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Precision matched solution of the coupled beam envelope equations for a periodic quadrupole lattice with space charge*

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Abstract

The coupled Kapchinskij - Vladimirskij (K-V) envelope equations for a charged particle beam transported by a periodic system of quadrupoles with self-consistent space charge force have previously been solved by various approximate methods, with accuracy ranging from 1% to 10%. A new method of solution is introduced here, which is based on a double expansion of the beam envelope functions in powers of the focal strength and either the beam's emittance or its dimensionless perveance. This method results in accuracy better than 0.1% for typical lattice and beam parameters when carried through one consistent level of approximation higher than employed in previous work. Several useful quantities, such as the values of the undepressed tune and the beam's perveance in the limit of vanishing emittance, are represented by very rapidly converging power series in the focal strength, with accuracy of .01% or better.

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I. Introduction

The matched (i.e. periodic) solution of the coupled Kapchinskij – Vladimirskij (K-V) beam envelope Equations [1,2] is used extensively in the design of quadrupole transport systems for intense charged particle beams. Although an accurate numerical solution for the beam envelope radii is easily obtained for specified beam and quadrupole lattice parameters, approximate analytical solutions continue to be useful for design studies, scaling, cost optimization, and physical understanding. Various analytical methods have been applied to solve these equations during the last twenty five years [3,4,5,6,7], with the degree of error decreasing from about 10% for the early smooth limit approximation to less than 1% using the small parameter expansions employed by Anderson [6] and Lee [7]. In the present work the error is reduced to less than 0.1% for typical system parameters, but one may question the value of this new work since several approximations have been made in deriving the K-V equations, which may produce errors much larger than 0.1%. These approximations include the neglect of third order geometric aberrations, non-linear components of quadrupole fringe fields, higher order magnetic multipoles, and deviations from the assumed flat space charge profile. A primary motivation for the present work is that the analyses of Anderson [6] and Lee [7] essentially employed expansions in powers of the quadrupole strength k alone. However, the matched K-V envelope equations, when written in dimensionless form, may actually

be expanded in powers of <u>two</u> <u>independent</u> dimensionless parameters. These parameters are proportional to k and either the beam's emittance ε or the beam's dimensionless perveance Q. While technically correct if done with sufficient care, expansion in k alone results in envelope expressions that are not easily used beyond the lowest non-trivial order, in contrast to the double expansion employed here. Second, the present approach is a conceptual simplification and yields a non-trivial limit when k=0 (corresponding to a self-pinched equilibrium). Third, the high (0.1%) accuracy of the present solution is of value in setting initial values of envelope radii for fundamental beam dynamics studies, where the effects of very small nonlinearities and mismatches are examined. Finally, the topic is of interest to workers in the field who make regular use of the K-V envelope equations and have often derived their own approximate solutions.

The K-V distribution function, which is a three dimensional shell in the four dimensional transverse phase space, is not used in the present work. In fact, it is obviously unrepresentative of real beam distributions. However, it is the only known distribution function that can be exactly matched in a periodic transport lattice. The K-V envelope equations for edge radii a and b are regarded as having a much greater validity as an approximate model because they are satisfied by rms radii $\left(\overline{x^2} \to a^2/4, \overline{y^2} \to b^2/4\right)$ under the assumptions of constant rms emittance and uniform elliptical space charge profile. These conditions are easily justifiable in the limit of either vanishing emittance or perveance.

The general problem may be stated as follows: For specified quadrupole strength K(z) with scale strength k (e.g. peak value) and period P, beam edge emittance ε, and dimensionless perveance Q, find the periodic envelope edge radii a(z) and b(z) of the matched elliptical beam profile in the transverse (x,y) plane. The depressed tune σ, which is a particle orbit's phase advance per period P, is then determined from the product of ε and the mean of $a(z)^{-2}$ according to Floquet theory. The condition of envelope periodicity results in a relation among k, ε and Q. Hence these three quantities cannot all be independently specified in their dimensionless forms. The mean radius $\bar{a}$ and period P, which at first appear to be forth and fifth parameters, get absorbed into combinations with k , ε and Q in this approach. As mentioned, the fundamentally new aspect of this work is that double power series expansions are used, which can be in either (k, ε) or (k, Q). Results from one expansion can be easily transformed to the other. A second new feature of the present work is the level of expansion. Here it is carried through five non-trivial orders instead of three orders as in Refs. 6 and 7 (e.g. k^5 instead of k^3). There is considerable apparent complexity in the additional orders, but it is greatly reduced with simplifying combinations of terms.

In section II the K-V equations are reduced to a mathematically convenient form which makes the expansion procedure nearly obvious. Section III gives the formal expansions of the coupled envelope equations in (k, ε). Their integration is formally

carried out in section IV along with an evaluation of the depressed tune. Results for the (k, Q) expansion are given in section V along with a variety of useful, very accurate, relations among Q, ε, σ, and the undepressed tune σ_v. Section VI tabulates results for the special case $K(z) = k \cos(2\pi z/P)$, with comparison to an exact numerical case. Section VII summarizes results through third order for the flat top quadrupole profile commonly used in conceptual design work. The main general results of the paper are Equations (50), (52), and (54) for high order terms of the beam radii, and Equations (70), (76), and (78) for relations among Q, ε, σ, and σ_v.

II. The K-V Envelope Equation

The x and y radii, a(z) and b(z), are assumed to satisfy the coupled K-V envelope equations [1]:

$$\frac{d^2a}{dz^2} = -K(z)a + \frac{\varepsilon^2}{a^3} + \frac{2Q}{a+b}, \tag{1}$$

$$\frac{d^2b}{dz^2} = +K(z)b + \frac{\varepsilon^2}{b^3} + \frac{2Q}{a+b}. \tag{2}$$

Here the quadrupole strength K(z) is the ratio of the transverse magnetic field gradient G(z) and particle rigidity [B ρ]= $\beta\gamma Mc/q$:

$$K(z) = \frac{G(z)}{[B\rho]}. \tag{3}$$

The perveance Q is a dimensionless parameter, which in the absence of any charge or current neutralization is related to beam current I by

$$Q = \frac{2qI}{(\beta\gamma)^3 Mc^3 4\pi\varepsilon_0}. \tag{4}$$

The un-normalized edge emittance $\varepsilon = \varepsilon_x = \varepsilon_y$ is the occupied two dimensional (x,dx/dz) phase space area divided by π. For the K-V distribution this is a uniformly filled ellipse. For electrostatic quadrulopes the K-V equations apply, but the formula for K(z) is modified by using the electric field gradient divided by βc.

By assumption

$$K(z+P) = K(z) = -K(z+P/2), \tag{5}$$

so the mean $\overline{K} = 0$. No additional symmetries are assumed (a minor generalization from previous work). The matched envelope radii then have the symmetries

$$a(z+P) = a(z), \quad b(z+P) = b(z), \tag{6}$$

$$a(z+P/2) = b(z). \tag{7}$$

In general a bar denotes the simple average over a full period, for example

$$\bar{a} = \frac{1}{P}\int_0^P dz\, a(z). \tag{8}$$

The depressed tune σ, or phase advance per lattice period (in either plane), is determined by the mean of the inverse beta function as defined by Courant and Snyder [8], $(\beta_x = a^2/\varepsilon)$; from Floquet theory (or the standard accelerator formalism for a particle subject to a periodic, linear, transverse force):

$$\sigma = \int_0^P \frac{dz}{\beta_x(z)} = P\varepsilon\overline{\frac{1}{a^2}} = P\varepsilon\overline{\frac{1}{b^2}}. \tag{9}$$

The tune has units of radians unless degrees are specified.

The approximate solution of Equations (1) and (2) is greatly aided by working with the sum and difference quantities

$$S(z) = \frac{a+b}{2}, \qquad D(z) = \frac{a-b}{2}. \tag{10}$$

It will be seen that S(z) is nearly constant and has period P/2, while D(z) carries most of the variation of a and b and has symmetry D(z+P/2) = -D(z). Equations (1) and (2) are added and subtracted to yield

$$\frac{d^2S}{dz^2} = -KD + \frac{\varepsilon^2}{2}\left[\frac{1}{(S+D)^3} + \frac{1}{(S-D)^3}\right] + \frac{Q}{S}, \tag{11}$$

$$\frac{d^2D}{dz^2} = -KS + \frac{\varepsilon^2}{2}\left[\frac{1}{(S+D)^3} - \frac{1}{(S-D)^3}\right]. \tag{12}$$

Since the desired solutions are periodic we have from Equation (11) the special condition

$$0 = -\overline{KD} + \frac{\varepsilon^2}{2}\left[\overline{\frac{1}{(S+D)^3}} + \overline{\frac{1}{(S-D)^3}}\right] + Q\overline{\frac{1}{S}}. \tag{13}$$

The tune is given by

$$\sigma = P\varepsilon\overline{\frac{1}{(S+D)^2}} = P\varepsilon\overline{\frac{1}{(S-D)^2}} = \frac{P\varepsilon}{2}\left[\overline{\frac{1}{(S+D)^2}} + \overline{\frac{1}{(S-D)^2}}\right]. \tag{14}$$

At this point it is clear that the mean edge radius $(\bar{a} = \bar{b} = \bar{S})$ can be absorbed by ε and Q; we define dimensionless functions s(z) and d(z):

$$S = \bar{a}[1 + s(z)], \qquad D = \bar{a}d(z), \tag{15}$$

so the edge radii are written

$$a = \bar{a}(1 + s + d), \qquad b = \bar{a}(1 + s - d). \tag{16}$$

Equations (11) – (13) become

$$\frac{d^2s}{dz^2} = -Kd + \left(\frac{\varepsilon^2}{\bar{a}^4}\right)\frac{1}{2}\left[\frac{1}{(1+s+d)^3} + \frac{1}{(1+s-d)^3}\right] + \left(\frac{Q}{\bar{a}^2}\right)\frac{1}{1+s}, \tag{17}$$

$$\frac{d^2 d}{dz^2} = -K(1+s) + \left(\frac{\varepsilon^2}{\bar{a}^4}\right)\frac{1}{2}\left[\frac{1}{(1+s+d)^3} - \frac{1}{1+s-d)^3}\right], \tag{18}$$

$$0 = -\overline{Kd} + \left(\frac{\varepsilon^2}{\bar{a}^4}\right)\frac{1}{2}\left[\overline{\frac{1}{(1+s+d)^3}} + \overline{\frac{1}{(1+s-d)^3}}\right] + \left(\frac{Q}{\bar{a}^2}\right)\overline{\frac{1}{(1+s)}}. \tag{19}$$

The evaluation of s(z) is simplified by subtracting Equation (19) from Equation (17), a procedure similar to one introduced by Anderson in 6. The equation for s(z) becomes

$$\frac{d^2 s}{dz^2} = \left(\overline{Kd} - Kd\right) + \left(\frac{\varepsilon^2}{\bar{a}^4}\right)\frac{1}{2}\left[\frac{1}{(1+s+d)^3} - \overline{\frac{1}{(1+s+d)^3}} + \frac{1}{(1+s-d)^3} - \overline{\frac{1}{(1+s-d)^3}}\right]$$
$$+ \left(\frac{Q}{\bar{a}^2}\right)\left[\frac{1}{(1+s)} - \overline{\frac{1}{(1+s)}}\right]. \tag{20}$$

In subsequent sections we solve Equations (18) – (20). Note that at this stage there is a specifiable function K(z), and the two parameters, $(\varepsilon/\bar{a}^2)$ and $(Q/\bar{a}^2)$, are related through Equation (19). The functions s(z) and d(z) are determined by second order ordinary differential equations that involve K(z) and the two (not independent) parameters. These equations are to be solved subject to the conditions of zero mean and periodicity:

$$\begin{aligned} &\bar{s} = \bar{d} = 0, \\ &s(z+P) = s(z), \quad d(z+P) = d(z). \end{aligned} \tag{21}$$

In fact it is also easily shown from Equations (18) and (20) that stronger symmetries apply:

$$s(z+P/2) = s(z), \qquad d(z+P/2) = -d(z). \tag{22}$$

The tune formula, Equation (14), may be written in the convenient form

$$\frac{\sigma \overline{a}^2}{P\varepsilon} = \frac{1}{2}\left[\overline{\frac{1}{(1+s+d)^2}} + \overline{\frac{1}{(1+s-d)^2}}\right]. \tag{23}$$

III. Method of solution

It is natural to attempt to solve Equations (18-20) by power series expansions in the amplitude of K(z). Formally we define

$$K(z) = kf(z/P), \tag{24}$$

where f is a specified dimensionless function and k is a variable parameter for expansion (normalization of f does not need to be specified here). The tricky issue is how to treat the parameters ε and Q. In Reference [7] they were formally expanded in k, with the ratio $Q\bar{a}^2/\varepsilon^2$ held fixed (independent of k). Alternatively, in Reference (6) both ε^2 and Q were regarded as being proportional to k^2 and the K-V equations were solved essentially by an iterative procedure. Both approaches lead to non-transparent results in all but the lowest orders of k and tend to submerge the important roles play by emittance and perveance. The latter defect is demonstrated by setting k=0 in Equations (18-20). A nontrivial matched solution is immediate:

$$s = d = 0, \tag{25}$$

$$\frac{\varepsilon^2}{\bar{a}^4} + \frac{Q}{\bar{a}^2} = 0. \tag{26}$$

Q must be negative in this case, which can be the result of sufficient space charge neutralization by plasma. The tune in this case is simply

$$\sigma = \frac{P\varepsilon}{\bar{a}^2} = \frac{P}{\bar{a}}\sqrt{-Q}. \tag{27}$$

While neutralized pinch equilibria are not the topic of this work, it is apparent that it is unnecessary and possibly misleading to strongly tie the values of both ε and Q to k from the beginning.

An examination of Equations (18-20) shows that ε appears more frequently than Q. Therefore it is reasonable to use Equation (19) to eliminate $Q/\bar{a}^2$ from the system and solve for $s(z)$ and $d(z)$ as double power series in k and ε. This procedure immediately yields useful results for the cold beam limit $(\varepsilon \rightarrow 0)$. However, for some applications an expansion in k and Q is actually more useful, for example in evaluating the undepressed tune σ_v (here subscript "v" denotes vacuum, since the usual designation σ_0 could be confused with an order of expansion). The transformation between the two expansions is derived in section V.

The magnitude of K(z) employed in accelerators and beam transport systems is limited by considerations of stability. If $Q = 0$ then single particle orbits are unstable at $\sigma = \sigma_v = 180°$, with a very broad stop band for increasing values of k. Finite Q results in a collective envelope instability (breathing mode) for sufficiently large σ_v. The unstable zone begins at $\sigma_v \approx 90°$ for $\sigma \approx \sigma_v$. For $\sigma << \sigma_v$ the unstable zone begins at $\sigma_v \approx 115°$. Consequently, design values of σ_v are usually set below $90°$. For the simple case $K(z) = k\cos(2\pi z/P)$ it is shown in section VI that this requires $|kP^2/(4\pi)| \leq 1.0$. A good convergence rate for expanded envelope parameters is found when this condition is

observed, and this is generally true for any periodic $K(z)$ when $\sigma_\nu < 90°$. The combination $\varepsilon^2/\bar{a}^4 + Q/\bar{a}^2$ is approximately equal to $\sigma_\nu^{\ 2}/P^2$ (the smooth limit); this also bounds the range for these parameters if $Q > 0$.

It is found that $|s| \leq .015$ and $|d| \leq .35$ for $\sigma_\nu < \pi/4$. A Taylor expansion of the functions of $s \pm d$ appearing in Equations (18-20) therefore appears to be of questionable value, e.g.

$$\frac{1}{(1+s+d)^3} = 1 - 3(s+d) + 6(s+d)^2 - 10(s+d)^3 + 15(s+d)^4 - \ldots \qquad (28)$$

may converge slowly at some values of z. However, it will be seen that convergence is actually quite rapid when the differential equations for s and d are integrated in z.

Equations (18-20) exhibit the symmetry

$$K \to -K, s \to s, d \to -d, \varepsilon \to \varepsilon, Q \to Q. \qquad (29)$$

We therefore expect expansions in $(\mathrm{k}, \varepsilon)$ to (schematically) contain powers of terms of the form

$$d \;\propto\; k + (k^3 + k\varepsilon^2) + (k^5 + k^3\varepsilon^2 + k\varepsilon^5) + \ldots, \qquad (30)$$

$$s \;\propto\; k^2 + \left(k^4 + k^2\varepsilon^2\right) + \ldots, \qquad (31)$$

$$\frac{Q}{\bar{a}^2} \;\propto\; (k^2 + \varepsilon^2) + (k^4 + k^2\varepsilon^2 + \varepsilon^4) + (k^6 + k^4\varepsilon^2 + k^2\varepsilon^4 + \varepsilon^6) + \ldots . \qquad (32)$$

Terms are grouped and designated here according to their combined powers of k and ε as first order, second order, third order and so on even though they contain mixed combinations of k and ε.

At this point it is convenient to introduce dimensionless variables and parameters. This is done by absorbing the period length P:

$$t = z/P, \tag{33a}$$

$$\kappa(t) = \mathrm{K}(z)P^2 = (kP^2)f(t), \tag{33b}$$

$$\mathrm{E} = \varepsilon P/\overline{a}^2, \tag{33c}$$

$$U = QP^2/\overline{a}^2. \tag{33d}$$

A lattice period has length equal to unity in t, and "prime" will denote d/dt.

To evaluate the terms of various orders appearing in s, d and U, we first expand Equations (18-20) in simple power series involving s and d:

$$d'' = -\kappa(1+s) + \mathrm{E}^2\left[-3d + 12sd - 10(3s^2d + d^3) + \ldots\right], \tag{34}$$

$$s'' = (\overline{\kappa d} - \kappa d) + \mathrm{E}^2\left[-3s + 6\left(s^2 - \overline{s^2}\right) + 6\left(d^2 - \overline{d^2}\right) - ..\right] + U(-s + ..), \tag{35}$$

$$U = \overline{\kappa d}(1 - \overline{s^2} + ..) - \mathrm{E}^2\left[1 + 6\left(\overline{s^2} + \overline{d^2}\right) - 10\left(\overline{s^3} + 3\overline{sd^2}\right) + 15\left(\overline{s^4} + 6\overline{s^2d^2} + \overline{d^4}\right) - ..\right] \tag{36}$$

$$\cdot\left(1 - \overline{s^2} + ..\right).$$

Following the schematic forms (30) and (31) we set

$$d = d_1 + d_3 + d_5 + .., \tag{37}$$

$$s = s_2 + s_4 + .., \tag{38}$$

where the subscript denotes the order of a term, e.g. d_3 contains both k^3 and $k\varepsilon^2$ contributions. Then Equation (36) yields

$$U = \left[\overline{\kappa d_1} - \mathrm{E}^2\right] + \left[\overline{Kd_3} - 6\mathrm{E}^2\overline{d_1^2}\right] + \left[\overline{\kappa d_5} - \overline{\kappa d_1}\,\overline{s_2^2} + \mathrm{E}^2\left(-5\overline{s_2^2} - 12\overline{d_1 d_3} + 30\overline{s_2 d_1^2} - 15\overline{d_1^4}\right)\right] + \ldots, \tag{39}$$

where we have displayed terms through 6th order.

From Equation (34) we get

$$d_1'' = -\kappa, \tag{40}$$

$$d_3'' = -\kappa s_2 - 3\mathrm{E}^2 d_1, \tag{41}$$

$$d_5'' = -\kappa s_4 + \mathrm{E}^2\left(-3d_3 + 12 s_2 d_1 - 10 d_1^3\right). \tag{42}$$

And from Equations (35) and (39)

$$s_2'' = \overline{\kappa d_1} - \kappa d_1, \tag{43}$$

$$s_4'' = \left(\overline{\kappa d_3} - \kappa d_3\right) + \mathrm{E}^2\left(-3s_2 + 6d_1^2 - 6\overline{d_1^2}\right) - \left(\overline{\kappa d_1} - \mathrm{E}^2\right)\left(s_2\right). \tag{44}$$

Equations (40)–(44) can be integrated for given $\kappa(t)$; this will be done in general terms in section IV The mean values $\overline{\kappa d_3}$ and $\overline{\kappa d_5}$ will be evaluated from lower order quantities and an expression for U evaluated from Equation (39). Finally, the depressed tune is given by Equation (23), which becomes

$$\frac{\sigma}{\mathrm{E}} = 1 + 3\left(\overline{s^2} + \overline{d^2}\right) - 4\left(\overline{s^3} + 3\overline{sd^2}\right) + 5\left(\overline{s^4} + 6\overline{s^2 d^2} + \overline{d^4}\right) - .. \tag{45}$$

$$= 1 + 3\overline{d_1^2} + \left(3\overline{s_2^2} + 6\overline{d_1 d_3} - 12\overline{s_2 d_1^2} + 5\overline{d_1^4}\right) + \ldots . \tag{46}$$

IV. Formal Solution

Equations (40) – (44), when solved for specified $\kappa(t)$, give the envelope functions expanded through fifth order terms. In each of these five equations it is necessary to integrate twice in t, with the two constants of integration determined by the requirements of zero mean and periodicity. As an example, consider the simple case $\kappa = kP^2\cos(2\pi t)$. From Equation (40) we have

$$d_1'' = -\kappa = -kP^2\cos(2\pi t), \tag{47a}$$

$$d_1 = \frac{1}{\pi}\left(\frac{kP^2}{4\pi}\right)\cos(2\pi t). \tag{47b}$$

From Equation (43)

$$s_2'' = \left(\overline{\kappa d_1} - \kappa d_1\right) = -2\left(\frac{kP^2}{4\pi}\right)^2\cos(4\pi t), \tag{48a}$$

$$s_2 = \frac{1}{8\pi^2}\left(\frac{kP^2}{4\pi}\right)^2\cos(4\pi t). \tag{48b}$$

These simple functions illustrate the symmetries and approximate magnitude of the lowest order terms of d and s; recall $(kP^2/4\pi) < 1$ to make $\sigma_v < \pi/2$.

In general $d_1(t)$ and $s_2(t)$ are determined from $\kappa(t)$ alone, by integration. To obtain d_3 we must integrate Equation (41) with conditions of periodicity and zero mean applied to each third order combination of terms separately; let

$$f_3'' = -\kappa s_2, \tag{49a}$$

$$e_1'' = -d_1. \tag{49b}$$

Then the formal solution is

$$d_3 = f_3 + 3\mathrm{E}^2 e_1, \tag{50}$$

with $f_3(t)$ and $e_1(t)$ being functionals of $\kappa(t)$, each satisfying the two integration conditions. The evaluation of s_4 requires the definition of four more functionals of κ:

$$g_4'' = \overline{\kappa f_3} - \kappa f_3, \tag{51a}$$

$$h_2'' = \overline{\kappa e_1} - \kappa e_1, \tag{51b}$$

$$i_2'' = -s_2, \tag{51c}$$

$$j_2'' = d_1^2 - \overline{d_1^2}, \tag{51d}$$

giving

$$s_4 = \left(g_4 + \overline{\kappa d_1} i_2\right) + \mathrm{E}^2\left(3h_2 + 2i_2 + 6j_2\right). \tag{52}$$

Similarly, evaluation of d_5 requires eight new functionals of κ:

$$l_5'' = -\kappa g_4, \tag{53a}$$

$$m_3'' = -\kappa h_2, \tag{53b}$$

$$n_3'' = -\kappa i_2, \tag{53c}$$

$$p_3'' = -\kappa j_2, \tag{53d}$$

$$q_3'' = -f_3, \tag{53e}$$

$$r_1'' = -e_1, \tag{53f}$$

$$u_3'' = s_2 d_1, \tag{53g}$$

$$v_3'' = -d_1^3, \tag{53h}$$

giving

$$d_5 = (l_5 + \overline{\kappa d}_1 n_3) + \mathrm{E}^2(3m_3 + 2n_3 + 6p_3 + 3q_3 + 12u_3 + 10v_3) + 9\mathrm{E}^4 r_1. \tag{54}$$

To obtain useful expressions for U and σ various mean values must be evaluated [see Equations (39) and (46)]. We need at least the set $[\overline{\kappa d}_1, \overline{\kappa d}_3, \overline{d_1^2}, \overline{\kappa d}_5, \overline{s_2^2}, \overline{d_1 d_3}, \overline{s_2 d_1^2}$, and $\overline{d_1 \kappa s_2}]$. Some means, such as $\overline{d_1^2}$ and $\overline{s_2^2}$, must be calculated directly from the relevant functions. However, there are useful relations among many means and shortcuts for evaluations; for the present work it is sufficient to solve for only $[d_1, s_2, f_3, e_1]$ to evaluate e.g. the complicated mean $\overline{\kappa d_5}$. Several examples follow to demonstrate techniques. First, note that we may write

$$\overline{\kappa d_1} = -\overline{d_1'' d_1} = \overline{d_1'^2}, \tag{55}$$

where we used Equation (40), integration by parts, and the condition of periodicity. Similarly

$$\overline{\kappa d_3} = -\overline{d_1''\ d_3} = -\overline{d_1 d_3''} = \overline{d_1 \kappa s_2} + 3\mathrm{E}^2 \overline{d_1^2}. \tag{56}$$

The mean $\overline{d_1 \kappa s_2}$ may be simplified using Equation (43):

$$\overline{d_1 \kappa s_2} = \overline{\left(\overline{d_1 \kappa} - s_2''\right) s_2} = -\overline{s_2''\ s_2} = \overline{s_2'^2}. \tag{57}$$

We also have the simple relations

$$\overline{\kappa f_3} = -\overline{d_1''\ f_3} = -\overline{d_1 f_3''} = \overline{d_1 \kappa s_2} = \overline{s_2'^2}, \tag{58}$$

$$\overline{\kappa e_1} = -\overline{d_1''\ e_1} = -\overline{d_1 e_1''} = \overline{d_1^2}, \tag{59}$$

$$\overline{d_1 e_1} = -\overline{e_1'' \; e_1} = \overline{e_1'^2}, \tag{60}$$

$$\overline{d_1 d_3} = \overline{d_1 f_3} + 3\mathrm{E}^2 \overline{d_1 e_1}. \tag{61}$$

The evaluation of $\overline{\kappa d_5}$ involves $\overline{d_1 \kappa s_4}$ at an intermediate step; this is readily shown to be

$$\overline{d_1 \kappa s_4} = \overline{\left(\overline{kd_1} - s_2''\right) s_4} = -\overline{s_2 s_4''} = \left(\overline{f_3'^2} + \overline{d_1'^2}\,\overline{s_2^{\,2}}\right) + \mathrm{E}^2\left(3\overline{f_3 d_1} + 2\overline{s_2^{\,2}} - 6\overline{s_2 d_1^{\,2}}\right). \tag{62}$$

We then have

$$\begin{aligned} \overline{\kappa d_5} &= -\overline{d_1'' \; d_5} = -\overline{d_1 d_5''} = \overline{d_1 \kappa s_4} + \mathrm{E}^2\left(3\overline{d_1 d_3} - 12\overline{s_2 d_1^{\,2}} + 10\overline{d_1^{\,4}}\right) \\ &= \left(\overline{f_3'^2} + \overline{d_1'^2} \;\; \overline{s_2^{\,2}}\right) + \mathrm{E}^2\left(6\overline{d_1 f_3} + 2\overline{s_2^{\,2}} - 18\overline{s_2 d_1^{\,2}} + 10\overline{d_1^{\,4}}\right) + 9\mathrm{E}^4 \overline{e_1'^2}. \end{aligned} \tag{63}$$

Equation (39) now provides the formal expansion of U through sixth order; after some straightforward substitution from the mean value formulas we get

$$U = \left(\overline{d_1'^2} - \mathrm{E}^2\right) + \left(\overline{s_2'^2} - 3\mathrm{E}^2 \overline{d_1^{\,2}}\right) + \left[\overline{f_3'^2} - \mathrm{E}^2\left(6\overline{d_1 f_3} + 3\overline{s_2^{\,2}} - 12\overline{s_2 d_1^{\,2}} + 5\overline{d_1^{\,4}}\right) - 27\mathrm{E}^4 \overline{e_1'^2}\right] + \ldots \tag{64}$$

Similarly, Equation (46) gives

$$\frac{\sigma}{\mathrm{E}} = 1 + 3\overline{d_1^{\,2}} + \left[\left(6\overline{d_1 f_3} + 3\overline{s_2^{\,2}} - 12\overline{s_2 d_1^2} + 5\overline{d_1^{\,4}}\right) + 18\mathrm{E}^2 \overline{e_1'^2}\right] + .. \tag{65}$$

While the second and fourth order terms of U and zeroth and second order terms of σ / E can be deduced from the work of Lee [7] and Anderson [6], the complicated higher order terms in brackets are new. Fortunately, they can be written in a very convenient form, to be given in section V.

V. Useful Relations

Much of the value of the present treatment of the envelope equations lies in the accurate relations that connect the parameters $\left(Q, P, \bar{a}, \varepsilon, \sigma, \sigma_v\right)$. To a considerable extent these relations are found to be nearly independent of the specific form of $\kappa(t)$. We expect to find two independent relations by manipulating Equations (64) and (65) for U and σ/E. We also obtain a formula for σ_v from these equations by taking the limit $U = 0$.

First consider the cold beam limit $(\mathrm{E} = 0)$; Equation (64) and (65) become

$$U = \overline{d_1'^2} + \overline{s_2'^2} + \overline{f_3'^2} + 8^{\text{th}} \text{ order} \equiv \lambda[\kappa(t)], \tag{66}$$

$$\frac{\sigma}{\mathrm{E}} = 1 + 3\overline{d_1^2} + \left(6\overline{d_1 f_3} + 3\overline{s_2^2} - 12\overline{s_2 d_1^2} + 5\overline{d_1^4}\right) + 6^{\text{th}} \text{ order} \equiv 1 + \psi[\kappa(t)]. \tag{67}$$

The functional λ appears to converge very rapidly and can be well approximated by its first two terms. The functional ψ also appears to converge rapidly, but less so that λ (see section VI. Due to the complexity of its forth order term it is desirable to avoid explicit evaluation of ψ if possible.

Using the functionals λ and ψ we now write Equations (64) and (65) in the convenient general forms $(\mathrm{E} \neq 0)$

$$U - \lambda + \mathrm{E}^2(1 + \psi) = -27\mathrm{E}^4\overline{e_1'^2} + 8^{\text{th}} \text{ order}, \tag{68}$$

$$\frac{\sigma}{\mathrm{E}} - (1 + \psi) = 18\mathrm{E}^2\overline{e_1'^2} + 6^{\text{th}} \text{ order}. \tag{69}$$

Only the three functionals $\left(\lambda, \psi, \overline{e_1'^2}\right)$ appear at the present level of approximation, instead of five as might have been expected. We may now eliminate ψ between Equations (68) and (69) to obtain the very useful and accurate relation

$$U - \lambda + \sigma \mathrm{E} = -9\mathrm{E}^4 \overline{e_1'^2} + 8^{\text{th}} \text{ order.} \tag{70}$$

So far power series expansions in (k, ε) have been used, but for some purposes we need expansions in (k, Q). These are obtained by solving Equation (68) for E^2 through 6th order and substituting it in previously derived equations for d, s and σ. In this procedure U is regarded as a term of second order, or equivalently we could regard the new expansion as being in the parameter $\sqrt{U}$. We have immediately

$$\mathrm{E}^2 = (\lambda_2 - U) + \left[\lambda_4 - \psi_2(\lambda_2 - U)\right] + \left[\lambda_6 - \psi_2\lambda_4 + \left(\psi_2{}^2 - \psi_4\right)(\lambda_2 - U) - 27\overline{e_1'^2}(\lambda_2 - U)^2\right] + \ldots, \tag{71}$$

where $\{\lambda_2, \lambda_4, \lambda_6, \psi_2, \psi_4\}$ are the respectively ordered terms in Equations (66) and (67) defining the functionals λ and ψ. The sixth order term for E^2 in Equation (71) is not needed to transform the expressions for $d(t)$ and $s(t)$ through 5th order. However, note that this substitution generates new higher order terms at every level. For example, the factor of E^2 in d_3 will contribute 5th and higher orders to the (k, Q) expansion of d. Substitution for E^2 in Equation (69) yields

$$\frac{\sigma}{\mathrm{E}} = 1 + \psi_2 + \left[\psi_4 + 18\overline{e_1'^2}(\lambda_2 - U)\right] + 6^{\text{th}} \text{ order.} \tag{72}$$

We are now able to obtain an expression for σ that does not explicitly contain E. From (71) and (72), after some algebra:

$$\sigma^2 = \mathrm{E}^2\left(\frac{\sigma}{\mathrm{E}}\right)^2 = (\lambda_2 - U) + \left[\lambda_4 + \psi_2(\lambda_2 - U)\right] + \left[\lambda_6 + \psi_2\lambda_4 + \psi_4(\lambda_2 - U) + 9\overline{e_1'^2}(\lambda_2 - U)^2\right] + 8\text{th order.} \tag{73}$$

This can be written in the convenient form

$$\sigma^2 = (\lambda - U)(1 + \psi) + 9(\lambda - U)^2\overline{e_1'^2} + 8^{\text{th}} \text{ order.} \tag{74}$$

An expression for the vacuum tune is obtained directly from Equation (73) by setting $U = 0$:

$$\sigma_\nu^2 = \lambda_2 + (\lambda_4 + \psi_2\lambda_2) + \left(\lambda_6 + \psi_2\lambda_4 + \psi_4\lambda_2 + 9\overline{e_1'^2}\lambda_2^{\,2}\right) + 8^{\text{th}} \text{ order.} \tag{75}$$

However, this is not the best available formula for σ_ν. If we expand $\cos(\sigma_\nu)$ a very rapidly converging series is obtained, which gives extraordinary accuracy using only the lowest order terms. This is

$$2[1 - \cos(\sigma_\nu)] = \sigma_\nu^{\,2} - \frac{\sigma_\nu^{\,4}}{12} + \frac{\sigma_\nu^{\,6}}{360} - .. = \lambda_2 + \left(\lambda_4 + \psi_2\lambda_2 - \frac{\lambda_2^{\,2}}{12}\right) + \left(\lambda_6 + \psi_2\lambda_4 + \psi_4\lambda_2 + 9\overline{e_1'^2}\lambda_2^2 - \frac{\lambda_2}{6}(\lambda_4 + \psi_2\lambda_2) + \frac{\lambda_2^{\,3}}{360}\right) + .. \tag{76}$$

Accuracy better that .01% is obtained from terms through 4th order, which written out explicitly are

$$2[1 - \cos(\sigma_\nu)] \approx \overline{d_1'^2} + \left[\overline{s_2'^2} + 3\overline{d_1^{\,2}}\,\overline{d_1'^2} - \frac{\left(\overline{d_1'^2}\right)^2}{12}\right]. \tag{77}$$

The rapid convergence of this series may be related to the fact that terms of higher order than k^2 vanish in the thin lens limit. Numerically, in equation (77) we find that the 4th order term is small ($\leq .01$) because of the approximate equality $3\overline{d_1^2} \approx \overline{d_1'^2}/12$.

A final useful relation is found by evaluating the difference between $\cos(\sigma_v)$ and $\cos(\sigma)$. This is expected to be approximately equal to $U/2$ based on the smooth limit; a very accurate version is

$$\left[2(1-\cos\sigma)-2(1-\cos\sigma_v)+\frac{U\sin\sigma}{\mathrm{E}}-\frac{U^2}{12}-\frac{U^3}{180}\right]=U^2\left(\frac{\psi_2}{6}-\frac{\lambda_2}{120}-9\overline{e_1'^2}\right)+8^{\text{th}}\text{order.} \quad (78)$$

The 6th order terms on the right of Equation (78) nearly cancel $\left(\leq .003U^2\right)$.

Equation (78) is clearly exact for the limit $U=0$. We can also make it exact for the special limit $k \to 0$ by substituting

$$\frac{U^2}{12}+\frac{U^3}{180} \to 2\left(1-\cosh\sqrt{U}\right)+\sqrt{U}\sinh\left(\sqrt{U}\right). \quad (79)$$

In this limit we also have $\sigma = \mathrm{E} = \sqrt{-U} = i\sqrt{U}$.

VI. The Case of K=k cos(2πz/P)

The fifth order solution for s and d and the relations, derived in general form in sections III-V, are tabulate here for the analytically simple and useful case of a quadrupole gradient approximated by a single cosine term. Recall the dimensionless forms

$$\kappa = KP^2 = \left(kP^2\right)\cos\left(2\pi z/P\right) \equiv 4\pi\beta\cos(2\pi t), \tag{80}$$

$$\mathrm{E} = \varepsilon P/\bar{a}^2, U = Q P^2/\bar{a}^2, \tag{81}$$

where $t = z/P$ and we define $\beta \equiv k P^2/4\pi$ for notational convenience. We have previously obtained [Equations (47), (48)]

$$d_1 = \frac{\beta}{\pi}\cos(2\pi t), \tag{82}$$

$$s_2 = \frac{\beta^2}{8\pi^2}\cos(4\pi t). \tag{83}$$

Equations (49a,b) are integrated and the resulting functions (f_3, e_1) inserted in Equation (50) to yield the third order term

$$d_3 = \frac{\beta^3}{16\pi^3}\left[\cos(2\pi t) + \frac{1}{9}\cos(6\pi t)\right] + \frac{3\mathrm{E}^2\beta}{4\pi^3}\cos(2\pi t) \quad . \tag{84}$$

Similarly, Equations $(51a-d)$ are integrated to obtain (g_4, h_2, i_2, j_2). These functions are inserted in Equation (52) to give

$$s_4 = \frac{\beta^4}{128\pi^4}\left[\frac{28}{9}\cos(4\pi t) + \frac{1}{36}\cos(8\pi t)\right] - \frac{5\mathrm{E}^2\beta^2}{64\pi^4}\cos(4\pi t)\ . \tag{85}$$

Finally Equations (53a-h) are integrated for $(l_5, m_3, n_3, p_3, q_3, r_1, u_3, v_3)$, which are inserted in Equation (54) to obtain

$$d_5 = \frac{\beta^5}{256\pi^5}\left[\frac{28}{9}\cos(2\pi t) + \frac{113}{324}\cos(6\pi t) + \frac{1}{900}\cos(10\pi t)\right] + \frac{\mathrm{E}^2\beta^3}{256\pi^5}\left[434\cos(2\pi t) + \frac{930}{81}\cos(6\pi t)\right] + \frac{9\mathrm{E}^4\beta}{16\pi^5}\cos(2\pi t). \tag{86}$$

Several features of the expansions are apparent. Clearly the size of the terms drops rapidly with decreasing focal strength $(\sim \beta)$ and emittance $(\sim \mathrm{E})$. However, β and E may be of order unity in a practical example, so the convergence of the expansion actually depends on the numerical coefficients, e.g. the factor of $9/(16\pi^5) = .00184$ in the last term of d_5. It is also clear that s_2 and s_4 are of minor significance compared with d_1 and d_3; the matched sum of $a(z)$ and $b(z)$ is constant within about 1%. Terms involving powers of emittance are generally larger than terms involving only powers of β. Therefore a truncation of the expansions of s and d is most accurate for space charge dominated beams.

The functionals $\lambda[\kappa]$ and $x[\kappa]$ are readily evaluated; from Equations (65) and (67):

$$\lambda = \overline{d_1'^2} + \overline{s_2'^2} + \overline{f_3'^2} + .. = 2\beta^2 + \frac{\beta^4}{8\pi^2} + \frac{5\beta^6}{576\pi^4} + .. \quad , \tag{87}$$

$$1 + \psi = 1 + 3\overline{d_1^2} + \left(6\overline{d_1 f_3} + 3\overline{s_2^2} - 12\overline{s_2 d_1^2} + 5\overline{d_1^4}\right) + .. = 1 + \frac{3\beta^2}{2\pi^2} + \frac{219\beta^4}{128\pi^4} + .. \ . \tag{88}$$

Note the "excellent" convergence of λ and "good" convergence of ψ. The function $e_1(t)$ is needed for the high order forms of Relations (68) – (78); from Equations (49b) and (82) we have in the present example

$$e_1 = \frac{\beta}{4\pi^3}\cos(2\pi t), \tag{89a}$$

$$\overline{e_1'^2} = \frac{\beta^2}{8\pi^4}. \tag{89b}$$

Then Equations (68) – (70) become

$$U - \lambda + \mathrm{E}^2(1+\psi) = \frac{-27\beta^2\mathrm{E}^4}{8\pi^4} + 8^{\text{th}} \text{ order}, \tag{90}$$

$$\frac{\sigma}{\mathrm{E}} - (1+\psi) = \frac{9\beta^2\mathrm{E}^2}{4\pi^4} + 6^{\text{th}} \text{ order}, \tag{91}$$

$$U - \lambda + \sigma\mathrm{E} = \frac{-9\beta^2\mathrm{E}^4}{8\pi^4} + 8^{\text{th}} \text{ order}, \tag{92}$$

where Equation (92) is derived from Equations (90) and (91) and is therefore not an independent result.

The direct expansions for σ and σ_ν, Equations (74) and (75), are found to converge poorly and are not recommended for numerical work. In the present example

$$\sigma_\nu^2 = 2\beta^2 + \frac{25}{8\pi^2}\beta^4 + \frac{1169}{144\pi^4}\beta^6 + \ldots, \tag{93}$$

but we have for $\cos(\sigma_\nu)$ the spectacular

$$2\big(1-\cos(\sigma_\nu)\big) = 2\beta^2 - (.016704635)\beta^4 + (.000019137)\beta^6 + \ldots . \tag{94}$$

Equation (78), which relates σ, u, and E, becomes

$$\left[2(1-\cos(\sigma)-2(1-\cos(\sigma_\nu)+\frac{U}{\mathrm{E}}\sin(\sigma)-\frac{U^2}{12}-\frac{U^3}{180}\right]=-(.002886)\beta^2U^2+8^{\text{th}}\text{ order.} \quad (95)$$

Equations (92), (93) and (94) are recommend for accurate numerical work and are used in the following example.

A single numerical case is presented to indicate the typical accuracy of the fifth order solution and relations. We choose parameters representative of an intense beam, where space charge dominates emittance by a factor of about 3: $\beta^2 = .5, \mathrm{E}^2 = .25$. For these parameters the lowest order (smooth limit) yields approximate values for the envelope maxima:

$$d(0) \approx d_1(0) = \frac{\beta}{\pi} \approx .2250791, \quad (96a)$$

$$s(0) \approx 0. \quad (96b)$$

From Equations (87), (90), (91), (93) we also find in lowest order

$$\lambda \approx \lambda_2 = \overline{d_1'^2} = 2\beta^2 = 1.0, \quad (96c)$$

$$U \approx \lambda_2 - \mathrm{E}^2 = .75, \quad (96d)$$

$$\sigma \approx \mathrm{E} = .5 \text{ radian} = 28.6479^\circ, \quad (96e)$$

$$\sigma_\nu \approx \sqrt{2\beta^2} = 1.0 \text{ radian} = 57.29578^\circ. \quad (96f)$$

The much more accurate Equation (94) gives for σ_ν in lowest order

$$\sigma_\nu \approx \cos^{-1}(1-\beta^2) = \cos^{-1}(.5) = 60.00000^\circ, \quad (96g)$$

These very approximate results are to be compared with the exact numerical solution

$$d(0) = .2308369, \tag{97a}$$

$$s(0) = .0062717, \tag{97b}$$

$$U = .7315774, \tag{97c}$$

$$\sigma = 31.06746^\circ, \tag{97d}$$

$$\sigma_v = 59.8618366^\circ. \tag{97e}$$

The smooth limit value of $d(0)$ is low by 2.5%, U is high by 2.5%, σ is low by 7.8%, and σ_v is low by 4.3%. However (96g) predicts σ_v high by only .23%.

Smooth limit formulas (96a-f) are seen to provide a rough evaluation of useful envelope quantities, but they are hardly suitable for accurate design or code initialization. The higher order approximations do provide this accuracy. We obtain from Equations (83)-(95):

$$d(0) \approx d_1(0) + d_3(0) + d_5(0) = .2250791 + .0050678 + .0005917 = .2307386, \tag{98a}$$

$$s(0) \approx s_2(0) + s_4(0) = .0063326 - .0000373 = .0062953, \tag{98b}$$

$$U \approx .7317614, \tag{98c}$$

$$\sigma \approx 31.06063^\circ, \tag{98d}$$

$$\sigma_v \approx 59.80688^\circ \text{ (from 93)}, \tag{98e}$$

$$\approx 59.8618366^\circ \text{ (from 94)}. \tag{98f}$$

From Equations (87-89) the three structure parameters are

$$\lambda \approx \lambda_2 + \lambda_4 + \lambda_6 = 1 + .0031663 + .0000111 = 1.0031774, \tag{98g}$$

$$\psi \cong \psi_2 + \psi_4 = .0759909 + .0043911 = .0803820, \tag{98h}$$

$$\overline{e_1'^2} = .0006416. \tag{98i}$$

The fifth order $d(0)$ is low by .043%, $s(0)$ is high by .37%, U is high by .025%, σ is low by .022%, and σ_v low by .092% if Equation (93) is used. Equation (94) gives σ_v exactly to seven places.

VII. Summary of Design Relations for a Flat Top FODO Quadrupole System

The simple quadrupole function analyzed in section (6), $K = k\cos(2\pi z/P)$ is mainly useful for fundamental theoretical studies. For conceptual design a less accurate third order envelope solution for the flat-top quadrupole function is much more relevant and convenient. The formulas which follow are derived for the simple "FODO" lattice with quadrupoles of alternating strength $\pm k$ centered at $z = 0, L, 2L, \ldots$. Here $L = P/2$ is the half lattice period length and ηL is the effective quadrupole field length. The drifts, which are of effective length $(1-\eta)\,L$, are centered at $L/2, 3L/2, \ldots$. We are primarily interested in the envelope radii at the middle of the x-focussing quadrupoles:

$$a(0) \approx \bar{a}[1 + s_2(0) + d_1(0) + d_3(0)], \tag{99a}$$

$$b(0) \approx \bar{a}[1 + s_2(0) - d_1(0) - d_3(0)]. \tag{99b}$$

We have from section IV:

$$d_1(0) = \frac{(\eta k L^2)}{4}\left(1 - \frac{\eta}{2}\right), \tag{100}$$

$$s_2(0) = \frac{(\eta k L^2)^2}{24}\left(\frac{1}{2} - \frac{13}{12}\eta + \frac{13}{16}\eta^2 - \frac{\eta^3}{5}\right), \tag{101}$$

$$\begin{aligned} d_3(0) = {} & \frac{(\eta k L^2)^3}{192}\left(1 - \frac{19}{6}\eta + \frac{463}{120}\eta^2 - \frac{511}{240}\eta^3 + \frac{9}{20}\eta^4\right) + \\ & \frac{(\eta k L^2)}{16}\left(1 - \frac{\eta^2}{2} + \frac{\eta^3}{8}\right)\left(\frac{\varepsilon L}{\bar{a}^2}\right)^2. \end{aligned} \tag{102}$$

The three formulas for $(\sigma, \sigma_v, Q, \varepsilon)$ from section 5 become

$$2(1-\cos\sigma_v) \approx (\eta k L^2)^2\left(1-\frac{2}{3}\eta\right)-(\eta k L^2)^4\left(\frac{\eta^2}{90}-\frac{\eta^3}{63}+\frac{\eta^4}{180}\right), \qquad (103)$$

$$\left(\frac{4QL^2}{\bar{a}^2}\right)+\left(\frac{2L\varepsilon\sigma}{\bar{a}^2}\right) \approx (\eta k L^2)^2\left(1-\frac{2\eta}{3}\right)+\frac{(\eta k L^2)^4}{3}\left(\frac{1}{16}-\frac{5}{24}\eta+\frac{191}{720}\eta^2-\frac{16}{105}\eta^3+\frac{\eta^4}{30}\right), \qquad (104)$$

$$2(1-\cos\sigma)-2(1-\cos\sigma_v)+\left(\frac{2QL}{\varepsilon}\right)\sin(\sigma)-\frac{4}{3}\frac{Q^2L^4}{\bar{a}^4} \approx 0. \qquad (105)$$

Equations (99-105) are accurate to better than 1.0% when $\sigma_v < 90°$.

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