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Multiple Arrivals using Liouville Equations

S. Fomel and J.A. Sethian

Computing Sciences Directorate

Mathematics Department

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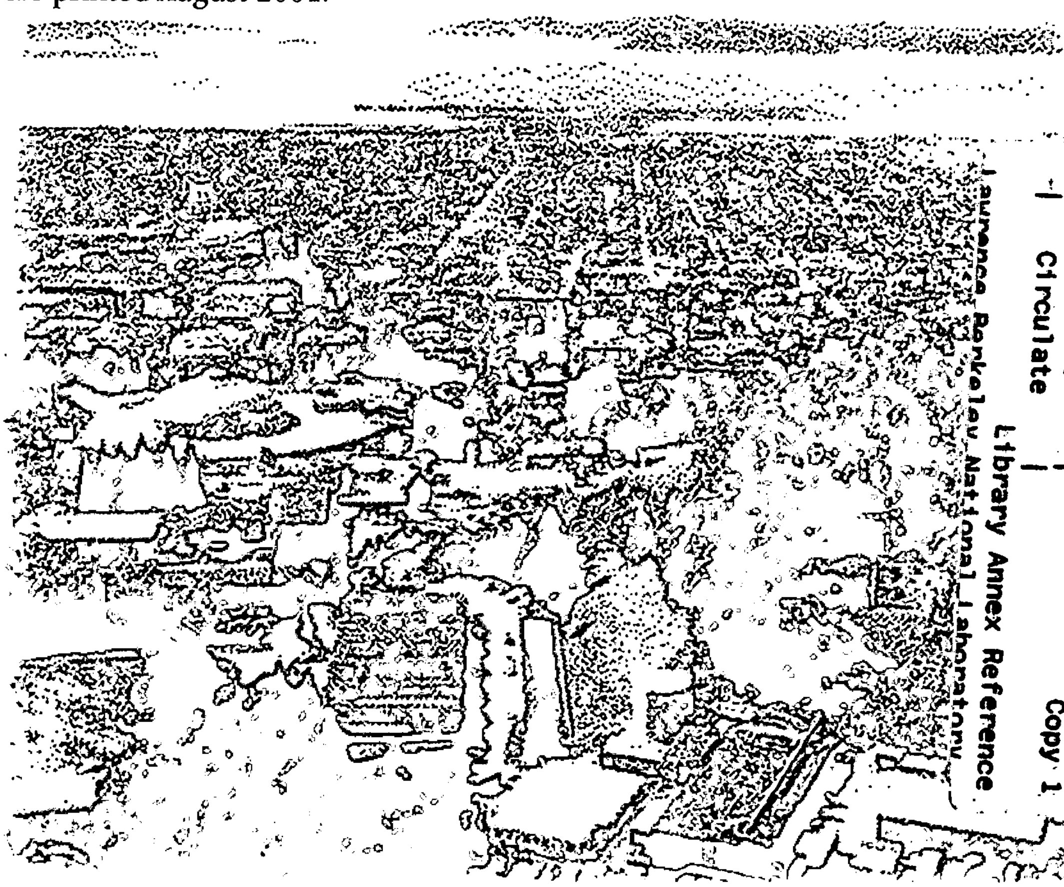

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April 2001

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Multiple Arrivals using Liouville Equations

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April 16, 2001[1]

Traveltime from a fixed source $\tau(\mathbf{x})$ in an isotropic medium ($\mathbf{x} \in \mathbf{R}^{\mathbf{N}}$) is governed by the eikonal equation

$$|\nabla\tau|^2 \; v^2(\mathbf{x}) = 1 \;, \tag{1}$$

where $v(\mathbf{x})$ is the velocity distribution.

The rays [characteristics of equation (1)] are defined by the system of Hamilton-Jacobi ordinary differential equations:

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= v^2(\mathbf{x})\,\mathbf{p} \;; & (2)\\ \frac{d\mathbf{p}}{dt} &= -\frac{1}{v(\mathbf{x})}\,\nabla v \;, & (3) \end{aligned}$$

where t has the meaning of the traveltime along the ray, and $\mathbf{p}$ corresponds to $\nabla\tau$ and is constrained by the Hamilton equation

$$|\mathbf{p}|^2 \; v^2(\mathbf{x}) = 1 \;, \tag{4}$$

equivalent to (1).

In the two-dimensional case, where $\mathbf{x} = \{\mathbf{x}, \mathbf{z}\}$, it is convenient to introduce the angle θ between the vertical and the slowness vector $\mathbf{p}$ such that $\mathbf{p} = \{\sin\theta/\mathbf{v}(\mathbf{x},\mathbf{z}), \cos\theta/\mathbf{v}(\mathbf{x},\mathbf{z})\}$. Equation (4) is then automatically satisfied, and we can rewrite system (2-3) in the form

$$\begin{aligned} \frac{dx}{dt} &= v(x,z)\,\sin\theta \;; & (5)\\ \frac{dz}{dt} &= v(x,z)\,\cos\theta \;; & (6)\\ \frac{d\theta}{dt} &= \frac{\partial v}{\partial z}\,\sin\theta - \frac{\partial v}{\partial x}\,\cos\theta \;. & (7) \end{aligned}$$

The initial conditions for solving system (5-7) consist of the initial point $\{x_0, z_0\}$ and the take-off angle θ_0.

The solution of system (5-7) as a function of time t and the initial conditions x_0, z_0, and θ_0 satisfies the Liouville partial differential equations:

$$\frac{\partial x}{\partial t} + v\,\sin\theta_0\,\frac{\partial x}{\partial x_0} + v\,\cos\theta_0\,\frac{\partial x}{\partial z_0} + \left(\frac{\partial v}{\partial z_0}\,\sin\theta_0 - \frac{\partial v}{\partial x_0}\,\cos\theta_0\right)\frac{\partial x}{\partial\theta_0} = 0 \tag{8}$$

[1] This Lawrence Berkeley National Laboratory Technical Report was written on April 16, 2001. Backup files listing from Department of Energy computers are included at the end of the report, as well as dated e-mail containing the correspondence between the two authors.

$$\frac{\partial z}{\partial t} + v \sin\theta_0 \frac{\partial z}{\partial x_0} + v \cos\theta_0 \frac{\partial z}{\partial z_0} + \left(\frac{\partial v}{\partial z_0} \sin\theta_0 - \frac{\partial v}{\partial x_0} \cos\theta_0\right) \frac{\partial z}{\partial \theta_0} = 0 \tag{9}$$

$$\frac{\partial \theta}{\partial t} + v \sin\theta_0 \frac{\partial \theta}{\partial x_0} + v \cos\theta_0 \frac{\partial \theta}{\partial z_0} + \left(\frac{\partial v}{\partial z_0} \sin\theta_0 - \frac{\partial v}{\partial x_0} \cos\theta_0\right) \frac{\partial \theta}{\partial \theta_0} = 0 \tag{10}$$

where the velocity v is evaluated at $\{x_0, z_0\}$. The appropriate initial conditions for system (8-10) are $\{x, z, \theta\} = \{x_0, z_0, \theta_0\}$ at $t = 0$.

Let us denote by $T(x, z, \theta)$ the time at which the ray that starts at point $\{x, z\}$ with the take-off angle θ first reaches the surface $z = 0$. Correspondingly, the emergence point and the emergence angle of this ray at the surface will be defined by functions $X(x, z, \theta)$ and $\Theta(x, z, \theta)$. Differentiating the condition

$$z(T(x_0, z_0, \theta_0), x_0, z_0, \theta_0) = 0 \, , \tag{11}$$

where $z(t, x_0, z_0, \theta_0)$ is the solution of equation (9), we find that, in the region where $\frac{\partial z}{\partial t}$ is different from zero, the function T has to satisfy the partial differential equation

$$v(x,z) \sin\theta \frac{\partial T}{\partial x} + v(x,z) \cos\theta \frac{\partial T}{\partial z} + \left(\frac{\partial v}{\partial z} \sin\theta - \frac{\partial v}{\partial x} \cos\theta\right) \frac{\partial T}{\partial \theta} = 1 \, . \tag{12}$$

with the boundary condition $T|_{z=0} = 0$. As follows from equations (12), (8), and (10), and the conditions

$$x(T(x_0, z_0, \theta_0), x_0, z_0, \theta_0) = X(x_0, z_0, \theta_0) \, ; \tag{13}$$

$$\theta(T(x_0, z_0, \theta_0), x_0, z_0, \theta_0) = \Theta(x_0, z_0, \theta_0) \, , \tag{14}$$

the functions $X(x, z, \theta)$ and $\Theta(x, z, \theta)$ additionally satisfy the orthogonal equations

$$v(x,z) \sin\theta \frac{\partial X}{\partial x} + v(x,z) \cos\theta \frac{\partial X}{\partial z} + \left(\frac{\partial v}{\partial z} \sin\theta - \frac{\partial v}{\partial x} \cos\theta\right) \frac{\partial X}{\partial \theta} = 0 \tag{15}$$

$$v(x,z) \sin\theta \frac{\partial \Theta}{\partial x} + v(x,z) \cos\theta \frac{\partial \Theta}{\partial z} + \left(\frac{\partial v}{\partial z} \sin\theta - \frac{\partial v}{\partial x} \cos\theta\right) \frac{\partial \Theta}{\partial \theta} = 0 \tag{16}$$

with the boundary conditions $X|_{z=0} = x$ and $\Theta|_{z=0} = \theta$.

We propose to apply equations (12), (15) and (16) for a numerical computations of traveltimes on a fixed x, z grid. Although both $T(x, z, \theta)$ and $X(x, z, \theta)$ functions are strictly single-valued, we can extract from them the possibly multi-valued traveltimes from every grid point x, z to a surface point y at $z = 0$. The extraction would simply amount to evaluating $T(x, z, \theta)$ at the level set of $X(x, z, \theta) = y$.

Dated E-mail Correspondence containing Technical Memo

```
From fomel@math.lbl.gov Mon Apr 16 14:00 PDT 2001
Received: from math.lbl.gov (math.lbl.gov [128.3.7.22])
by math.berkeley.edu (8.9.3/8.9.3) with ESMTP id OAA04060
for <sethian@math.berkeley.edu>; Mon, 16 Apr 2001 14:00:25 -0700 (PDT)
Received: from dniepr (dniepr.lbl.gov [128.3.3.153])
by math.lbl.gov (8.10.2/8.10.2) with SMTP id f3GLOPD27256
for <sethian@math.berkeley.edu>; Mon, 16 Apr 2001 14:00:25 -0700 (PDT)
Message-Id: <200104162100.f3GLOPD27256@math.lbl.gov>
Date: Mon, 16 Apr 2001 14:01:16 -0700 (PDT)
From: Sergey Fomel <fomel@math.lbl.gov>
Reply-To: Sergey Fomel <fomel@math.lbl.gov>
Subject: Re: Your Message Sent on Tue, 10 Apr 2001 09:41:07 -0700 (PDT)
To: sethian@math.berkeley.edu
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>I would think that we should indeed link them together. Can you send me
>a very short latex file with the equations, pointing to the exact
>point you mean - I want to think about it while I am traveling....

Jamie,

Please find the latex file enclosed. The explanation is very raw and will
probably need some refining, but the main equations are there.

I have been reading some more literature on the subject. Apparently, the idea of
computing traveltimes in the (x,z,theta) space is not new. This idea is the
essence of Maslov's asymptotic ray theory. Asymptotic theoreticians (like
Hormander) use it to construct uniform asymptotics of the ray-theoretical
solution near the caustics. However, the idea to use Liouville's PDE and the
corresponding numerical scheme look like an entirely new computational approach.

Have a nice trip. I plan to be in the office on Wednesday.

Sergey


--Band_of_Gorillas_852_000
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Traveltime from a fixed source $\tau(\bold{x})$ in an isotropic medium
($\bold{x} \in R^N$) is governed by the eikonal equation
\begin{equation}
  \label{eq:eikonal}
  \left|\nabla \tau\right|^2\,v^2(\bold{x}) = 1\;,
\end{equation}
where $v(\bold{x})$ is the velocity distribution.

The rays [characteristics of equation~(\ref{eq:eikonal})] are defined
by the system of Hamilton-Jacobi ordinary differential equations:
\begin{eqnarray}
  \label{eq:xray}
  \frac{d \bold{x}}{d t} & = & v^2(\bold{x})\,\bold{p}\;; \\
```

```latex
  \label{eq:pray}
  \frac{d \bold{p}}{d t} & = & - \frac{1}{v(\bold{x})}\,\nabla v\;,
\end{eqnarray}
where $t$ has the meaning of the traveltime along the ray, and
$\bold{p}$ corresponds to $\nabla \tau$ and is constrained by the
Hamilton equation
\begin{equation}
  \label{eq:hamilton}
  \left|\bold{p}\right|^2\,v^2(\bold{x}) = 1\;,
\end{equation}
equivalent to~(\ref{eq:eikonal}).

In the two-dimensional case, where $\bold{x} = \{x,z\}$, it is
convenient to introduce the angle~$\theta$ between the vertical and
the slowness vector~$\bold{p}$ such that $\bold{p} =
\left\{\sin{\theta}/v(x,z),\cos{\theta}/v(x,z)\right\}$.
Equation~(\ref{eq:hamilton}) is then automatically satisfied, and we
can rewrite system~(\ref{eq:xray}-\ref{eq:pray}) in the form
\begin{eqnarray}
  \label{eq:xt}
  \frac{d x}{d t} & = & v(x,z)\,\sin{\theta}\;; \\
  \label{eq:zt}
  \frac{d z}{d t} & = & v(x,z)\,\cos{\theta}\;; \\
  \label{eq:thetat}
  \frac{d \theta}{d t} & = &
  \frac{\partial v}{\partial z}\,\sin{\theta} -
  \frac{\partial v}{\partial x}\,\cos{\theta}\;.
\end{eqnarray}
The initial conditions for solving
system~(\ref{eq:xt}-\ref{eq:thetat}) consist of the initial
point~$\{x_0,z_0\}$ and the take-off angle~$\theta_0$.

The solution of system~(\ref{eq:xt}-\ref{eq:thetat}) as a function of
time $t$ and the initial conditions $x_0$, $z_0$, and $\theta_0$
satisfies the Liouville partial differential equations:
\begin{eqnarray}
 \label{eq:xl}
 \frac{\partial x}{\partial t}   + v\,\sin{\theta_0}\,
 \frac{\partial x}{\partial x_0} + v\,\cos{\theta_0}\,
 \frac{\partial x}{\partial z_0} + \left(
   \frac{\partial v}{\partial z_0}\,\sin{\theta_0} -
   \frac{\partial v}{\partial x_0}\,\cos{\theta_0}\right)\,
 \frac{\partial x}{\partial \theta_0} & = & 0\;; \\
 \label{eq:zl}
 \frac{\partial z}{\partial t}   + v\,\sin{\theta_0}\,
 \frac{\partial z}{\partial x_0} + v\,\cos{\theta_0}\,
 \frac{\partial z}{\partial z_0} + \left(
   \frac{\partial v}{\partial z_0}\,\sin{\theta_0} -
   \frac{\partial v}{\partial x_0}\,\cos{\theta_0}\right)\,
 \frac{\partial z}{\partial \theta_0} & = & 0\;; \\
 \label{eq:thetal}
 \frac{\partial \theta}{\partial t}   + v\,\sin{\theta_0}\,
 \frac{\partial \theta}{\partial x_0} + v\,\cos{\theta_0}\,
 \frac{\partial \theta}{\partial z_0} + \left(
   \frac{\partial v}{\partial z_0}\,\sin{\theta_0} -
   \frac{\partial v}{\partial x_0}\,\cos{\theta_0}\right)\,
 \frac{\partial \theta}{\partial \theta_0} & = & 0\;,
\end{eqnarray}
where the velocity $v$ is evaluated at $\{x_0,z_0\}$. The appropriate
initial conditions for system~(\ref{eq:xl}-\ref{eq:thetal}) are
$\{x,z,\theta\} = \{x_0,z_0,\theta_0\}$ at $t=0$.

Let us denote by $T(x,z,\theta)$ the time at which the ray that starts
at point $\{x,z\}$ with the take-off angle $\theta$ first reaches the
surface $z=0$.  Correspondingly, the emergence point and the emergence
angle of this ray at the surface will be defined by functions
$X(x,z,\theta)$ and $\Theta(x,z,\theta)$. Differentiating the
condition
```

```
\begin{equation}
  \label{eq:zsurface}
  z(T(x_0,z_0,\theta_0),x_0,z_0,\theta_0) = 0\;,
\end{equation}
where $z(t,x_0,z_0,\theta_0)$ is the solution of
equation~(\ref{eq:z1}), we find that, in the region where
$\frac{\partial z}{\partial t}$ is different from zero, the function
$T$ has to satisfy the partial differential equation
\begin{equation}
  \label{eq:tmarch}
  v(x,z)\,\sin{\theta}\,
 \frac{\partial T}{\partial x} + v(x,z)\,\cos{\theta}\,
 \frac{\partial T}{\partial z} + \left(
   \frac{\partial v}{\partial z}\,\sin{\theta} -
   \frac{\partial v}{\partial x}\,\cos{\theta}\right)\,
 \frac{\partial T}{\partial \theta} = 1\;.
\end{equation}
with the boundary condition $\left.T\right|_{z=0} = 0$. As follows
from equations~(\ref{eq:tmarch}), (\ref{eq:x1}),
and~(\ref{eq:theta1}), and the conditions
\begin{eqnarray}
  \label{eq:xsurface}
  x(T(x_0,z_0,\theta_0),x_0,z_0,\theta_0) & = &
  X(x_0,z_0,\theta_0)\;; \\
  \label{eq:thetasurface}
  \theta(T(x_0,z_0,\theta_0),x_0,z_0,\theta_0) & = &
  \Theta(x_0,z_0,\theta_0)\;,
\end{eqnarray}
the functions $X(x,z,\theta)$ and $\Theta(x,z,\theta)$ additionally
satisfy the orthogonal equations
\begin{eqnarray}
  \label{eq:xmarch}
  v(x,z)\,\sin{\theta}\,
  \frac{\partial X}{\partial x} + v(x,z)\,\cos{\theta}\,
  \frac{\partial X}{\partial z} + \left(
    \frac{\partial v}{\partial z}\,\sin{\theta} -
    \frac{\partial v}{\partial x}\,\cos{\theta}\right)\,
  \frac{\partial X}{\partial \theta} & = & 0\;. \\
  \label{eq:thetamarch}
  v(x,z)\,\sin{\theta}\,
  \frac{\partial \Theta}{\partial x} + v(x,z)\,\cos{\theta}\,
  \frac{\partial \Theta}{\partial z} + \left(
    \frac{\partial v}{\partial z}\,\sin{\theta} -
    \frac{\partial v}{\partial x}\,\cos{\theta}\right)\,
  \frac{\partial \Theta}{\partial \theta} & = & 0\;.
\end{eqnarray}
with the boundary conditions $\left.X\right|_{z=0} = x$ and
$\left.\Theta\right|_{z=0} = \theta$.

We propose to apply equations~(\ref{eq:tmarch}), (\ref{eq:xmarch}) and
(\ref{eq:thetamarch}) for a numerical computations of traveltimes on a
fixed ${x,z}$ grid. Although both $T(x,z,\theta)$ and $X(x,z,\theta)$
functions are strictly single-valued, we can extract from them the
possibly multi-valued traveltimes from every grid point ${x,z}$ to a
surface point $y$ at $z=0$. The extraction would simply amount to
evaluating $T(x,z,\theta)$ at the level set of $X(x,z,\theta) = y$.


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Computing Information Services - Lawrence Berkeley National Laboratory
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Restore completed: August 01, 2001

Traveltime from a fixed source $\tau(\bold{x})$ in an isotropic medium
($\bold{x} \in R^N$) is governed by the eikonal equation
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  \label{eq:eikonal}
  \left|\nabla \tau\right|^2\,v^2(\bold{x}) = 1\;,
\end{equation}
where $v(\bold{x})$ is the velocity distribution.

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by the system of Hamilton-Jacobi ordinary differential equations:
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  \label{eq:xray}
  \frac{d \bold{x}}{d t} & = & v^2(\bold{x})\,\bold{p}\;; \\
  \label{eq:pray}
  \frac{d \bold{p}}{d t} & = & - \frac{1}{v(\bold{x})}\,\nabla v\;,
\end{eqnarray}
where $t$ has the meaning of the traveltime along the ray, and
$\bold{p}$ corresponds to $\nabla \tau$ and is constrained by the
Hamilton equation
\begin{equation}
  \label{eq:hamilton}
  \left|\bold{p}\right|^2\,v^2(\bold{x}) = 1\;,
\end{equation}
equivalent to~(\ref{eq:eikonal}).

In the two-dimensional case, where $\bold{x} = \{x,z\}$, it is
convenient to introduce the angle~$\theta$ between the vertical and
the slowness vector~$\bold{p}$ such that $\bold{p} =
\left\{\sin{\theta}/v(x,z),\cos{\theta}/v(x,z)\right\}$.
Equation~(\ref{eq:hamilton}) is then automatically satisfied, and we
can rewrite system~(\ref{eq:xray}-\ref{eq:pray}) in the form
\begin{eqnarray}
  \label{eq:xt}
  \frac{d x}{d t} & = & v(x,z)\,\sin{\theta}\;; \\
  \label{eq:zt}
  \frac{d z}{d t} & = & v(x,z)\,\cos{\theta}\;; \\
  \label{eq:thetat}
  \frac{d \theta}{d t} & = &
  \frac{\partial v}{\partial z}\,\sin{\theta} -
  \frac{\partial v}{\partial x}\,\cos{\theta}\;.
\end{eqnarray}
The initial conditions for solving
system~(\ref{eq:xt}-\ref{eq:thetat}) consist of the initial
point~$\{x_0,z_0\}$ and the take-off angle~$\theta_0$.

The solution of system~(\ref{eq:xt}-\ref{eq:thetat}) as a function of
time $t$ and the initial conditions $x_0$, $z_0$, and $\theta_0$
satisfies the Liouville partial differential equations:
\begin{eqnarray}
 \label{eq:xl}
 \frac{\partial x}{\partial t}   + v\,\sin{\theta_0}\,
 \frac{\partial x}{\partial x_0} + v\,\cos{\theta_0}\,
 \frac{\partial x}{\partial z_0} + \left(
   \frac{\partial v}{\partial z_0}\,\sin{\theta_0} -
   \frac{\partial v}{\partial x_0}\,\cos{\theta_0}\right)\,
 \frac{\partial x}{\partial \theta_0} & = & 0\;; \\
 \label{eq:zl}
 \frac{\partial z}{\partial t}   + v\,\sin{\theta_0}\,
 \frac{\partial z}{\partial x_0} + v\,\cos{\theta_0}\,
 \frac{\partial z}{\partial z_0} + \left(
```

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   \frac{\partial v}{\partial z_0}\,\sin{\theta_0} -
   \frac{\partial v}{\partial x_0}\,\cos{\theta_0}\right)\,
 \frac{\partial z}{\partial \theta_0} & = & 0\;; \\
 \label{eq:thetal}
 \frac{\partial \theta}{\partial t}   + v\,\sin{\theta_0}\,
 \frac{\partial \theta}{\partial x_0} + v\,\cos{\theta_0}\,
 \frac{\partial \theta}{\partial z_0} + \left(
   \frac{\partial v}{\partial z_0}\,\sin{\theta_0} -
   \frac{\partial v}{\partial x_0}\,\cos{\theta_0}\right)\,
 \frac{\partial \theta}{\partial \theta_0} & = & 0\;,
\end{eqnarray}
where the velocity $v$ is evaluated at $\{x_0,z_0\}$. The appropriate
initial conditions for system~(\ref{eq:xl}-\ref{eq:thetal}) are
$\{x,z,\theta\} = \{x_0,z_0,\theta_0\}$ at $t=0$.

Let us denote by $T(x,z,\theta)$ the time at which the ray that starts
at point $\{x,z\}$ with the take-off angle $\theta$ first reaches the
surface $z=0$.  Correspondingly, the emergence point and the emergence
angle of this ray at the surface will be defined by functions
$X(x,z,\theta)$ and $\Theta(x,z,\theta)$. Differentiating the
condition
\begin{equation}
  \label{eq:zsurface}
  z(T(x_0,z_0,\theta_0),x_0,z_0,\theta_0) = 0\;,
\end{equation}
where $z(t,x_0,z_0,\theta_0)$ is the solution of
equation~(\ref{eq:zl}), we find that, in the region where
$\frac{\partial z}{\partial t}$ is different from zero, the function
$T$ has to satisfy the partial differential equation
\begin{equation}
  \label{eq:tmarch}
  v(x,z)\,\sin{\theta}\,
 \frac{\partial T}{\partial x} + v(x,z)\,\cos{\theta}\,
 \frac{\partial T}{\partial z} + \left(
   \frac{\partial v}{\partial z}\,\sin{\theta} -
   \frac{\partial v}{\partial x}\,\cos{\theta}\right)\,
 \frac{\partial T}{\partial \theta} = 1\;.
\end{equation}
with the boundary condition $\left.T\right|_{z=0} = 0$. As follows
from equations~(\ref{eq:tmarch}), (\ref{eq:xl}),
and~(\ref{eq:thetal}), and the conditions
\begin{eqnarray}
  \label{eq:xsurface}
  x(T(x_0,z_0,\theta_0),x_0,z_0,\theta_0) & = &
  X(x_0,z_0,\theta_0)\;; \\
  \label{eq:thetasurface}
  \theta(T(x_0,z_0,\theta_0),x_0,z_0,\theta_0) & = &
  \Theta(x_0,z_0,\theta_0)\;,
\end{eqnarray}
the functions $X(x,z,\theta)$ and $\Theta(x,z,\theta)$ additionally
satisfy the orthogonal equations
\begin{eqnarray}
  \label{eq:xmarch}
  v(x,z)\,\sin{\theta}\,
  \frac{\partial X}{\partial x} + v(x,z)\,\cos{\theta}\,
  \frac{\partial X}{\partial z} + \left(
    \frac{\partial v}{\partial z}\,\sin{\theta} -
    \frac{\partial v}{\partial x}\,\cos{\theta}\right)\,
  \frac{\partial X}{\partial \theta} & = & 0\;. \\
  \label{eq:thetamarch}
  v(x,z)\,\sin{\theta}\,
  \frac{\partial \Theta}{\partial x} + v(x,z)\,\cos{\theta}\,
  \frac{\partial \Theta}{\partial z} + \left(
    \frac{\partial v}{\partial z}\,\sin{\theta} -
    \frac{\partial v}{\partial x}\,\cos{\theta}\right)\,
  \frac{\partial \Theta}{\partial \theta} & = & 0\;.
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\end{eqnarray}
with the boundary conditions $\left.X\right|_{z=0} = x$ and
$\left.\Theta\right|_{z=0} = \theta$.

We propose to apply equations~(\ref{eq:tmarch}), (\ref{eq:xmarch}) and
(\ref{eq:thetamarch}) for a numerical computations of traveltimes on a
fixed ${x,z}$ grid. Although both $T(x,z,\theta)$ and $X(x,z,\theta)$
functions are strictly single-valued, we can extract from them the
possibly multi-valued traveltimes from every grid point ${x,z}$ to a
surface point $y$ at $z=0$. The extraction would simply amount to
evaluating $T(x,z,\theta)$ at the level set of $X(x,z,\theta) = y$.
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Roger J. Cocher
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