

Quadratic Function Approaching Method for Magnetotelluric Sounding Data Inversion

Liangjun Yan* and Wenbao Hu, Department of Geophysics and Petroleum Resources, Yangtze University, Keni Zhang, Lawrence Berkeley National Laboratory

Summary

The quadratic function approaching method (QFAM) is introduced for magnetotelluric sounding (MT) data inversion. The method takes the advantage of that quadratic function has single extreme value, which avoids leading to an inversion solution for local minimum and ensures the solution for global minimization of an objective function. The method does not need calculation of sensitivity matrix and not require a strict initial earth model. Examples for synthetic data and field measurement data indicate that the proposed inversion method is effective.

Introduction

Most well-known approaches for MT data non-linear inversion, such as Newton's method and conjugate gradients-based methods, are based on an objective function ($\Phi(\mathbf{m})$) approximated with its first-order Taylor expansion about the reference model (\mathbf{m}_k). This is because that the objective function in the vicinity of extreme point can be approximated by a quadratic function. Inversion problems in geophysics do not search for the extreme value of objective function, but the minimum value or global extreme value of it. However, most MT inversion algorithms, such as OCCAM (deGroot-Hedlin and Constable, 1990), RRI (Smith and Booker, 1991) and NLCG (Rodi and Mackie, 2001), are limited to solving for the "extreme" value. These nonlinear inversion methods use partial differentiation information (Jacobian matrix) to determine the iteration search direction, which may lead to a solution for local minimization. Computation of Jacobian matrix is time consuming and limits the speed of MT data inversion. In this paper, the quadratic function approaching method (QFAM) is applied to the inversion of one-dimensional MT data. The QFAM method is originally used in the non-linear optimization study. This approach avoids the calculation of gradient or second order derivative by using the function value only, and does not need calculation of sensitivity matrix which greatly simplifies the inversion procedure. Additionally, the proposed method does not require a strict initial earth model.

Theory

The QFAM is implemented by constructing a quadratic function through several control points in the model space. At the control points, quadratic function is given the same

value as the corresponding objective function. The minimum of objective function is approximated by minimum value of the quadratic function. An optimization algorithm developed by Zhao (2000) for searching control points is used to update quadratic function and improve the determination of minimum location. An iterative procedure is followed until the optimized global minimum point has been reached. The solution is considered to be the best approximation of the real earth model.

Objective Function

MT data and earth model has a very complex nonlinear relationship. We can write the inverse problem (N layers) as

$$\mathbf{d} = F(\mathbf{m}) + \mathbf{e}, \quad (1)$$

where \mathbf{d} is a data vector [d^1, d^2, \dots, d^N], \mathbf{m} is a model vector, $\mathbf{m}=[m^1, m^2, \dots, m^N]$, \mathbf{e} is an error vector, and $F(\mathbf{m})$ is a forward modeling function. In this study, log values for the data vector and model vector are used.

Inverse problems in geophysics are undetermined. We find a specific model by minimizing a model objective function subject to data constraints. In an iterative inverse procedure, it is not enough by only matching field measured data. The matching approach may lead to a very complex earth model, or a false structure. One of the effective approaches for reducing uncertainty of inversion is by defining an objective function with parameters of the model structure, and the inverse problem is solved by minimizing the model objective function subject to adequately fitting data. We solve the problem following Tikhonov and Arsenin (1977), taking a regularized solution to be a model minimizing an objective function, Φ , defined by

$$\Phi(\mathbf{m}) = (\mathbf{d} - F(\mathbf{m}))^T \mathbf{V}^{-1} (\mathbf{d} - F(\mathbf{m})) + \lambda \mathbf{m}^T \mathbf{L}^T \mathbf{L} \mathbf{m} \quad (2)$$

where λ is regularization factor and a positive number. The positive-definite matrix \mathbf{V} plays the role of the variance of the error vector \mathbf{e} . The second term of equation (2) defines a stabilizing functional on the model space. The matrix \mathbf{L} is a second-difference operator. For a flattest model norm matrix \mathbf{L} , we adopt the equations by Routh and Oldenburg (1999).

In solving inverse problem, the earth model is discretized into 30-40 layers in a constant layer thickness (in log

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value). This scheme can avoid the requirement of layer thickness in calculation.

Quadratic function approaching method

In the model space, $(N+1)(N+2)/2$ points are selected to construct a quadratic function which is used to determine global minimum. The function is defined as

$$\Omega(\mathbf{m}) = \frac{1}{2} \mathbf{m}^T \mathbf{H} \mathbf{m} + \mathbf{b}^T \mathbf{m} + c \quad (3)$$

where \mathbf{m} is an n -dimensional model vector, $\mathbf{H}(h_{ij})$ is a symmetric square matrix with a size of $N \times N$, and the vector $\mathbf{b}=(b_j)_{1 \times N}$. From equation (3), it is easy to find that the $\Omega(\mathbf{m})$ is exclusively determined by matrix \mathbf{H} , vector \mathbf{b} and the constant c . Total $(N+1)(N+2)/2$ coefficients need to be determined for the function. We adopt the scheme developed by Zhao (2000) for selecting the $(N+1)(N+2)/2$ control points. This scheme ensures

$$\Omega(\mathbf{m}_i) = \Phi(\mathbf{m}_i), \quad i = 0, 1, 2, \dots, \frac{1}{2}(N+1)(N+2) - 1$$

and provides a convenient computation method for gradient and second order derivative of the function. In the model space, $N+1$ initial model $\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_n$ are randomly selected. And then an affine transformation is applied to these vectors based on equation (4).

$$\mathbf{m}' = \mathbf{A}^{-1}(\mathbf{m} - \mathbf{m}_0) \quad (4)$$

where

$$\mathbf{A} = \begin{bmatrix} m_1^1 - m_0^1, m_2^1 - m_0^1, \dots, m_N^1 - m_0^1 \\ m_1^2 - m_0^2, m_2^2 - m_0^2, \dots, m_N^2 - m_0^2 \\ \cdot \\ \cdot \\ \cdot \\ m_1^N - m_0^N, m_2^N - m_0^N, \dots, m_N^N - m_0^N \end{bmatrix}$$

After transformation, $\mathbf{m}_0, \mathbf{m}_1 - \mathbf{m}_0, \mathbf{m}_2 - \mathbf{m}_0, \dots$, and $\mathbf{m}_N - \mathbf{m}_0$ become $\mathbf{0}, \xi_1, \xi_2, \dots$, and ξ_N , respectively. In the new coordinate system, ξ_1, ξ_2, \dots , and ξ_n are unit vectors and the distances from $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_N$ to \mathbf{m}_0 equal to 1.

Additional $N(N+1)/2$ model vectors can be determined by

$$\mathbf{m}_{ij} = \frac{1}{2}(\mathbf{m}_i + \mathbf{m}_j), \quad i=0,1,2,\dots,N; j=0,1,2,\dots,N; i \neq j \quad (5)$$

Apparently, \mathbf{m}_{ij} and \mathbf{m}_{ji} are equal. We can also apply affine transformation by equation (4) to these vectors. In the transformed coordinate system, the distance between \mathbf{m}'_{ij} and \mathbf{m}'_{0j} or \mathbf{m}'_{i0} is 0.5.

To determine the coefficients of the quadratic function, the objective functions corresponding to $(N+1)(N+2)/2$ model vectors are noted as

$$\begin{aligned} \Phi_i &= \Phi(\mathbf{m}_i) = \Phi(\mathbf{m}'_i), & i = 0, 1, 2, \dots, N \\ \Phi_{ij} &= \Phi(\mathbf{m}_{ij}) = \Phi(\mathbf{m}'_{ij}), & i = 0, 1, 2, \dots, N; j = 0, 1, 2, \dots, N; i \neq j \end{aligned}$$

It is supposed that the quadratic function equals to the objective function at the control points. We have:

$$\begin{aligned} \Omega_i &= \Phi_i, & i = 0, 1, 2, \dots, N \\ \Omega_{ij} &= \Phi_{ij}, & i = 0, 1, 2, \dots, N; j = 0, 1, 2, \dots, N; i \neq j \end{aligned}$$

The coefficients can be easily determined using the following relationships:

$$\begin{cases} c = \Phi_0 \\ h_{ii} = 4(\Phi_i - 2\Phi_{0i} + \Phi_0) \\ h_{ij} = 4(\Phi_{ij} - \Phi_{0j} - \Phi_{0i} + \Phi_0), \\ b_j = 4(\Phi_{0j} - \Phi_j + \Phi_0) \end{cases}, i = 1, 2, \dots, N; j = 1, 2, \dots, N.$$

The minimum model solutions based on these coefficients are models under the new coordinate system. We need to transform these solutions back to its original coordinate system, that is done by

$$\mathbf{m}_{N+1} = \mathbf{m}_0 - \mathbf{A}\mathbf{H}^{-1}\mathbf{b}, \quad (6)$$

A new minimum model \mathbf{m}_{N+1} is obtained from equation (6). It is the best approximation for global minimum model at this stage. Together with initial models, $N+2$ model solutions are now available. In the next iteration, the worst model is discarded. The left $N+1$ models are used to generate $N(N+1)/2$ new model vectors by following equation (5). Using the same procedure as discussed above, new quadratic function coefficients are determined and then again a new minimum model is obtained by using the equation (6). This iteration procedure is repeated until a satisfied solution is reached.

Example with synthetic data

The QFAM is applied to inversion of synthetic data obtained from two earth models. The first model consists of three layers (Figure 1), and the second model (Figure 2) consists of five layers. Figure 1 and Figure 2 show the inversion results for the two models respectively. Both results match their corresponding Earth model quite well.

Field data Inversion

In this section, we demonstrate the application of the QFAM to real field MT data. The data was collected at an oil field in the Northeast China. The MT measuring line is parallel to a seismic exploration line, which is about 2 kilometers away.

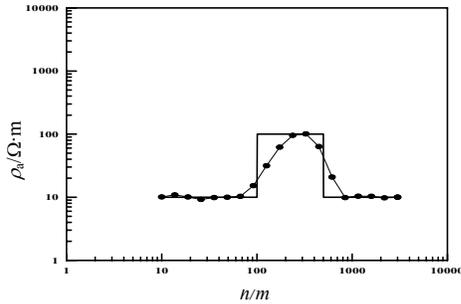


Figure 1 Inversion result of 3-layer model. Solid line: Model; Dot line: Inversion result.
 Model parameters :
 $\rho_1 = 10\Omega \cdot m, \rho_2 = 100\Omega \cdot m, \rho_3 = 10\Omega \cdot m$
 $h_1 = 100m, h_2 = 400m$

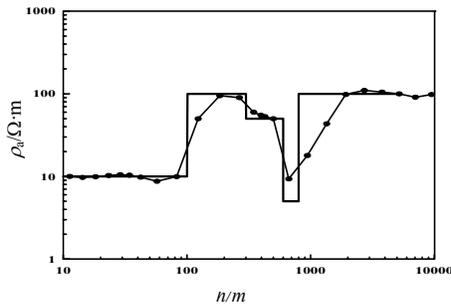


Figure 2 Inversion result of 5-layer model. Solid line: Model; Dot line: Inversion result.
 Model parameters:
 $\rho_1 = 10\Omega \cdot m, \rho_2 = 100\Omega \cdot m, \rho_3 = 50\Omega \cdot m, \rho_4 = 100\Omega \cdot m, \rho_5 = 100\Omega \cdot m$
 $h_1 = 100m, h_2 = 200m, h_3 = 300m, h_4 = 200m$

MT Data inversion for single point measurement

To calibrate rock properties of the geological layers at surveying area, a single point measurement was conducted near station 65 of the measuring line. The apparent resistivity curves from both TE and TM model indicate that there is almost no influence of static shift, and also, the curves show the characteristics approximating to the response of one-dimensional earth structure. Therefore, this problem can be treated as a one-dimensional inversion problem and solved using the QFAM. Figure 2 shows comparison between inverted resistivity distribution and

electrical logs at the borehole WS#1. The two curves match each other quite well.

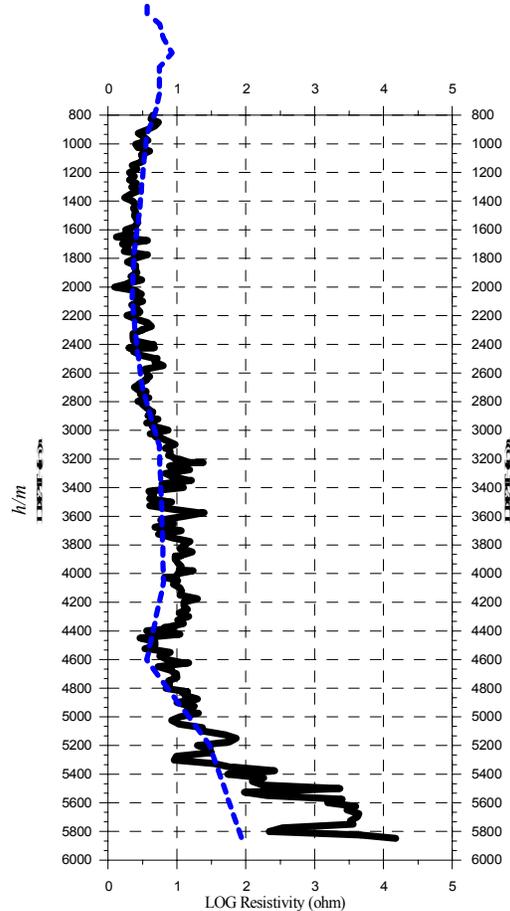


Figure 3 Comparison of inversion result and electrical logs. (Solid line: log; Dash line: inversion result)

Figure 4 shows inversion results of the 2D resistivity cross-section along the measuring line. Layering distribution of resistivity is very clear at the top layers (for the top 2 kilometers). A thick layer with lower resistivity lies underneath the top layers. The base rock demonstrates high resistivity. Depth of the top surface of the base rock is about 5 kilometer at the left side of the cross-section and is less than 2 kilometer at the right side (Figure 4(a)). By comparing to seismic time profile (Figure 4(b)) measured at the site, we can find that the both results have very similar layering distribution. MT inversion and seismic survey produce almost identical results for top surface of the base rock. MT inversion results also provide detail information of resistivity changes inside the base rock.

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Conclusions

The QFAM realizes global non-linear inversion and reduces the possibility of solution uncertainty. This method avoids the calculation of gradient or second order derivative by using the function value only, and does not need calculation of sensitivity matrix which greatly simplifies the inversion procedure. Additionally, the proposed method does not require a strict initial earth model. Examples for synthetic data and field data inversion indicate that the QFAM is an effective method for non-linear MT data inversion.

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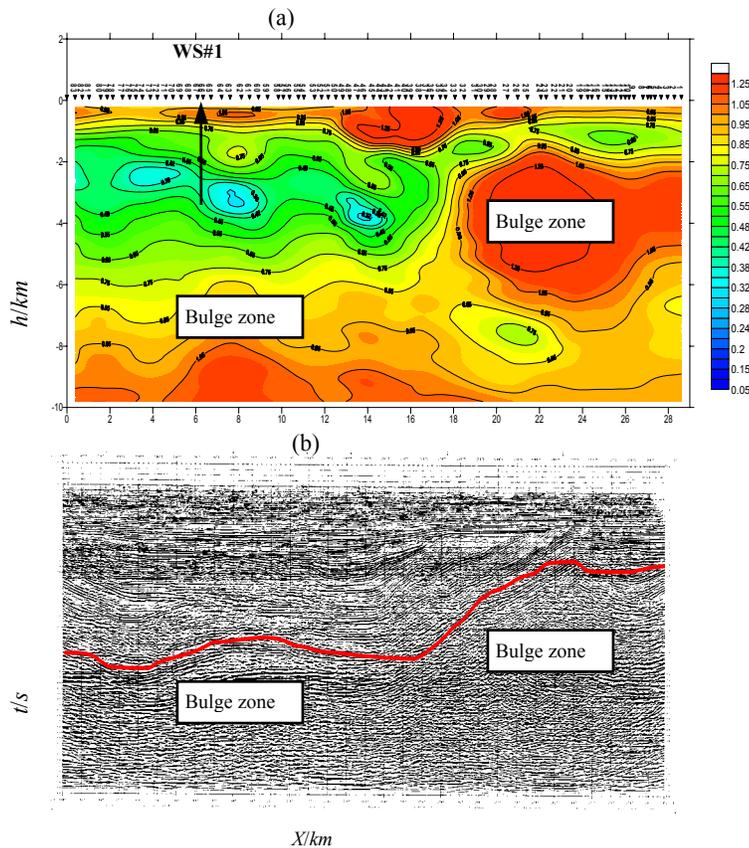


Figure 4 Comparison of inversion results and seismic time profile. (a) Inverted resistivity profile, (b) Seismic time profile