the dilaton field.

In this paper we will not try to incorporate the dilaton field in our action. Instead we will stay at small but finite u (we may think that we put our system in a box) and find a correction to the tachyon potential.

Let us come back now to the consideration of background u_i (where u_i's are not necessarily equal to each other) for D directions where we assume $D \geq 3$ and all u_i's are away from zero. As before the remaining $26-D$ directions are compactified in a box with sides R_i. To derive the correction to the potential we have to match the terms at $1/\sqrt{u}$ as $u \to 0$ in the expression coming from the partition function (49) and the one coming from the space-time action (50). The partition function (19) being plugged into the second term in (49) gives

$$\mathcal{S}^{(1)} \sim \frac{V_{26-D}}{(\sqrt{\alpha'\pi})^{26-D}} e^{-2a}(1+\frac{D}{2}+2a) \frac{1}{(\sqrt{8\pi})^{26}} \left(\prod_{i=1}^{D} \frac{1}{\sqrt{u_i \alpha'/2}}\right) \int_0^\infty \frac{dt}{2\pi} e^{2t} f^{-24}(t) \tag{53}$$

where $V_{26-D} = \prod_i R_i$ and we restored the factors of α'. On the other hand by inserting the quadratic profile (51) into the potential density in (50) that has to be of the form

$$k \cdot e^{-2T}(1+2T)$$

where k is some constant, we obtain

$$\mathcal{S}^{(1)} \sim T_{25} k V_{26-D} e^{-2a}(1+\frac{D}{2}+2a)\prod_{i=1}^{D}\sqrt{\frac{2\pi}{u_i}}. \tag{54}$$

Equating (53) and (54) we obtain the following expression for the 1-loop effective potential.

$$V_{1\text{loop}}(T) = e^{-T}(1+T) + gZe^{-2T}(1+2T) \tag{55}$$

where Z is up to normalization the analytically continued Neumann partition function with the dilaton tadpole subtracted which we can formally write as

$$Z = \frac{1}{(\sqrt{4\pi})^{26}}\int_0^\infty \frac{dt}{2\pi} e^{2t} f^{-24}(t).$$

Note that the procedure of matching (53) and (54) is quite sensitive to the normalization of the partition function (19). In particular if we had a different normalization of the zero modes for the Neumann boundary conditions we would find that the coefficient k in (54) depends on D which is clearly an inconsistency. The fact that we obtained the correct normalization based on purely world sheet considerations seems to be quite encouraging.

From the above expression for the correction to the potential we can read off the one-loop corrected D25 brane tension

$$T_{25}^{1l} = T_{25}^{tree}(1+gZ). \tag{56}$$

The imaginary part of Z is due to the open string tachyon and is given by the expression (29) for $D=0$, multiplied by $1/(4\pi)^{13}$. In the proper definition of the tension (56) one should consider only the real part. The imaginary part specifying the decay rate is a separate piece of information. See [30] for a detailed discussion of how it works in field theory.

10 Loop corrected Dp-brane tensions

In this section we would like to study the ratio of the brane tensions by finding the limiting value of the effective action (49) in a similar way to how it was done in [35].

We begin as in [35] by extremizing the action $S(a,u_i)$ with respect to a. To simplify the formulas let us assume that $Z_0(u)$ denotes the complete tree level partition function for D boundary conditions with u_i and the $D-26$ Neumann ones. Differentiating the expression (49) with respect to a and equating the result to zero we obtain the following equation for $a^* = a^*(u)$

$$\begin{aligned} Z_0(u) + 2ge^{-a^*}Z_1(u) &= (a^* + \sum_i \frac{\alpha'}{2}u_i - u_i\frac{\partial}{\partial u_i} + 1)Z_0(u) + \\ &\quad 2ge^{-a^*}(2a^* + \sum_i \alpha' u_i - u_i\frac{\partial}{\partial u_i} + 1)Z_1^{a.c.}(u) \end{aligned}$$

Let us represent now a^* as a sum $a^* = a_0^* + \Delta a^*$ where

$$a_0^* = -\sum_i \frac{\alpha'}{2} u_i + \sum_i u_i \frac{\partial}{\partial u_i} \ln Z_0(u)$$

is the tree level solution. Let us also use the fact that $Z_1^{a.c.}(u)$ has the form $Z_1^{a.c.}(u) = Z_0^2(u)\tilde{Z}_1(u)$ (19).

Then we have the following equation for the correction term Δa^*

$$-\Delta a^* = g\left(e^{-a_0^*} Z_0(u)\right) e^{-\Delta a^*} (4\Delta a^* - 2\sum_i u_i \frac{\partial}{\partial u_i}) \tilde{Z}_1(u) .$$

Note that as $u \to \infty$ the factor $\left(e^{-a_0^*} Z_0(u)\right)$ monotonically decreases to the value $(\sqrt{2\pi})^D$. The above equation for Δa^* is exact. However it is more consistent, in view of higher loop corrections, to keep in it only terms of the first order in g. This gives

$$\Delta a^* = 2g\left(e^{-a_0^*} Z_0(u)\right) \sum_i u_i \frac{\partial}{\partial u_i}) \tilde{Z}_1(u) .$$

It follows from this equation that $lim_{u\to\infty} \Delta a^* = 0$ if the following assumptions on $\tilde{Z}_1(u)$ are true:

$$lim_{u\to\infty} \tilde{Z}_1(u) = Const , \qquad lim_{u\to\infty} \sum_i u_i \frac{\partial}{\partial u_i} \tilde{Z}_1(u) = 0 .$$

Both assumptions can be easily shown to be true for the "bulk" part of the $\tilde{Z}_1(u)$, i.e. for the part where integration over the modulus has cutoffs on the two ends. Moreover it follows from our considerations in section 7 that these assumptions are also true for the contribution of the closed string channel boundary ($t \to \infty$, $T \to 0$). As for the open string channel part we have to leave it at the level of conjecture.

With these assumptions being true we can safely plug in the tree level solution $a^* = a_0^*$ into equation (49) and take the limit $u \to \infty$. We obtain (restoring the explicit volume factor at the tree level partition function)

$$\lim_{u\to\infty} S(u, a^*(u)) = \lim_{u\to\infty} \Big[e^{-a^*(u)} Z_0(u) \prod_{i=D+1}^{26} \frac{R_i}{\sqrt{2\pi\alpha'}} +$$

$$g(e^{-a^*(u)} Z_0(u))^2 \tilde{Z}_1(u) \prod_{i=D+1}^{26} \frac{R_i}{\sqrt{\pi\alpha'}} \Big] =$$

$$(2\pi)^{D/2} \prod_{i=D+1}^{26} \frac{R_i}{\sqrt{2\pi\alpha'}} + g(2\pi)^D \frac{1}{(\sqrt{8\pi})^{26}} (\sqrt{2})^D \int_0^\infty \frac{dt}{2\pi} t^{-D/2} e^{2t} f^{-24}(t) \prod_{i=D+1}^{26} \frac{R_i}{\sqrt{\pi\alpha'}} .$$

Dropping the volume factor the last expression can be rewritten as

$$T_p^{1l} = \lim_{u\to\infty} S(u, a^*(u)) = T_p^{tree}(1 + g\tilde{Z}_p) \tag{57}$$

where

$$T_p^{tree} = \frac{1}{g}(\sqrt{2\pi\alpha'})^{-13}(2\pi\sqrt{\alpha'})^D$$

is the tree level tension of a Dp-brane with $p = 25 - D$ and

$$\tilde{Z}_p = \frac{1}{(\sqrt{4\pi})^{26}} \int_0^\infty \frac{dt}{2\pi} \left(\frac{2\pi}{t}\right)^{(25-p)/2} e^{2t} f^{-24}(t) \,. \tag{58}$$

Thus the equations (57), (58) give the one-loop corrected tensions of Dp-branes. Again the normalization of this result seems to be quite meaningful. In particular for $D = 0$ we recover (56). Up to possible differences in normalization it agrees with the expected one-loop correction based on on-shell string theory considerations [32].

11 Discussion

In this paper we have computed the one loop correction to the tachyonic potential in order to investigate its contribution to the tree level tachyonic condensation process. This calculation consisted of two steps: The first step, which was well defined and unambigous, was the calculation of a string amplitude as a function of the modulus. The second step, more arbitrary and open to question, was the determination of the modular measure and the use of formula (5) at the one-loop level. The recipe we have used for this part of the problem, although simple and natural, is somewhat arbitrary and lacks firm foundation. Clearly, more work is needed to put these results on a sounder foundation.

Another question that needs further investigation is the treatment of divergences. We have chosen to avoid the divergences due to the presence of tachyons by a suitable analytic continuation. This method generates a complex tachyon potential, which is to be expected on the grounds of vacuum instability. An alternative possibility is to appeal to Fischler-Susskind mechanism [31]. Again, more work is needed to clarify the situation. There is also the problem of the divergence due to the dilaton when $D < 3$, which we did not treat in this paper.

Our computations lead to corrections to the tachyon potential (55) and to brane tensions (57), (58) that look quite meaningful. In particular based on our considerations we may give the following qualitative argument on the nature of higher loop corrections to the process describing the reduction of $D25$ brane into a lower dimensional brane. If the picture with the boundary states discussed in section 2 is correct then the n-loop correction will have a factor of $(e^{-a}Z_0(u))^{n-1}$. Furthermore it looks plausible that similar to the one-loop case the corrections to the $a^*(u)$, i.e., to the value of a extremizing the action, will be negligible as $u \to \infty$. In that case we will get a correction that up to a constant factor coincides with the appropriate n-loop partition function with Dirichlet and Neumann boundary conditions. Thus it looks like in this situation nothing happens to the effective string coupling constant. The processes we considered describe only some descent relations between branes. Of course even if one starts with a single D0 brane there are relevant perturbations that drive the

system further to the bottom of the tachyon potential. As discussed in [35], [36], [37] the constant perturbation, which one may always switch on as long as the world sheet has a boundary, will on one hand keep the effective string coupling small and on the other hand will damp any open string amplitude. It is not clear to us that this apparent damping factor will dominate over any other possible relevant perturbation. There may be a growing factor in the boundary state similar to the $Z_0(u)$ that will compensate e^{-a}.

Finally one should also understand if normalizations of the one-loop corrected Dp-brane tensions are in accord with the on-shell string theory considerations. We leave these questions for a future investigation.

Acknowledgements

A. K. wants to acknowledge a useful discussion with Barton Zwiebach.

References

[1] K. Bardakci, *Dual models and spontaneous symmetry breaking*, Nucl. Phys. **B68** (1974) 331.

[2] K. Bardakci and M. Halpern, *Dual M-models*, Nucl. Phys. **B73**, (1974) 295.

[3] K. Bardakci, *Spontaneous symmetry breakdown in the standard dual string model*, Nucl. Phys. **B133**, (1978) 297.

[4] V. A. Kostelecky and S. Samuel, *The static tachyon potential in the open bosonic string theory*, Phys. Lett. **B207** (1988) 169.

[5] V. A. Kostelecky and S. Samuel, *On a nonperturbative vacuum for the open bosonic string*, Nucl. Phys. **B336**, 263 (1990).

[6] A. Sen, *Tachyon condensation on the brane antibrane system*, JHEP **9808**, 012 (1998) [hep-th/9805170].

[7] A. Sen, *SO(32) Spinors of Type I and Other Solitons on Brane-Antibrane Pair*, JHEP **9809** (1998) 023 [hep-th/9808141].

[8] A. Sen, *Descent relations among bosonic D-barnes*, Int. J. Mod. Phys. A14 (1999) 4061 [hep-th/9902105].

[9] A. Sen, *Universality of the tachyon potential*, JHEP **9912** (1999) 027 [hep-th/9911116].

[10] A. Sen and B. Zwiebach, *Tachyon condensation in string field theory*, JHEP **0003** (2000) 002 [hep-th/9912249].

[11] N. Moeller and W. Taylor, *Level truncation and the tachyon in open bosonic string field theory*, Nucl. Phys. **B583** (2000) 105-144 [hep-th/0002237].

[12] E. Witten, *On Background Independent Open-String Field Theory*, Phys. Rev. **D46** (1992) 5467-5473; hep-th/9208027.

[13] E. Witten, *Some Computations in Background Independent Open-String Field Theory*, Phys. Rev. **D47** (1993) 3405-3410; hep-th/9210065.

[14] K. Li and E. Witten, *Role of Short Distance Behavior in Off-Shell Open-String Field Theory*, Phys. Rev. **D48** (1993) 853-860; hep-th/9303067.

[15] S. Shatashvili, *Comment on the Background Independent Open String Theory*, Phys. Lett. **B311** (1993) 83-86; hep-th/9303143.

[16] S. Shatashvili, *On the Problems with Background Independence in String Theory*, hep-th/9311177.

[17] A. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, Commun. Math. Phys. **155** (1993) 249-260; hep-th/9205088.

[18] H. Hata and B. Zwiebach, *Developing the covariant Batalin-Vilkovisky approach to string theory*, Annals of Physics, Vol. 229. (1994) 177-216.

[19] B. Zwiebach, *Closed string field theory: quantum action and the B-V master equation*, Nucl. Phys. **B390** (1993) 33-152; hep-th/9206084.

[20] E. S. Fradkin and A. A. Tseytlin, *Quantum string theory effective action*, Nucl. Phys. **B261** (1985) 1; *Effective field theory from quantized strings*, Phys. Lett. **B158** (1985) 316.

[21] A. A. Tseytlin, *Sigma model approach to string theory*, Int. Jr. of Modern Physics A Vol. 4, No. 6 (1989) 1257-1318.

[22] A. A. Tseytlin, *Renormalization group and string loops*, Int. Jr. of Modern Physics A Vol. 5, No. 4 (1990) 589-658.

[23] A. A. Tseytlin, *Sigma model approach to string theory effective actions with tachyons*, hep-th/0011033.

[24] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, *Adding holes and crosscaps to the superstring*, Nucl. Phys. **B293** (1987) 83-113.

[25] A. Abouelsaood, C. G. Callan, C. R. Nappi and S. A. Yost, *Open strings in background gauge fields*, Nucl. Phys. **B280** (1987) 599-624.

[26] S. Weinberg, *Cancellation of one-loop divergences in $SO(8192)$ string theory*, Phys. Lett. **187B** 278.

[27] N. Marcus, *Unitarity and regularized divergences in string amplitudes*, Phys. Lett. **219B**, 265.

[28] J. Polchinski, *String theory*, Vol. 1

[29] C. G. Callan and S. Coleman, *Fate of the false vacuum. II. First quantum corrections*, Phys. Rev. **16D** (1977) p. 1762.

[30] E. J. Weinberg and A. Wu, *Understanding complex perturbative effective potentials*, Phys. Rev. **36D** (1987) p. 2474.

[31] W. Fischler and L. Susskind, *Dilaton tadpoles, string condensates and scale invariance II*, Phys. Lett. **173B** (1986) 262.

[32] D. Berenstein, R. Corrado, W. Fischler, S. Paban and M. Rozali, *Virtual D-Branes*, Phys.Lett. B384 (1996) 93-97; hep-th/9605168.

[33] I. Affleck and A. Ludwig, *Universal noninteger "ground state degeneracy" in critical quantum systems*, Phys. Rev. Lett. **67** (1991) 161; *Exact conformal field theory results on multichannel Kondo effect: single fermion Green's function, self-energy, and resistivity*, Phys. Rev. **B48** (1993) 7297.

[34] A. Gerasimov and S. Shatashvili, *On Exact Tachyon Potential in Open String Field Theory*, JHEP **0010** (2000) 034; hep-th/0009103.

[35] D. Kutasov, M. Marino and G. Moore, *Some Exact Results on Tachyon Condensation in String Field Theory*, JHEP **0010** (2000) 045; hep-th/0009148.

[36] A. Gerasimov and S. Shatashvili, *Stringy Higgs Mechanism and the Fate of Open Strings*, JHEP **0101** (2001) 019; hep-th/0011009.

[37] M. Kleban, A. Lawrence and S. Shenker, *Closed strings from nothing*, hep-th/0012081.

[38] S. Frolov, *On off-shell structure of open string sigma model*, hep-th/0104042.

[39] D. Ghoshal and A. Sen, *Normalization of the Background Independent Open String Field Theory Action*, JHEP **0011** (2000) 021; hep-th/0009191.

[40] O. Andreev, *Some computations of partition functions and tachyon potentials in background independent off-shell string theory*, hep-th/0010218.

[41] O. Andreev, *More about partition function of open bosonic string in background fields and string theory effective action*, hep-th/0104061.

[42] S.P. de Alwis, *Boundary String Field Theory, the Boundary State Formalism and D-Brane Tension*, Phys.Lett. **B505** (2001) 215-221; hep-th/0101200.

[43] R. Rashkov, K. S. Viswanathan, Y. Yang, *Background Independent Open String Field Theory with Constant B field On the Annulus*, hep-th/0101207.

[44] T. Suyama, *Tachyon condensation and spectrum of strings on D-branes*, hep-th/0102192.

[45] K. S. Viswanathan and Y. Yang, *Tachyon condensation and background independent superstring field theory*, hep-th/0104099.

[46] M. Alishahiha, *One-loop correction of the tachyon action in boundary superstring field theory*, hep-th/0104164.

[47] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Springer 1966.

[48] S. Nakamura, *Closed-string tachyon condensation and on-shell effective action of open-string tachyons*, hep-th/0105054.

[49] B. Craps, P. Kraus and F. Larsen, *Loop Corrected Tachyon Condensation*, hep-th/0105227.

[50] E. T. Akhmedov, M. Laidlaw and G. W. Semenoff, *On a Modification of the Boundary State Formalism in Off-shell String Theory*, hep-th/0106033.

Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098