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THEORY OF NUCLEON-NUCLEON SCATTERING IN THE STRIP APPROXIMATION TO THE MANDELSTAM REPRESENTATION

John M. Charap, Elihu Lubkin, and Antonio Scotti

December 19, 1961

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Contents

Abstract v

I. Introduction 1

II. Kinematics

A. Kinematical Variables 8

B. The Mandelstam Representation 10

C. The Various Amplitudes 14

III. Asymptotic Upper Bounds to the χ Amplitudes 24

IV. The Allowed Single Spectral Functions 28

A. The Cases $i = +, -, 5$ 32

B. The Cases $i = 1, 2$ 36

C. Further Restrictions Implied by the Regularity of the G Amplitudes 40

D. Further Restrictions Imposed by the Dominance of Diffraction Scattering 42

V. The Dynamically Independent Single-Particle States 44

VI. Unitarity in the Nucleon-Antinucleon Channel 47

VII. Unitarity in the Nucleon-Nucleon Channel 59

VIII. Equations for the Spectral Functions 80

IX. The Yukawa Poles and the Simplest Box Diagram 84

Acknowledgments 87

Appendix A. Crossing 88

Appendix B. Derivations of the Unitarity Equations 95

1. Extraction of the Azimuthal Angles from a Product of Matrix Elements 97

Appendix C. Calculations Pertaining to the Unitarity Equations 99

Appendix D. The Angular Integrations 114

References 120

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ABSTRACT

The nucleon-nucleon scattering amplitude is discussed within the framework of the strip approximation. Asymptotic bounds to the behavior of the amplitudes are derived, and are applied to limit the number of "allowed" single spectral functions to six and correspondingly to limit the types of dynamically independent one-particle states. In particular, it is found that the pion is in this sense allowed, whilst the deuteron is not. The unitarity equations, in which only two-particle intercalated states are retained, are explicitly derived in both the N-N and N-$\overline{N}$ channels. The N-$\overline{N}$ equations express a a portion of the double spectral functions in terms of the π-N amplitudes; the N-N equations express another portion of these functions through coupled integral equations.

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I. INTRODUCTION

In recent years the study of strongly interacting systems of particles has abandoned the paths of conventional field theory, and sought to build a theory on the basis of an assumed analyticity property of the S matrix, together with its unitarity. In particular, a framework of approximation has been proposed (1) on the basis of the Mandelstam representation (2), and an approximate form of unitarity in which only one- and two-particle intercalated states are retained, and this has been applied to the study of pion-pion scattering (1) and of pion-nucleon scattering (3).

It is the purpose of the present paper to extend this technique to nucleon-nucleon scattering. This problem has already been studied within the framework of S-matrix theory (4, 5). In GGMW[1] however, the emphasis was on a study of the individual partial waves. This approach, while admirable for an understanding of low-energy phenomena, is severely limited at high energies when many partial waves contribute. However, at these high energies the most interesting phenomenon in the physical region is the pronounced diffraction peak, a phenomenon common to all known scattering cross sections for the strongly interacting family of particles. Since the interesting physics is confined to a portion of the physical region in a strip around the border thereof, it is hoped that evaluating the singularities of the amplitude in a region forming a similar, nearby strip in the unphysical region (Fig. 1) might lead to a good approximate description of the amplitude. This is the basis of the "strip approximation" of Chew and Frautschi (1).

The double spectral functions in the strips are determined by a consideration of unitarity. For the strips parallel to $t = 0$, $u = 0$ in Fig. 1 (the precise meaning of the notions and quantities mentioned here will be made clear in the sequel), the relevant unitarity condition is that for nucleon-antinucleon scattering, and in particular those contributions which arise from two-pion intercalated states. This introduces the absorptive parts of the pion-nucleon scattering amplitudes, and these are considered as "given."

The remaining strips parallel to $s = 0$ are determined by a solution of the integral equations arising from unitarity in the nucleon-nucleon channel. Alternatively, the previously determined strips can be used to define a generalised potential (6) which can be inserted into a Schrödinger equation for a determination of the low-energy part of the nucleon-nucleon scattering amplitude.[2]

The main purpose of this paper is to set up the unitarity equations in both the nucleon-antinucleon and nucleon-nucleon channels which determine the double spectral functions in the strip region. Before we can proceed to this, however, it is necessary to discuss the kinematics in some detail. This is done in Section II, where we also introduce a number of different sets of scalar functions that can be used to specify the amplitudes.

The crossing relations between the amplitudes describing nucleon-nucleon scattering and those describing nucleon-antinucleon scattering are derived in Appendix A. These relations could have been taken directly from GGMW; however, we felt it would be of interest to present an alternative derivation that leans less heavily on conventional field theory. Also in Section II, on the basis of results derived by GGMW, we express the analyticity of the amplitudes. This at first leads us to consider the general subtracted form of the Mandelstam representation (Eq. 2.5) for the amplitudes.

This form, with its many independent spectral functions is clearly cumbersome for our purpose. It may also be dangerous. Perhaps the simplest way to understand this is to consider the effect of an extra or redundant subtraction. This is to introduce a new subtraction term which is completely and uniquely determined by the weight function of the integral in which the redundant subtraction was made. Since a priori we know only the general form of the Mandelstam representation, the number of terms and their weight functions being unknown, we are faced with a dilemma. For if we postulate "too large" a number of subtractions compared to the "actual" number required, the supernumerary weight functions, which should be correlated with the others, will appear as independent quantities to be calculated; in fact, the set of weight functions is overdetermined, a particularly dangerous situation in any approximate scheme of calculation. On the other hand, if we

choose "too few" subtractions, the integrals we write down will be divergent.

Froissart (7) has shown for scalar particles that a weak form of unitarity can severely restrict the number of "independent" single spectral functional and polynomial terms in the Mandelstam representation in the following sense: any additional such terms ("allowed terms") must be determined completely and uniquely by the double spectral function. The allowed terms may always be determined by considering the unitarity requirement in a suitable number of individual partial waves, and a solution of the resulting N/D equations (8), but only up to the inherent ambiguity of the CDD poles (9).

This has led us to consider, in Sections III and IV, modifications to the Froissart argument appropriate to a consideration of the nucleon-nucleon problem. To do this, we first place asymptotic bounds on the amplitude (Section III). We use a partial-wave expansion of the helicity amplitudes and assume a maximum range of appreciable interaction, and also assume that diffraction scattering dominates the elastic scattering at high energies.[3] Then in Section IV we apply these bounds and conclude that the allowed single spectral functions are six in number, and that there are no allowed polynomial terms.

In the nucleon-antinucleon channel there are four allowed single spectral integrals. The partial-wave unitarity equations relevant to their determination are the $J = 0$ singlet and triplet equations in each isospin channel. As we have observed, their solutions are ambiguous because of the possibility of CDD poles; this corresponds to the possibility of dynamically independent mesons being present, the Born terms arising from exchange of which give just such poles. Our restriction to just four allowed single spectral integrals in the nucleon-antinucleon channel limits us then

to just four kinds of allowed dynamically independent mesons of nucleon-number zero, and they turn out to be the scalar, pseudoscalar, isoscalar, or isovector mesons. This is discussed in Section V.

We observe that the pseudoscalar isovector pion fits into the class of "allowed" particles. All particles of spin higher than zero are excluded, however, an extension of results previously derived for the pion-pion and pion-nucleon problems. In particular, this means that it is not legitimate to simulate the exchange of a cluster of resonating pions in a $J = 1$ state (the ρ, ω, or η "particles") by including the corresponding Born terms in the amplitude, uncorrelated with the form of the double spectral functions. The only consistent way to take account of such exchange is to use the recently proposed technique (10) of utilising the Regge (11) continuation of the amplitudes (11) in the complex angular momentum. Such resonances are now represented by "Regge poles", or poles in the complex angular momentum plane.

We are left with the two single spectral integrals in the nucleon-nucleon channel. These contribute only to the $^1S_0$ and $^3P_0$ partial waves of the isovector part of the amplitude. First we observe that the two have nothing to do with the deuteron: as we might have anticipated, the deuteron can be nothing but a dynamically dependent particle—we might say loosely that it cannot be considered an elementary particle.

In Section VII we find that the structure of the unitarity equations for nucleon-nucleon scattering forces the presence of single spectral integrals in just these two partial waves. Specifically, Eqs. (7.34) and (7.35) are inconsistent with the vanishing of the single spectral function terms contained in $h_{1,5}(p^2)$. We are at present unable to exclude the possibility of CDD ambiguities in these two partial waves. However, it would be most reasonable to insist that no extra parameter be introduced, so that there are no such

CDD poles, since it is expected that no independent parameters enter into the final equations of S-matrix theory.

It is interesting to observe that again in this problem, as previously in the pion-pion and pion-nucleon problems, it is possible to exclude dynamically independent particles with $J \geqslant 1$. About particles with $J < 1$ we can as yet say nothing. Furthermore, there is a chain of consistency: if we assume that the pion-nucleon amplitude has single spectral functions corresponding to $J = 0$ in the crossed channels, which are allowed by Froissart type of argument (3), one is forced to assume the presence of similar terms in the crossed channel of nucleon-nucleon scattering.

We shall then adopt the following philosophy. The Mandelstam representation will be written as a sum of unsubtracted double spectral integrals, together with just those single spectral integrals allowed by the arguments of Section IV, Eqs. (8.1) and (8.2). The single spectral functions in the nucleon-antinucleon channel are in principle determined directly from pion-nucleon scattering. They will in particular contain the δ-functions corresponding to the one-pion pole.

The double spectral functions are to be determined from the equations of Section VIII. The contributions from the nucleon-nucleon channel result from the solution of coupled integral equations. It may happen that the so-determined double spectral functions lead to divergent integrals. In this case, one is faced with the possibility of making subtractions so that the double spectral integrals converge, but one would then have to resort to N/D type calculations to obtain the subtraction terms, with the resultant CDD ambiguities. Alternatively one could use the analysis in terms of "Regge poles" (10) giving a unique meaning to these formally divergent integrals, hoping to obtain in this way a unique determination.

This situation will presumably arise in the nucleon-nucleon problem for the case of the deuteron. We have already indicated that it cannot be introduced as a CDD pole, since its origin is dynamical, it having nothing to do with allowed single spectral functions. Its introduction as a "Regge pole" would seem to be the most satisfactory. Since in this paper the solution of the equations for the double spectral functions is not attempted, we shall not pursue this point further.

In Section VI the unitarity equations for the nucleon-antinucleon channel are derived in the two-meson approximation, and in Section VII similar equations for nucleon-nucleon unitarity are obtained. In Section VIII the equations for the double spectral functions are discussed and summarized.

Finally, the Yukawa poles and the simplest box-diagram contributions are derived in Section IX. The simplest box-diagram contributions calculated from either nucleon-nucleon or nucleon-antinucleon unitarity with two-particle intercalated states are shown to coincide in the region where both apply, thus affording an internal consistency check.

II. KINEMATICS

A. Kinematical Variables

We are primarily concerned here with the elastic scattering of two nucleons.[4] Because of the substitution law we will have also to consider the amplitudes for the elastic scattering of a nucleon and an antinucleon.

The four-momenta in either of these processes will be denoted as p_1, p_2, p_1', p_2'. These will all be sensed into the scattering diagram, so that if $p = (p_0, \mathbf{p})$, then $\pm p_0$ is the energy, $\pm \mathbf{p}$ is the momentum of the corresponding particle, the + sign applies to an incoming particle, the − sign to an outgoing particle; and conservation of 4-momentum reads

$$p_1 + p_2 + p_1' + p_2' = 0 . \tag{2.1}$$

Each of these p's squares[5] to m^2. We neglect the proton-neutron mass difference.

We define the customary three scalar invariants as

$$\begin{aligned} s &= (p_1 + p_2)^2 , \\ t &= (p_1 + p_1')^2 , \\ u &= (p_1 + p_2')^2 ; \\ s + t + u &= 4m^2 . \end{aligned} \tag{2.2}$$

This constraint will always be understood, even when s, t, u are considered as complex variables.

The nucleon-nucleon scattering process is described by incoming momenta p_1, p_2, and outgoing momenta $-p_1'$, $-p_2'$ (Fig. 2). The Mandelstam parameters are related to the common parameters for nucleon-nucleon scattering by

$$s = 4E^2 = 4(m^2 + p^2),$$

$$t = -2p^2(1-z),$$

$$u = -2p^2(1+z),$$

$$\underset{\sim}{p}_i \cdot \underset{\sim}{p}_f = p^2 z,$$

$$z = \cos\theta,$$

$$y = (1-z^2)^{\frac{1}{2}} = \sin\theta \geqslant 0,$$

$$p_1 = (E, \underset{\sim}{p}_i),\ p_2 = (E, -\underset{\sim}{p}_i),\ p_1' = (-E, -\underset{\sim}{p}_f),\ p_2' = (-E, \underset{\sim}{p}_i),$$

where $\underset{\sim}{p}_i$, $\underset{\sim}{p}_f$, p, E, θ are the c.m. momentum of the incident particle "1," of the scattered particle "1," their absolute value, the c.m. energy of one particle, and the c.m. scattering angle, from $\underset{\sim}{p}_i$ to $\underset{\sim}{p}_f$. The physical region for this process, $E \geqslant m$, $-1 \leqslant z \leqslant 1$, is given by $t \leqslant 0$, $u \leqslant 0$, in Mandelstam parameters. This range of variables and the nucleon-nucleon process will in consequence be frequently designated as "the s channel".

The nucleon-antinucleon scattering process is related by crossing to the nucleon-nucleon process if an outgoing nucleon line is converted to an incoming antinucleon line, and an incoming nucleon line is converted to an outgoing antinucleon line—the details are discussed briefly in Section II-B below, and more extensively in Appendix A. Which nucleon lines are so converted is a matter of indifference, owing to the Pauli principle. We follow GGMW, and choose p_1 to be the 4-momentum of the incoming nucleon, p_2' the 4-momentum of the incoming antinucleon, $-p_1'$ the 4-momentum of the outgoing nucleon, and $-p_2$ the 4-momentum of the outgoing antinucleon (Fig. 3). Thus, the subscript 2 designates the antinucleon. The "line" of particles 1 is "unchanged"; that of particles 2 is described "backwards in time" in the sense of Feynman. Another mnemonic advantage of this convention for crossing is that t is still the momentum-transfer variable:

$$
\begin{aligned}
u &= 4\overline{E}^2 = 4(m^2 + \overline{p}^2) , \\
t &= -\, 2\overline{p}^2 (1 - \overline{z}), \\
s &= -\, 2\overline{p}^2 (1 + \overline{z}), \\
\overline{\underset{\sim}{p}}_i \cdot \overline{\underset{\sim}{p}}_f &= \overline{p}^2 \overline{z}, \\
\overline{z} &= \cos \overline{\theta}, \\
\overline{y} &= (1 - \overline{z}^2)^{\frac{1}{2}} = \sin \overline{\theta} \geqslant 0,
\end{aligned}
\tag{2.4}
$$

and

$$p_1 = (\overline{E}, \overline{\underset{\sim}{p}}_i), \; p_2' = (\overline{E}, -\overline{\underset{\sim}{p}}_i) , \; p_1' = (-\overline{E}, -\overline{\underset{\sim}{p}}_f), \; p_2 = (-\overline{E}, \overline{\underset{\sim}{p}}_f),$$

where $\overline{\underset{\sim}{p}}_i, \overline{\underset{\sim}{p}}_f, \overline{p}, \overline{E}$, and $\overline{\theta}$ are the c. m. momentum of the incident nucleon, that of the emerging nucleon, their absolute value, the c. m. energy of one particle, and the c. m. scattering angle from $\underset{\sim}{p}_i$ to $\underset{\sim}{p}_f$. The physical region for this process, $\overline{E} \geqslant m$, $-1 \leqslant \overline{z} \leqslant 1$, is given through the above transformation by $s \leqslant 0$, $t \leqslant 0$, in Mandelstam parameters, a range which we will describe as the "u channel."

The region $s \leqslant 0$, $u \leqslant 0$, or "t channel," also describes nucleon-antinucleon scattering, as may be seen by the Pauli principle, G-parity reflection, or direct employment of a different convention for crossing.

B. The Mandelstam Representation

We follow Mandelstam(2) and GGMW[6] in postulating analytic properties for a set of basic amplitudes for the 4-nucleon-line processes of nucleon-nucleon and nucleon-antinucleon scattering. In the following subsection several alternative sets of such amplitudes will be defined. Their analytic properties will all be deduced from the GGMW result that those of one such set, the "F" amplitudes, satisfy a Mandesltam representation.

The "Mandelstam amplitudes" are functions of the complex variables s, t, u (subject to the constraint $s + t + u = 4\, m^2$), the singularities of which correspond to the thresholds of physical processes. For our problem, with

four external nucleon lines, they have representations of the form

$$
\begin{aligned}
\mathcal{A}(s,t,u) = {} & \frac{s^N t^N}{\pi^2} \int ds' \int dt' \; \frac{\rho_{st}(s',t')}{s'^N t'^N (s'-s)(t'-t)} \\
& + \frac{t^N u^N}{\pi^2} \int dt' \int du' \; \frac{\rho_{tu}(t',u')}{t'^N u'^N (t'-t)(u'-u)} \\
& + \frac{u^N s^N}{\pi^2} \int du' \int ds' \; \frac{\rho_{us}(u',s')}{u'^N s'^N (u'-u)(s'-s)} \\
& + \sum_{p=0}^{M} \left\{ \frac{s^M t^p}{\pi} \int ds' \; \frac{\rho_s(s')}{s'^M (s'-s)} + \frac{t^M u^p}{\pi} \int dt' \; \frac{\rho_t(t')}{t'^M (t'-t)} \right. \\
& \left. + \frac{u^M s^p}{\pi} \int du' \; \frac{\rho_u(u')}{u'^M (u'-u)} \right\} \\
& + \sum_{i,j=0}^{L} \rho_{ij} \, s^i t^j .
\end{aligned}
\tag{2.5}
$$

We have written the most general subtracted form (7) consistent with the above-stated analyticity property. The weight functions $\rho_{st}, \rho_{tu}, \rho_{us}$, the double spectral functions (dsfs), are real and nonzero in regions asymptotically bounded by $s' = 4m^2$, $t' = 4\mu^2$, $u' = 4\mu^2$, where μ is the pion mass. The single spectral functions (ssfs) ρ_s, ρ_t, ρ_u are also real, and may be nonzero for $s' \geq m_D^2$, $t' \geq \mu^2$, $u' \geq \mu^2$ where m_D is the mass of the lightest state of nucleon-number 2 (physically, of course, the deuteron).

In Sections VI and VII we present unitarity relations that determine the contributions to the dsfs coming from Landau (13) - Cutkowsky (14) diagrams which have two-particle intermediate states. For the nucleon-nucleon channel, these are of the general form of Fig. 4, and for the nucleon-antinucleon of the form of Fig. 5. We note that the simplest box diagram[7]

(Fig. 6) has two-particle intermediate states in both channels.

In Fig. 7 the stippled regions A of the dsfs have contributions from Landau-Cutkowsky diagrams that have more than two-particle intermediate states in all channels: we call these the "inner regions." The regions B with vertical cross-hatching have contributions from diagrams of the form of Fig. 4, i.e., those with two nucleons in the intercalated state in nucleon-nucleon scattering. The regions C with horizontal cross-hatching have contributions from diagrams of the form of Fig. 5, i.e., those with two mesons in the intercalated state of nucleon-antinucleon scattering. The simplest box diagrams (Fig. 6) make contributions throughout the region of nonvanishing dsfs, including the unshaded, crescent-like regions of Fig. 7.

The ssfs will include the δ-function contributions from one-particle intermediate states in Landau-Cutkowsky diagrams of the form of Fig. 8. In particular, there will be δ-function contributions corresponding to the one-pion states in ρ_t and ρ_u at $t' = \mu^2$, $u' = \mu^2$; and, in principle at any rate, the deuteron δ-function at $s' = m_D^2$ in ρ_s.

The deuteron term has been discussed in Section I. In Section IX we give the one-pion contributions, and also discuss the simplest box diagrams. These are of interest because, as we have seen, their contributions to the dsfs may be calculated by applying the unitarity conditions with two-particle intercalated states in either the nucleon-nucleon, or the nucleon-antinucleon, channels; the unshaded crescentlike regions of Fig. 7 occur in the strips parallel to both $s = 0$ and $u = 0$ (or $t = 0$). Thus it has to be confirmed that the two derivations of the simplest box-diagram contributions to the dsfs agree.

A complete solution of the nucleon-nucleon problem would require complete knowledge of the dsfs and of the ssfs. However, we hope that the nature of the dsfs in the inner regions has only a small influence on physical nucleon-nucleon scattering when either the energy or the momentum transfer (direct or exchange) is not too large (1); i.e., we hope that near the boundaries of the physical regions the amplitudes in the physical regions are controlled by the behavior of the dsfs in the "strip" regions to good approximation, namely, those dsf regions, complementary to the mysterious inner regions, where in at least one channel the unitarity condition with only two-particle intercalated states is correct. Were we to know the absorptive parts corresponding to the "blobs" in Figs. 4 and 5, these parts of the dsfs could be calculated precisely from the unitarity equations (8.4, 8.5, 8.6).

For nucleon-antinucleon unitarity, the appropriate absorptive parts are for pion-nucleon scattering, and we shall regard these to be "given" input data. For nucleon-nucleon unitarity, however, the absorptive parts are still for nucleon-nucleon scattering. If we suppose that the inner regions of the dsfs may be neglected even when calculating the relevant absorptive parts, we can set up a set of coupled integral equations for the dsfs in the strips parallel to $s = 0$.

The ssfs ρ_t, ρ_u will be considered as given, since they are determined by the amplitudes for nucleon-antinucleon annihilation into one pion.

The ssfs ρ_s are in principle determined by integral equations derived from the nucleon-nucleon unitarity relation. Especially, the emergence of the δ-function corresponding to the deuteron should be indicated; we have touched on this subject in the introduction.

C. The Various Amplitudes

The S matrix for nucleon-nucleon scattering is

$$S = 1 + iR, \tag{2.6}$$

$$R = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_1' + p_2') m^2 E^{-2} \mathcal{T} . \tag{2.7}$$

Following GGMW, we write[8]

$$\begin{aligned} \mathcal{T} = [& F_1^0(S-\tilde{S}) + F_2^0(T+\tilde{T}) + F_3^0(A-\tilde{A}) + F_4^0(V+\tilde{V}) \\ & + F_5^0(P-\tilde{P})] \mathfrak{B}^0 \\ & + [F_1^1(S-\tilde{S}) + F_2^1(T+\tilde{T}) + F_3^1(A-\tilde{A}) + F_4^1(V+\tilde{V}) \\ & + F_5^1(P-\tilde{P})] \mathfrak{B}^1 , \end{aligned} \tag{2.8a}$$

where

$$\begin{aligned} S &= \{\bar{u}(-p_2')u(p_2)\} \{\bar{u}(-p_1')u(p_1)\} , \\ T &= \tfrac{1}{2} \{\bar{u}(-p_2') (1/2i)[\gamma_\mu, \gamma_\nu] u(p_2)\} \{\bar{u}(-p_1') (1/2i)[\gamma_\mu, \gamma_\nu] u(p_1)\} , \\ A &= \{\bar{u}(-p_2')i\gamma_5\gamma_\mu u(p_2)\} \{\bar{u}(-p_1')i\gamma_5\gamma_\mu(p_1)\} , \\ V &= \{\bar{u}(-p_2')\gamma_\mu u(p_2)\}\{\bar{u}(-p_1')\gamma_\mu u(p_1)\} , \\ P &= \{\bar{u}(-p_2')\gamma_5 u(p_2)\} \{\bar{u}(-p_1')\gamma_5 u(p_1)\} , \end{aligned} \tag{2.8b}$$

and

$$\tilde{S} = \{\bar{u}(-p_1')u(p_2)\}\{\bar{u}(-p_2')u(p_1)\} , \quad \text{and so on;} \tag{2.8c}$$

that is, the expressions for $\tilde{S}, \tilde{T}$, etc., are obtained from those for S, T, etc., by interchanging the spinors under "bars." This is equivalent to the linear transformation of Fierz:

$$\begin{bmatrix} \tilde{S} \\ \tilde{T} \\ \tilde{A} \\ \tilde{V} \\ \tilde{P} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & -2 & 0 & 0 & 6 \\ 4 & 0 & -2 & 2 & -4 \\ 4 & 0 & 2 & -2 & -4 \\ 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} S \\ T \\ A \\ V \\ P \end{bmatrix} . \tag{2.9}$$

The isotopic spin projection operators in the s channel are

$$\mathfrak{B}^0 = \tfrac{1}{4}(1 - \underset{\sim}{\tau}^{(1)} \cdot \underset{\sim}{\tau}^{(2)}) ,$$
$$\mathfrak{B}^1 = \tfrac{1}{4}(3 + \underset{\sim}{\tau}^{(1)} \cdot \underset{\sim}{\tau}^{(2)} . \tag{2.10}$$

We will in the sequel employ the notation Γ_i for the Dirac operators 1, $(1/2i\sqrt{2})[\gamma_\mu, \gamma_\nu]$, $i\gamma_5\gamma_\mu$, γ_μ, γ_5, in that order, with $i = 1, 2, 3, 4, 5$, and contractions understood, so that we may write (2.8a) in the more compact form

$$\mathcal{T} = \bar{u}(-p_2')\, u(-p_1')\, \Sigma_{ijI}\, F_i^I(\delta_{ij} + (-)^i\, \mathfrak{F}_{ij})\, \Gamma_j^{(1)} \Gamma_j^{(2)} \mathfrak{B}^I\, u(p_1)\, u(p_2) , \tag{2.11}$$

where $\mathfrak{F}$ is the Fierz matrix of (2.9).

The helicities, spinor phase conventions, and 3-components of isotopic spin which specify a given S-matrix element of clear physical meaning determine the choice of the spinors u which enter into the forms S, T, etc., whereas the $F_i^I(s,t,u)$ are ten Mandelstam-amplitude "4-point function form factors" dissociated from the particular spinors u.

The Pauli principle—namely, that the S-matrix element be odd under interchange of all quantum numbers of the two particles in either the initial or in the final state—assumes the form

$$F_i^I(A) = (-)^{i+I} F_i^I(B), \tag{2.12a}$$

if A and B are Pauli-conjugate points on the Mandelstam (s, t, u) diagram; i.e., for

$$s(A) = S(B), \quad t(A) = u(B), \quad u(A) = t(B); \tag{2.12b}$$

in brief,

$$F_i^I(s,t,u) = (-)^{i+I} F_i^I(s,u,t) . \tag{2.12c}$$

Equation (2.12c) is easily obtained in the s channel, and then follows generally

by analytic continuation for pairs of (s, t, u) points (A, B), where A is reached by a path in the complex (s, t, u) space, B by a concurrently described path through Pauli-conjugate points, and where one and hence both paths do not meet a singularity of the relevant amplitudes. Thus when the points A and B have reached values in the physical regions of the u and t channels respectively, Eq. (2.12c) supplies a symmetry condition for nucleon-anti-nucleon scattering: this is the same symmetry as that implied by G parity.

GGMW argue in Section III of their paper that the F are, in fact, Mandelstam amplitudes.

We shall usually employ the G amplitudes of GGMW, defined by

$$G_i^I = (G\ F)_{ij}\ F_j^I\ , \tag{2.13a}$$

where the matrix (G F) is given by[9]

$$(G\ F) = \frac{1}{4\pi} \begin{bmatrix} 1 & 0 & -4 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 4 & 0 & 1 \end{bmatrix} . \tag{2.13b}$$

Since the G amplitudes are related to the Mandelstam amplitudes F by a numeric matrix, they also are Mandelstam amplitudes. Since (G F) does not involve I, and has zero entries for all i+j odd, the conditions of Pauli symmetry are unaltered:

$$G_i^I(s,t,u) = (-)^{i+I} G_i^I(s,u,t)\ . \tag{2.14}$$

The complicated relationship of the G and F amplitudes to the helicity quantum numbers is clarified by the introduction of helicity amplitudes ϕ(15). If we have

$$\mathcal{T} = (4\pi\ E/m^2)\ \underline{\overline{\phi}}\ , \tag{2.15}$$

then in the c.m. system we obtain

$$\left\langle \lambda_1', \lambda_2'; -p_1' - p_2' \,|\, \Phi \,|\, \lambda_1, \lambda_2; p_1, p_2 \right\rangle$$

$$= \left\langle \lambda_1', \lambda_2'; \underset{\sim}{p}_f \,|\, \Phi \,|\, \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle, \tag{2.16}$$

and

$$d\sigma\,(\lambda_1', \lambda_2'; \underset{\sim}{p}_f; \lambda_1, \lambda_2; \underset{\sim}{p}_i)/d\Omega$$

$$= \left| \left\langle \lambda_1', \lambda_2'; \underset{\sim}{p}_f \,|\, \Phi \,|\, \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle \right|^2 \tag{2.17}$$

is the differential cross section per unit c.m.-system solid angle for the process in which λ_1, λ_2 are the initial helicities, λ_1', λ_2' are the final helicities; $\underset{\sim}{p}_i$, $-\underset{\sim}{p}_i$ are the initial momenta, and $\underset{\sim}{p}_f$, $-\underset{\sim}{p}_f$ the final physical momenta. We will take the plane from which azimuthal angles are measured to be the plane of $\underset{\sim}{p}_i$ and $\underset{\sim}{p}_f$, and in fact assign Euler angles (0, 0, 0) to the initial state, $(0, \theta, 0)$ to the final state, to define our ϕ amplitudes:

$$\left\langle \lambda_1', \lambda_2'; \underset{\sim}{p}_f \,|\, \Phi \,|\, \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle \equiv (\lambda_1', \lambda_2' \,|\, \phi(p,\theta) \,|\, \lambda_1, \lambda_2), \tag{2.18}$$

i.e., the ϕ amplitudes are obtained by using spinors u of definite helicity in Eq. (2.11), if the factor $m^2/4\pi E$ is prefixed for the purpose of giving an amplitude which simply squares to $d\sigma/d\Omega$. If the factor $m^2/4\pi E$ is not prefixed, we have "τ" amplitudes. In either case, we use an appropriate linear superposition of isotopic-spin spinors to produce a definite total isotopic spin I in the s channel.

The sixteen choices of helicities yield sixteen ϕ amplitudes for each total s-channel isotopic spin $I = 0, 1$. Of these, only five of each sixteen are kinematically independent, if account be taken of the invariance properties of the nucleon-nucleon system, including the parity invariance.[10] After GGMW, we use the names $\phi_1, \cdots, \phi_5$ for these amplitudes:

$$\phi_1 = (\tfrac{1}{2}, \tfrac{1}{2}|\phi|\tfrac{1}{2}, \tfrac{1}{2}) = (-\tfrac{1}{2}, -\tfrac{1}{2}|\phi|-\tfrac{1}{2}, -\tfrac{1}{2});$$

$$\phi_2 = (\tfrac{1}{2}, \tfrac{1}{2}|\phi| - \tfrac{1}{2}, -\tfrac{1}{2}) = (-\tfrac{1}{2}, -\tfrac{1}{2}|\phi|\tfrac{1}{2}, \tfrac{1}{2});$$

$$\phi_3 = (\tfrac{1}{2}, -\tfrac{1}{2}|\phi|\tfrac{1}{2}, -\tfrac{1}{2}) = (-\tfrac{1}{2}, \tfrac{1}{2}|\phi| - \tfrac{1}{2}, \tfrac{1}{2});$$

$$\phi_4 = (\tfrac{1}{2}, -\tfrac{1}{2}|\phi| - \tfrac{1}{2}, \tfrac{1}{2}) = (-\tfrac{1}{2}, \tfrac{1}{2}|\phi|\tfrac{1}{2}, -\tfrac{1}{2});$$

$$\begin{aligned}\phi_5 &= (\tfrac{1}{2}, \tfrac{1}{2}|\phi|\tfrac{1}{2}, -\tfrac{1}{2}) = (-\tfrac{1}{2}, \tfrac{1}{2}|\phi|\tfrac{1}{2}, \tfrac{1}{2}) \\ &= (-\tfrac{1}{2}, -\tfrac{1}{2}|\phi|\tfrac{1}{2}, -\tfrac{1}{2}|\phi| - \tfrac{1}{2}, -\tfrac{1}{2}) \\ &= -(\tfrac{1}{2}, \tfrac{1}{2}|\phi| - \tfrac{1}{2}, \tfrac{1}{2}) = -(\tfrac{1}{2}, -\tfrac{1}{2}|\phi|\tfrac{1}{2}, \tfrac{1}{2}) \\ &= -(-\tfrac{1}{2}, -\tfrac{1}{2}|\phi| -\tfrac{1}{2}, \tfrac{1}{2}) = -(\tfrac{1}{2}, -\tfrac{1}{2}|\phi| - \tfrac{1}{2}, -\tfrac{1}{2}) .\end{aligned}$$

By inserting the spinors of appropriate helicity into (2.8), and inserting the prefixing factor $(\phi\ \tau) = m^2/4\pi E$, GGMW give the ϕ linearly in terms of the F; but we quote the ϕ in terms of the G:

$$(\phi\ G) = \frac{1}{2E}\begin{bmatrix} E^2 & m^2 z & m^2 & m^2 z & -p^2 \\ -E^2 & (E^2+p^2)z & -m^2 & m^2 z & -p^2 \\ 0 & m^2(1+z) & -p^2(1+z) & E^2(1+z) & 0 \\ 0 & m^2(1-z) & p^2(1-z) & E^2(1-z) & 0 \\ 0 & -mEy & 0 & -mEy & 0 \end{bmatrix} . \qquad (2.20)$$

This matrix introduces kinematical singularities, such as the branch point owing to the factor $1/E$, so that the ϕ are not Mandelstam amplitudes. It is easy, nevertheless, to remove these singularities from the helicity amplitudes one by one, if we allow different correction factors for different ϕ_i, to produce Mandelstam amplitudes χ_i diagonally related to the helicity amplitudes:

$$\chi_1 = Ep^2\phi_1 ,$$
$$\chi_2 = Ep^2\phi_2 ,$$
$$\chi_3 = Em^2(1+z)^{-1}\phi_3 , \tag{2.21}$$
$$\chi_4 = Em^2(1-z)^{-1}\phi_4 ,$$
$$\chi_5 = m^3y^{-1}\phi_5 .$$

We have here suppressed the label I of total isotopic spin, and this obtains hereafter for operations that proceed analogously and separately for the two values I = 0, 1.

By computing $(\chi\ G) = (\chi\ \phi)\ (\phi\ G)$ from Eqs. (2.21) and (2.20), one does in fact obtain a matrix of elements analytic in s, t, u:

$$(\chi\ G) = \frac{1}{32}\begin{bmatrix} -s(t+u) & 4m^2(t-u) & -4m^2(t+u) & 4m^2(t-u) & -(t+u)^2 \\ s(t+u) & (s-t-u)\ (t-u) & 4m^2(t+u) & 4m^2(t-u) & -(t+u)^2 \\ 0 & (4m^2)^2 & 4m^2(t+u) & 4m^2s & 0 \\ 0 & (4m^2)^2 & -4m^2(t+u) & 4m^2s & 0 \\ 0 & -(4m^2)^2 & 0 & -(4m^2)^2 & 0 \end{bmatrix}. \tag{2.22}$$

This matrix, though of analytic elements, is not numeric. It bears nonzero entries with odd i + j, so that the symmetry conditions imposed by the Pauli principle are somewhat more complicated. If we define

$$\chi_\pm^I = \chi_3^I \pm \chi_4^I , \tag{2.23}$$

then

$$\chi_i^I(s,t,u) = (-)^I \epsilon_i\ \chi_i^I(s;u,t), \tag{2.24}$$

with

$$\epsilon_i = -, -, +, -, +$$

respectively for (2.25)

$$i = 1, 2, +, -, 5 .$$

We take some care in the sequel to indicate in each section whether our χ amplitudes include χ_3 and χ_4, or χ_+ and χ_-.

Another consequence of the fact that the elements of $(\chi\ G)$ are not numeric is that many of the elements of $(\chi\ G)^{-1}$ have poles. Thus, although a Mandelstam representation for the G will assure one for the χ, a Mandelstam representation for the χ does not in itself ensure one for the G. In the sequel, we will find the χ more useful for discussion of the asymptotic behavior of amplitudes than the G, because of the clearer relation to physical assumptions about cross sections. So we will nevertheless make use of the requirement of a Mandelstam representation, though weaker, for the χ amplitudes, and only after that impose the condition that the G be Mandelstam amplitudes too.

The S matrix for nucleon-antinucleon scattering is

$$\overline{S} = 1 + i\,\overline{R}, \tag{2.26}$$

$$\overline{R} = (2\pi)^4\,\delta^{(4)}(p_1 + p_2' + p_1' + p_2)\,(m^2/\overline{E}^2)\,\overline{\mathcal{T}}. \tag{2.27}$$

$$\overline{\mathcal{T}} = (4\pi\,\overline{E}/m^2)\,\overline{\underline{\phi}}\ . \tag{2.28}$$

$$d\overline{\sigma}(\lambda', \overline{\lambda}'; \underset{\sim}{\overline{p}}_f; \lambda, \overline{\lambda}; \underset{\sim}{\overline{p}}_i)/d\overline{\Omega} = \left|\ \left\langle \lambda', \overline{\lambda}'; \underset{\sim}{\overline{p}}_f \middle| \overline{\underline{\phi}} \middle| \lambda, \overline{\lambda}; \underset{\sim}{\overline{p}}_i \right\rangle\right|^2; \tag{2.29}$$

$$\left\langle \lambda', \overline{\lambda}'; \underset{\sim}{\overline{p}}_f \middle| \overline{\underline{\phi}} \middle| \lambda, \overline{\lambda}; \underset{\sim}{\overline{p}}_i \right\rangle \equiv (\lambda', \overline{\lambda}' | \overline{\phi}\ (\overline{p}, \overline{\theta}) | \lambda, \overline{\lambda}); \tag{2.30}$$

$$\overline{\phi}_1 = (\tfrac{1}{2}, \tfrac{1}{2} | \overline{\phi}\ | \tfrac{1}{2}, \tfrac{1}{2}) = (-\tfrac{1}{2}, -\tfrac{1}{2} | \overline{\phi}\ | -\tfrac{1}{2}, -\tfrac{1}{2}), \text{ etc.}, \tag{2.31}$$

as in Eq. (2.19), and

$$\overline{\tau} = (4\pi\overline{E}/m^2)\ \overline{\phi}\ . \tag{2.32}$$

The phases of these matrix elements are defined by computing the matrix elements $\overline{\tau}$ of the operator $\overline{\mathcal{T}}$ in the manner of Appendix A, where u and v are spinors of definite helicity, with isotopic spinors in a linear combination of definite total initial and final (and hence u channel) isotopic spin I, are applied to the same matrix M(F) (Eq. A.1) used for the computation of nucleon-nucleon amplitudes.

If one uses positive-frequency spinors u even for the antinucleons, and compensates for this by defining new amplitudes $\overline{F}$ by imposing the condition that the u-spinor result computed from $M(\overline{F})$ agree with the u- and v-spinor results computed from M(F), one obtains the GGMW crossing matrix $(\overline{F}\ F)$ after the GGMW calculation.

We define amplitudes $\overline{\chi}$ by

$$\overline{\chi} = (\overline{\chi}\ \overline{\phi})\ \overline{\phi}\ , \tag{2.33}$$

where the coefficients $(\chi\ \phi)$ are the same functions of $\overline{E}$, $\overline{p}^2$, $\overline{z}, \overline{y}$ as the $(\chi\ \phi)$ of Eq. (2.21) are of E, p^2, z, y.

From these definitions, it is possible to compute the crossing matrix $(\overline{\chi}\ \chi) = BZ^{-1}$ (see Appendix A) and its inverse $(\chi\ \overline{\chi}) = BZ$; it is $(\chi\ \overline{\chi})$ rather than $(\overline{\chi}\ \chi)$ which we will need so that physical arguments on the forms of the nucleon-antinucleon amplitudes $\overline{\chi}$ will bear on the mathematical forms of the χ.

Thus we have

$$B = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}; \quad B^2 = 1\ . \tag{2.34}$$

From Eq. (A.14) we have

$$
Z = \begin{bmatrix}
\frac{-2m^2 t}{(s+t)^2} & \frac{-2m^2 t}{(s+t)^2} & \frac{-s}{4m^2}\left[\frac{t+u}{s+t} + \frac{su}{(s+t)^2}\right] & \frac{-t^2}{(s+t)} & \frac{stu}{m^2(s+t)^2} \\[2ex]
\frac{-2m^2 t}{(s+t)^2} & \frac{1}{2}\left[\frac{t+u}{s+t} + \frac{su}{(s+t)^2}\right] & \frac{st}{(s+t)^2} & \frac{-t^2}{(s+t)^2} & \frac{stu}{m^2(s+t)^2} \\[2ex]
-m^2\left[\frac{t+u}{u(s+t)} + \frac{s}{(s+t)^2}\right] & \frac{4m^4 t}{u(s+t)^2} & \frac{-2m^2 st}{u(s+t)^2} & \frac{2m^2 t^2}{u(s+t)^2} & \frac{-2st}{(s+t)^2} \\[2ex]
\frac{-4m^4}{(s+t)^2} & \frac{-4m^4}{(s+t)^2} & \frac{2m^2 s}{(s+t)^2} & \frac{t+u}{s+t} - \frac{2m^2 t}{(s+t)^2} & \frac{2su}{(s+t)^2} \\[2ex]
\frac{4m^4}{(s+t)^2} & \frac{4m^4}{(s+t)^2} & \frac{-2m^2 s}{(s+t)^2} & \frac{2m^2 t}{(s+t)^2} & \frac{4m^2 t - su}{(s+t)^2}
\end{bmatrix} \tag{2.35}
$$

Both χ and $\bar{\chi}$ amplitudes are here understood to bear the subscripts 1, 2, 3, 4, 5.

If we extend the list of barred amplitudes to $\bar{G}$ and $\bar{F}$ by putting $(\bar{G}\,\bar{\chi})\,(s,t,u) = (G\,\chi)\,(u,t,s)$, and $(F\,G) = (\bar{F}\,\bar{G})$, a numeric matrix, then the $\bar{F}$ coincide with those defined in the manner remarked on above, by GGMW. We have seen this by observing that $(\bar{\chi}\;\bar{F})\,(\bar{F}\;F)\,(F\,\chi)$, computed from the simple numeric matrix $(\bar{F}\,F)$, as quoted in GGMW,[11] agrees with our matrix $(\bar{\chi}\;\chi)$. In particular, the numeric character of $(\bar{F}\,F)$ and $(\bar{G}\,G)$ trivially ensure that the $\bar{F}$ and the $\bar{G}$ are Mandelstam amplitudes, so that the analyticity of the elements of $(\bar{\chi}\;\bar{G})$ confirms that the $\bar{\chi}$ are Mandelstam amplitudes—the matrix $(\bar{\chi}\,\chi)$ contains poles that would otherwise put this conclusion in question.

We quote the crossing matrix $(G\,\bar{G}) = (\bar{G}\,G)$:

$$(G\,\bar{G}) = B\Delta,$$

$$\Delta = \frac{1}{4}\begin{bmatrix} -1 & 6 & 4 & -4 & -1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 2 & -1 \\ -1 & 0 & 2 & 2 & 1 \\ -1 & 6 & -4 & 4 & -1 \end{bmatrix} ; \quad \Delta^2 = 1 . \tag{2.36}$$

III. ASYMPTOTIC UPPER BOUNDS TO THE χ AMPLITUDES

The partial-wave expansion GGMW (4.8) for the helicity amplitudes ϕ, and (χ ϕ), Eq. (2.21), give the partial-wave expansions for the χ:

$$\begin{aligned}
\chi_1 &= p^2(E/p)\Sigma_J(2J+1)\, T_1^J\, d_{0,0}^J\ , \\
\chi_2 &= p^2(E/p)\Sigma_J(2J+1)\, T_2^J\, d_{0,0}^J\ , \\
\chi_3 &= m^2(E/p)\Sigma_J(2J+1)\, T_3^J\, d_{1,1}^J\ /(1+z)\ , \\
\chi_4 &= m^2(E/p)\Sigma_J(2J+1)\, T_4^J\, d_{-1,1}^J\ /(1-z), \\
\chi_5 &= m^2(m/p)\Sigma_J(2J+1)\, T_5^J\, d_{1,0}^J\ /(1-z^2)^{\frac{1}{2}}\ .
\end{aligned} \tag{3.1}$$

Upper bounds for the χ may be obtained through unitarity,

$$|T_k^J| \leq 1, \tag{3.2}$$

if a finite range R of appreciable interaction is assumed. We introduce the simplest form for such an assumption, namely, that the Σ_J runs, for fixed large p, to $J_{max} = pR \propto \frac{1}{2}p \sim s^{\frac{1}{2}}$. This, together with the explicit expressions for the $d_{m,n}^J(z)$,

$$\begin{aligned}
d_{0,0}^J(z) &= P_J(z); \\
d_{1,1}^J(z) &= (1+z)\,/\,(J(J+1))\,[P_J'(z) - (1-z)\,P_J''(z)]\ ; \\
d_{-1,1}^J(z) &= (1-z)/(J(J+1))\,[P_J'(z) + (1+z)\,P_J''(z)]\ ; \\
d_{1,0}^J(z) &= -(1-z^2)^{1/2}/\,(J(J+1))^{1/2} P_J'(z)\ ,
\end{aligned} \tag{3.3}$$

yield the following asymptotic bounds:

For fixed z, $-1 < z < 1$, as $s \to \infty$,

$$\begin{aligned}
\chi_i^I &= O(s^{7/4}),\ i = 1,2,3, \\
\chi_i^I &= O(s^{3/4}),\ i = 4,5;
\end{aligned} \tag{3.4}$$

for $z = 1$, as $s \to \infty$, or, more precisely, as $s \to \infty$ while t is fixed < 0,

$$\chi_i^I = O(s^2), \; i = 1, 2, 4,$$
$$\chi_i^I = O(s), \; i = 3, 5. \tag{3.5}$$

The situation for negative z, and in particular for $z = -1$, is deducible from that for positive z by means of the Pauli principle, Eq. (2.24). As our amplitude forms satisfy Eq. (2.24) identically, the asymptotic conditions for $z = -1$ which follow from (3.5) and (2.24) need not be explicitly considered. They are given, however, for completeness.

We assume what seems physically reasonable, viz., that at very high energies the elastic scattering is primarily diffraction scattering. This makes it reasonable to strengthen the above bounds so that $O(s^a)$ is replaced by $o(s^a)$ except for those amplitudes which may contribute to the coherent forward diffraction peak.

The only amplitudes on which the bounds will not be strengthened, then, are those at $z = 1$ for which there is no spin flip or helicity flip--these notions coincide for $z = 1$; specifically, these amplitudes are χ_1^I and χ_3^I. The amplitudes for nucleon-nucleon scattering at $z = -1$ depend on those at $z = 1$ in virtue of the Pauli principle, and may be said to provide an alternative description of forward scattering.

The isolation of individual helicity amplitudes available through use of the χ is crucial in completely expressing physical assumptions of diffraction scattering in their strongest form, because the coherent processes are well separated; linear combinations of amplitudes with distinct asymptotic bounds need be limited by only the weakest bound.

The asymptotic upper bounds are listed in Table I, which includes the redundant results for $z = -1$ for ready reference below.

The limits on the nucleon-antinucleon scattering amplitudes as the energy variable $u \to \infty$, for fixed $\bar{z}$, are derived similarly, and can in fact be read from Table I. The only change needed aside from the use of $(u, \bar{z})$ in place of (s, z) arises from the fact that the $\bar{z} = \pm 1$ amplitudes are no longer kinematically dependent, so that the assumption of diffraction scattering introduces o's at $\bar{z} = -1$ for all the $\bar{\chi}_i^I$, as seen in Table II.

The behavior of the χ as $u \to \infty$ follows from Table II and the crossing matrix $(\chi\ \bar{\chi})$, Eq. (2.34). The isotopic spin matrix B is a nonsingular numeric matrix which does not mix the different i, so that it does not influence asymptotic behavior: the isotopic spin index may be suppressed. The asymptotic behavior of the elements of the other factor Z in $(\chi\ \bar{\chi}) = BZ$, as $u \to \infty$, in the general, forward, and backward limits, is readily obtained from Eq. (2.34), and is as follows:

For $\bar{z}$ fixed, $-1 < z < 1$, s and t are both proportional to u, and

$$Z_{ij} = O \begin{bmatrix} u^{-1} & u^{-1} & u & 1 & u \\ u^{-1} & 1 & 1 & 1 & u \\ u^{-1} & u^{-2} & u^{-1} & u^{-1} & 1 \\ u^{-2} & u^{-2} & u^{-1} & 1 & 1 \\ u^{-2} & u^{-2} & u^{-1} & u^{-1} & 1 \end{bmatrix} . \tag{3.6a}$$

For $\bar{z} = 1$, or, more precisely, for t fixed, $s \sim -u$,

$$Z_{ij} = O \begin{bmatrix} u^{-2} & u^{-2} & u & u^{-2} & 1 \\ u^{-2} & 1 & u^{-1} & u^{-2} & 1 \\ u^{-1} & u^{-3} & u^{-2} & u^{-2} & u^{-1} \\ u^{-2} & u^{-2} & u^{-1} & 1 & 1 \\ u^{-2} & u^{-2} & u^{-1} & u^{-2} & 1 \end{bmatrix} . \tag{3.6b}$$

For $\bar{z} = -1$, or, more precisely, for a fixed, $t \sim -u$,

$$Z_{ij} = 0 \begin{bmatrix} u^{-1} & u^{-1} & u^{-1} & 1 & 1 \\ u^{-1} & u^{-1} & u^{-1} & 1 & 1 \\ u^{-2} & u^{-2} & u^{-2} & u^{-1} & u^{-1} \\ u^{-2} & u^{-2} & u^{-2} & u^{-1} & u^{-1} \\ u^{-2} & u^{-2} & u^{-2} & u^{-1} & u^{-1} \end{bmatrix} . \tag{3.6c}$$

The resulting asymptotic bounds on the χ are listed in Table III. Tables I and III are presented in one diagram, in Fig. 9. The dependencies at $t \to \infty$ are deduced from those as $u \to \infty$ by the Pauli symmetry. These asymptotic bounds will henceforth be designated simply as "the asymptotic bounds."

IV. THE ALLOWED SINGLE SPECTRAL FUNCTIONS

As in Froissart, (7), the difference $\Delta\chi_i^I$ of two sets of χ_i^I, both limited by the asymptotic bounds, both satisfying general kinematical properties of the nucleon-nucleon scattering amplitudes, and with equal dsfs, will be shown to have a special form involving only a finite number of terms. Specifically, the result, Eqs. (4.31), (4.32), and (4.43) involves no polynomial and six ssfs: two in the isovector s channel, and two in the Pauli-symmetric t and u channels, for each value of the total isotopic spin. The above mentioned "general kinematical properties" are that the χ be Mandelstam amplitudes in the sense that they be given in terms of subtracted dsf and ssf integrals and a polynomial in s, t, u, and, further, that the corresponding G or F amplitudes also be Mandelstam amplitudes, which necessitates that four linear combinations of the χ vanish at $s = 0$ or at $s = 4m^2$ (see Eq. 4.30). These conditions may be understood without direct reference to the G or F amplitudes if one requires the $\bar{\chi}$ amplitudes to be Mandelstam amplitudes (see Appendix A).

The original general subtracted form of the $\Delta\chi_i^I$ can be written

$$\Delta\chi_i^I = \sum_{p=0}^{M} \left(\frac{t^p s^M}{\pi} \int \frac{\Delta\rho_{sp}^{iI}(s')ds'}{s'^M (s' - s)} + \frac{u^p t^M}{\pi} \int \frac{\Delta\rho_{tp}^{iI}(t')dt'}{t'^M (t' - t)} \right.$$

$$\left. + \frac{s^p u^M}{\pi} \int \frac{\Delta\rho_{up}^{iI}(u')du'}{u'^M (u' - u)} \right) + \sum_{p,q=0}^{L} t^p s^q \Delta\rho_{pq}^{iI} \quad . \qquad (4.1)$$

The asymptotic bounds at general fixed angle limit this form to that of Eq. (4.2) for $i = 3, 4, 5$, and to that of Eq. (4.3) for $i = 1, 2$. The argument for $i = 3, 4, 5$ is precisely that given in Section 5 of Froissart's paper, and will not be given here. The result is

$$\Delta\chi_i^{\,I}(s,t,u) = \sum_{p=0}^{M} \left(\frac{t^p s^{1-p}}{\pi} \int \frac{ds' \Delta\rho_{sp}^{iI}(s')}{(s'-s)s'^{\,1-p}} + \frac{u^p t^{1-p}}{\pi} \int \frac{dt' \Delta\rho_{tp}^{iI}(t')}{(t'-t)t'^{\,1-p}} \right.$$

$$\left. + \frac{s^p u^{1-p}}{\pi} \int \frac{du' \Delta\rho_{up}^{iI}(u')}{(u'-u)u'^{\,1-p}} \right) + \Delta\rho_{00}^{iI}, \quad i = 3, 4, 5, \tag{4.2a}$$

with the various $\Delta\rho_p(x) = o(x^{3/4 - p})$. (4.2b)

The result for $i = 1, 2$ is

$$\Delta\chi_i^{\,I}(s,t,u) = \sum_{p=0}^{M} \left(\frac{t^p s^{2-p}}{\pi} \int \frac{ds' \Delta\rho_{sp}^{iI}(s')}{(s'-s)s'^{\,2-p}} + \frac{u^p t^{2-p}}{\pi} \int \frac{dt' \Delta\rho_{tp}^{iI}(t')}{(t'-t)t'^{\,2-p}} \right.$$

$$\left. + \frac{s^p u^{2-p}}{\pi} \int \frac{du' \Delta\rho_{up}^{iI}(u')}{(u'-u)u'^{\,2-p}} \right) + \Delta\rho_{00}^{iI} + \Delta\rho_{10}^{iI} s + \Delta\rho_{01}^{iI} t, \tag{4.3a}$$

with the various $\Delta\rho_p(x) = o(x^{7/4-p})$, (4.3b)

The argument for $i = 1, 2$ differs from that for the other i only in that the weaker asymptotic bounds--7/4 powers--admit a few more terms easily guessed. The detailed argument follows:

$\mathrm{Im}\, \Delta\chi_i^{\,I} = \sum_{p=0}^{M} t^p \Delta\rho_{sp}^{iI}(s)$ in the s channel (i.e., for $t<0$, $u<0$). If the convenient variable λ is introduced by $z = \cos\theta = 1 - 2\lambda$, then $t = (4m^2 - s)\lambda$, and

$$\mathrm{Im}\, \Delta\chi_i^{\,I} = \sum_{p=0}^{M} (4m^2 - s)^p \lambda^p \Delta\rho_{sp}(s) = o(s^{7/4}),$$

as $s \to \infty$ for fixed θ, i.e., for fixed λ. Each coefficient of λ^p must also be $o(s^{7/4})$, whence

$$\Delta\rho_{sp}(s) = o(s^{7/4 - p}).$$

It follows that the integrals in

$$\sum_{p=0}^{M} \frac{t^p s^{2-p}}{\pi} \int \frac{ds' \Delta\rho^{iI}_{sp}(s')}{(s'-s)s'^{2-p}}$$

converge, with the pth term $o(s^{7/4})$, and share a common imaginary part with the $\Delta\chi_i^I$ in the s channel. Similar argument for the t and u channels establishes that

$$\Delta\chi_i^I - \sum_{p=0}^{M} \left(\frac{t^p s^{2-p}}{\pi} \int \frac{ds' \Delta\rho^{iI}_{sp}(s')}{(s'-s)s'^{2-p}} + \frac{u^p t^{2-p}}{\pi} \int \frac{dt' \Delta\rho^{iI}_{tp}(t')}{(t'-t)t'^{2-p}} + \frac{s^p u^{2-p}}{\pi} \int \frac{du' \Delta\rho^{iI}_{up}(u')}{(u'-u)u'^{2-p}} \right) \tag{4.4}$$

has no imaginary part in any physical region, so that it is a polynomial augmented by possible pole terms, and

$$\Delta\rho^{iI}_{(s,t,u)p}(x) = o(x^{7/4-p}) .$$

The explicit coefficient of t^p, for example, bears an s^{2-p} factor, which is a pole term for $p \geq 3$. The expression (4.4) is, then, of the form

$$\Sigma_{p=3}^{M} \Sigma_{q=2-p}^{-1} (t^p s^q a_{pq} + u^p t^q b_{pq} + s^p u^q c_{pq}) + \Sigma_{p,q=0}^{L} \Delta\rho_{pq} t^p s^q , \tag{4.5}$$

where the possible pole terms have been gathered together in the first sum. These all have $p + q \geq 2$, so that at a fixed angle, with s, t, u mutually proportional, they all go $> o(s^{7/4})$. If one again puts $t = (4m^2 - s)\lambda$, and $u = (4m^2 - s)(1 - \lambda)$, it may be seen that the dependencies on λ of the different pole terms and of the terms in the final sum with $p + q \geq 2$ are sufficiently different that each term must separately be $o(s^{7/4})$, and hence zero, if the sum is to be $o(s^{7/4})$. The sum must be $o(s^{7/4})$, because the integrals in the original form (4.4) of the expression and the $\Delta\chi_i^I$ are all $o(s^{7/4})$. Thus, (4.5) must reduce to

$$\Delta\rho_{00} + \Delta\rho_{10}s + \Delta\rho_{01}t,$$

which establishes (4.3).

We argue that the terms of order $\geqslant s^2$ in (4.5) must in fact separately vanish because of their different dependence on λ. The nonvanishing terms of maximum value $m \geqslant 2$ of $p + q$, if they exist, are

$$\Sigma_{p=0}^{M}(t^p s^{m-p} a_{p,m-p} + u^p t^{m-p} b_{p,m-p} + s^p u^{m-p} c_{p,m-p}) + \Sigma_{p=0}^{L} \Delta\rho_{p,m-p} t^p s^{m-p}$$

$$= \Sigma_{p=0}^{M} \left((4m^2 - s)^p s^{m-p} \lambda^p a_p + (4m^2 - s)^m (1-\lambda)^p \lambda^{m-p} b_p \right.$$

$$\left. + s^p (4m^2 - s)^{m-p} (1-\lambda)^{m-p} c_p \right) + \Sigma_{p=0}^{L} \Delta\rho_p (4m^2 - s)^p s^{m-p} \lambda^p ,$$

where the subscript $q = m - p$ has been dropped for brevity. The coefficient of s^m must vanish, for all λ:

$$\Sigma_{p=m+1}^{M} \left((-\lambda^p a_p + (-)^m (1-\lambda)^p \lambda^{m-p} b_p + (-)^m (1-\lambda)^{m-p} c_p \right)$$

$$+ \Sigma_{p=0}^{L} \Delta\rho_p (-\lambda)^p = 0 .$$

Since $q = m - p \leqslant -1$ in the first sum, we have $m < p$ there, and this has caused the lower limit of the first sum to be $m + 1$, instead of 0. Since $q = m - p \geqslant 0$ in the second sum, $p \leqslant m$, which has cut the upper limit of the second sum to m. The vanishing of the principal part of this function of λ at $\lambda = 0$ requires that the b_p vanish; the vanishing of the principal part at $\lambda = 1$ requires that the c_p vanish; and the vanishing of the remaining polynomial requires that the remaining coefficients, the a_p and the $\Delta\rho_p$, vanish. It follows that there is no nonvanishing term with $p + q \geqslant 2$.

The asymptotic bounds in the various forward and backward directions, Fig. 9, and Pauli symmetrization, will now be applied. The cases $i = +, -, 5$, and $i = 1, 2$ are discussed separately.

A. The Cases $i = +, -, 5$

In this subsection, Eq. (4.2) is further specialized.

It is convenient to consider

$$\chi_{\pm} = \chi_3 \pm \chi_4 \tag{4.6}$$

in place of χ_3 and χ_4 because the Pauli symmetrization is simpler; see Eq. (2.24).

However, the $\chi_{\pm}$ in each asymptotic limit need satisfy only the weakest of the conditions satisfied by χ_3, χ_4, so that some information is conceivably lost, and if lost should be imposed at some later stage. The final $\chi_{\pm}$ nevertheless happen to be limited by the strongest of these asymptotic bounds, without any such further imposition of asymptotic bounds--compare Fig. 9 and Eq. (4.18). This point will therefore receive no further mention.

Except insofar as the asymptotic conditions to be imposed are Pauli-symmetric, Pauli symmetrization will be deferred to the end, and until then the isotopic spin indix will be dropped. In this subsection, i will be understood to run over the values $+, -, 5$ only, and $i = +, -, 5$ will be handled together as far as possible, for brevity.

The condition ($u \to \infty$, s fixed < 0) will be abbreviated U_s, and similarly for the other conditions. Order-of-magnitude assertions will be understood to apply in the limit as the relevant variable tends to infinity.

In the limit U_s, the variables lie in the u physical region, $\chi_i = o(1)$, and so

$$\mathrm{Im}\,\Delta\chi_i = \Sigma_{p=0}^{M} s^p \Delta\rho_{up}^i (u) = o(1)\,, \tag{4.2b}$$

whence each $\Delta\rho_{up}^i (u) = o(1)$, which is, however, weaker than (4.2b) except when $p = 0$:

$$\Delta\rho_{u0}^i (u) = o(1)\,. \tag{4.7}$$

The limit T_s also produces one new condition:

$$\mathrm{Im}\,\Delta\chi_i = \Sigma_{p=0}^{M} u^p \Delta\rho_{tp}^i (t) = o(1);$$

$$\Sigma_{p=0}^{M} (-t)^p \Delta\rho_{tp}^i (t) = o(1)\,. \tag{4.8}$$

The new conditions (4.7) and (4.8) may be used to reduce the number of subtractions. Thus, by virtue of (4.7), the $\Delta\rho_{u0}^i$ term in (4.2) will be replaced by

$$\frac{1}{\pi}\int\frac{du'}{u'-u}\,\Delta\rho_{u0}^i (u') + \text{increment of } \Delta\rho_{00}^i\,. \tag{4.9}$$

The $\Delta\rho_{tp}^i$ sum is first transformed so as to isolate the largest powers of t:

$$\frac{1}{\pi}\sum_{p=0}^{M} u^p t^{1-p}\int\frac{dt'\Delta\rho_{tp}^i (t')}{(t'-t)t'^{\,1-p}}$$

$$= \frac{1}{\pi}\sum_{p=0}^{M} (-t)^p t^{1-p}\int\frac{dt'\Delta\rho_{tp}^i (t')}{(t'-t)t'^{\,1-p}} \tag{4.10}$$

$$+ \frac{1}{\pi}\sum_{p=1}^{M} [(4m^2 - s - t)^p - (-t)^p]\int\frac{dt'\Delta\rho_{tp}^i (t')}{(t'-t)t'^{\,1-p}}\,,$$

the first line of which may then be written

$$\frac{t}{\pi}\int\frac{dt'}{t'-t}\,\frac{1}{t'}\sum_{p=0}^{M} (-t')^p \Delta\rho_{tp}^i (t')$$

$$= \frac{1}{\pi}\int\frac{dt'}{t'-t}\sum_{p=0}^{M} (-t')^p \Delta\rho_{tp}^i (t') + \text{increment of } \Delta\rho_{00}^i\,, \tag{4.11}$$

by virtue of (4.8); the $\Delta\rho^i_{tp}$ sum will be replaced by (4.11) and the second line of (4.10) .

The altered $\Delta\rho^i_{u0}$ term (4.9) and the new $\Delta\rho^i_{tp}$ sum are now o(1) in the limit T_s . The possible bad behavior has been moved into $\Delta\rho^i_{00}$. The o(1) bound for T_s is therefore simplified to

$$\frac{1}{\pi}\sum_{p=0}^{M} t^p s^{1-p}\int\frac{ds'\Delta\rho^i_{sp}(s')}{(s'-s)s'^{1-p}} + \Delta\rho^i_{00} = o(1) . \tag{4.12}$$

Taking the imaginary part of the left-hand side, one sees that each $\Delta\rho^i_{sp}(s)$ vanishes, it being the coefficient of a polynomial in t which tends to zero. Then it follows that $\Delta\rho^i_{00}$ also vanishes.

The modified $\Delta\chi_i$ are, at this point,

$$\begin{aligned}\Delta\chi_i = {} & \frac{1}{\pi}\int\frac{dt'}{t'-t}\sum_{p=0}^{M}(-t')^p\Delta\rho^i_{tp}(t') \\ & + \frac{1}{\pi}\sum_{p=1}^{M}\left((4m^2-s-t)^p-(-t)^p\right)\int\frac{dt'\Delta\rho^i_{tp}(t')}{(t'-t)t'^{1-p}} \\ & + \frac{1}{\pi}\frac{du'}{u'-u}\Delta\rho^i_{u0}(u') \\ & + \frac{1}{\pi}\sum_{p=1}^{M} s^p u^{1-p}\int\frac{du'\Delta\rho^i_{up}(u')}{(u'-u)u'^{1-p}} .\end{aligned} \tag{4.13}$$

The first, third, and fourth lines of (4.13) meet $o(s^2)$ and even o(s) behavior in the limit S_t; they are respectively O(1), o(1), and $o(s^{3/4})$. Consequently, all the terms in the second line with $p \geq 2$ must vanish. For i = 5, where o(s) behavior is required, the p = 1 term also must vanish. The bounds for S_u are met by the first three lines, so that the $p \geq 2$ terms in the last line must vanish, and so must even the p = 1 term for i = 5.

The redefinition

$$4m^2\Delta\rho^i_{t1}(t') + \Sigma^M_{p=0}(-t')^p\Delta\rho^i_{tp}(t') \to \Delta\rho^i_{t0}(t') = o(1) , \tag{4.14}$$

by virtue of (4.8) and (4.2b), then leads to

$$\Delta\chi_i{}^I(s,t,u) = \frac{1}{\pi}\int\frac{dt'}{t'-t}\,\Delta\rho^{iI}_{t0}(t') + \frac{1}{\pi}\int\frac{du'}{u'-u}\,\Delta\rho^{iI}_{u0}(u') \tag{4.15}$$

$$- \frac{s}{\pi}\int\frac{dt'}{t'-t}\,\Delta\rho^{iI}_{t1}(t') + \frac{s}{\pi}\int\frac{du'}{u'-u}\,\Delta\rho^{iI}_{u1}(u') ,$$

where the isotopic spin index I has been restored, in preparation for Pauli symmetrization.

The analytically continued condition of Pauli symmetry, (2.24),

$$s(A) = S(B) = s,$$

$$u(A) = t(B) = x,$$

$$A \in u \text{ channel}, \quad B \in t \text{ channel},$$

$$\chi_i{}^I(A) = \epsilon_i(-)^I\chi_i{}^I(B) ,$$

implies

$$\mathrm{Im}\,\Delta\chi_i{}^I(A) = \epsilon_i(-)^I\,\mathrm{Im}\,\Delta\chi_i{}^I(B) , \tag{4.16}$$

or

$$\Delta\rho^{iI}_{u0}(x) + s\,\Delta\rho^{iI}_{u1}(x) = \epsilon_i(-)^I(\Delta\rho^{iI}_{t0}(x) - s\,\Delta\rho^{iI}_{t1}(x)) ,$$

for a range of s. Therefore, we have

$$\Delta\rho^{iI}_{u0}(x) = \epsilon_i(-)^I\,\Delta\rho^{iI}_{t0}(x) \text{ and } \Delta\rho^{iI}_{u1}(x) = -\,\epsilon_i(-)^I\Delta\rho^{iI}_{t1}(x) , \tag{4.17}$$

whence the $\Delta\chi_i{}^I$ for $i = +, -, 5$ may be represented as

$$\Delta\chi_i{}^I = \frac{1}{\pi}\int dx(\Delta\rho^{iI}_0(x) + s\Delta\rho^{iI}_1(x))\left(\frac{1}{x-t} + \epsilon_i(-)^I\,\frac{1}{x-u}\right) , \tag{4.18a}$$

where

$$\Delta\rho_0^{iI}(x) \equiv \Delta\rho_{t0}^{iI}(x) = o(1) , \tag{4.18b}$$

$$\Delta\rho_1^{iI}(x) \equiv \Delta\rho_{u1}^{iI}(x) = o(x^{-1/4}),$$

and the term proportional to s is not present for $i = 5$; i.e.,

$$\Delta\rho_1^{5I} = 0. \tag{4.18c}$$

Each term in (4.18) meets all the asymptotic bounds and the condition of Pauli symmetrization, in consequence of which the form may not be specialized any further by these conditions.

B. <u>The Cases $i = 1, 2$</u>

In this subsection, Eq. (4.3) is further specialized.

The index i will be understood to assume the values 1, 2, only. Both values are dealt with together under the weaker asymptotic bounds pertaining to $i = 1$, as long as possible, for the sake of brevity.

The equation for $\mathrm{Im}\,\Delta\chi_i$ in the limit U_s is

$$\Sigma_{p=0}^{M} s^p \Delta\rho_{up}^{i}(u) = o(u) ,$$

which yields the new information

$$\Delta\rho_{u0}^{i}(u) = o(u) . \tag{4.19}$$

T_s yields

$$\Sigma_{p=0}^{M} (-t)^p \Delta\rho_{tp}^{i}(t) = o(t) . \tag{4.20}$$

Equation (4.19) implies that the $\Delta\rho_{u0}^{i}$ term in (4.3) may be replaced by

$$\frac{u}{\pi} \int \frac{du'}{(u'-u)\,u'} \; \Delta\rho_{u0}^{i}(u') , \tag{4.21}$$

if the coefficients in the polynomial are altered.

The $\Delta\rho^i_{tp}$ sum may be written

$$\frac{t}{\pi}\int\frac{dt'}{(t'-t)t'}\sum_{p=0}^{M}(-t')^p\Delta\rho^i_{tp}(t') + \text{(increment to polynomial)} \tag{4.22}$$

$$+\frac{1}{\pi}\sum_{p=1}^{M}\left[(4m^2-s-t)^p-(-t)^p\right]t^{2-p}\int\frac{dt'\,\Delta\rho^i_{tp}(t')}{(t'-t)t'^{\,2-p}},$$

where the largest powers of t' were collected in the first line, which was then "unsubtracted" by virtue of (4.20).

If the replacements indicated by (4.21) and (4.22) are made in (4.3), with the appropriate alteration of the coefficients in the polynomial, it is easily seen that the $\Delta\rho^i_{tp}$ and $\Delta\rho^i_{up}$ sums meet the asymptotic bound o(t) for T_s. The remaining terms in (4.3) constitute a polynomial in t of order o(t), whose terms therefore vanish, except for the terms independent of t, and

$$\begin{aligned}\Delta\chi_i = {}& \frac{s^2}{\pi}\int\frac{ds'\,\Delta\rho^i_{s0}(s')}{(s'-s)s'^{\,2}} + \Delta\rho^i_{00} + \Delta\rho^i_{01}s \\ &+\frac{t}{\pi}\int\frac{dt'}{(t'-t)t'}\sum_{p=0}^{M}(-t')^p\,\Delta\rho^i_{tp}(t') \\ &+\frac{1}{\pi}\sum_{p=1}^{M}\left[(4m^2-s-t)^p-(-t)^p\right]t^{2-p}\int\frac{dt'\,\Delta\rho^i_{tp}(t')}{(t'-t)t'^{\,2-p}} \\ &+\frac{u}{\pi}\int\frac{du'}{(u'-u)u'}\,\Delta\rho^i_{u0}(u') \\ &+\frac{1}{\pi}\sum_{p=1}^{M}s^p\,u^{2-p}\int\frac{du'\,\Delta\rho^i_{up}(u')}{(u'-u)u'^{\,2-p}}.\end{aligned} \tag{4.23}$$

All terms except those indexed $p \geq 2$ in the third line are $o(s^{7/4})$ in the limit S_t. The bound $O(s^2)$ therefore eliminates the terms indexed $p > 2$ in the third line; that with $p = 2$ is also eliminated, in the case $i = 2$, by the bound $o(s^2)$.

The simplified third line is

$$\frac{4m^2 - s}{\pi}\, t \int \frac{dt'\, \Delta\rho^i_{t1}(t')}{(t'-t)t'} + \frac{(4m^2 - s)^2 - 2t(4m^2 - s)}{\pi} \int \frac{dt' \Delta\rho^i_{t2}(t')}{t'-t} . \tag{4.24}$$

The terms with $p > 2$ in the fifth line of (4.23), and that with $p = 2$ in the case $i = 2$, likewise vanish if we estimate the terms in the limit S_u.

The $\Delta\rho^i_{t1}(t')$ may be redefined consistent with (4.3b) so that the term with coefficient $-2t(4m^2 - s)$ in (4.24) is subsumed in the $\Delta\rho^i_{t1}$ expression. A term

$$\frac{t}{\pi} \int \frac{dt'}{(t'-t)t'}\, \Delta\rho^i_{t0}(t'), \tag{4.25}$$

$$\Delta\rho^i_{t0}(t') = o(t') ,$$

where $\Delta\rho^i_{t0}(t')$ is redefined, incorporates the second line of (4.23) by virtue of (4.20), and the remaining terms independent of s in (4.24), provided that $\frac{16m^4}{\pi} \int \frac{dt' \Delta\rho^i_{t2}(t')}{t'-t}$ is rewritten as $\frac{16m^4}{\pi}\, t \int \frac{dt' \Delta\rho^i_{t2}(t')}{(t'-t)t'}$ + constant ("unnecessary subtraction"), and the constant cast into the polynomial. The terms left in (4.24) are then,

$$-\frac{st}{\pi} \int \frac{dt' \Delta\rho^i_{t1}(t')}{(t'-t)t'} , \quad \frac{-8m^2 s + s^2}{\pi} \int \frac{dt' \Delta\rho^i_{t2}(t')}{t'-t} . \tag{4.26}$$

The $-8m^2 s$ term may be cast into the form of the first term by another unnecessary subtraction, if a term linear in s is cast into the polynomial. Finally, $\Delta\rho^i_{t1}$ is redefined to encompass this, its sign is reversed, and

$$\Delta\chi_i^{\,I}(s,t,u) = \frac{s^2}{\pi} \int \frac{ds' \Delta\rho^{iI}_{s0}(s')}{(s'-s)s'^2} + \Delta\rho^{iI}_{00} + \Delta\rho^{iI}_{01}\, s \tag{4.27a}$$

$$+ \frac{t}{\pi} \int \frac{dt'}{(t'-t)t'}\, \Delta\rho^{iI}_{t0}(t') + \frac{st}{\pi} \int \frac{dt'}{(t'-t)t'}\, \Delta\rho^{iI}_{t1}(t')$$

$$+ \frac{s^2}{\pi} \int \frac{dt'}{t'-t} \Delta\rho_{t2}^{iI}(t') + \frac{u}{\pi} \int \frac{du'}{(u'-u)u'} \Delta\rho_{u0}^{iI}(u')$$

$$+ \frac{su}{\pi} \int \frac{du'}{(u'-u)u'} \Delta\rho_{u1}^{iI}(u') + \frac{s^2}{\pi} \int \frac{du'}{u'-u} \Delta\rho_{u2}^{iI}(u') , \qquad (4.27a \text{ cont.})$$

for $i = 1, 2$; with

$$\Delta\rho_{t2}^{2I} = \Delta\rho_{u2}^{2I} = 0 , \qquad (4.27b)$$

and where the $\Delta\rho_{s0}^{iI}(s') = o(s'^{7/4})$, the other $\Delta\rho_0^{iI}(x) = o(x)$, and the other $\Delta\rho_p^{iI}(x) = o(x^{7/4 - p})$, as before. Each term now meets all the asymptotic bounds.

The Pauli principle, applied in the manner of Eq. (2.24), now yields

$$\Delta\rho_{u0}^{iI}(x) + s\Delta\rho_{u1}^{iI}(x) + s^2\Delta\rho_{u2}^{iI}(x) = -(-)^I[\Delta\rho_{t0}^{iI}(x) + s\Delta\rho_{t1}^{iI}(x) + s^2\Delta\rho_{t2}^{iI}(x)] , \qquad (4.28)$$

which provides three equations employed to eliminate the $\Delta\rho_u$, to yield an expression in place of (4.27a) that is Pauli symmetric, except for the function of s that appears in the first line of (4.27a). This function is correctly symmetrized by prefixing the factor $\frac{1-(-)^I}{2}$. Thus, the form

$$\Delta\chi_i^I(s,t,u) = \frac{1-(-)^I}{2} \left(\frac{s^2}{\pi} \int \frac{ds'\Delta\rho_{s0}^{iI}(s')}{(s'-s)s'^2} + \Delta\rho_{00}^{iI} + \Delta\rho_{01}^{iI} s \right)$$

$$+ \frac{1}{\pi} \int \frac{dx}{x} \Delta\rho_0^{iI}(x) \left(\frac{t}{x-t} - (-)^I \frac{u}{x-u} \right)$$

$$+ \frac{s}{\pi} \int \frac{dx}{x} \Delta\rho_1^{iI}(x) \left(\frac{t}{x-t} - (-)^I \frac{u}{x-u} \right) \qquad (4.29a)$$

$$+ \frac{s^2}{\pi} \int dx \, \Delta\rho_2^{iI}(x) \left(\frac{1}{x-t} - (-)^I \frac{1}{x-u} \right) ,$$

where

$$\Delta\rho_2^{2I} = 0 , \qquad (4.29b)$$

and

$$\Delta\rho_{s0}^{iI}(s') = o(s'^{7/4}) ,$$
$$\Delta\rho_{0}^{iI}(x) = o(x),$$
$$\Delta\rho_{1}^{iI}(x) = o(x^{3/4}) , \qquad (4.29c)$$
$$\Delta\rho_{2}^{iI}(x) = o(x^{-1/4}) ,$$

incorporates the asymptotic bounds and the Pauli principle. Some superfluous subscripts t have been dropped.

C. Further Restrictions Implied by the Regularity of the G Amplitudes

The matrix $(G\chi)$, inverse to Eqs. (2.22), and (4.30) below, possesses some poles at $s = 0$ and $s = 4\,m^2$. It follows that appropriate linear combinations of χ amplitudes must possess corresponding zeros, so that the G amplitudes can be Mandelstam amplitudes. Explicitly,

$$G_1 = -\frac{4}{m^2 s\,(4\,m^2 - s)}\left[4\,m^2(\chi_1 - \chi_2 + \chi_-) + (u - t)\,\chi_+ + (u - t)\,\frac{s}{2m^2}\,\chi_5\right], \qquad (4.30a)$$

$$G_2 = \frac{4}{m^2(4\,m^2 - s)}\left[\chi_+ + \frac{s}{2m^2}\,\chi_5\right], \qquad (4.30b)$$

$$G_3 = \frac{4}{m^2\,(4\,m^2 - s)}\,\chi_- \qquad (4.30c)$$

$$G_4 = -\frac{4}{m^2\,(4\,m^2 - s)}\,(\chi_+ + 2\,\chi_5), \qquad (4.30d)$$

$$G_5 = -\frac{4}{m^2\,(4\,m^2 - s)^2}\left[4\,m^2\,(\chi_1 + \chi_2) + (t - u)\,\chi_+ + (t - u)\,\frac{s + 4\,m^2}{2m^2}\,\chi_5\right], \qquad (4.30e)$$

are to be regular where, as before,

$$\chi_{\pm} = \chi_3 \pm \chi_4 .$$

Thus, e.g., χ_- has to vanish at $s = 4\,m^2$, identically in t. The imposition of these restrictions leads, via arguments of the same general form as those in subsections A and B above, to integral representations for the ΔG of the form

$$\Delta G_i^I = \frac{1 + (-1)^{I+i}}{2\pi} \int \frac{ds'}{s' - s} \; \Delta\gamma_s^{iI}(s')$$

$$+ \frac{1}{\pi} \int dx \left[\frac{1}{x-t} + (-1)^{I+i} \frac{1}{x-u} \right] \Delta\gamma^{iI}(x), \qquad (4.31)$$

where we have introduced new weight functions $\Delta\gamma$ to simplify the notation;

$$\Delta\gamma_s^{iI}(s') = 0, \text{ for } i = 2, 3, 4 ; \qquad (4.32a)$$

$$\Delta\gamma_s^{iI}(s') = o(s'^{1/4}) ; \qquad (4.32b)$$

$$\Delta\gamma^{iI}(x) = o(x^{-1/4}), \text{ for } i = 3, 4; \qquad (4.32c)$$

and

$$\Delta\gamma^{iI}(x) = o(1), \text{ for all } i . \qquad (4.32d)$$

All these $\Delta\gamma$ are independent functions (before the introduction of dynamics in the form of unitarity equations), except for the linear relations

$$\tfrac{1}{4}\left[\Delta\gamma^{1I}(x) + \Delta\gamma^{5I}(x)\right] = \Delta\gamma^{2I}(x) , \qquad (4.32e)$$

and the asymptotic conditions

$$\Delta\gamma^{1I}(x) - \Delta\gamma^{5I}(x) = o(x^{-1/4}) . \qquad (4.32f)$$

D. Further Restrictions Imposed by the Dominance of Diffraction Scattering

We have made the assumption that at high energies the scattering is dominated by diffraction scattering; i.e., that at high energies only those amplitudes that correspond to coherent processes in the forward direction can attain their maximal bounds. This enabled us to write $o(x^a)$ in place of $O(x^a)$ for many of the bounds in Fig. 9. Further information can be extracted from this assumption. One consequence is that the coherent forward scattering amplitudes become pure imaginary in the high-energy limit.

Now, in nucleon-nucleon scattering, for each isotopic spin state there are two total cross sections, corresponding to the two possibilities of equal or opposite helicities. The optical theorem gives expressions for these cross sections:

$$\sigma_{++}^{I}(p) = \frac{4\pi}{p} \operatorname{Im} \phi_1^I (p^2, z = 1), \tag{4.33a}$$

$$\sigma_{+-}^{I}(p) = \frac{4\pi}{p} \operatorname{Im} \phi_3^I (p^2, z = 1) . \tag{4.33b}$$

If we assume that these cross sections approach constants σ_{++}^{I} and σ_{+-}^{I}, then

$$\operatorname{Im} \phi_i^I (p^2, z = 1) \sim \frac{p}{4\pi} \sigma_{++}^{I} , \tag{4.34a}$$

and

$$\operatorname{Im} \phi_3^I (p^2, z = 1) \sim \frac{p}{4\pi} \sigma_{+-}^{I} , \tag{4.34b}$$

and by continuity, for fixed $t < 0$ the same condition applies. By virtue of our assumption of the dominance of diffraction scattering, for fixed $t < 0$,

$$\operatorname{Re} \phi_1^I = o(p), \text{ as } p \to \infty ; \tag{4.35a}$$

$$\operatorname{Re} \phi_3^I = o(p), \text{ as } p \to \infty . \tag{4.35b}$$

If $\phi = (\phi G)G$ be substituted in Eq. (4.35), and this be done for two sets of proposed G amplitudes with the same dsfs, one obtains

$$p^{-1}(\phi\, G)_{1i}\ \mathrm{Re}\ \Delta G_i^I\ (s,t,u) \to 0 \quad \text{as} \quad s \to \infty, \quad t \text{ fixed} \quad < 0; \tag{4.36a}$$

and

$$p^{-1}(\phi\, G)_{3i}\ \mathrm{Re}\ \Delta G_i^I\ (s,t,u) \to 0 \quad \text{as} \quad s \to \infty, \quad t \text{ fixed} < 0; \tag{4.36b}$$

where ΔG has the same significance as always. In particular, Eq. (4.31) implies that, in the limit S_t,

$$\Delta G_i^I\ (s,t,u) = \frac{1}{\pi}\int \frac{dx}{x-t}\ \Delta\gamma^{i\ I}(x) + o\ (1), \tag{4.37}$$

and

$$\mathrm{Re}\ \ \Delta G_i^I\ (s,t,u) \sim \frac{1}{\pi}\int \frac{dx}{x-t}\ \ \Delta\gamma^{i\ I}(x)\ . \tag{4.38}$$

Conditions (4.36a) and (4.36b) then reduce to

$$\frac{1}{\pi}\int \frac{dx}{x-t}\ [\ \Delta\gamma^{1\ I}(x) - \Delta\ \gamma^{5I}(x)\ ] = 0, \text{ for a range of } t < 0; \tag{4.39a}$$

and

$$\frac{1}{\pi}\int \frac{dx}{x-t}[\ \Delta\gamma^{3I}(x) - \Delta\gamma^{4I}(x)\ ] = 0, \quad \text{for a range of } t < 0. \tag{4.39b}$$

Thus we obtain

$$\Delta\gamma^{1I}(x) = \Delta\gamma^{5I}(x) = \Delta\gamma^{2I}(x), \tag{4.40a}$$

and

$$\Delta\gamma^{3I}(x) = \Delta\gamma^{4I}(x)\ . \tag{4.40b}$$

Similarly, in the limit U_t

$$\mathrm{Re}\ \overline{\phi}_1^I = o\ (\overline{p}), \tag{4.41a}$$

$$\mathrm{Re}\ \overline{\phi}_3^I = o\ (\overline{p})\ , \tag{4.41b}$$

or

$$\Sigma_{jJi}\overline{p}^{-1}\ (\overline{\phi}\ \overline{G})_{1i}\ B_{IJ}\ \Delta_{ij}\ \mathrm{Re}\ \Delta G_j^J\ (s,t,u) \to 0, \tag{4.42a}$$

and

$$\Sigma_{jJi}\overline{p}^{\ -1}\ (\overline{\phi}\ \overline{G})_{3i}\ B_{IJ}\ \Delta_{ij}\ \mathrm{Re}\ \Delta G_j^J\ (s,t,u) \to 0, \tag{4.42b}$$

giving precisely the same conditions as before.

In accordance with the philosophy of the Introduction, we conclude this section with the Mandelstam representation, properly Pauli-symmetrized, for the G amplitudes as a sum of the unsubtracted double spectral integrals and the Δ G of Eq. (4.31) above:

$$G_i^I(s,t,u) = \frac{1}{\pi^2} \int \frac{ds'}{s'-s} \int dx \left[\frac{1}{x-t} + (-1)^{i+I} \frac{1}{x-u} \right] \rho_{ix}^I (s',x)$$

$$+ \frac{1}{\pi^2} \int \frac{dt'}{t'-t} \int \frac{du'}{u'-u} \frac{1}{2} \left[\rho_i^I (t',u') + (-1)^{i+I} \rho_i^I (u',t') \right]$$

$$+ \Delta G_i^I (s,t,u) . \tag{4.43}$$

V. THE DYNAMICALLY INDEPENDENT SINGLE-PARTICLE STATES

In this section we take up the connection between the allowed ssfs and the contributions to the amplitudes from single-particle states. These contributions should be obtained from a suitable continuation of the unitarity equations. A general and systematic discussion of this point in the framework of the S-matrix dynamical theory still presents some difficulties(16), but it is clear that such contributions are formally the same as the renormalized Born terms of conventional field theory. A single-particle state with the same quantum numbers as those of a given channel then givesrise to poles in the energy variable of that channel, i.e., to a δ-function contribution to the corresponding ssfs.

If these ssfs are not allowed, the parameters (position and residue of the pole) of such a single-particle state are determined uniquely, in principle at least, as discussed in the Introduction. Such states will then correspond to particles whose origin is dynamical. We shall see that the deuteron, as might be expected, is an example of this kind. On the other hand, if the ssfs are allowed, the CDD ambiguities which arise in their

determination allow these parameters to be considered as free. Such is the case with the pion.

The ssfs corresponding to particles of nucleon-number two will be in the s channel; those of nucleon-number zero in the t-u channel. In either case, states corresponding to particles of spin $\geqslant 1$ cannot be dynamically independent. The derivation goes as follows:

The allowed ssf terms are given by Eq. (4.31), subject to the conditions (4.32), (4.40). The nucleon-number two ssfs, $\Delta\gamma_s^{iI}$, contribute only to $G^1{}_1$ and $G^1{}_5$, and their contributions are independent of the scattering angle θ in the s channel. It follows immediately from the analysis of the G amplitudes into partial waves in the s channel, given by GGMW in their equations (4.23) and (4.25), that these terms contribute only to the $J = 0$ partial waves, and trivially only in the $I = 1$ channel.

Similarly the nucleon-number zero ssfs $\Delta\gamma^{iI}$ contribute only to $\overline{G}_1{}^I$ and $\overline{G}_5{}^I$, this time with terms independent of $\overline{\theta}$, the scattering angle in the u channel. We conclude that only the $J = 0$ partial waves in the u channel are affected.

It is perhaps worth noting that the contribution to the G amplitudes from an exchange of a single particle of nucleon-number zero and mass μ in the t-u channels with isospin I and tensorial coupling $i(i = 1, 2, 3, 4, 5$ for $S, T, A, V, P)$[12] is

$$G_j^J = [(\mu^2 - t)^{-1} + (-1)^{j+J}(\mu^2 - u)^{-1}]\,(4\pi)^{-1} g^2$$
$$\times [\delta_{J,0}(\delta_{I,0} - 3\delta_{I,1}) + \delta_{J,1}(\delta_{I,0} + \delta_{I,1})]\, A_{ji}\,, \qquad (5.1)$$

where the matrix A is given by

$$A = \frac{1}{8\pi} \begin{bmatrix} 1 & -6 & -4 & 4 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & -2 & -1 \\ 1 & 0 & 2 & 2 & -1 \\ 1 & -6 & 4 & -4 & 1 \end{bmatrix} . \tag{5.2}$$

Then conditions (4.40) require that in a particular column the first, second, and last elements be equal, and that the third and fourth elements be equal. This condition is satisfied only for the first and last columns, i.e., only for S and P coupling.

Similarly in the s channel, the exchange of a particle of nucleon-number two, tensorial coupling i, and isospin I contributes to the amplitudes a term[13]

$$G_j^J = - (4\pi^2)^{-1} g^2 (\mu^2 - s)^{-1} \delta_{I,J} A'_{ji}, \tag{5.3}$$

where

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{5.4}$$

Conditions (4.32) exclude all but the I = 1, S, and P cases.

To summarize the results of this section: Any particles, other than scalar or pseudoscalar ones, that can interact with nucleons cannot be dynamically independent, else the Born terms corresponding to single-particle exchange lead to a violation of the bounds imposed by the arguments of Section III. Specifically, the deuteron cannot be dynamically independent; the pion may be.

This result agrees with results obtained earlier in considerations of the pion-nucleon and pion-pion problems; when it was found by essentially similar methods that particles of spin greater than or equal to unity had to be dynamically dependent.

VI. UNITARITY IN THE NUCLEON-ANTINUCLEON CHANNEL

In order to write the unitarity equations for nucleon-antinucleon scattering in the approximation in which only the lowest-mass one- and two-particle intermediate states are retained, we need representations for the amplitudes for annihilation of a nucleon-antinucleon pair into one or two pions. The former is trivial and leads to the one-pion exchange pole in the amplitudes, discussed in detail by GGMW and in Section VIII. For the latter, we refer to Frazer and Fulco (17), who write the c. m. S matrix for $\pi+\pi \rightarrow N+\bar{N}$ as

$$S = -(2\pi)^4 \, i \, \delta^{(4)}(p_2 + p_1' + q_1 + q_2) \frac{m}{2\bar{E}\omega} \tau , \qquad (6.1)$$

where q_1 and q_2 are the 4-momenta of the ingoing pions. (See Fig. 10). If the helicities of the nucleon and antinucleon are $\lambda, \bar{\lambda}$, then

$$\tau_{\lambda\bar{\lambda}} = \bar{u}_\lambda(-p_1') \, T \, v_{\bar{\lambda}}(-p_2) . \qquad (6.2)$$

Introducing the isospin indices,

$$T_{\beta\alpha} = \delta_{\beta\alpha} T^{(+)} + \frac{1}{2} [\tau_\beta, \tau_\alpha] T^{(-)} , \qquad (6.3)$$

and

$$T^{(\pm)} = -A^{(\pm)} + \frac{1}{2} i \gamma (q_1 - q_2) B^{(\pm)} , \qquad (6.4)$$

where $A^{(\pm)}$, $B^{(\pm)}$ are the conventional amplitudes for π-N scattering.

The differential cross section for $\pi+\pi \rightarrow N+\bar{N}$ with helicities $\lambda\bar{\lambda}$ in the final state is

$$\frac{d\sigma(\pi+\pi \to N+\bar{N})}{d\Omega} = \frac{\bar{p}}{q} \left| \mathcal{F}_{\lambda\bar{\lambda}} \right|^2 \tag{6.5}$$

where

$$\mathcal{F}_{\lambda\bar{\lambda}} = \frac{m}{2\bar{E}} \frac{\tau_{\lambda\bar{\lambda}}}{4\pi} \tag{6.6}$$

The amplitudes $\mathcal{F}_{\lambda\bar{\lambda}}$ are just the Jacob-Wick (15) helicity amplitudes; a direct evaluation leads, with our conventions for the spinors, to

$$\mathcal{F}_{++} \equiv \mathcal{F}_{\frac{1}{2},\frac{1}{2}} = \frac{mi}{8\pi\bar{E}} \left(-\frac{\bar{p}}{m} A + Bq z_q\right),$$

$$\mathcal{F}_{--} \equiv \mathcal{F}_{-\frac{1}{2},-\frac{1}{2}} = \mathcal{F}_{++},$$

$$\mathcal{F}_{+-} \equiv \mathcal{F}_{+\frac{1}{2},-\frac{1}{2}} = \frac{mi}{8\pi\bar{E}} \left(\frac{\bar{E}}{m} Bq y_q\right), \tag{6.7}$$

$$\mathcal{F}_{-+} \equiv \mathcal{F}_{-\frac{1}{2},+\frac{1}{2}} = -\mathcal{F}_{+-} ;$$

where

$$\bar{\underset{\sim}{p}} \cdot \underset{\sim}{q} = \bar{p}\, q \cos\theta_q, \quad z_q = \cos\theta_q, \text{ and } y_q = \sin\theta_q . \tag{6.8}$$

If we define the scalar invariants u, v, w for the $\pi+\pi \to N+\bar{N}$ reaction by

$$u = (p_2 + p_1')^2 = 4\bar{E}^2 = 4(m^2 + \bar{p}^2) = 4\omega^2 = 4(\mu^2 + q^2),$$

$$v = (p_2 + q_2)^2 = m^2 + \mu^2 - 2\bar{E}\omega + 2\bar{p}\, q\, z_q,$$

$$w = (p_2 + q_1)^2 = m^2 + \mu^2 - 2\bar{E}\omega - 2\bar{p}\, q\, z_q,$$

Then $A^{(\pm)}(u, v, w)$ and $B^{(\pm)}(u, v, w)$ satisfy Mandelstam representation as functions of their argument.

According to Singh and Udgaonkar, (3), we can write

$$A^{(\pm)}(u,v,w) \equiv A^{(\pm)}(\bar{p}^2, z_q)$$

$$= A_0^{(\pm)}(p^2) + \frac{z_q}{\pi} \int_{\lambda_0}^{\infty} \frac{d\lambda}{\mu} a^{(\pm)}(\lambda, \bar{p}^2) \left[\frac{1}{\lambda - z_q} \mp \frac{1}{\lambda + z_q}\right], \tag{6.10}$$

$$A_0^{(-)}(\bar{p}^2) = 0 , \tag{6.11}$$

and

$$B^{(\pm)}(u,v,w) \equiv B^{(\pm)}(\bar{p}^2, z_q)$$

$$= \frac{1}{\pi} \int_{\lambda_0}^{\infty} d\lambda\, b^{(\pm)}(\lambda, \bar{p}^2) \left[\frac{1}{\lambda - z_q} \mp \frac{1}{\lambda + z_q}\right]. \tag{6.12}$$

The lower limit of these integrations, λ_0, is a function of $\bar{p}$:

$$\lambda_0 = \frac{v_0 + \bar{p}^2 + q^2}{2\bar{p}q} . \tag{6.13}$$

For the A amplitudes, v_0 is given by the asymptote to the boundary curve for the (u, v) double spectral function in the Mandelstam representation of the amplitude, i.e., $v_0 = (m + \mu)^2$. The B amplitudes, however, also contain the nucleon pole term, and this corresponds to a δ funxtion in b at a value of λ corresponding to $v = m^2$. Accordingly, we will take

$$v_0 = m^2 \tag{6.14}$$

and remember that a and b are in fact zero up to $\lambda = \lambda'_0$, corresponding to $v_0' = (m + \mu)^2$ apart from a term $\pi g^2 \delta(\lambda_0 - \lambda)\,(2\bar{p}q)^{-1}$ in b.

Substituting in (6.9), we obtain

$$\mathcal{F}^{(\pm)}_{++}(\overline{p},\theta_q) = \frac{mi}{8\pi\overline{E}}\left[a_0^{(\pm)} + \frac{z_q}{\pi}\int_{\lambda_0}^{\infty}\frac{d\lambda}{\lambda}\, a^{(\pm)}(\lambda)\left(\frac{1}{\lambda-z_q} \mp \frac{1}{\lambda+z_q}\right)\right], \tag{6.15}$$

and

$$\mathcal{F}^{(\pm)}_{+-}(\overline{p},\theta_q) = \frac{mi}{8\pi\overline{E}}\,\frac{y_q}{\pi}\int_{\lambda_0}^{\infty} d\lambda\, \beta^{(\pm)}(\lambda)\left(\frac{1}{\lambda-z_q} \mp \frac{1}{\lambda+z_q}\right), \tag{6.16}$$

where

$$a_0^{(\pm)} = -\frac{\overline{p}}{m} A_0^{(\pm)},$$

$$\alpha^{(\pm)}(\lambda) = -\frac{\overline{p}}{m} a^{(\pm)}(\lambda) + \lambda q\, b^{(\pm)}(\lambda), \tag{6.17}$$

and

$$\beta^{(\pm)}(\lambda) = \frac{\overline{E}}{m} q\, b^{(\pm)}(\lambda),$$

and we have suppressed the $\overline{p}^2$ dependence of the weight functions.

We can write the amplitudes for states of definite total isotopic spin

$$\mathcal{F}^{(0)} = \sqrt{6}\,\mathcal{F}^{(+)},$$
$$\mathcal{F}^{(1)} = 2\,\mathcal{F}^{(-)}. \tag{6.18}$$

After these preliminaries we are in a position to write down the contribution to $\mathrm{Im}\,\overline{\phi}_i^{\,I}$ arising from the two-pion intermediate state in the unitarity equations:

$$\mathrm{Im}_{2\pi}\left\langle \lambda', \overline{\lambda}'; \underset{\sim}{p}_f \,|\, \overline{\overline{\phi}} \,|\, \lambda, \overline{\lambda}; \underset{\sim}{\overline{p}}_i \right\rangle$$

$$= q(4\pi)^{-1}\int d\Omega_q \left\langle \underset{\sim}{q} \,|\, \mathcal{F} \,|\, \lambda', \overline{\lambda}'; \underset{\sim}{\overline{p}}_f \right\rangle^{*} \left\langle \underset{\sim}{q} \,|\, \mathcal{F} \,|\, \lambda, \overline{\lambda}; \underset{\sim}{\overline{p}}_i \right\rangle \tag{6.19a}$$

$$= q(4\pi)^{-1}\int d\Omega_q\, \mathcal{F}^{*}_{\lambda'\overline{\lambda}'}(-\theta_{2q})\,\mathcal{F}_{\lambda\overline{\lambda}}(\theta_{1q})$$

$$\times \exp i\left[(\lambda-\overline{\lambda})\,\phi_{1q} + (\lambda'-\overline{\lambda}')\,\phi_{2q}\right]. \tag{6.19b}$$

This equation is derived in Appendix B. We have

$$e^{i\phi_{2q}} = y_{2q}^{-1} \left[\bar{y}\, z_{1q} - \bar{z}\, y_{1q} \cos\phi_{1q} + i\, y_{1q} \sin\phi_{1q} \right],$$

$$z_{2q} = z_{1q}\, \bar{z} + y_{1q}\, \bar{y} \cos\phi_{1q}, \tag{6.20}$$

and

$$d\Omega_q = d\, z_{1q}\, d\phi_{1q}.$$

We recall that $\bar{\theta}$ is the angle through which the nucleon is scattered in the nucleon-antinucleon scattering, and that $\bar{z}, \bar{y}$ are the cosine and sine of $\bar{\theta}$.

If we substitute the appropriate helicities in (6.19) for the evaluation of the $\bar{\phi}_i$, and express the $\mathfrak{F}$ in terms of the weight functions α and β through (6.15) and (6.16), the angular integrations can be performed (Appendix C). The results are:[14, 15]

$$\begin{aligned}
\mathrm{Im}_{2\pi}\, \bar{\phi}_1^{\,I} = (5 \pm 1)\, \frac{q}{4\pi} \left(\frac{m}{8\pi\, \bar{E}}\right) \Big\{ & 4\pi \left| a_0^{(\pm)} \right|^2 \\
& + \frac{1}{\pi} \int_{\lambda_0}^{\infty} d\lambda\, 2\, \mathrm{Re} \left[a_0^{(\pm)*} a^{(\pm)}(\lambda) \right] \frac{2}{\lambda} \left[\lambda L(\lambda) - 4\pi \right] \\
& + \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} \frac{d\lambda}{\lambda_0} \int_{\lambda_0}^{\infty} \frac{d\mu}{\mu} (1 \pm 1) \left[4\pi - \lambda L(\lambda) - \mu L(\mu) \right] a^{(\pm)*}(\lambda)\, a^{(\pm)}(\mu) \\
& + \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\, \frac{1}{\pi} \int \frac{4\pi^2 dz'}{\left[K(\lambda, \mu, z') \right]^{\frac{1}{2}}} \\
& \left[\frac{1}{z' - \bar{z}} \pm \frac{1}{z' + \bar{z}} \right] a^{(\pm)*}(\lambda)\, a^{\pm}(\mu) \Big\},
\end{aligned} \tag{6.21a}$$

$$\mathrm{Im}_{2\pi}\, \bar{\phi}_2^{\,I} = \mathrm{Im}_{2\pi}\, \bar{\phi}_1^{\,I}, \tag{6.21b}$$

$$\frac{1}{1+\bar{z}} \quad \mathrm{Im}_{2\pi}\, \bar{\phi}_3{}^{I} - \frac{1}{1-\bar{z}} \quad \mathrm{Im}_{2\pi}\, \bar{\phi}_4{}^{I}$$

$$= -(5 \pm 1)\, \frac{q}{4\pi} \left(\frac{m}{8\pi \bar{E}}\right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\, \beta^{(\pm)*}(\lambda)$$

$$\times\ \beta^{(\pm)}(\mu)\, \frac{1}{\pi} \int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \left[\frac{1}{z'-\bar{z}} \mp \frac{1}{z'+\bar{z}}\right] 2 \left[\frac{1+z'^2}{(1-z'^2)^2}(\lambda^2+\mu^2-2\lambda\mu z') - \frac{1+\lambda\mu z'}{1-z'^2}\right],$$

(6.21c)

$$\frac{1}{1+\bar{z}} \quad \mathrm{Im}_{2\pi}\, \bar{\phi}_3{}^{I} - \frac{1}{1-\bar{z}} \quad \mathrm{Im}_{2\pi}\, \bar{\phi}_4{}^{I}$$

$$= (5 \pm 1)\, \frac{q}{4\pi} \left(\frac{m}{8\pi \bar{E}}\right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda$$

$$\times \int_{\lambda_0}^{\infty} d\mu\, \beta^{(\pm)*}(\lambda)\beta^{(\pm)}(\mu)\ \frac{1}{\pi} \int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \left[\frac{1}{z'-\bar{z}} \pm \frac{1}{z'+\bar{z}}\right] 2 \left[\frac{2z'}{(1-z'^2)^2}(\lambda^2+\mu^2-2\lambda\mu z') - \frac{\lambda\mu+z'}{1-z'^2}\right];$$

(6.21d)

and if we use $\bar{\phi}_5 = \left\langle \frac{1}{2}, \frac{1}{2} \mid \bar{\phi} \mid \frac{1}{2}, -\frac{1}{2} \right\rangle$,

$$\frac{i1}{\bar{y}} \, \mathrm{Im}_{2\pi} \, \bar{\phi}_5{}^{I} = -(5 \pm 1) \, \frac{q}{4\pi} \left(\frac{m}{8\pi \bar{E}}\right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda$$

$$\times \int_{\lambda_0}^{\infty} d\mu \, a^{(\pm)*}(\lambda) \, \beta^{(\pm)}(\mu) \, \frac{1}{\pi} \int \frac{4\pi^2 dz'}{[K(\lambda, \mu z')]^{\frac{1}{2}}}$$

$$\times \left[\frac{1}{z' - \bar{z}} \mp \frac{1}{z' + \bar{z}}\right] \frac{(\lambda - \mu z')}{1 - z'^2} \quad . \tag{6.21e}$$

Using instead $\bar{\phi}_5 = \left\langle -\frac{1}{2}, \frac{1}{2} \mid \bar{\phi} \mid \frac{1}{2}, \frac{1}{2} \right\rangle$, we are led to a right-hand side that is the formal complex conjugate of Eq. (6.21e). However, the right-hand side is in fact <u>real,</u> as it has to be since it is the imaginary part of an amplitude. This reality is a consequence of the final-state interaction theorem for the reaction $N+\bar{N} \to \pi-\pi$, which imposes a phase relationship between the A and the B amplitude most easily expressed in terms of the partial waves. We will write $\mathrm{Re}\, a^* \beta$ in the right-hand side of (6.21e) rather than $a^* \beta$, to ensure that in an approximate calculation quantities that should be real are indeed real. A similar situation will be seen to arise in Section VII where we discuss nucleon-nucleon unitarity.

We can now use

$$\bar{G} = (\bar{G}\, \bar{\phi})\, \bar{\phi} \, , \tag{6.22a}$$

where

$$(\overline{G}\,\overline{\phi}) = \frac{1}{\overline{E}\,\overline{p}^2}\begin{bmatrix} \overline{p}^2 & -\overline{p}^2 & \dfrac{m^2-\overline{p}^2\overline{z}}{1+\overline{z}} & \dfrac{-m^2-\overline{p}^2\overline{z}}{1-\overline{z}} & -2\dfrac{\overline{E}}{m}\overline{p}^2\dfrac{\overline{z}}{\overline{y}} \\ 0 & 0 & \dfrac{-\overline{E}^2}{1+\overline{z}} & \dfrac{-\overline{E}^2}{1-\overline{z}} & -2\dfrac{\overline{E}}{m}\overline{E}^2\dfrac{1}{\overline{y}} \\ 0 & 0 & \dfrac{-\overline{E}^2}{1+\overline{z}} & \dfrac{\overline{E}^2}{1-z} & 0 \\ 0 & 0 & \dfrac{\overline{E}^2}{1+\overline{z}} & \dfrac{\overline{E}^2}{1-\overline{z}} & 2\dfrac{\overline{E}}{m}m^2\dfrac{1}{\overline{y}} \\ -\overline{E}^2 & -\overline{E}^2 & \dfrac{-\overline{E}^2\overline{z}}{1+\overline{z}} & \dfrac{-\overline{E}^2\overline{z}}{1-\overline{z}} & -2\dfrac{\overline{E}}{m}(\overline{E}^2+m^2)\dfrac{\overline{z}}{\overline{y}} \end{bmatrix} \tag{6.22b}$$

to derive expressions for $\mathrm{Im}_{2\pi}\,\overline{G}_i^{\,I}$; this is done in Appendix C. These expressions appear to have poles at $\overline{p}^2 = 0$, which would be inconsistent with the Mandelstam representation. However, we must recall how α and β are related to a and b through (6.17), and when this substitution is made a cancellation occurs to remove these spurious singularities.

The resulting equations are:[15]

$$\mathrm{Im}_{2\pi}\,\overline{G}_i^{\,I}(\overline{p}^2,\overline{z}) = \mathrm{Im}_{2\pi}\,\overline{G}_i^{\,I}(\overline{p}^2,0)$$

$$+\frac{q}{\overline{E}}\,\frac{1}{4\pi}\left(\frac{m}{8\pi\overline{E}}\right)^2(5\pm1)\,\frac{2}{\pi^2}\int_{\lambda_0}^{\infty}d\lambda\int_{\lambda_0}^{\infty}d\mu\,\frac{\overline{z}}{\pi}\int\frac{4\pi^2dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}$$

$$\times\left[\frac{1}{z'-\overline{z}}\mp\frac{1}{z'+\overline{z}}\right]\frac{1}{z'}\left[k_{i11}(\lambda,\mu,z')\,a^{(\pm)^*}(\overline{p}^2,\lambda)\,a^{(\pm)}(\overline{p}^2,\mu)\right.$$

$$+ k_{i12}(\lambda, \mu, z')\, a^{(\pm)*}(\bar{p}^2, \lambda)\, b^{(\pm)}(\bar{p}^2, \mu)$$

$$+ k_{i21}(\lambda, \mu, z')\, b^{(\pm)*}(\bar{p}^2, \lambda)\, a^{(\pm)}(\bar{p}^2, \mu)$$

$$+ k_{i22}(\lambda, \mu, z')\, b^{(\pm)*}(\bar{p}^2, \lambda)\, b^{(\pm)}(\bar{p}^2, \mu)\,]$$

for $i = 1, 5$; and

$$\mathrm{Im}_{2\pi}\, \bar{G}_i^{\,I}(\bar{p}^2, z) = \frac{q}{\bar{E}}\, \frac{1}{4\pi} \left(\frac{m}{8\pi\bar{E}}\right)^2 (5 \pm 1)\, \frac{2}{\pi^2}$$

$$\times \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\, \frac{1}{\pi} \int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left[\frac{1}{z' - \bar{z}} \mp (-1)^i \frac{1}{z' + \bar{z}}\right]$$

$$\times [\, k_{i11}\, a^* a + k_{i12}\, a^* b + k_{i21}\, b^* a + k_{i22}\, b^* b\,] \,, \tag{6.23}$$

for $i = 2, 3, 4$; where

$$k_1(\lambda, \mu, z) = \begin{bmatrix} 0 & -\dfrac{\bar{E}^2}{m^2}\, \dfrac{q}{m}\, \bar{p}\, z\, \dfrac{(\lambda - \mu z)}{1 - z^2} \\ -\dfrac{\bar{E}^2}{m^2}\, \dfrac{q}{m}\, \bar{p}\, z\, \dfrac{(\mu - \lambda z)}{1 - z^2} & -\dfrac{\bar{E}^2 q^2}{m^2} \left\{ \dfrac{2m^2}{\bar{p}^2} \left[\dfrac{\lambda u + z}{1 - z^2} - \dfrac{2z}{(1 - z^2)^2} \times (\lambda^2 + \mu^2 - 2\lambda\mu z) \right] + z \left[\dfrac{\lambda^2 + \mu^2 + 2}{1 - z^2} - \dfrac{4}{(1 - z^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z) \right] \right\} \end{bmatrix}; \tag{6.24a}$$

$$k_2(\lambda, \mu, z) = \begin{bmatrix} 0 & -\dfrac{\overline{E}^4 q}{\overline{p}\, m^3} \dfrac{(\lambda - \mu z)}{1 - z^2} \\ -\dfrac{\overline{E}^4 q}{\overline{p}\, m^3} \dfrac{(\mu - \lambda z)}{1 - z^2} & \dfrac{\overline{E}^4 q^2}{\overline{p}^2 m^2} \left[\dfrac{4}{(1 - z^2)^2} (\lambda^2 + \mu^2 - 2\lambda \mu z) - \dfrac{\lambda^2 + \mu^2 + 2}{1 - z^2} \right] \end{bmatrix}; \tag{6.24b}$$

$$k_3(\lambda, \mu, z) = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{2\overline{E}^4 q^2}{\overline{p}^2 m^2} \left[\dfrac{\lambda\mu + z}{1 - z^2} - \dfrac{2z}{(1 - z^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z) \right] \end{bmatrix} \tag{6.24c}$$

$$k_4(\lambda, \mu, z) = \begin{bmatrix} 0 & \dfrac{\overline{E}^2}{\overline{p}} \dfrac{q}{m} \dfrac{(\lambda - \mu z)}{1 - z^2} \\ \dfrac{\overline{E}^2}{\overline{p}} \dfrac{q}{m} \dfrac{(\mu - \lambda z)}{1 - z^2} & \dfrac{\overline{E}^2 q^2}{m^2} \left\{ - \dfrac{\overline{E}^2}{\overline{p}^2} \left[\dfrac{4}{(1 - z^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z) - \dfrac{\lambda^2 + \mu^2 + 2}{1 - z^2} \right] + \left[\dfrac{\lambda^2 + \mu^2 - 2\lambda\mu z}{1 - z^2} \right] \right\} \end{bmatrix}; \tag{6.24d}$$

$$k_5(\lambda,\mu,z)=\begin{bmatrix} -2\,\dfrac{\overline{E}^2}{m^2} & \dfrac{q}{m}\,\dfrac{\overline{E}^2}{\overline{p}}\left[\dfrac{2(\mu-\lambda z)}{1-z^2}-\dfrac{\overline{p}^2}{m^2}\,\dfrac{z(\lambda-\mu z)}{1-z^2}\right] \\ \dfrac{q}{m}\,\dfrac{\overline{E}^2}{\overline{p}}\left[\dfrac{2(\lambda-\mu z)}{1-z^2}-\dfrac{\overline{p}^2}{m^2}\,\dfrac{z(\mu-\lambda z)}{1-z^2}\right] & 2\,\dfrac{\overline{E}^2 q^2}{\overline{p}^2}\left\{\dfrac{\overline{p}^2 z}{2m^2}\left[\dfrac{4(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2}-\dfrac{\lambda^2+\mu^2+2}{1-z^2}\right]+\left[\dfrac{2z}{(1-z^2)^2}\,(\lambda^2+\mu^2-2\lambda\mu z)-\dfrac{\lambda\mu+z}{1-z^2}\right]\right\} \end{bmatrix} ; \tag{6.24e}$$

$$\mathrm{Im}_{2\pi}\,\overline{G}_i^{(1)}(\overline{p}^2,0)=0\,,\qquad \text{for } i=1,5; \tag{6.25}$$

$$\mathrm{Im}_{2\pi}\overline{G}_1^{(0)}(\overline{p}^2,0)=-\frac{q}{\overline{E}}\,\frac{1}{4\pi}\left(\frac{m}{8\pi\overline{E}}\right)^2\frac{\overline{E}^2q^2}{\overline{p}^2}\,\frac{48}{\pi^2}$$

$$\times\int_{\lambda_0}^{\infty}d\lambda\int_{\lambda_0}^{\infty}d\mu\; b^{(+)^*}(\overline{p}^2,\lambda)\,b^{(+)}(\overline{p}^2,\mu)$$

$$\times\left[\,4\pi-\lambda\,L(\lambda)-\mu\,L(\mu)+\lambda\mu\,J(\lambda,\mu,0)\,\right], \tag{6.26a}$$

and

$$\mathrm{Im}_{2\pi}\overline{G}_5^{(0)}(\overline{p}^2,0)=-\frac{q}{\overline{E}}\,\frac{1}{4\pi}\left(\frac{m}{8\pi\overline{E}}\right)^2\frac{\overline{E}^2}{\overline{p}^2}\;24\left\{2\pi\,\frac{\overline{p}^2}{m^2}\,\left|A_0^{(+)}(\overline{p}^2)\right|^2\right.$$

$$+\frac{1}{\pi}\int_{\lambda_0}^{\infty}d\lambda\,[\,\lambda\,L(\lambda)-4\pi]\,\frac{\overline{p}}{m}\,2\mathrm{Re}\,[\,A_0^{(+)^*}(\overline{p}^2)\left(\frac{\overline{p}}{m}\frac{1}{\lambda}\,a^{(+)}(\overline{p}^2,\lambda)\right.$$

$$\left.\left.+\,q\,b^{(+)}(p^2,\lambda)\right)\,]\right\} \tag{6.26b cont.}$$

$$+ \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu \left[\frac{\bar{p}}{m} \frac{1}{\lambda} a^{(+)}(\bar{p}^2, \lambda) - q\, b^{(+)}(\bar{p}^2, \lambda) \right]^*$$

$$\times \left[\frac{\bar{p}}{m} \frac{1}{\mu} a^{(+)}(\bar{p}^2, \mu) - q\, b^{(+)}(\bar{p}^2, \mu) \right]$$

$$\left[4\pi - \lambda L(\lambda) - \mu L(\mu) + \lambda\mu J(\lambda, \mu, 0) \right] . \qquad (6.26b)$$

These equations, although derived by a consideration of unitarity in the nucleon-antinucleon channel, where physically of course $\bar{p}^2 > 0$, may be analytically continued down to the threshold where $q^2 = 0$ and $\bar{p}^2 = \mu^2 - m^2 < 0$. It might be thought that difficulties of interpretation will arise at $\bar{p}^2 = 0$ over which branch of $\sqrt{\bar{p}^2}$ is to be chosen. However, an examination of (6.24) and (6.26) shows that this is not the case.

The easiest way to establish this is to transform the variables so that

$$\lambda = \frac{(v' + \bar{p}^2 + q^2)}{2\,\bar{p}\,q} ,$$

and

$$\mu = \frac{(v'' + \bar{p}^2 + q^2)}{2\,\bar{p}\,q} , \qquad (6.27)$$

before making the continuation. It is then clear that $\bar{p}$ enters the equations only as $\bar{p}^2$.

VII. UNITARITY IN THE NUCLEON-NUCLEON CHANNEL

In this section we derive equations that express the unitarity of the S matrix for nucleon-nucleon scattering below the pion-production threshold. These equations will be used in the following section to derive integral equations for the dsfs in the strip regions parallel to $s = 0$ (Fig. 1), as outlined in Section II-B and the Introduction.

From the unitarity of the S matrix

$$S^{\dagger} S = 1 , \tag{7.1}$$

in the energy region $4m^2 < s < (2m + \mu)^2$, we deduce

$$\frac{1}{2i} \left[\left\langle \lambda_1' , \lambda_2' ; \underset{\sim}{p}_f \left| \underline{\Phi} \right| \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle - \left\langle \lambda_1', \lambda_2'; \underset{\sim}{p}_f \left| \underline{\Phi}^{\dagger} \right| \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle \right]$$

$$= p(4\pi)^{-1} \Sigma_{\mu_1 \mu_2} \int d\Omega_p \left\langle \lambda_1', \lambda_2' ; \underset{\sim}{p}_f \left| \underline{\Phi}^{\dagger} \right| \mu_1, \mu_2; \underset{\sim}{p} \right\rangle$$

$$\times \left\langle \mu_1, \mu_2; \underset{\sim}{p} \left| \underline{\Phi} \right| \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle, \tag{7.2}$$

(cf. Appendix B).

Now we obtain

$$\left\langle \lambda_1', \lambda_2'; \underset{\sim}{p}_f \left| \underline{\Phi}^{\dagger} \right| \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle = \left\langle \lambda_1, \lambda_2; \underset{\sim}{p}_i \left| \underline{\Phi} \right| \lambda_1', \lambda_2'; \underset{\sim}{p}_f \right\rangle^*$$

$$= \left\langle \lambda_1', \lambda_2'; \underset{\sim}{p}_f \left| \underline{\Phi} \right| \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle^* , \tag{7.3}$$

since for just those combinations of helicities for which interchanging λ_1', λ_2' with λ_1, λ_2 introduces a sign change to the amplitude, so also does interchanging $\underset{\sim}{p}_f$ with $\underset{\sim}{p}_i$ (i.e., the amplitude ϕ_5 is odd in θ). Thus the left-hand side of (7.2) is

$$\mathrm{Im} \left\langle \lambda_1', \lambda_2' ; \underset{\sim}{p}_f \left| \underline{\Phi} \right| \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle .$$

For $s > (2m + \mu)^2$ Eq. (7.2) is no longer valid, since additional states may now contribute on the right-hand side. We may, however, use it to define the elastic contribution to the imaginary part of the amplitude. If the angular dependence of the amplitudes is extracted as described in Appendix B, we may write

$$\mathrm{Im}_{el}(\lambda'_1, \lambda'_2 \,|\phi\,(p,\theta)\,|\lambda_1, \lambda_2)$$

$$= p(4\pi)^{-1}\Sigma\,\mu_1\mu_2 \int d\Omega_1\,(\mu_1\mu_2\,|\,\phi\,(p,-\theta_2)\,|\lambda'_1, \lambda'_2)^*$$

$$\times(\mu_1, \mu_2|\,\phi\,(p,\theta_1)\,|\lambda_1\lambda_2) \times (p\cdot f)\,, \tag{7.4}$$

where the phase factor $(p \cdot f)$ contains the dependence on the azimuthal angles and is given by

$$(p\cdot f) = \exp i\left[(\lambda_1-\lambda_2)\,\phi_1 + (\lambda'_1 - \lambda'_2)\,\phi_2 + (\mu_1-\mu_2)\,\phi_3\right]\,. \tag{7.5}$$

The angles are connected by the trigonometrical relations

$$\cos\theta_2 = \cos\theta_1\cos\theta + \sin\theta_1\sin\theta\cos\phi_1\,,$$

$$e^{i\phi_2} = (\sin\theta_2)^{-1}\left[\sin\theta\cos\theta_1 - \cos\theta\sin\theta_1\cos\phi_1 + i\sin\theta_1\sin\phi_1\right]\,,$$

$$e^{i\phi_3} = (\sin\theta_2)^{-1}\left[-\sin\theta_1\cos\theta + \cos\theta_1\sin\theta\cos\phi_1 - i\sin\theta\sin\phi_1\right]\,,$$

$$d\Omega_1 = d(\cos\phi_1)\,d\phi_1\,. \tag{7.6}$$

Each of the matrix elements appearing in (7.6) is one of the amplitudes ϕ_i defined in (2.19), and identifying them appropriately enables us to write

$$\mathrm{Im}_{el}\,\phi_i\,(p,\theta) = p(4\pi)^{-1}\,\Sigma_{jk}\int d\Omega_1\,\phi_j^*(p,-\theta_2)\,\phi_k(p,\theta_1)\,A_{ijk}(\phi_1,\phi_2,\phi_3)\,. \tag{7.7}$$

The 5 by 5 by 5 matrix A_{ijk} includes the phase factor (p. f.) and also the (numerical) factors that arise when going from the labeling of the amplitudes by the helicities to the labeling by the index i. Explicitly, we have

$$A_{1jk} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\cos\phi_3 \end{bmatrix} ; \tag{7.8a}$$

$$A_{2jk} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\cos\phi_3 \end{bmatrix} ; \tag{7.8b}$$

$$A_{3jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\phi_3} & 0 & 0 \\ 0 & 0 & 0 & e^{-i\phi_3} & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \exp i\,(\phi_2 + \phi_1) ; \tag{7.8c}$$

$$A_{4jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi_3} & 0 \\ 0 & 0 & e^{-i\phi_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \exp i\,(\phi_2 - \phi_1) ; \tag{7.8d}$$

$$A_{5jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{i\phi_3} & e^{-i\phi_3} & 0 \end{bmatrix} \exp i\,\phi_1 \,. \tag{7.8e}$$

In obtaining A_{5jk} we used the definition

$$\phi_5 \equiv \left(\tfrac{1}{2}, \tfrac{1}{2} \,|\, \phi \,|\, \tfrac{1}{2}, -\tfrac{1}{2} \right) . \tag{7.9}$$

Had we used instead

$$\phi_5 = \left(-\tfrac{1}{2}, \tfrac{1}{2} \,|\, \phi \,|\, \tfrac{1}{2}, \tfrac{1}{2} \right) , \tag{7.10}$$

we would have obtained

$$A_{5'jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\phi_3} \\ 0 & 0 & 0 & 0 & -e^{i\phi_3} \\ -1 & -1 & 0 & 0 & 0 \end{bmatrix} \exp\left[-i\phi_2\right] . \tag{7.8e'}$$

The equality of (7.9) with (7.10) is a consequence of time-reversal invariance, without which there would be six, and not five, independent amplitudes. We shall return to this point later.

In order to perform the angular integrations in (7.7), it is necessary explicitly to display the dependence of the $\phi_i\,(p, \theta)$ on the angle θ. This we do by relating the ϕ_i to a set of Mandelstam amplitudes. In II-C we have defined two such sets of amplitudes, viz., the G_i and χ_i, and either could now be used. Although it is for the G_i that the "primitive" Mandelstam relations are written down, it turns out to be simpler to use the χ_i. Accordingly, we write

$$\mathrm{Im}_{el}\, \phi_i(p, \theta) = p(4\pi)^{-1} \Sigma_{jk} \; d\Omega_1\, \chi_j^{\,*}(p^2, z_2) \chi_k(p^2, z_1)\, B_{ijk} \,, \tag{7.11}$$

where

$$B_{1jk} = \begin{bmatrix} (Ep^2)^{-2} & 0 & 0 & 0 & 0 \\ 0 & (Ep^2)^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2m^{-6}y_1y_2 \cos\phi_3 \end{bmatrix}, \quad (7.12a)$$

$$B_{2jk} = \begin{bmatrix} 0 & (Ep^2)^{-2} & 0 & 0 & 0 \\ (Ep^2)^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2m^{-6}y_1y_2 \cos\phi_3 \end{bmatrix}, \quad (7.12b)$$

$$B_{3jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (Em^2)^{-2}(1+z_1)(1+z_2)e^{i\phi_3} & 0 & 0 \\ 0 & 0 & 0 & (Em^2)^{-2}(1-z_1)(1-z_2)e^{-i\phi_3} & 0 \\ 0 & 0 & 0 & 0 & -2m^{-6}y_1y_2 \end{bmatrix} e^{i(\phi_2+\phi_1)}, \quad (7.12c)$$

$$B_{4jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (Em^2)^{-2}(1-z_1)(1+z_2)e^{i\phi_3} & 0 \\ 0 & 0 & (Em^2)^{-2}(1+z_1)(1-z_2)e^{-i\phi_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2m^{-6}y_1y_2 \end{bmatrix} e^{i(\phi_2-\phi_1)}, \quad (7.12d)$$

$$B_{5jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & (Ep^2m^3)^{-1}y_1 \\ 0 & 0 & 0 & 0 & (Ep^2m^3)^{-1}y_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (Em^5)^{-1}y_2(1+z_1)e^{i\phi_3} & -(Em^5)^{-1}y_2(1-z_1)e^{-i\phi_3} & 0 \end{bmatrix} e^{i\phi_1}, \tag{7.12e}$$

$$B'_{5jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (Em^5)^{-1}y_1(1+z_2)e^{-i\phi_3} \\ 0 & 0 & 0 & 0 & -(Em^5)^{-1}y_1(1-z_2)e^{i\phi_3} \\ (Ep^2m^3)^{-1}y_2 & (Ep^2m^3)^{-1}y_2 & 0 & 0 & 0 \end{bmatrix} e^{-i\phi_2}. \tag{7.12e'}$$

Here and after, we will use $z_i = \cos\theta_i$, and $y_i = \sin\theta_i$.

From (4.43) and (4.32) it follows that we may write

$$G_i^I(s,t,u) = \frac{z}{\pi}\int_{\lambda_1}^{\infty} d\lambda\, \frac{1}{\lambda}\left[\frac{1}{\lambda-z} + (-1)^I \frac{1}{\lambda+z}\right] A_i^I(p^2,\lambda)$$

$$+[1-(-1)^I]\, h_i^I(p^2), \quad \text{for } i = 1,5; \tag{7.13a}$$

and

$$G_i^I(s,t,u) = \frac{1}{\pi}\int_{\lambda_1}^{\infty} d\lambda \left[\frac{1}{\lambda-z} + (-1)^{I+i} \frac{1}{\lambda+z}\right] A_i^I(p^2,\lambda),$$

$$\text{for } i = 2,3,4; \tag{7.31b}$$

where

$$A_i^I(p^2,\lambda) = \frac{1}{\pi}\int_0^{\infty} \frac{dp'^2}{p'^2-p^2}\, \rho_{ix}^I\left(4p'^2 + 4m^2,\ -2p^2(1-\lambda)\right)$$

$$+ \frac{1}{\pi}\int \frac{dz'}{z'-\lambda}\, \frac{1}{2}\left[\rho_i^I\left(-2p^2(1+z'),\ -2p^2(1-\lambda)\right)\right.$$

$$\left. + (-1)^{i+I}\rho_i^I\left(-2p^2(1-\lambda),\ -2p^2(1+z')\right)\right] + \Delta\gamma^{iI}\left(-2p^2(1-\lambda)\right). \tag{7.14}$$

The terms $h^1_{1,5}(p^2) = G^1_{1,5}(p^2, z = 0)$ contain the s-channel single spectral terms involving $\Delta\gamma_s^{11}$ and $\Delta\gamma_s^{51}$, and would have been present even if the subtraction in (7.13a) had not been made.

The lower limit of the integrations,

$$\lambda_1 = 1 + \mu^2/2p^2 , \tag{7.15}$$

is again a formal one: apart from the δ functions in A_i^I corresponding to the one-pion pole in the amplitudes, the lower limit could be given by

$$\lambda'_1 = 1 + 2\,\mu^2/p^2 .$$

Now we have

$$\chi = (\chi G)\, G \tag{7.16}$$

where

$$(\chi\, G) = \frac{1}{2}\begin{bmatrix} E^2p^2 & m^2p^2z & m^2p^2 & m^2p^2z & -p^4 \\ -E^2p^2 & (E^2+p^2)p^2z & -m^2p^2 & m^2p^2z & -p^4 \\ 0 & m^4 & -m^2p^2 & m^2E^2 & 0 \\ 0 & m^4 & m^2p^2 & m^2E^2 & 0 \\ 0 & -m^4 & 0 & -m^4 & 0 \end{bmatrix}. \tag{7.17}$$

Therefore we obtain

$$\chi_{1,2}{}^I(p^2, z) = \frac{z}{\pi}\int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda}\left[\frac{1}{\lambda - z} + (-1)^I \frac{1}{\lambda + z}\right] f_{1,2}{}^I(p^2, \lambda) + g_{1,2}{}^I(p^2),$$

$$\chi_3(p^2, z) = \frac{1}{\pi}\int_{\lambda_1}^{\infty} d\lambda \left[\frac{1}{\lambda - z}\, f_{(+)}{}^I(p^2, \lambda) + (-1)^I \frac{1}{\lambda + z}\, f_{(-)}{}^I(p^2, \lambda)\right],$$

$$\chi_4(p^2, z) = \frac{1}{\pi}\int_{\lambda_1}^{\infty} d\lambda \left[\frac{1}{\lambda - z}\, f_{(-)}{}^I(p^2, \lambda) + (-1)^I \frac{1}{\lambda + z}\, f_{(+)}{}^I(p^2, \lambda)\right],$$

$$\chi_5(p^2, z) = \frac{1}{\pi}\int_{\lambda_1}^{\infty} d\lambda \left[\frac{1}{\lambda - z} + (-1)^I \frac{1}{\lambda + z}\right] f_5{}^I(p^2, \lambda); \tag{7.18}$$

where

$$f_1(p^2, \lambda) = \frac{1}{2} p^2 [E^2 A_1 (p^2, \lambda) + m^2 \lambda A_2(p^2, \lambda) + m^2 A_3(p^2, \lambda) + m^2 \lambda A_4(p^2, \lambda) - p^2 A_5(p^2, \lambda)] ,$$

$$f_2(p^2, \lambda) = \frac{1}{2} p^2 [-E^2 A_1(p^2, \lambda) + (E^2+p^2)\lambda A_2(p^2, \lambda) - m^2 A_3(p^2, \lambda) + m^2 \lambda A_4(p^2, \lambda) - p^2 A_5(p^2, \lambda)] ,$$

$$f_{(\pm)}(p^2, \lambda) = \frac{1}{2} m^2 [m^2 A_2(p^2, \lambda) \mp p^2 A_3(p^2, \lambda) + E^2 A_4(p^2, \lambda)] ,$$

$$f_5(p^2, \lambda) = - \frac{1}{2} m^4 [A_2(p^2, \lambda) + A_4(p^2, \lambda)] ; \tag{7.19}$$

and

$$g_1{}^I(p^2) = \frac{1}{2} [1 - (-1)^I] p^2 [E^2 h_1{}^I(p^2) - p^2 h_5{}^I(p^2)] ;$$

$$g_2{}^I(p^2) = \frac{1}{2} [1 - (-1)^I] p^2 [-E^2 h_1{}^I(p^2) - p^2 h_5{}^I(p^2)] . \tag{7.20}$$

We now substitute (7.18) into (7.11) and perform the angular integrations. The details are given in Appendix B. We give the results here:[15]

$$\mathrm{Im}_{el}\, \phi_1{}^I(p,\theta) = \frac{p}{4\pi} \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \frac{z}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \frac{1}{z'}$$

$$\times \left[\frac{1}{z' - z} + (-1)^I \frac{1}{z' - z} \right] \frac{1}{E^2 p^4} [f_1^*(\lambda) f_1(\mu) + f_2^*(\lambda) f_2(\mu)]$$

$$- \frac{2}{m^6} f_5^*(\lambda) f_5(\mu) (\lambda\mu - z')$$

$$+ \frac{p}{4\pi} [1 - (-1)^I] \left\{ \frac{1}{E^2 p^4} [2\pi (|g_1|^2 + |g_2|^2) \right.$$

$$+ \frac{1}{\pi} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} 2\mathrm{Re} (f_1^*(\lambda) g_1 + f_2^*(\lambda) g_2) (\lambda L(\lambda) - 4\pi)]$$

$$+ \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \int_{\lambda_1}^{\infty} \frac{d\mu}{\mu} [4\pi - \lambda L(\lambda) - \mu L(\mu) + \lambda\mu J(\lambda, \mu, 0)]$$

(7.21a cont.)

$$\times \left[\frac{1}{E^2 p^4} \left(f_1^*(\lambda)\, f_1(\mu) + f_2^*(\lambda)\, f_2(\mu) \right) - \frac{2}{m^6} \lambda\mu\, f_5^*(\lambda)\, f_5(\mu) \right] \Bigg\} \tag{7.21a}$$

$$\mathrm{Im}_{el}\, \phi_2^{\,I}(p,\theta) = \frac{p}{4\pi} \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\, \frac{z}{\pi} \int \frac{4\pi^2 dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}} \frac{1}{z'}$$

$$\times \left[\frac{1}{z'-z} + (-1)^I \frac{1}{z'+z} \right] \Bigg\{ \frac{1}{E^2 p^4} \left[f_1^*(\lambda) f_2(\mu) + f_2^*(\lambda) f_1(\mu) \right]$$

$$- \frac{2}{m^6} f_5^*(\lambda)\, f_5(\mu)\, (\lambda\mu - z') \Bigg\}$$

$$+ \frac{p}{4\pi} \left[1 - (-1)^I \right] \Bigg\{ \frac{1}{E^2 p^4} \left[2\pi\, 2\, \mathrm{Re}(g_1^* g_2) \right.$$

$$+ \frac{1}{\pi} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda}\, 2\, \mathrm{Re} \left(f_1^*(\lambda) g_2 + f_2^*(\lambda)\, g_1 \right) \left(\lambda\, L(\lambda) - 4\pi \right) \Bigg]$$

$$+ \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \int_{\lambda_1}^{\infty} \frac{d\mu}{\mu} \left[4\pi - \lambda\, L(\lambda) - \mu\, L(\mu) + \lambda\mu\, J(\lambda,\mu,0) \right]$$

$$\times \Bigg[\frac{1}{E^2 p^4} \left(f_1^*(\lambda) f_2(\mu) + f_2^*(\lambda) f_1(\mu) \right)$$

$$- \frac{2}{m^6} \lambda\mu\, f_5^*(\lambda) f_5(\mu) \Bigg] \Bigg\}, \tag{7.21b}$$

$$\frac{1}{1+z}\ \mathrm{Im}_{el}\ \phi_3(p,\theta) + \frac{1}{1-z}\ \mathrm{Im}_{el}\ \phi_4\ (p,\theta)$$

$$= \frac{p}{4\pi}\ \frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\ \frac{1}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}\left[\frac{1}{z'-z} + (-1)^I \frac{1}{z'+z}\right]$$

$$\times\left\{\frac{1}{E^2 m^4}\left[f^*_{(+)}(\lambda) f_{(+)}(\mu) + f^*_{(-)}(\lambda)\ f_{(-)}(\mu)\right]\left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1+z')^2} + \frac{\lambda\mu + z'}{1+z'}\right]\right.$$

$$+ \frac{1}{E^2 m^4}\left[f^*_{(+)}(\lambda)\ f_{(+)}(\mu) - f^*_{(-)}(\lambda)\ f_{(-)}(\mu)\right]\frac{\lambda+\mu}{1+z'},$$

$$+ \frac{1}{E^2 m^4}\left[f^*_{(+)}(\lambda)\ f_{(-)}(\mu) + f^*_{(-)}(\lambda)\ f_{(+)}(\mu)\right]\left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1-z')^2} - \frac{\lambda\mu+z'}{1-z'}\right]$$

$$+ \frac{1}{E^2 m^4}\left[f^*_{(+)}(\lambda)\ f_{(-)}(\mu) - f^*_{(-)}(\lambda) f_{(+)}(\mu)\right]\frac{\lambda-\mu}{1-z'}$$

$$\left. - \frac{4}{m^6}\ f^*_5(\lambda)\ f_5(\mu)\left[\frac{(\lambda^2+\mu^2-2\lambda\mu z')\ (1+z'^2)}{(1-z'^2)^2} - \frac{1+\lambda\mu z'}{1-z'^2}\right]\right\},$$

(7.21c)

$$\frac{1}{1+z}\ \mathrm{Im}_{el}\ \phi_3(p,\theta) - \frac{1}{1-z}\ \mathrm{Im}_{el}\ \phi_4(p,\theta)$$

$$= \frac{p}{4\pi}\ \frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\ \frac{1}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}\left[\frac{1}{z'-z} + (-1)^{I+1}\frac{1}{z'+z}\right]$$

$$\times\ \frac{1}{E^2 m^4}\left[f^*_{(+)}(\lambda) f_{(+)}(\mu) + f^*_{(-)}(\lambda)\ f_{(-)}(\mu)\right]\left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1+z')^2} + \frac{\lambda\mu + z'}{1+z'}\right]$$

$$+ \frac{1}{E^2 m^4}\left[f^*_{(+)}(\lambda) f_{(+)}(\mu) - f^*_{(-)}(\lambda) f_{(-)}(\mu)\right]\frac{\lambda+\mu}{1+z'}$$

$$- \frac{1}{E^2 m^4} \left[f^*_{(+)}(\lambda)\, f_{(-)}(\mu) + f^*_{(-)}(\lambda)\, f_{(+)}(\mu) \right] \left[\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1-z')^2} - \frac{\lambda\mu + z'}{1-z'} \right]$$

$$- \frac{1}{E^2 m^4} \left[f^*_{(+)}(\lambda)\, f_{(-)}(\mu) - f^*_{(-)}(\lambda)\, f_{(+)}(\mu) \right] \frac{\lambda-\mu}{1-z'}$$

$$+ \frac{4}{m^6} f^*_5(\lambda)\, f_5(\mu) \left[\frac{(\lambda^2+\mu^2-2\lambda\mu z')2z'}{(1-z'^2)^2} - \frac{\lambda\mu + z'}{1-z'^2} \right]. \qquad (7.21d)$$

Using B_{5jk}, we obtain

$$\frac{1}{y} \operatorname{Im}_{el} \phi_5(p,\theta) = \frac{p}{4\pi} \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \frac{1}{\pi} \int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \left[\frac{1}{z'-z} + (-1)^I \frac{1}{z'+z} \right]$$

$$\times \frac{1}{1-z'^2} \left\{ \frac{1}{E\, p^2 m^3} [f^*_1(\lambda) + f^*_2(\lambda)]\, f_5(\mu)\, (\lambda - \mu z') \right.$$

$$+ \frac{1}{E\, m^5} f^*_5(\lambda)\, f_{(+)}(\mu) [(\mu - \lambda z')(1+\mu) + (1-z'^2) - (\lambda^2+\mu^2-2\lambda\mu z')]$$

$$\left. - \frac{1}{E\, m^5} f^*_5(\lambda)\, f_{(-)}(\mu) [(\mu - \lambda z')(1-\mu) - (1-z'^2) + (\lambda^2+\mu^2-2\lambda\mu z')] \right\}. \qquad (7.21e)$$

Had we used B'_{5jk} we would have obtained on the right-hand side the formal complex conjugate of (7.21e). Now formally, the right-hand side of (7.21e) is not real, but since it is the imaginary part of a function, we know it must be real, and the equality of the two derivations of $\operatorname{Im}_{el} \phi_5$ follows. That the left-hand side of (7.21e) is indeed the imaginary part of ϕ_5 (rather

than the anti-Hermitian part of an amplitude) is a consequence of time reversal invariance, another consequence of which, as we have already remarked, is the equality of the two amplitudes $(\frac{1}{2}, \frac{1}{2} | \phi | \frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2} | \phi | \frac{1}{2}, \frac{1}{2})$ which lead to the two derivations of $\mathrm{Im}_{el}\, \phi_5$. Since in any <u>approximate</u> derivation of the amplitudes there is nothing to ensure that the correlations between the phases of the $f_i(\lambda)$ which lead to this reality condition, and which are the expression of time-reversal invariance, we will write

$$\frac{1}{y} \mathrm{Im}_{el}\, \phi_5(p,\theta) = \mathrm{Re}[\text{ right-hand side of (7.21e)}] \,, \tag{7.22}$$

and will impose the constraint condition which comes from the vanishing of the imaginary part of (7.21e), <u>viz.</u>,

$$\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \; \frac{1}{[K(\lambda,\mu,z)]^{\frac{1}{2}}} \Big\{ m^2 \,\mathrm{Im}\,[(f_1^*(\lambda) + f_2^*(\lambda))\, f_5(\mu)]\,(\lambda-\mu z)$$

$$+ p^2 \,\mathrm{Im}\,[f_5^*(\lambda)\, f_{(+)}(\mu)]\,[(\mu-\lambda z)(1+\mu) + (1-z^2) - (\lambda^2+\mu^2 - 2\lambda\mu z)]$$

$$+ p^2 \,\mathrm{Im}\,[f_5^*(\lambda)\, f_{(-)}(\mu)]\,[(\mu-\lambda z)(1-\mu) - (1-z^2) + (\lambda^2+\mu^2-2\lambda\mu z)]\Big\} = 0 \tag{7.23}$$

For simplicity in Eqs. (7.21-23) we have suppressed the dependence on p^2 of the absorptive parts of the f_i and g_i, and also dropped their index I.

These equations are still not in a useful form for calculations. The dependence of their right-hand sides as $p^2 \to 0$ would appear not to be consistent with the known threshold behavior of the amplitudes. However, as a consequence of (7.19) and (7.20) certain cancellations should be made, and

these will eliminate the apparent inconsistencies. Accordingly we substitute these expressions for the $f(\lambda)$ and the g, and using

$$G = (G\,\phi)\,\phi\,, \tag{7.24}$$

take the appropriate combinations of the $\mathrm{Im}_{el}\,\phi_i$ to obtain

$$\mathrm{Im}_{el}\,G_i^{\,I} = \frac{p}{E}\,\frac{1}{4\pi}\,\frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\,\frac{z}{\pi}\int \frac{4\pi^2\,dz'}{[\,K(\lambda,\mu,z')]^{\frac{1}{2}}}$$

$$\times\left[\frac{1}{z'-z} + (-1)^I\,\frac{1}{z'+z}\right]\frac{1}{z'}$$

$$\times\,\mathcal{K}_{ijk}(\lambda,\mu,z')\,A_j^{\,I*}(p^2,\lambda)\,A_k^{\,I}(p^2,\mu) + [\,1-(-1)^I]\,\mathrm{Im}\,h_i^{\,I}(p^2),$$

$$\text{for } i = 1,5;$$

and

$$\mathrm{Im}_{el}\,G_i^{\,I} = \frac{p}{E}\,\frac{1}{4\pi}\,\frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\,\frac{1}{\pi}\int \frac{4\pi^2 dz'}{[\,K(\lambda,\mu,z')]^{\frac{1}{2}}}$$

$$\times\left[\frac{1}{z'-z} + (-1)^{I+1}\,\frac{1}{z'+z}\right]$$

$$\times\,\mathcal{K}_{ijk}(\lambda,\mu,z')\,A_j^{\,I}(p^2,\lambda)\,A_k^{\,I}(p^2,\mu), \quad \text{for } i = 2,3,4\,. \tag{7.25}$$

We have $\mathcal{K}_{ijk}^{(\lambda,\mu,z)} =$

$$
\begin{bmatrix}
E^2 & -p^2\mu & m^2 & 0 & 0 \\
-p^2\lambda & \begin{array}{l} \frac{1}{E^2}\left[p^4\lambda\mu + m^2p^2z - m^4\left(\frac{\lambda\mu+z}{1-z^2}\right)\right. \\ \quad + \frac{1}{2}zp^4\left(\frac{\lambda^2+\mu^2-2\lambda\mu z}{1-z^2}\right) \\ \quad \left. - \frac{1}{2}m^2p^2z\left(\frac{\lambda^2+\mu^2+2}{1-z^2}\right)\right] + 2m^2z\left(\frac{\lambda^2+\mu^2-2\lambda\mu z}{(1-z^2)^2}\right) \end{array} & -m^2\left(\frac{\lambda-\mu z}{1-z^2}\right) - \frac{1}{2}p^2z\left(\frac{\mu-\lambda z}{1-z^2}\right) & \frac{1}{2}p^2z & -\frac{1}{2}p^2z\left(\frac{\mu-\lambda z}{1-z^2}\right) \\
m^2 & -m^2\left(\frac{\mu-\lambda z}{1-z^2}\right) - \frac{1}{2}p^2z\left(\frac{\lambda-\mu z}{1-z^2}\right) & \begin{array}{l} \frac{m^4}{E^2} + \frac{m^2p^2}{E^2}\left(\frac{\lambda^2+\mu^2-2\lambda\mu z}{1-z^2}\right) \\ \quad -p^2z\left[\frac{-2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} + \frac{\lambda\mu+z}{1-z^2}\right] \end{array} & -m^2\left(\frac{\mu-\lambda z}{1-z^2}\right) + \frac{1}{2}p^2z\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 \\
0 & \frac{1}{2}p^2z & -m^2\left(\frac{\lambda-\mu z}{1-z^2}\right) + \frac{1}{2}p^2z\left(\frac{\mu-\lambda z}{1-z^2}\right) & \begin{array}{l} m^2\left[-\frac{2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} + \frac{\lambda\mu+z}{1-z^2}\right] \\ \quad -\frac{1}{2}p^2z\left[\frac{4(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda^2+\mu^2+2}{1-z^2}\right] \end{array} & -\frac{1}{2}p^2z\left(\frac{\mu-\lambda z}{1-z^2}\right) \\
0 & -\frac{1}{2}p^2z\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 & -\frac{1}{2}p^2z\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0
\end{bmatrix}
\quad ; \quad (7.26a)
$$

$$\mathcal{K}_{2jk}^{(\lambda,\mu,z)} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & m^2\left[\frac{2(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda^2+\mu^2+z^2}{1-z^2}\right] + \frac{1}{2}E^2\left(\frac{\lambda^2+\mu^2}{1-z^2}\right) - p^2\frac{\lambda\mu z}{1-z^2} & \frac{1}{2}(m^2-p^2)\left(\frac{\mu-\lambda z}{1-z^2}\right) & \frac{1}{2}E^2 & -\frac{1}{2}E^2\left(\frac{\mu-\lambda z}{1-z^2}\right) \\ 0 & \frac{1}{2}(m^2-p^2)\left(\frac{\lambda-\mu z}{1-z^2}\right) & p^2\left[\frac{2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda\mu+z}{1-z^2}\right] & \frac{1}{2}E^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 \\ 0 & \frac{1}{2}E^2 & \frac{1}{2}E^2\left(\frac{\mu-\lambda z}{1-z^2}\right) & -\frac{1}{2}E^2\left[\frac{4(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda^2+\mu^2+2}{1-z^2}\right] & -\frac{1}{2}E^2\left(\frac{\mu-\lambda z}{1-z^2}\right) \\ 0 & -\frac{1}{2}E^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 & -\frac{1}{2}E^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 \end{bmatrix} \quad ; \ (7.26b)$$

$$\mathcal{K}_{3jk}^{(\lambda,\mu,z)} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -m^2\left[\frac{2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda\mu+z}{1-z^2}\right] & m^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 & 0 \\ 0 & m^2\left(\frac{\mu-\lambda z}{1-z^2}\right) & -p^2\left[\frac{2(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} + \frac{\lambda\mu z-\lambda^2-\mu^2-z^2}{1-z^2}\right] & E^2\left(\frac{\mu-\lambda z}{1-z^2}\right) & 0 \\ 0 & 0 & E^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & E^2\left[\frac{2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda\mu+z}{1-z^2}\right] & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \ (7.26c)$$

$$\mathcal{K}_{4jk}^{(\lambda,\mu,z)} =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}m^2\left[-4\frac{(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2}+\frac{\lambda^2+\mu^2+2}{1-z^2}\right] & -\frac{1}{2}m^2\left(\frac{\mu-\lambda z}{1-z^2}\right) & \frac{1}{2}m^2 & \frac{1}{2}m^2\left(\frac{\mu-\lambda z}{1-z^2}\right) \\
0 & -\frac{1}{2}m^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & -p^2\left[\frac{2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2}-\frac{\lambda\mu+z}{1-z^2}\right] & -\left(E^2-\frac{1}{2}m^2\right)\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 \\
0 & \frac{1}{2}m^2 & -\left(E^2-\frac{1}{2}m^2\right)\left(\frac{\mu-\lambda z}{1-z^2}\right) & \begin{array}{c}\frac{m^2}{2}\left(\frac{\lambda^2+\mu^2}{1-z^2}\right)+p\frac{2\lambda\mu z}{1-z^2} \\ +E^2\left[\frac{2(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2}-\frac{\lambda^2+\mu^2+z^2}{1-z^2}\right]\end{array} & \frac{1}{2}m^2\left(\frac{\mu-\lambda z}{1-z^2}\right) \\
0 & \frac{1}{2}m^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 & \frac{1}{2}m^2\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0
\end{bmatrix} \quad ; \quad (7.26d)$$

$$\mathcal{K}_{5jk}^{(\lambda,\mu,z)} =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \begin{array}{c} m^2\left[\frac{2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda\mu+z}{1-z^2}\right] \\ +\frac{1}{2}p^2 z\,\frac{\lambda^2+\mu^2}{1-z^2} - p^2\,\frac{\lambda\mu}{1-z^2} \end{array} & -\frac{1}{2}p^2 z\left(\frac{\mu-\lambda z}{1-z^2}\right) & \frac{1}{2}p^2 z & \begin{array}{c} E^2\left(\frac{\lambda-\mu z}{1-z^2}\right) \\ +\frac{1}{2}p^2 z\left(\frac{\mu-\lambda z}{1-z^2}\right) \end{array} \\
0 & -\frac{1}{2}p^2 z\left(\frac{\lambda-\mu z}{1-z^2}\right) & p^2 z\left[\frac{2z(\lambda^2+\mu^2-2\lambda\mu z)}{(1-z^2)^2} - \frac{\lambda\mu+z}{1-z^2}\right] & \frac{1}{2}p^2 z\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 \\
0 & \frac{1}{2}p^2 z & \frac{1}{2}p^2 z\left(\frac{\mu-\lambda z}{1-z^2}\right) & \begin{array}{c} -E^2\left[\frac{2z(\lambda^2+\mu^2-2\lambda\mu z}{(1-z^2)^2} - \frac{\lambda\mu+z}{1-z^2}\right] \\ +\frac{1}{2}p^2 z\,\frac{(\lambda^2+\mu^2)}{1-z^2} - p^2\,\frac{\lambda\mu}{1-z^2} \end{array} & \begin{array}{c} m^2\left(\frac{\lambda-\mu z}{1-z^2}\right) \\ -\frac{1}{2}p^2 z\left(\frac{\mu-\lambda z}{1-z^2}\right) \end{array} \\
0 & E^2\left(\frac{\mu-\lambda z}{1-z^2}\right) + \frac{1}{2}p^2 z\left(\frac{\lambda-\mu z}{1-z^2}\right) & 0 & m^2\left(\frac{\mu-\lambda z}{1-z^2}\right) - \frac{1}{2}p^2 z\left(\frac{\lambda-\mu z}{1-z^2}\right) & -p^2
\end{bmatrix} \quad ; \ (7.26e)$$

To calculate $\mathrm{Im}\, h^{I}_{1,5}(p^2)$ we need to use

$$\mathrm{Im}\, G_1 \Big|_{z=0} = \frac{1}{Ep^2}\left[p^2\, \mathrm{Im}\,(\phi_1 - \phi_2) + m^2\, \mathrm{Im}\,(\phi_3 - \phi_3)\right]\Big|_{z=0}, \tag{7.27}$$

and

$$\mathrm{Im}\, G_5 \Big|_{z=0} = -\frac{1}{Ep^2} E^2\, \mathrm{Im}\,(\phi_1 + \phi_2)\Big|_{z=0}, \tag{7.28}$$

$$\begin{aligned}
\mathrm{Im}\,(\phi_1 - \phi_2)\Big|_{z=0} &= \frac{p}{4\pi}\left[1 - (-1)^I\right]\Big\{ 8\pi E^2 |h_1|^2 \\
&+ \frac{2}{\pi}\int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda}\, 2\mathrm{Re}\left[h_1^* \left(E^2 A_1(\lambda) - p^2\lambda A_2(\lambda) + m^2 A_3(\lambda)\right)\right] \\
&\qquad \times \left[\lambda L(\lambda) - 4\pi\right] \\
&+ \frac{2}{\pi^2}\int_{\lambda_1}^{\infty}\frac{d\lambda}{\lambda}\int_{\lambda_1}^{\infty}\frac{d\mu}{\mu}\left[\lambda\mu\, J(\lambda,\mu,0) - \lambda L(\lambda) - \mu L(\mu) + 4\pi\right] \\
&\times \frac{1}{E^2}\left[E^2 A_1^*(\lambda) - p^2\lambda A_2^*(\lambda) + m^2 A_3^*(\lambda)\right]\left[E^2 A_1(\mu) - p^2\mu A_2(\mu) + m^2 A_3(\mu)\right]\Big\},
\end{aligned} \tag{7.29}$$

$$\begin{aligned}
\mathrm{Im}\,(\phi_1 + \phi_2)\Big|_{z=0} &= \frac{p}{4\pi}\left[1 - (-1)^I\right]\Big\{ 8\pi \frac{p^4}{E^2} |h_5|^2 \\
&- \frac{p^2}{E^2}\,\frac{2}{\pi}\int_{\lambda_1}^{\infty}\frac{d\lambda}{\lambda}\, 2\mathrm{Re}\left[h_5^*\left(E^2\lambda A_2(\lambda) + m^2\lambda A_4(\lambda) - p^2 A_5(\lambda)\right)\right] \\
&\qquad \times \left[\lambda L(\lambda) - 4\pi\right] \\
&+ \frac{2}{\pi^2}\int_{\lambda_1}^{\infty}\frac{d\lambda}{\lambda}\int_{\lambda_1}^{\infty}\frac{d\mu}{\mu}\left[\lambda\mu\, J(\lambda,\mu,0) - \lambda L(\lambda) - \mu L(\mu) + 4\pi\right] \\
&\times \frac{1}{E^2}\left[E^2\lambda A_2^*(\lambda) + m^2\lambda A_4^*(\lambda) - p^2 A_5^*(\lambda)\right]\left[E^2\mu A_2(\mu) \right. \\
&\qquad \left. + m^2\mu A_4(\mu) - p^2 A_5(\mu)\right] \\
&- m^2\lambda\mu\left[A_2^*(\lambda) + A_4^*(\lambda)\right]\left[A_2(\mu) + A_4(\mu)\right],
\end{aligned} \tag{7.30}$$

$$\mathrm{Im}\,(\phi_3 - \phi_4)\Big|_{z=0} = \frac{p}{4\pi}\,[\,1-(-1)^I\,]\,\frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\,\frac{p^2}{E^2}$$

$$\times\{[\lambda\mu\, J(\lambda,\mu,0) - \lambda\, L(\lambda) - \mu\, L(\mu) + 4\pi]\,[-m^2 A_2^*(\mu)A_2(\mu) + E^2 A_4^*(\lambda)A_4(\mu)]$$

$$+ [\,(\lambda^2+\mu^2)\, J(\lambda,\mu,0) - \mu\, L(\lambda) - \lambda\, L(\mu)\,]\; p^2\, A_3^*(\lambda)\, A_3(\mu)$$

$$-[\,\lambda\, J(\lambda,\mu,0) - L(\mu)]\; A_3^*(\lambda)\,[\,m^2 A_2(\mu) + E^2\, A_4(\mu)\,]$$

$$-[\,\mu\, J(\lambda,\mu,0) - L(\lambda)\,]\,[\,m^2 A_2^*(\lambda) + E^2\, A_4^*(\mu)\,]\, A_3(\mu)\}\,. \qquad (7.31)$$

In deriving this last equation we have used the useful relationships

$$\frac{1}{\pi}\int \frac{4\pi^2\, dz}{[\,K(\lambda,\mu,z)\,]^{\frac{1}{2}}}\;\frac{1}{1\pm z} = \frac{1}{\lambda\pm\mu}\,[\,L(\lambda) \pm L(\mu)] \qquad (7.32)$$

and

$$\frac{1}{\pi}\int \frac{4\pi^2 dz}{[\,K(1\pm z)]^{\frac{1}{2}}}\;\frac{1}{(1\pm z)^2} = \frac{1}{(\lambda\pm\mu)^2}\left\{-4\pi \pm \frac{\lambda\mu \pm 1}{\lambda\pm\mu}\,[\,L(\lambda) \pm L(\mu)]\right\}. \qquad (7.33)$$

It now follows that

$$\mathrm{Im}\, h_1^{\,I}(p^2) = \frac{p}{E}\;\frac{1}{4\pi}\;\{\,8\pi E^2\,|\,h_1\,|^2$$

$$+ \frac{2}{\pi}\int_{\lambda_1}^{\infty}\frac{d\lambda}{\lambda}\;\; 2\,\mathrm{Re}\,[\,h_1^*\,(E^2 A_1(\lambda) - p^2\lambda\, A_2(\lambda) + m^2\, A_3(\lambda)\,)\,]$$

$$\times\,[\,\lambda\, L(\lambda) - 4\pi]$$

$$+ \frac{1}{E^2}\;\frac{2}{\pi^2}\int_{\lambda_1}^{\infty}\frac{d\lambda}{\lambda}\int_{\lambda_1}^{\infty}\frac{d\mu}{\mu}\,[\,\lambda\mu\, J(\lambda,\mu,0) - \lambda\, L(\lambda) - \mu L(\mu) + 4\pi]$$

(7.34 cont.)

$$\times [\left(E^2 A_1^*(\lambda) - p^2 \lambda A_2^*(\lambda) + m^2 A_3^*(\lambda)\right) \left(E^2 A_1(\mu) - p^2 \mu A_2(\mu) + m^2 A_3(\mu)\right)$$

$$+ m^2 E^2 A_4^*(\lambda) A_4(\mu) - m^4 A_2^*(\lambda) A_2(\mu)]$$

$$+ m^2 p^2 [(\lambda^2 + \mu^2) J(\lambda, \mu, 0) - \mu L(\lambda) - \lambda L(\mu)] A_3^*(\lambda) A_3(\mu)$$

$$- m^2 [\lambda J(\lambda, \mu, 0) - L(\mu)] A_3^*(\lambda) [m^2 A_2(\mu) + E^2 A_4(\mu)]$$

$$- m^2 [\mu J(\lambda, \mu, 0) - L(\lambda)] [m^2 A_2^*(\lambda) + E^2 A_4^*(\lambda)] A_3(\mu)\} , \qquad (7.34)$$

$$\mathrm{Im}\, h_5^I(p^2) = \frac{p}{E} \frac{1}{4\pi} \{ -8\pi p^2 | h_5 |^2$$

$$+ \frac{2}{\pi} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \; 2\,\mathrm{Re}\, [h_5^* \left(E^2 \lambda A_2(\lambda) + m^2 \lambda A_4(\lambda) - p^2 A_5(\lambda)\right)]$$

$$\times [\lambda L(\lambda) - 4\pi]$$

$$+ \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \int_{\lambda_1}^{\infty} \frac{d\mu}{\mu} [\lambda\mu J(\lambda, \mu, 0) - \lambda L(\lambda) - \mu L(\mu) + 4\pi]$$

$$\times [m^2 \lambda\mu A_4^*(\lambda) A_4(\mu) - E^2 \lambda\mu A_2^*(\lambda) A_2(\mu) - p^2 A_5^*(\lambda) A_5(\mu)$$

$$+ A_5^*(\lambda) \left(E^2 \mu A_2(\mu) + m^2 \mu A_4(\mu)\right) + \left(E^2 \lambda A_2^*(\lambda) + m^2 \lambda A_4^*(\lambda)\right) A_5(\mu)]\} . \qquad (7.35)$$

The constraint condition of Eq. (7.23) now reads

$$\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \; \frac{1}{[K(\lambda,\mu,z)]^{\frac{1}{2}}} \; (\lambda^2-\mu^2)\,[A_2^*(\lambda)\,A_2(\mu) - A_4^*(\lambda)A_4(\mu)]$$

$$+ (1-z^2)\,\mathrm{Im}\,A_2^*(\lambda)\,A_4(\mu) + (\lambda-\mu z)\mathrm{Im}\,[(A_3^*(\lambda) - A_5^*(\lambda))$$

$$\times (A_2(\mu) + A_4(\mu))\,] = 0. \tag{7.36}$$

We should mention that Eqs. (7.34) and (7.35) are essentially the unitarity equations for the ${}^1s_0$ and ${}^3p_0$ partial waves, f_0^0 and f_{11}^0 respectively. For according to GGMW,

$$f_0^0 = pE^{-1}\,\frac{1}{2}\int dz\,[E^2\,G_1 - z\,p^2\,G_2 + m^2\,G_3] \tag{7.37a}$$

and

$$f_{11}^0 = pE^{-1}\,\frac{1}{2}\int dz\,[zE^2\,G_2 + zm^2\,G_4 - p^2\,G_5\,], \tag{7.37b}$$

and these are the <u>only</u> partial waves to which terms in G_1, G_5, independent of z contribute, in particular to which the ssfs $\Delta\gamma_s^{11}$, $\Delta\gamma_s^{51}$ contribute. Thus the ambiguities introduced by the two s-channel ssfs can affect only the two $J = 0$ partial waves, in confirmation of the results of Section V. We cannot at present exclude such ambiguities; nor can similar ambiguities for $J < 1$ be excluded from any channel of either of the other two problems considered so far — π-π and π-N scattering.

VIII. EQUATIONS FOR THE SPECTRAL FUNCTIONS

In this short section we collect together for convenience the equations which, in the strip approximation, determine the spectral function.

Let us first recall the Mandelstam representation for the G amplitudes (cf. 4.43):

$$G_i^I(s,t,u) = \frac{1}{\pi^2} \int \frac{ds'}{s'-s} \int dx \left[\frac{1}{x-t} + (-1)^{i+I} \frac{1}{x-u} \right] \rho_{ix}^I(s',x)$$

$$+ \frac{1}{\pi^2} \int \frac{dt'}{t'-t} \int \frac{du'}{u'-u} \frac{1}{2} \left[\rho_i^I(t',u') + (-1)^{i+I} \rho_i^I(u',t') \right]$$

$$+ \frac{1+(-1)^{I+i}}{2\pi} \int \frac{ds'}{s'-s} \Delta\gamma_s^{iI}(s')$$

$$+ \frac{1}{\pi} \int dx \left[\frac{1}{x-t} + (-1)^{I+i} \frac{1}{x-u} \right] \Delta\gamma^{iI}(x) , \tag{8.1}$$

where $\Delta\gamma_s^{iI}(s') = 0$, for $i = 2, 3, 4$; (8.2a)

and

$$\Delta\gamma^{1I}(x) = \Delta\gamma^{5I}(x) = \Delta\gamma^{2I}(x) \tag{8.2b}$$

$$\Delta\gamma^{3I}(x) = \Delta\gamma^{4I}(x) \tag{8.2c}$$

In Eqs. (6.23) we have given expressions for the imaginary parts of the $\overline{G}$ amplitudes in the approximation in which only the Landau-Cutkowsky diagrams with two-pion intermediate states are retained (Fig. 5): if now we apply the matrix $(G\,\overline{G}) = B\Delta$ (2.36), we obtain expressions for

$\mathrm{Im}_{u>0}\, G$ to the same approximation, i.e., we have

$$\mathrm{Im}_{u>0}\, G_i^I(s,t,u) \equiv \frac{1}{\pi} \int \frac{ds'}{s'-s} (-1)^{i+I} \rho_{ix}^I(s',u)$$

(8.3 cont.)

$$+ \frac{1}{\pi} \int \frac{dt'}{t'-t} \frac{1}{2} [\rho_i^I (t', u) + (-1)^{i+I} \rho_i^I (u, t')] + (-1)^{i+I} \Delta\gamma^{iI}(u)$$

$$= \Sigma_{jJ} B_{IJ} \Delta_{ij} \; \mathrm{Im}_{2\pi} \; \overline{G}_j^J \; . \tag{8.3}$$

If we now take the imaginary part of this equation when $s > 0$, we have, referring to (6.23).

$$\rho_{ix}^I (s, u) \Big|_{2\pi} = (-1)^{i+I} \Sigma_{jJ} \Delta_{ij} \; \mathrm{Im}_{s>0} \; \mathrm{Im}_{2\pi} \overline{G}_j^J$$

$$= (-1)^{i+I} \Sigma_{jJ} B_{IJ} \Delta_{ij} \; \mathrm{Im}_{\overline{z} < -1} \; \mathrm{Im}_{2\pi} \overline{G}_j^J$$

$$= (-1)^{i+I} \Sigma_{jJ} B_{IJ} \Delta_{ij} (-1)^{j+J+1}$$

$$\times \frac{q}{\overline{E}} \frac{1}{4\pi} \left(\frac{m}{8\pi\overline{E}}\right)^2 (5 + (-1)^J) \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu$$

$$\times \frac{4\pi^2}{[K(\lambda, \mu, \overline{z})]^{\frac{1}{2}}} [k_{j11} a^* a + k_{j12} a^* b + k_{j21} b^* a$$

$$+ k_{j22} b^* b] \; . \tag{8.4}$$

Similarly, taking the imaginary part for $t > 0$, we have

$$\frac{1}{2} [\rho_i^I (t, u) + (-1)^{i+I} \rho_i^I (u, t)] \Big|_{2\pi} = \Sigma_{jJ} B_{IJ} \Delta_{ij} \frac{q}{\overline{E}} \frac{1}{4\pi} \left(\frac{m}{8\pi\overline{E}}\right)^2$$

$$\times (5 + (-1)^J) \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu \times \frac{4\pi^2}{[K(\lambda, \mu, z)]^{\frac{1}{2}}} [k_{j11} a^* a$$

$$+ k_{j12} a^* b + k_{j21} b^* a + k_{j22} b^* b] \; . \tag{8.5}$$

Finally, by putting $\bar{z} = 0$, we could obtain equations for the $\Delta\gamma^{iI}\Big|_{2\pi}$ which are given explicitly in terms of the pion-nucleon amplitudes. The one-pion contributions to the $\Delta\gamma^{iI}$, the δ-function terms, are discussed in the next section.

Let us return to the equations for the two-pion contributions to the dsfs, (8.4) and (8.5). If on their right-hand sides we exclude from the b absorptive parts the δ-function corresponding to the one-nucleon pole in pion-nucleon scattering, i.e., if we exclude the simplest box diagram (Fig. 6) from the Landau-Cutkowsky diagrams with two-pion intermediate states (Fig. 5), the remaining portions of the two-pion contributions to the dsfs are nonzero in the regions C and the inner regions of Fig. 7. In particular, the two-pion contributions are the <u>only</u> nonzero contributions in the strips parallel to $t = 0$, $u = 0$. Equations (8.4) and (8.5), then give explicitly the dsfs in these strips in terms of the pion-nucleon absorptive parts. This is in accordance with our statements in Section II-B and the Introduction. Furthermore, we have shown how the ssfs $\Delta\gamma^{iI}$ are to be derived in the strip approximation.

We now turn to the strips of the dsfs parallel to $s = 0$, and refer to (7.24) and (7.25). If we take the imaginary parts of these equations for $t > 0$ $(z > 1)$, we will, comparing with (8.1), obtain

$$\rho^{I}_{ix}(s,t)\Big|_{el} = \frac{p}{E}\,\frac{1}{4\pi}\,\frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\, \frac{4\pi^2}{[K(\lambda,\mu,z)]^{\frac{1}{2}}}$$

$$\times\, \Sigma_{jk}\, \mathcal{K}_{ijk}(\lambda,\mu,z)\, A^{I}_{j}(p^2,\lambda)^{*}\, A^{I}_{k}(p^3,\mu)\,. \tag{8.6}$$

Unlike the equations (8.4) and (8.5), the absorptive parts A^{I}_{i} on the right-hand side of (8.6) are not "given," but are to be derived from (7.14). This requires a prior knowledge of the dsfs. However, in the strip

approximation we suppose that the inner region of the dsfs is not important, even for the determination of the absorptive parts: accordingly it is enough to take for the dsfs the sum of $\rho\,|_{el}$ and $\rho\,|_{2\pi}$, taking care not to count the contribution of the simplest box diagram twice. This then allows (8.6) and (7.14) to be solved as coupled integral equations determining $\rho^{I}_{ix}\,|_{el}$.

We again observe that it is possible to separate out from $\rho^{I}_{ix}\,|_{el}$ the part coming from the simplest box diagram. This comes from the δ functions in $\Delta\gamma^{iI}$ corresponding to one-pion exchange in the absorptive parts. The remainder of $\rho^{I}_{ix}\,|_{el}$ is nonzero in the regions B (and the inner regions) of Fig. 7, and in particular (again apart from the simplest box diagram contribution) is the <u>only</u> nonzero part of the dsfs in the strips parallel to $s = 0$.

Thus we have shown how the unitarity equations lead to a determination, in the strip approximation, of those parts of the dsfs which, we hope, dominate the low-energy or low momentum-transfer parts of the physical scattering amplitudes.

The ssfs $\Delta\gamma^{iI}_{s}$ in the s channel are determined by application of (7.34) and (7.35). Similar equations could also be used to derive any other ssfs which arose from subtraction, subject to the caveat about undetermined CDD poles mentioned in the Introduction.

IX. THE YUKAWA POLES AND THE SIMPLEST BOX DIAGRAM

We have observed that one-particle intermediate states in Landau-Cutkowsky diagrams give rise to poles in the amplitudes, and in particular the one-pion interemediate states in the nucleon-antinucleon channels give rise to poles at $u = \mu^2$, $t = \mu^2$. These are given in GGMW and are[16]

$$G_i^I \Big|_{I\pi} = - [(\mu^2 - t)^{-1} + (-1)^{i+I} (\mu^2 - u)^{-1}] \, g^2 \, \eta_i \, (-3\delta_{I,1} + \delta_{I,0}), \tag{9.1}$$

where

$$\eta_i = (1, 1, -1, -1, 1) . \tag{9.2}$$

Let us now consider the simplest box diagram. This is of interest for reasons discussed in the Introduction and amplified in Section VIII. The equations are represented schematically by Fig. 5; the "blobs" include all states with the same quantum numbers as a nucleon plus a pion, in particular the one-nucleon state. If we wish to isolate the contribution from this state we must include in a and b, the absorptive parts of pion-nucleon scattering, only the terms arising from the one-nucleon pole. We have already seen that there is no contribution to a, and a contribution of $\pi g^2 \delta(\lambda_0 - \lambda)(2 \bar{p} q)^{-1}$ to b, the rest of b, and the whole of a, with thresholds at λ_0' given by $v_0 = (m+\mu)^2$ coming from interemediate states with more than just a nucleon in the pion-nucleon scattering. If then we take just this one-nucleon term in the "blobs" of Fig. 5; i.e., if we take the contribution associated with Fig. 6, the simplest box diagram, to Im $\bar{G}$, we obtain[14]

$$\mathrm{Im}\, \bar{G}_i^{(I)} (\bar{p}^2, \bar{z}) \Big|_{\mathrm{box}} = \mathrm{Im}\, \bar{G}^{(I)} (\bar{p}^2, 0) \Big|_{\mathrm{box}}$$

$$+ \frac{q}{\bar{E}} \frac{1}{4\pi} \left(\frac{m}{8 \pi \bar{E}} \right)^2 (5 \pm 1) \frac{2}{\pi^2} \frac{\bar{z}}{\pi} \int \frac{4 \pi^2 \, dz'}{[K(\lambda_0, \lambda_0, z')]^{\frac{1}{2}}} \left[\frac{1}{z' - \bar{z}} \mp \frac{1}{z' + \bar{z}} \right] \frac{1}{\bar{z}} \tag{9.3 cont.}$$

$$\times \frac{\pi^2 g^4}{4\bar{p}^2 q^2} \quad k_{i22}(\lambda_0, \lambda_0, z'), \qquad \text{for } i = 1, 5;$$

$$\mathrm{Im}\, \bar{G}_i^{(I)}(\bar{p}^2, \bar{z})\Big|_{\text{box}} = \frac{q}{\bar{E}} \frac{1}{4\pi} \left(\frac{m}{8\pi\bar{E}}\right)^2 (5 \pm 1) \frac{2}{\pi^2} \frac{1}{\pi}$$

$$\times \int \frac{4\pi^2 dz'}{[K(\lambda_0, \lambda_0, z')]^{\frac{1}{2}}} \left[\frac{1}{z' - \bar{z}} \mp (-1)^i \frac{1}{z' + \bar{z}} \right] \frac{\pi^2 g^4}{4\bar{p}^2 q^2} k_{i22}(\lambda_0, \lambda_0, z'),$$

$$\text{for } i = 2, 3, 4; \tag{9.3}$$

$$\mathrm{Im}\, \bar{G}_i^{(1)}(\bar{p}^2, 0)\Big|_{\text{box}} = 0, \qquad \text{for } i = 1, 5;$$

$$\mathrm{Im}\, \bar{G}_i^{(0)}(\bar{p}^2, 0)\Big|_{\text{box}} = -\frac{q}{\bar{E}} \frac{1}{4\pi} \left(\frac{m}{8\pi\bar{E}}\right)^2 \frac{\bar{E}^2 q^2}{\bar{p}^2} \frac{48}{\pi^2}$$

$$\times \frac{\pi^2 g^4}{4\bar{p}^2 q^2} \left[4\pi - 2\lambda_0 L(\lambda_0) + \lambda_0^2 J(\lambda_0, \lambda_0, 0) \right],$$

$$\text{for } i = 1, 5. \tag{9.4}$$

By taking the imaginary part of the absorptive parts given by (9.3) for $\bar{z} < -1$ it is possible to arrive at a determination of the contribution of the (s, u) double-spectral function associated with the simplest box diagram. For we have

$$\rho_{ix}^{I}(s, u)\Big|_{\text{box}} = (-1)^{i+I} \mathrm{Im}_{s>0} \mathrm{Im}_{u>0} G_i^I(s, t, u)\Big|_{\text{box}}$$

$$= (-1)^{i+I} \sum_{jI'} B_{II'} \Delta_{ij} \mathrm{Im}_{\bar{z}<-1} \mathrm{Im}_{u>0} \bar{G}_j^{I'}(s, t, u)\Big|_{\text{box}}$$

$$= (-1)^{i+I} \sum_{jI'} B_{II'} \Delta_{ij} (-1)^{j+I'+1}$$

$$\times \frac{q}{\bar{E}} \frac{1}{4\pi} \left(\frac{m}{8\pi\bar{E}}\right)^2 (5 + (-1)^{I'}) \frac{\pi^2 g^4}{4\bar{p}^2 q^2} \frac{2}{\pi^2} 4\pi^2 \frac{k_{j22}(\lambda_0, \lambda_0, -\bar{z})}{[K(\lambda_0, \lambda_0, -\bar{z})]^{\frac{1}{2}}}. \tag{9.5}$$

Now it is also possible to calculate the (s, u) double-spectral function associated with the simplest box diagram by considering nucleon-nucleon unitarity. We have now to include in the absorptive parts A that enter in Eqs. (8.6) only the contributions of the one-pion poles. These we have already stated and it follows from (9.1) that

$$A_i^I(p^2,\lambda)\Big|_{1\pi} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \eta_i \frac{g^2}{8} \frac{1}{2p^2} \delta(\lambda-\lambda_1) . \tag{9.6}$$

The other contributions to A, arising from two-pion states, etc., have their threshold at $\lambda_1' = 1 + 2\mu^2/p^2$.

If we use (9.6) to calculate the (s, u) double-spectral function associated with the simplest box diagram, we obtain, using Eq. (8.6),

$$\rho_{ix}^I(s,u)\Big|_{box} = \begin{pmatrix} 9 \\ 1 \end{pmatrix} \frac{p}{E} \frac{1}{4\pi} \frac{2}{\pi^2} \frac{4\pi^2}{[K(\lambda_1,\lambda_1,-z)]^{\frac{1}{2}}}$$

$$\times \sum_{jk} \eta_j \eta_k \frac{g^4}{64} \frac{1}{4p^4} \mathcal{K}_{ijk}(\lambda_1,\lambda_1,-z) . \tag{9.7}$$

Of course the two determinations (9.5) and (9.7) must be consistent with each other. If we notice that

$$\sum_{I'} (-1)^I B_{II'} (-1)^{I'+1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}, \tag{9.8}$$

this consistency reduces to the set of identities

$$m^2 \bar{E}^{-2} \sum_j (-1)^i \Delta_{ij} (-1)^j k_{j22}(\lambda_0,\lambda_0,-\bar{z})$$

$$= \sum_{\ell m} \eta_\ell \eta_m \mathcal{K}_{i\ell m}(\lambda_1,\lambda_1,-z) , \tag{9.9}$$

which are indeed verified.

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APPENDIX A; CROSSING

The crossing matrix $(\bar{\chi}\ \chi)$ relating the $\bar{\chi}_i^{\,I}(s,t,u)$ to the $\chi_j^{\,J}(s,t,u)$ may be obtained from the matrices $(\bar{\chi}\ \bar{F})\ (s,t,u) = (\chi\ F)\ (u,t,s)$, $(\chi\ F)$, and the crossing matrix $(\bar{F}F)$, all of which appear in GGMW. For clarity, however, the work is described in detail below, in slightly different language that leans less heavily on conventional field theory. The resulting Eqs. (A.7, A.10, A.14), coincide with that obtained from the matrices in GGMW, and thus serve as an algebraic check.

The same Feynman-diagram "black box", represented by a matrix $M(F(s,t,u))$, where $s = (p_1 + p_2)^2$, $t = (p_1 + p_1')^2$, $u = (p_1 + p_2')^2$, is attached to nucleon spinors for nucleons of definite helicity and definite z component of isotopic spin, to give nucleon-nucleon amplitudes (Fig. 2):

$$\tau_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(s,t,u) = \bar{u}_{\lambda_1'}(-p_1')[\bar{u}_{\lambda_2'}(-p_2')M(F(s,t,u))\,u_{\lambda_2}(p_2)]u_{\lambda_1}(p_1), \tag{A.1}$$

or to nucleon and antinucleon spinors to give nucleon-antinucleon amplitudes (Fig. 3):

$$\bar{\tau}_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(s,t,u) = -\,\bar{u}_{\lambda_1'}(-p_1')[\bar{v}_{\lambda_2}(p_2')M(F(s,t,u))\,v_{\lambda_2'}(-p_2)]u_{\lambda_1}(p_1). \tag{A.2}$$

The minus sign in (A.2) comes from a part of the Feynman rules not usually stated; namely, there is a minus sign for each antiparticle in the final state, except for antiparticles corresponding to totally noninteracting antiparticle lines—but the noninteracting line case contributes only to the unit matrix in $S = 1 + iT$ in the case of two-particle elastic scattering, so that this exception is not relevant here. By agreeing to take states where particle 1 is created first, particle 2 next; 1' first and 2' next in the nucleon-nucleon case, and where the nucleon is created first, the antinucleon next in both

states of the nucleon-antinucleon case, one determines that there is no relative sign due to permutations of fermions in initial and final states; and, of course, as there are equal numbers of closed loops in comparable diagrams, there is no relative sign from the minus signs for closed loops.

That the same matrix function of the $F((p_1 + p_2)^2, (p_1 + p_1')^2, (p_1 + p_2')^2)$ may be used for both cases really requires a detailed argument. To convince the skeptical reader of this, we draw attention to the fact that since the nucleon-nucleon process involves (s, t, u) in the region where $t < 0$, $u < 0$, whereas the nucleon-antinucleon process as described above involves (s, t, u) in the region where $t < 0$, $s < 0$, the use of the "same" $F(s, t, u)$ for both processes involves the concept of analytic continuation, and the knowledge that the $F(s, t, u)$ are well-behaved amplitudes; i.e., Mandelstam amplitudes. That the $F(s, t, u)$ may reasonably to expected to be Mandelstam amplitudes is argued in the first two paragraphs of Sec. III of GGMW.[17]

The matrix M, as given by an imagined sum over Feynman diagrams based on interactions with all the usually assumed symmetries of strong interactions, may indeed be reduced to the form $M(F(s, t, u))$, a superposition of numeric Dirac matrices with coefficients functions only of the Lorentz invariants formed from the 4-momenta, if all the symmetry restrictions are in fact imposed, in addition to the Dirac equations in the four external 4-momenta (see GGMW). The same matrix is involved in the nucleon-antinucleon process, and the same symmetry restrictions apply; Pauli symmetry in one case and G-conjugation symmetry in the other lead to the same conditions. However, two of the Dirac equations involve negatives of physical momenta—but the convention of using reversed momenta for outgoing particles means that the same literal p variables enter the algebra. Since the reduction of the

common original matrix to a linear combination involving five amplitudes proceeds thus by parallel algebraic operations, the $M\{F(s,t,u)\}$ are the same, with the same functions F, differing only in the numerical values of the arguments s, t, u. That the functions are the same in the sense of a sequence of parallel algebraic and Feynman-integral operations on their arguments means that they are related by analytic continuation in the sense of Mandelstam; that no singularity that would make this ambiguous is introduced in the operations follows from the argument given in GGMW to show that the F are Mandelstam amplitudes.

If some symmetry restrictions are dropped, in order, for example, to apply the discussion to weak interactions, then (A.1) and (A.2) must be modified only to the extent that the common matrix M involve formal inner products of Dirac matrices with external 4-momenta; this follows from the simple observation that the number of independent scalar coefficients would then exceed five. The manipulation indicated in Eq. (A.6) below would still be the central step in the explicit expression of crossing, but the use of a different expression for M would lead to algebraic details distinct from those which follow from (A.3): A similar remark applies if all the present symmetries are assumed, but if M is by choice resolved into five amplitudes in a way that does not eliminate all inner products of Dirac matrices with external 4-momenta, as is done, e.g., in Ref. (5).

Now we have

$$\phi_{\lambda_1'\lambda_2'\lambda_1\lambda_2} = (4\pi E)^{-1}\, m^2\, \tau_{\lambda_1'\lambda_2'\lambda_1\lambda_2'} ,$$

and the ϕ_i are particular $\phi_{\lambda_1'\lambda_2'\lambda_1\lambda_2}$; similarly, $\bar{\phi} = (4\pi\bar{E})^{-1}\, m^2\, \bar{\tau}$, and the χ, $\bar{\chi}$ are obtained by simple diagonal matrices which remove the singularities from ϕ, $\bar{\phi}$, as has been discussed in Section II-C. Thus, $\bar{\chi} = (\bar{\chi}\bar{\tau})\,\bar{\tau}$,

$\chi = (\chi\ \tau)\ \tau$, where $(\bar{\chi}\ \bar{\tau})\ (s,t,u) = (\chi\ \tau)\ (u,t,s)$. The χ and $\bar{\chi}$ of this Appendix bear subscripts 1, 2, 3, 4, 5; the subscripts + and -, occasionally used elsewhere, do not apply here.

It is convenient to define new Mandelstam amplitudes X_i^I, such that

$$M(F) = \Sigma_{i,I}\ X_i^I\ (s,t,u)\,\Gamma_i^{(1)}\ \Gamma_i^{(2)}\ \mathfrak{B}^I\ , \tag{A.3}$$

and to develop $(\bar{\chi}\ \chi)$ from

$$\bar{\chi} = (\bar{\chi}\ \bar{\tau})\ (\bar{\tau}\ X)\ (X\ \tau)\ (\tau\ \chi)\chi\ . \tag{A.4}$$

By utilizing the definition

$$v_{\lambda a}(p) = -\ iC\tau_2\ \bar{u}_{\lambda a}(p)^T,\ \ C = i\ \gamma_2\gamma_4\ , \tag{A.5}$$

for the charge-conjugate or antiparticle spinor (in the sense of G parity), it is easy to show by transposing the entire (1 by 1) matrix that

$$\bar{v}_{\lambda_2}(p_2')\ M\ v_{\lambda_2'}\ (p_2) = -\ \bar{u}_{\lambda_2'}\ (p_2)\ \tau_2 C M^T C \tau_2 u_{\lambda_2}(p_2')\ . \tag{A.6}$$

By utilizing $C\Gamma_j^T C = (-)^{j-1}\ \Gamma_j$ (where $j = 1, 2, 3, 4, 5$ corresponds to S, T, A, V, P, respectively), and $\tau_2 \mathfrak{B}^J\ \tau_2 = \Sigma_I\ B^{IJ}\mathfrak{B}^I$, with

$$B^{IJ} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}, \tag{A.7}$$

one finds that

$$\bar{\mathcal{F}}^I_{\lambda_1'\lambda_2'\lambda_1\lambda_2} = -\ B^{IJ} u_{\lambda_1'}\ (-p_1')u_{\lambda_2'}(-p_2)\Sigma(-)^j X_j^J \Gamma_j^{(1)}\Gamma_j^{(2)}\mathfrak{B}^I$$

$$\times u_{\lambda_2}(p_2')u_{\lambda_1}(p_1')\ , \tag{A.8}$$

where the use of a linear combination of spinors to yield given total isotopic spin I is understood but not written.

By comparing (A.8) with (A.1, A.3) one sees that since

$$\tau_i^{\ I} = (\tau\, X)_{ij}\ X_j^{\ I},$$

then

$$\bar{\tau}_i^{\ I} = -\ B^{IJ}\ (\bar{\tau}\ \bar{X})_{ij}\ (-)^j\ \chi_j^{\ J}\ , \tag{A.9}$$

where $(\bar{\tau}\ \bar{X})_{ij}(s,t,u) \equiv (\tau\ X)_{ij}(u,t,s)$, inasmuch as this matrix factor arises from exactly the same spinors, save that the arguments p_2 and p_2' are interchanged.

Hence,

$$\bar{\chi} = -\ B(\bar{\chi}\ \bar{\tau})\ (\bar{\tau}\ \bar{X})\left[(-)^j\right](X\ \tau)\ (\tau\ \chi)\chi \equiv BZ^{-1}\chi\ . \tag{A.10}$$

The matrix $(\tau\ \chi)$ can be worked out from Eq. (A.1) and (A.3) by multiplying the spinors, but GGMW have already worked out (τF), so one need only find $(F\ X)$ and use $(\tau\ X) = (\tau\ F)\ (F\ X)$:

$$\Sigma_j\ F_j\ (\Gamma_j^{(1)}\Gamma_j^{(2)} + (-)^j\tilde{\Gamma}_j^{(1)}\tilde{\Gamma}_j^{(2)}) = \Sigma_k\ X_k\ \Gamma_k^{(1)}\Gamma_k^{(2)} \tag{A.11}$$

readily gives $(F\ X)$ in terms of the S, T, A, V, P-adapted Fierz matrix $\mathfrak{F}$:

$$\Sigma_{j,k}\ F_j\ (\delta_{ij} + (-)^j\ \mathfrak{F}_{jk})\Gamma_k^{(1)}\ \Gamma_k^{(2)} = \Sigma_k\ X_k\Gamma_k^{(1)}\ \Gamma_k^{(2)}$$

yields

$$X_k = \Sigma_j\ F_j\ (\delta_{jk} + (-)^j\ \mathfrak{F}_{jk}) = (X\ F)_{kj}F_j\ . \tag{A.12}$$

Then, by matrix inversion,

$$(F\ X) = \tfrac{1}{2}\begin{bmatrix} 1 & -3 & 0 & -2 & 0 \\ \tfrac{1}{2} & 1 & 0 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & -1 & 0 \\ \tfrac{1}{2} & 0 & 1 & 1 & -\tfrac{1}{2} \\ 0 & -3 & 0 & 2 & 1 \end{bmatrix} \tag{A.13}$$

Note that

$$(Z(s,t,u))^{-1} = Z(u,t,s) \,, \tag{A.14a}$$

a convenient check on the matrix multiplications. The result for B has already been given in Eq. (A.7); the result for Z^{-1} is given as (A.14b) below, and Z appears as Eq. (2.35) in the text:

$$Z^{-1} = \begin{bmatrix} -\frac{2m^2 t}{(t+u)^2} & -\frac{2m^2 t}{(t+u)^2} & -\frac{u}{4m^2}\left[\frac{s+t}{t+u} + \frac{su}{(s+t)^2}\right] & -\frac{t^2}{(t+u)^2} & \frac{stu}{m^2(t+u)^2} \\ -\frac{2m^2 t}{(t+u)^2} & \frac{s+t}{2(t+u)} + \frac{su}{2(s+t)^2} & \frac{tu}{(t+u)^2} & -\frac{t^2}{(t+u)^2} & \frac{stu}{m^2(t+u)^2} \\ -\frac{m^2(s+t)}{s(t+u)} - \frac{m^2 u}{(t+u)^2} & \frac{4m^4 t}{s(t+u)^2} & -\frac{2m^2 tu}{s(t+u)^2} & \frac{2m^2 t^2}{s(t+u)^2} & -\frac{2tu}{(t+u)^2} \\ -\frac{4m^4}{(t+u)^2} & -\frac{4m^4}{(t+u)^2} & \frac{2m^2 u}{(t+u)^2} & \frac{s+t}{t+u} - \frac{2m^2 t}{(t+u)^2} & \frac{2su}{(t+u)^2} \\ \frac{4m^4}{(t+u)^2} & \frac{4m^4}{(t+u)^2} & -\frac{2m^2 u}{(t+u)^2} & \frac{2m^2 t}{(t+u)^2} & \frac{4m^2 t - su}{(t+u)^2} \end{bmatrix}. \tag{A.14b}$$

The equations $\chi_i^I = (\chi\ G)_{ij}\ G_j^I$ require that the χ be Mandelstam amplitudes with zeros imposed by the matrix $(\chi\ G)$, which is not simply a numeric matrix. These conditions are applied in Section IV-C. Since the χ amplitudes are Mandelstam amplitudes with a physical meaning more transparent than that of the G, inasmuch as they are very simply related to the helicity amplitudes, it would perhaps be annoying to have no alternative derivation of the information about zeros. But in fact, the crossing matrix gives this information directly, provided that one indeed has prior information confirming the regularity of the χ and of the $\overline{\chi}$.

$$(\bar{\tau}\,\bar{X})^{-1}\,(\bar{\chi}\,\bar{\tau})^{-1}\,\bar{\chi} = -\,B\left[(-)^{j}\right](\tau\,X)^{-1}\,(\chi\,\tau)^{-1}\,\chi \tag{A.15}$$

The right-hand side of (A.15) bears the poles of $(\tau\,X)^{-1}\,(\chi\,\tau)^{-1}$; the left-hand side bears the poles of $(\bar{\tau}\,\bar{X})^{-1}\,(\bar{\chi}\,\bar{\tau})^{-1}$. No pole can be cancelled from the equation, inasmuch as the poles on the right-hand side appear at s and $s-4M^2$, whereas those on the left-hand side appear at u and $u-4M^2$. The regularity of χ therefore imposes u and $u-4M^2$ zeros on appropriate linear combinations of the $\bar{\chi}$, whereas the regularity of the $\bar{\chi}$ imposes s and $s-4M^2$ zeros on appropriate linear combinations of the χ. Since $\chi = (\chi\,\tau)\,(\tau\,X)X$, and since X and G are related by a numeric matrix, these imposed zeros are equivalent to those imposed by regularity of $G = (\chi\,G)^{-1}\chi$.

That the χ and $\bar{\chi}$ are Mandelstam amplitudes in fact is, of course, a result that follows from prior knowledge that the X or F or G are Mandelstam amplitudes, which in turn follows from arguments outlined in GGMW, Sec. III. That the $\bar{\chi}$ are Mandelstam amplitudes follows immediately from $\bar{\chi} = -\,B\,(\bar{\chi}\,\bar{\tau})\,(\bar{\tau}\,\bar{X})\left[(-)^{j}\right]X$, inasmuch as $(\chi\,\tau)$ was originaly devised so that $\chi = (\chi\,\tau)\,(\tau\,X)X$ would be regular; i.e., $(\chi\,\tau)\,(\tau\,X)\,(s,t,u)$ has all elements regular, and therefore $(\bar{\chi}\,\bar{\tau})\,(\bar{\tau}\,\bar{X})\,(s,t,u) = (\chi\,\tau)\,(\tau\,X)\,(u,t,s)$ also has all elements regular.

It has been remarked in the text that if $\bar{F}$ amplitudes are defined by $\bar{F} = (\bar{F}\,\bar{\chi})\bar{\chi}$, where $(\bar{F}\,\bar{\chi})\,(s,t,u) \equiv (F\,\chi)\,(u,t,s)$, then $(\bar{F}\,F)$, the crossing matrix of GGMW, is a numeric matrix, and the related matrix $(G\,\bar{G})$ is explicitly quoted. $(\bar{F}\,F)$ may be computed directly from the above as $(\bar{F}\,\bar{\chi})\,(\bar{\chi}\,\chi)\,(\chi\,F)$, but as GGMW show, it is not in fact necessary to go to the trouble of handling the messy cancellations of functions of s, t, u that this would involve.

This may be seen directly from Eq. (A.10). If we define new amplitudes $\overline{X}$ by $(\overline{X}\ X) = -B\left[(-)^j\right]$, then $(\overline{F}\ F) = (\overline{F}\ \overline{X})\ (\overline{X}\ X)\ (X\ F)$, where $(\overline{X}\ X)$ is an extremely simple "crossing matrix", $(X\ F)$ is a numeric matrix (See Eq. (A.12)), and the point is made if $(\overline{F}\ \overline{X})\ (s,t,u) = (F\ X)\ (u,t,s)$ can be established, inasmuch as $(F\ X)$ is, of course, numeric. Now, $(\overline{F}\ \overline{X}) = (\overline{F}\ \overline{\chi})\ (\overline{\chi}\ X)\ (\overline{X}\ X)^{-1}$, in terms of previously examined transformations; or $(\overline{F}\ \overline{X}) = (\overline{F}\ \overline{\chi})\ (\overline{\chi}\ \overline{\tau})\ (\overline{\tau}\ \overline{X})\ (\overline{X}\ X)\ (\overline{X}\ X)^{-1} = (\overline{F}\ \overline{\chi})\ (\overline{\chi}\ \overline{\tau})\ (\overline{\tau}\ \overline{X})$. We have $(\overline{F}\ \overline{\chi})\ (s,t,u) = (F\ \chi)\ (u,t,s)$ by definition of $\overline{F}$; $(\overline{\chi}\ \overline{\tau})\ (s,t,u) = (\chi\ \tau)\ (u,t,s)$ by definition of $\overline{\chi}$; and $(\overline{\tau}\ \overline{X})\ (s,t,u) = (\tau\ X)\ (u,t,s)$ by definition of the symbol $(\overline{\tau}\ \overline{X})$, whence $(\overline{F}\ \overline{X})\ (s,t,u) = (F\ \chi)\ (\chi\ \tau)\ (\tau\ X)\ (u,t,s) = (F\ X)\ (u,t,s) = (F\ X)$ is indeed numeric.

APPENDIX B. DERIVATION OF THE UNITARITY EQUATIONS

For nucleon-nucleon scattering, where

$$S = 1 + i(2\pi)^4\ \delta^{(4)}\ (p_2 + p_2' + p_1 + p_1')\ \frac{m^2}{E^2}\ \frac{4\pi E}{m^2}\ \underline{\overline{\Phi}} \tag{B.1}$$

the unitarity condition,

$$S^\dagger\ S = 1\ , \tag{B.2}$$

leads to

$$\begin{aligned}
&\frac{1}{2i}\ [\ \langle \lambda_1',\ \lambda_2';\ -p_1',\ -p_2'\ |\ \underline{\overline{\Phi}}\ |\ \lambda_1, \lambda_2';\ p_1,\ p_2 \rangle \\
&-\langle \lambda_1',\ \lambda_2';\ -p_1',\ -p_2'\ |\ \underline{\overline{\Phi}}^\dagger\ |\ \lambda_1, \lambda_2;\ p_1,\ p_2\ \rangle\ ] \\
&= (2\pi)^4\ \frac{m^2}{E^2}\ \frac{4\pi E}{m^2}\ \frac{1}{2} \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\ \delta^{(4)}(k_1 + k_2 + p_1' + p_2') \\
&\times \Sigma_{\mu_1\mu_2} \langle \lambda_1',\ \lambda_2';\ -p_1',\ -p_2'\ |\ \underline{\overline{\Phi}}^\dagger\ |\ \mu_1, \mu_2;\ k_1,\ k_2 \rangle \\
&\times \langle \mu_1, \mu_2;\ k_1,\ k_2\ |\ \underline{\overline{\Phi}}\ |\ \lambda_1, \lambda_2;\ p_1,\ p_2 \rangle\ ,
\end{aligned} \tag{B.3}$$

where we have included only the two-nucleon states on the right-hand side. In the c.m. system $\underset{\sim}{p}_1' + \underset{\sim}{p}_2' = 0$, so that if, say, the $\underset{\sim}{k}_2$ integration is performed, the result of the spatial part of the δ function is just to set $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = 0$. If we write $\underset{\sim}{k}_1 = \underset{\sim}{p} = -\underset{\sim}{k}_2$, the right-hand side reduces to

$$(2\pi E)^{-1} \int d\Omega_p \int p^2\, dp\, \delta[2(p^2 + m^2)^{1/2} - 2E]$$

$$\times \Sigma \mu_1 \mu_2 \left\langle \lambda_1', \lambda_2'; \underset{\sim}{p}_f \,|\, \underline{\Phi}^\dagger \,|\, \mu_1, \mu_2; \underset{\sim}{p} \right\rangle$$

$$\times \left\langle \mu_1, \mu_2; \underset{\sim}{p} \,|\, \underline{\Phi} \,|\, \lambda_1, \lambda_2; \underset{\sim}{p}_i \right\rangle . \tag{B.4}$$

Performing the integration over p we obtain Eq. (7.2) .

For the nucleon-antinucleon unitarity, we have formally, if we include only two-pion intercalated states,

$$\frac{1}{2i} \left\langle N' \bar{N}' \,|\, (R - R^\dagger) \,|\, N \bar{N} \right\rangle$$

$$= \frac{1}{2} \Sigma_{\pi\pi} \left\langle N' \bar{N}' \,|\, R^\dagger \,|\, \pi\pi \right\rangle \left\langle \pi\pi \,|\, R \,|\, N \bar{N} \right\rangle , \tag{B.5}$$

which leads immediately to

$$\frac{1}{2i} [\left\langle \lambda', \bar{\lambda}'; -p_1', -p_2 \,|\, \underline{\bar{\Phi}} \,|\, \lambda, \bar{\lambda}; p_1, p_2' \right\rangle$$

$$- \left\langle \lambda', \bar{\lambda}'; -p_1', -p_2 \,|\, \underline{\bar{\Phi}}^\dagger \,|\, \lambda, \bar{\lambda}; p_1, p_2' \right\rangle]$$

$$= (2\pi)^4 \frac{\bar{E}^2}{m^2} \frac{m^2}{4\pi\bar{E}} \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} \delta^{(4)}(p_1' + p_2 + q_1 + q_2) \left(\frac{4\pi}{\omega}\right)^2$$

$$\times \left\langle \lambda', \bar{\lambda}'; -p_1', -p_2 \,|\, \mathcal{F}^\dagger \,|\, q_1, q_2 \right\rangle$$

$$\times \left\langle q_1, q_2 \,|\, \mathcal{F} \,|\, \lambda, \bar{\lambda}; p_1, p_2' \right\rangle \tag{B.6}$$

An argument parallel to that which led from (B.3) to (7.2) now gives (6.19a).

1. Extraction of the Azimuthal Angles from a Product of Matrix Elements

In this subsection the dependence of a product of matrix elements of rotationally invariant operators on azimuthal angles of the intercalated state is explicitly displayed. Namely,

$$\begin{aligned}&\left\langle f(0,\theta,0)\,|\,B^{\dagger}\,|\,n\,(\phi_1,\theta_1,\psi_1)\right\rangle\left\langle n(\phi_1,\theta_1,\psi_1)\,|A|\,i\right\rangle\\ &= e^{i\overline{\Phi}}\left\langle n(0,\,-\theta_2,\,0)\,|\,B\,|\,f_0\right\rangle^{*}\left\langle\,n(0,\,\theta_1,\,0)\,|\,A|\,i_0\right\rangle,\end{aligned} \tag{B.7}$$

where A and B are operators which commute with rotation operators; where the states are obtained from $|\,i_0\rangle$, $|n_0\rangle$, $|\,f_0\rangle$, eigenstates of J_z with eigenvalues m_i, m_n, m_f, respectively, as follows:

$$\begin{aligned}&|\,i\rangle = |\,i_0\rangle,\\ &|\,n\,(\phi_1,\theta_1,\psi_1)\rangle = e^{-iJ_z\phi_1}\;e^{-iJ_y\theta_1}\;e^{-iJ_z\psi_1}\;|\,n_0\rangle,\\ &|\,f\,(0,\theta,0)\rangle = e^{-iJ_y\theta}\;|\,f_0\rangle,\end{aligned} \tag{B.8}$$

in which active rotations are written explicitly in terms of the angular momentum operators $\underset{\sim}{J}$; where the angle θ_2 is the angle from a vector at the direction $(\theta_1,\,\phi_1)$ associated with the intercalated state to one at the direction $(\theta, 0)$ associated with the final state, so that

$$\cos\theta_2 = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos\phi_1; \tag{B.9}$$

and where

$$e^{i\overline{\Phi}} = \exp i\,[\,m_i\,\phi_1 + m_f\,\phi_2 + m_n\,\phi_3\,] \tag{B.10}$$

and

$$\begin{aligned}\sin\theta_2\,e^{i\phi_2} &= \sin\theta\cos\theta_1 - \cos\theta\sin\theta_1\cos\phi_1 + i\sin\theta\sin\phi_1,\\ \sin\theta_2\,e^{i\phi_3} &= -\cos\theta\sin\theta_1 + \sin\theta\cos\theta_1\cos\phi_1 - i\sin\theta\sin\phi_1.\end{aligned} \tag{B.11}$$

The application of this theorem in the text is to the two-particle states $|\lambda_1, \lambda_2, \underset{\sim}{p}\rangle$ of Jacob and Wick (15), constructed by the active rotation of fiducial states $|\lambda_1, \lambda_2, p\rangle$. In a fiducial state, the first particle, i.e., that whose creation operator acts first on the vacuum in the notation of conventional field theory, moves in the +z direction and has helicity λ_1, and the second particle moves in the opposite direction and has helicity λ_2, so that a fiducial state is an eigenstate of J_z with eigenvalue $m = \lambda_1 - \lambda_2$. The absolute value of the momentum of either particle is p.

The result (B.7, B.9—B.11) is derived as follows:

$$\begin{aligned}
&\langle f(0,\theta,0) \mid B^\dagger \mid n(\phi_1,\theta_1,\psi_1)\rangle\langle n(\phi_1,\theta_1,\psi_1)\mid A\mid i\rangle \\
&= \langle f_0 \mid e^{iJ_y\theta} B^\dagger e^{-iJ_z\phi_1} e^{-iJ_y\theta_1} e^{-J_z\psi_1} \mid n_0\rangle\langle n_0 \mid e^{iJ_z\psi_1} e^{iJ_y\theta_1} e^{iJ_z\phi_1} A \mid i_0\rangle \\
&= \langle f_0 \mid e^{iJ_y\theta} e^{-iJ_z\phi_1} e^{-iJ_y\theta_1} B^\dagger \mid n_0\rangle\langle n_0 \mid e^{iJ_y\theta_1} A \mid i_0\rangle\, e^{im_i\phi_1},
\end{aligned} \tag{B.12}$$

where $B^\dagger$ and A have been commuted with rotation operators, and where J_z has been replaced by its eigenvalue where obviously possible; note that the third Euler angle ψ has dropped out.

The 3-parameter rotation that appears as a succession of y, z, then y rotations, is now rewritten in the more conventional form of a succession of z, y, then z rotations:

$$e^{iJ_y\theta} e^{-iJ_z\psi_1} e^{-iJ_y\theta_1} = e^{iJ_z\phi_2} e^{iJ_y\theta_2} e^{iJ_z\phi_3} . \tag{B.13}$$

The new parameters ϕ_2, θ_2, ϕ_3 are given in (B.9, B.11). They are readily obtained if Eq. (B.13) between rotation group elements (the double-valued or covering group, for maximal generality), is replaced by the analogous equation in the faithful spin $\frac{1}{2}$ representation. Then $e^{i\underset{\sim}{J}\cdot\underset{\sim}{n}\theta}$ is replaced by $e^{i\frac{1}{2}\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\theta} = \cos\theta/2 + i\,\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\,\sin\theta/2$, for $\underset{\sim}{n}\cdot\underset{\sim}{n} = 1$, which gives a 2 by 2 matrix

equation that is simplified by multiplying Pauli matrices, to yield four trigonometric equations, and Eqs. (B.9, B.11).

Substitution of (B.13) into (B.12) produces the expression

$$e^{i\overline{\Phi}} \left\langle f_0 \,|\, e^{iJ_y\theta_2} B^\dagger \,|\, n_0 \right\rangle \left\langle n(0,\theta_1,0) \,|A|\, i_0 \right\rangle ,$$

with $e^{i\overline{\Phi}}$ given by (B.10), when $e^{iJ_z\phi_3}$ is commuted through $B^\dagger$, and the J_z are replaced by the appropriate eigenvalues. The final form (B.7) follows from $\left\langle f_0 \,|\, e^{iJ_y\theta_2} B^\dagger \,|\, n_0 \right\rangle = \left\langle n_0 \,|\, B\, e^{-iJ_y\theta_2} \,|\, f_0 \right\rangle^* = \left\langle n(0,\ -\theta_2, 0) \,|B|\, f_0 \right\rangle^*$. In particular, it can be readily seen that Eq. (6.19b) follows from (6.19a), and Eq. (7.4) from (7.2).

APPENDIX C: CALCULATIONS PERTAINING TO THE UNITARITY EQUATIONS

In Section VI, the steps leading from (6.19) to (6.21) were omitted. We give them here in some detail.

$$\mathrm{Im}_{2\pi}\, \overline{\phi}_1^{(I)} \equiv \mathrm{Im}_{2\pi} \left\langle \tfrac{1}{2}\,,\ \tfrac{1}{2}\,;\, \underset{\sim}{\overline{p}}_f \,|\, \overline{\Phi}^{(I)} \,|\, \tfrac{1}{2}\,,\ \tfrac{1}{2}\,;\, \underset{\sim}{\overline{p}}_i \right\rangle$$

$$= q(4\pi)^{-1} \int d\Omega_q\, (5 \pm 1)\, \mathcal{F}_{++}^{(\pm)^*}(-\theta_{2q})\ \mathcal{F}_{++}^{(\pm)}(\theta_{1q})$$

$$= (5 \pm 1)\, \frac{q}{4\pi} \left(\frac{m}{8\,\pi\,\overline{E}}\right)^2 \int d\Omega_q \left[a_0^{(\pm)^*} + \frac{z_{2q}}{\pi} \int_{\lambda_0}^{\infty} \frac{d\lambda}{\lambda}\ a^{(\pm)^*}(\lambda) \times \left(\frac{1}{\lambda - z_{2q}} \mp \frac{1}{\lambda + z_{2q}} \right) \right]$$

$$\times \left[a_0^{(\pm)} + \frac{z_{1q}}{\pi} \int_{\lambda_0}^{\infty} \frac{d\mu}{\mu}\ a^{(\pm)}(\mu) \left(\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}} \right) \right]$$

(C.1 cont.)

$$= (5 \pm 1) \frac{q}{4\pi} \left(\frac{m}{8\pi \bar{E}}\right)^2 \int d\Omega_q \left\{ |a_0^{(\pm)}|^2 \right.$$

$$+ \frac{z_{1q}}{\pi} \int_{\lambda_0}^{\infty} d\lambda \, 2\mathrm{Re}[a_0^{(\pm)*} a^{(\pm)}(\lambda)] \left[\frac{1}{\lambda - z_{1q}} \mp \frac{1}{\lambda + z_{1q}}\right]$$

$$+ \frac{z_{1q} z_{2q}}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu \, a^{(\pm)*}(\lambda) \, a^{(\pm)}(\mu)$$

$$\left. \times \left[\frac{1}{\lambda - z_{2q}} \mp \frac{1}{\lambda + z_{2q}}\right]\left[\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}}\right] \right\}$$

(C.1)

The angular integrals are given in Appendix D, and lead to (6.21a).

Similarly we have

$$\mathrm{Im}_{2\pi} \bar{\Phi}_2^{(I)} \equiv \mathrm{Im}_{2\pi} \left\langle \frac{1}{2}, \frac{1}{2}; \bar{\underset{\sim}{p}}_f \,|\, \bar{\underline{\Phi}}^{(I)} \,|\, -\frac{1}{2}, -\frac{1}{2}; \bar{\underset{\sim}{p}}_i \right\rangle$$

$$= q(4\pi)^{-1} \int d\Omega_q \, (5 \pm 1) \mathcal{F}_{++}^{(\pm)*}(-\theta_{2q}) \mathcal{F}_{--}^{(\pm)}(\theta_{1q}) = \mathrm{Im}_{2\pi} \bar{\Phi}_1^{(I)},$$

(C.2)

by virtue of the second equation of (6.7), leading to (6.21b):

$$\mathrm{Im}_{2\pi} \bar{\Phi}_3^{(I)} \equiv \mathrm{Im}_{2\pi} \left\langle \frac{1}{2}, -\frac{1}{2}; \bar{\underset{\sim}{p}}_f \,|\, \bar{\underline{\Phi}}^{(I)} \,|\, \frac{1}{2}, -\frac{1}{2}; \bar{\underset{\sim}{p}}_i \right\rangle$$

$$= q(4\pi)^{-1} \int d\Omega_q \, (5 \pm 1) \mathcal{F}_{+-}^{(\pm)*}(-\theta_{2q}) \, \mathcal{F}_{+-}^{(\pm)}(\theta_{1q}) \exp i(\phi_{1q} + \phi_{2q})$$

$$= -(5 \pm 1) \frac{q}{4\pi} \left(\frac{m}{8\pi \bar{E}}\right)^2 \int d\Omega_{1q} \frac{y_{1q} y_{2q}}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu$$

$$\times \beta^{(\pm)*}(\lambda) \, \beta^{(\pm)}(\mu) \left[\frac{1}{\lambda - z_{2q}} \mp \frac{1}{\lambda + z_{2q}}\right]\left[\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}}\right]$$

(C.3 cont.)

$$\times \exp\left[i\,\phi_{1q}\right] \frac{1}{y_{2q}} \left[\,\bar{y}\, z_{1q} - \bar{z}\, y_{1q} \cos\phi_{1q} + i y_{1q} \sin\phi_{1q}\right]$$

$$= -\,(5 \pm 1)\ \frac{q}{4\pi} \left(\frac{m}{8\,\pi\,\bar{E}}\right)^2 \frac{1}{\pi^2} \int d\Omega_{1q} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\ \beta^{(\pm)^*}(\lambda)\ \beta^{(\pm)}(\mu)$$

$$\times \left[\frac{1}{\lambda - z_{2q}} \mp \frac{1}{\lambda + z_{2q}}\right]\left[\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}}\right]$$

$$\times \left[\frac{1}{1+\bar{z}}\ (z_{1q}^{\ 2} + z_{2q}^{\ 2} - 2\bar{z}\ z_{1q}\, z_{2q}) - (1 - z_{1q}\, z_{2q})\right] , \tag{C.3}$$

and then, from (D.22),

$$\mathrm{Im}_{2\pi}\, \bar{\Phi}_3^{(I)} = -\,(1+\bar{z})\ (5 \pm 1)\ \frac{q}{4\pi} \left(\frac{m}{8\,\pi\,\bar{E}}\right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\ \beta^{(\pm)^*}(\lambda)\ \beta^{(\pm)}(\mu)$$

$$\times\ \frac{1}{\pi} \int \frac{4\pi^2\, dz'}{[\,K(\lambda, \mu, z')]^{\frac{1}{2}}} \left\{ \frac{1}{z' - \bar{z}} \left[\frac{\lambda^2 + \mu^2 - 2\lambda\,\mu z'}{(1 + z')^2} - \frac{1 - \lambda\mu}{1 + z'}\right]\right.$$

$$\left. \mp\ \frac{1}{z' + \bar{z}} \left[\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1 - z')^2} - \frac{1 + \lambda\,\mu}{1 - z'}\right]\right\} . \tag{C.4}$$

Similarly we have

$$\mathrm{Im}_{2\pi}\, \bar{\Phi}_4^{(I)} \equiv \mathrm{Im}_{2\pi} \left\langle \tfrac{1}{2}, -\tfrac{1}{2}; \bar{\mathbf{p}}_f \,\middle|\, \bar{\Phi}^{(I)} \,\middle|\, -\tfrac{1}{2}, \tfrac{1}{2}; \bar{\mathbf{p}}_i \right\rangle$$

$$= -\,q(4\pi)^{-1} \int d\Omega_q\ (5 \pm 1)\, \mathfrak{F}_{+-}^{(\pm)^*}(-\theta_{2q})\, \mathfrak{F}_{+-}^{(\pm)}(\theta_{1q})\ \exp i\,(\phi_{2q} - \phi_{1q})$$

$$= -\,(5 \pm 1)\ \frac{q}{4\pi} \left(\frac{m}{8\,\pi\,\bar{E}}\right)^2 \frac{1}{\pi^2} \int d\Omega_{1q} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\ \beta^{(\pm)^*}(\lambda)\ \beta^{(\pm)}(\mu)$$

$$\times \left[\frac{1}{\lambda - z_{2q}} \mp \frac{1}{\lambda + z_{2q}}\right] \left[\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}}\right]$$

(C.5 cont.)

$$\times \left[\frac{1}{1-\bar{z}} \left(z_{1q}^{\ 2} + z_{2q}^{\ 2} - 2\bar{z}\, z_{1q}\, z_{2q} \right) - \left(1 + z_{1q}\, z_{2q} \right) \right], \qquad \text{(C.5)}$$

and when we use (D.23),

$$\mathrm{Im}_{2\pi}\, \bar{\Phi}_4^{(I)} = -\,(1-\bar{z})\,(5 \pm 1)\, \frac{q}{4\pi} \left(\frac{m}{8\,\pi\,\bar{E}} \right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\, \beta^{(\pm)^*}(\lambda)\beta^{(\pm)}(\mu)$$

$$\times \frac{1}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}} \left\{ \frac{1}{z'-\bar{z}} \left[\frac{\lambda^2+\mu^2-2\,\lambda\mu z'}{(1-z')^2} - \frac{1+\lambda\mu}{1-z'} \right] \right.$$

$$\left. \mp \frac{1}{z'+z} \left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1+z')^2} - \frac{1-\lambda\mu}{1+z'} \right] \right\}. \qquad \text{(C.6)}$$

Equations (6.21c) and (6.21d) follow.

Similarly, we have

$$\mathrm{Im}_{2\pi}\, \bar{\Phi}_5^{(I)} = \mathrm{Im}_{2\pi} \left\langle \tfrac{1}{2}, \tfrac{1}{2}, \bar{\underset{\sim}{p}}_f \,\middle|\, \bar{\underline{\Phi}}^{(I)} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{2}, \bar{\underset{\sim}{p}}_i \right\rangle$$

$$= q(4\pi)^{-1} \int d\Omega_{1q}\, (5 \pm 1)\, \mathfrak{J}_{++}^{(\pm)^*}(-\theta_{2q})\, \mathfrak{J}_{+-}^{(\pm)}(\theta_{1q}) \exp i\,\phi_{1q}$$

$$= -\,(5 \pm 1)\, \frac{q}{4\pi} \left(\frac{m}{8\,\pi\,\bar{E}} \right)^2 \int d\Omega_q\, y_{1q}\, e^{i\phi_{1q}}$$

$$\times \left[a_0^{(\pm)^*} + \frac{z_{2q}}{\pi} \int_{\lambda_0}^{\infty} \frac{d\lambda}{\lambda}\, a^{(\pm)^*}(\lambda) \left(\frac{1}{\lambda - z_{2q}} \mp \frac{1}{\lambda + z_{2q}} \right) \right]$$

$$\times \frac{1}{\pi} \int_{\lambda_0}^{\infty} d\mu\, \beta^{(\pm)}(\mu) \left(\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}} \right)$$

$$= -\,(5 \pm 1)\, \frac{q}{4\pi} \left(\frac{m}{8\,\pi\,\bar{E}} \right)^2 \int d\Omega \left\{ a_0^{(\pm)^*} \frac{1}{\pi} \int_{\lambda_0}^{\infty} d\mu\, \beta^{(\pm)}(\mu) \right.$$

$$\times (z_{2q} - \bar{z}\, z_{1q}) \left[\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}} \right]$$

(C.7 cont.)

$$+ \frac{1}{\bar{y}} (z_{2q} - \bar{z}\, z_{1q})\, z_{2q} \frac{1}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu \frac{1}{\lambda} a^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu)$$

$$\times \left[\frac{1}{\lambda - z_{2q}} \mp \frac{1}{\lambda + z_{2q}}\right]\left[\frac{1}{\mu - z_{1q}} \mp \frac{1}{\mu + z_{1q}}\right] \Bigg\}, \qquad (C.7)$$

and with the aid of (D.7) and (D.15), (6.21e) follows.

If now we make the substitution of (6.22),

$$\bar{G} = (\bar{G}\,\bar{\phi})\,\bar{\phi}$$

$$\mathrm{Im}_{2\pi}\, \bar{G}_1^{(I)} = \frac{1}{\bar{E}\,\bar{p}^2} (5 \pm 1) \frac{q}{4\pi} \left(\frac{m}{8\pi\bar{E}}\right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu \frac{1}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \Bigg\{ 2m^2\, \beta^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu) \left[\frac{1}{z' - \bar{z}} \pm \frac{1}{z' + \bar{z}}\right] \left[\frac{2z'}{(1-z'^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z') - \frac{\lambda\mu + z'}{1 - z'^2}\right]$$

$$+ 2\bar{p}^2\, \bar{z}\, \beta^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu) \left[\frac{1}{z' - \bar{z}} \mp \frac{1}{z' + \bar{z}}\right] \left[\frac{1 + z'^2}{(1-z'^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z') - \frac{1 + \lambda\mu z'}{1 - z'^2}\right]$$

$$+ 2\, \frac{\bar{E}}{m}\, \bar{p}^2\, \bar{z}\, \mathrm{Re}\left[a^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu)\right] \left[\frac{1}{z' - \bar{z}} \mp \frac{1}{z' + \bar{z}}\right] \left[\frac{\lambda - \mu z'}{1 - z'^2}\right] \Bigg\}$$

$$= \frac{1}{\bar{E}\,\bar{p}^2} (5 \pm 1) \frac{q}{4\pi} \left(\frac{m}{8\pi\bar{E}}\right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu \frac{\bar{z}}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \frac{1}{z'}$$

$$\times \left[\frac{1}{z' - \bar{z}} \mp \frac{1}{z' + \bar{z}}\right] \Bigg\{ 2m^2\, \beta^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu) \left[\frac{2z'}{(1-z'^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z') - \frac{\lambda\mu + z'}{1 - z'^2}\right]$$

$$+ 2p^2\, \beta^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu)\, z' \left[\frac{1 + z'^2}{(1-z'^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z') - \frac{1 + \lambda\mu z'}{1 - z'^2}\right]$$

(C.8 cont.)

$$+ 2 \frac{\overline{E}}{m} \overline{p}^2 \operatorname{Re} \left[a^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu)\right] z' \frac{\lambda - \mu z'}{1 - z'^2} \Bigg\}$$

$$+ \frac{1}{\overline{E}\,\overline{p}^2} (5 \pm 1) \frac{q}{4\pi} \left(\frac{m}{8\pi\overline{E}}\right)^2 (1 \pm 1)\, m^2 \frac{4}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\, \beta^{(\pm)^*}(\lambda) \beta^{(\pm)}(\mu)$$

$$\times \left[-4\pi + \lambda L(\lambda) + \mu L(\mu) - \lambda\mu J(\lambda, \mu, 0)\right], \tag{C.8}$$

where

$$J(\lambda, \mu, 0) = \frac{4\pi}{\left[\lambda^2 + \mu^2 - 1\right]^{\frac{1}{2}}} \ln \left[\frac{\lambda\mu + \left[(\lambda^2 + \mu^2 - 1)\right]^{\frac{1}{2}}}{\lambda\mu - \left[(\lambda^2 + \mu^2 - 1)\right]^{\frac{1}{2}}} \right]. \tag{C.9}$$

We have used the identities

$$\frac{1}{\pi} \int \frac{4\pi^2 dz'}{\left[K(\lambda, \mu, z')\right]^{\frac{1}{2}}} \frac{1}{(1 - z'^2)^2} \left[(\lambda^2 + \mu^2 - 2\lambda\mu z')(1 + z'^2) - (1 + \lambda\mu z')(1 - z'^2)\right] = -4\pi,$$

and

$$\frac{1}{\pi} \int \frac{4\pi^2 dz'}{\left[K(\lambda, \mu, z')\right]^{\frac{1}{2}}} \frac{1}{1 - z'^2} (\lambda - \mu z') = L(\lambda). \tag{C.10}$$

Similarly we have

$$\operatorname{Im}_{2\pi} \overline{G}_2^{(I)} = \frac{1}{\overline{E}\,\overline{p}^2} (5 + 1) \frac{q}{4\pi} \left(\frac{m}{8\pi\overline{E}}\right)^2 \frac{2}{\pi^2} \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu \frac{1}{\pi} \int \frac{4\pi^2 dz'}{\left[K(\lambda, \mu, z')\right]^{\frac{1}{2}}}$$

$$\times \left[\frac{1}{z' - \overline{z}} \mp \frac{1}{z' + \overline{z}} \right]$$

$$\times \Bigg\{ 2\overline{E}^2 \beta^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu) \left[\frac{1 + z'^2}{(1 - z'^2)^2} (\lambda^2 + \mu^2 - 2\lambda\mu z') - \frac{1 + \lambda\mu z'}{1 - z'^2} \right]$$

$$+ 2 \frac{\overline{E}}{m} \overline{E}^2 \operatorname{Re}\left[a^{(\pm)^*}(\lambda)\, \beta^{(\pm)}(\mu)\right] \frac{\lambda - \mu z'}{1 - z'^2} \Bigg\}, \tag{C.11}$$

$$\mathrm{Im}_{2\pi}\,\overline{G}_3^{(I)} = \frac{1}{\overline{E}\,\overline{p}^2}\,(5 \pm 1)\quad \frac{q}{4\pi}\left(\frac{m}{8\,\pi\,\overline{E}}\right)^2 \frac{2}{\pi^2}\int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\,\frac{1}{\pi}\int \frac{4\pi^2 dz'}{[\,K(\lambda,\mu,z')\,]^{\frac{1}{2}}}$$

$$\times\left[\frac{1}{z'-\overline{z}} \pm \frac{1}{z'+\overline{z}}\right]$$

$$\times\; 2\,\overline{E}^2\;\beta^{(\pm)^*}(\lambda)\;\beta^{(\pm)}(\mu)\left[\frac{\lambda\mu + z'}{1-z'^2} - \frac{2z'}{(1-z'^2)^2}(\lambda^2 + \mu^2 - 2\lambda\mu z')\right], \tag{C.12}$$

$$\mathrm{Im}_{2\pi}\,\overline{G}_4^{(I)} = \frac{1}{\overline{E}\,\overline{p}^2}\,(5 + 1)\quad \frac{q}{4\pi}\left(\frac{m}{8\,\pi\,\overline{E}}\right)^2 \frac{2}{\pi^2}\int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\,\frac{1}{\pi}\int \frac{4\pi^2 dz'}{[\,K(\lambda,\mu,z')\,]^{\frac{1}{2}}}$$

$$\times\left[\frac{1}{z'-\overline{z}} \mp \frac{1}{z'+\overline{z}}\right]$$

$$\times\left\{-2\overline{E}^2\;\beta^{(\pm)^*}(\lambda)\;\beta^{(\pm)}(\mu)\left[\frac{1+z'^2}{(1-z'^2)^2}\;(\lambda^2 + \mu^2 - 2\lambda\mu z') - \frac{1+\lambda\mu z'}{1-z'^2}\right]\right.$$

$$\left. - 2\,\frac{\overline{E}}{m}\,m^2\,\mathrm{Re}\,[\,\alpha^{(\pm)^*}(\lambda)\;\beta^{(\pm)}(\mu)]\;\frac{\lambda - \mu z'}{1-z'^2}\right\}, \tag{C.13}$$

and

$$\mathrm{Im}_{2\pi}\,\overline{G}_5^{(I)} = -\,\frac{1}{\overline{E}\,\overline{p}^2}\,(5 + 1)\quad \frac{q}{4\pi}\left(\frac{m}{8\,\pi\,\overline{E}}\right)^2 \frac{2}{\pi^2}\int_{\lambda_0}^{\infty} d\lambda \int_{\lambda_0}^{\infty} d\mu\,\frac{\overline{z}}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}\,\frac{1}{z'}$$

$$\times\left[\frac{1}{z'-\overline{z}} \mp \frac{1}{z'+\overline{z}}\right]$$

$$\times\left\{2\overline{E}^2\;\alpha^{(\pm)^*}(\lambda)\;\alpha^{(\pm)}(\mu)\right.$$

$$- 2\overline{E}^2\;\beta^{(\pm)^*}(\lambda)\;\beta^{(\pm)}(\mu)\;z'\left[\frac{1+z'^2}{(1-z'^2)^2}\;(\lambda^2 + \mu^2 - 2\lambda\mu z') - \frac{1+\lambda\mu z'}{1 - z'^2}\right]$$

$$-2\frac{\bar{E}}{m}(\bar{E}^2+m^2)\,\mathrm{Re}\left[a^{(\pm)*}(\lambda)\,\beta^{(\pm)}(\mu)\right]\; z'\;\frac{\lambda-\mu z'}{1-z'^2}\Bigg\}$$

$$-\frac{1}{\bar{E}\,\bar{p}^2}(5\pm 1)\;\frac{q}{4\pi}\left(\frac{m}{8\pi\bar{E}}\right)^2 2\bar{E}^2\Bigg\{4\pi\,|a_0^{(\pm)}|^2$$

$$+\frac{2}{\pi}\int_{\lambda_0}^{\infty}\frac{d\lambda}{\lambda}\;2\,\mathrm{Re}[\,a_0^{(\pm)*}a^{(\pm)}(\lambda)]\;[\,\lambda\,L(\lambda)-4\pi]$$

$$+\frac{2}{\pi^2}(1\pm 1)\int_{\lambda_0}^{\infty}d\lambda\int_{\lambda_0}^{\infty}d\mu\;a^{(\pm)*}(\lambda)\,a^{(\pm)}(\mu)\,\frac{1}{\lambda\mu}\,[\,4\pi-\lambda\,L(\lambda)-\mu\,L(\mu)$$

$$+\,J(\lambda,\mu,0)]\Bigg\}\,. \qquad \text{(C.14)}$$

These equations, (C.8, C.11—14), have apparent singularities at $\bar{p}^2 = 0$, as is observed in Section VI. However the substitution of a and b for α and β through (6.17) leads to a cancellation and to the results of Eqs. (6.23—6.26).

In a similar way we go through the derivation of (7.21) in the nucleon-nucleon unitarity. From (7.11) we have

$$\mathrm{Im}_{el}\,\phi_1 = p(4\pi)^{-1}\int d\Omega\Bigg\{(Ep^2)^{-2}\left[\chi_1^*(z_2)\,\chi_1(z_1)+\chi_2^*(z_2)\chi_2(z_1)\right]$$

$$-\,2m^{-6}y_1y_2\cos\phi_3\,\chi_5^*(z_2)\chi_5(z_1)\Bigg\}, \qquad \text{(C.15)}$$

and

$$\mathrm{Im}_{el}\,\phi_2 = p(4\pi)^{-1}\int d\Omega\Bigg\{(E\,p^2)^{-2}\left[\chi_1^*(z_2)\chi_2(z_1)+\chi_2^*(z_2)\,\chi_1(z_1)\right]$$

$$-2m^{-6}\,y_1y_2\cos\phi_3\,\chi_5^*(z_2)\chi_5(z_1)\Bigg\}. \qquad \text{(C.16)}$$

Now we have

$$\int d\Omega\left[\chi_1^*(z_2)\chi_1(z_1)\right]$$

(C.17 cont.)

$$= \int d\Omega \left\{ \frac{z_2}{\pi} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \left[\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2} \right] f_1^*(\lambda) + g_1^* \right\}$$

$$\times \frac{z_1}{\pi} \int_{\lambda_1}^{\infty} \frac{d\mu}{\mu} \left[\frac{1}{\mu - z_1} + (-1)^I \frac{1}{\mu + z_1} \right] f_1(\mu) + g_1 \Big\} = |g_1|^2 \int d\Omega \; 1$$

$$+ \frac{1}{\pi} \int_{\lambda_1}^{\infty} d\lambda \; 2 \, \mathrm{Re} \left[f_1^*(\lambda) \, g_1 \right] \frac{1}{\lambda} \int d\Omega \, z_2 \left[\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2} \right]$$

$$+ \frac{1}{\pi^2} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \int_{\lambda_1}^{\infty} \frac{d\mu}{\mu} \; f_1^*(\lambda) \, f_1(\mu) \int d\Omega \, z_1 z_2 \left[\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2} \right]$$

$$\times \left[\frac{1}{\mu - z_1} + (-1)^I \frac{1}{\mu + z_1} \right] ,$$

and using (D.4, D.6, and D.14), we deduce

$$\int d\Omega \left[\chi_1^*(z_2) \, \chi_1(z_1) \right] = |g_1|^2 \, 4\pi$$

$$+ \frac{1}{\pi} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \; 2 \, \mathrm{Re} \left[f_1^*(\lambda) \, g_1 \right] \left[1 + (-1)^{I+i} \right] \left[\lambda \, L(\lambda) - 4\pi \right]$$

$$+ \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda} \int_{\lambda_1}^{\infty} \frac{d\mu}{\mu} \; f_1^*(\lambda) \, f_1(\mu) \left[1 + (-1)^{I+1} \right] \left[4\pi - \lambda \, L(\lambda) - \mu L(\mu) \right]$$

$$+ \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \; f_1^*(\lambda) \, f_1(\mu) \; \frac{1}{\pi} \int \frac{4\pi^2 \, dz'}{\left[K(\lambda, \mu, z') \right]^{\frac{1}{2}}} \left[\frac{1}{z' - z} + (-1)^{I+1} \frac{1}{z' + z} \right] \tag{C.17}$$

Similar expressions may be obtained for $\int d\Omega \, \chi_2^* \chi_2$, $\int d\Omega \, \chi_1^* \chi_2$ and $\int d\Omega \, \chi_2^* \chi_1$. Further, we obtain

$$\int d\Omega [\, y_1 y_2 \cos\phi_3 \, \chi_5^*(z_2)\chi_5(z_1)\,]$$

$$= \frac{1}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\, f_5^*(\lambda)\, f_5(\mu) \int d\Omega\, (z_2 z_1 - z)\left[\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} + (-1)^I \frac{1}{\mu + z_1}\right],$$

which from (D.12, D.14) gives

$$\int d\Omega\, [\, y_1 y_2 \cos\phi_3\, \chi_5^*(z_2)\, \chi_5(z_1)\,]$$

$$= \frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\, f_5^*(\lambda)\, f_5(\mu) \Bigg\{ [\,1 + (-1)^{I+1}\,]\, [\,4\pi - \mu\, L(\mu) - \lambda\, L(\lambda)]$$

$$+ \frac{1}{\pi}\int \frac{4\pi^2\, dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}\left[\frac{1}{z'-z} + (-1)^{I+1}\frac{1}{z'+z}\right]\lambda\mu$$

$$- \frac{z}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}\left[\frac{1}{z'-z} + (-1)^{I}\frac{1}{z'+z}\right]\Bigg\}$$

$$= \frac{2}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\, f_5^*(\lambda)\, f_5(\mu) \Bigg\{ [\,1 + (-1)^{I+1}\,]\, [\,4\pi - \mu\, L(\mu) - \lambda L(\lambda) + \lambda\mu\, J(\lambda,\mu,0)]$$

$$+ \frac{z}{\pi}\int \frac{4\pi^2\, dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}\left[\frac{1}{z'-z} + (-1)^{I}\frac{1}{z'+z}\right]\left[\frac{\lambda\mu - z'}{z'}\right]\Bigg\}\,. \qquad \text{(C.18)}$$

Equations (7.21a, b) follow. We next consider

$$\mathrm{Im}_{el}\, \phi_3\,(p,z) = \frac{p}{4\pi}\int d\Omega \Bigg\{ (E\, m^2)^{-2}\, (1+z_1)\, (1+z_2) e^{i\phi_3}\, \chi_3^*(z_2)\chi_3(z_1)$$

$$+ (E\, m^2)^{-2}\, (1-z_1)\, (1-z_2)\, e^{-i\phi_3}\, \chi_4^*(z_2)\, \chi_4(z_1) - 2m^{-6}\, y_1 y_2 \chi_5^*(z_2)\chi_5(z_1)\Bigg\}$$

$$\times \exp i\,(\phi_2 + \phi_1)\,. \qquad \text{(C.19)}$$

Now, we have

$$\int d\Omega\,(1 \pm z_1)\,(1 + z_2)\,\exp[\pm i\,\phi_3]\,\chi^*_{\frac{3}{4}}(z_2)\chi_{\frac{3}{4}}(z_1)\,\exp i\,(\phi_2 + \phi_1)$$

$$= \int d\Omega\,\frac{1}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\,(1\pm z_1)\,(1\pm z_2)\,\frac{1}{y_2}\,[-y_1 z + z_1 y\cos\phi_1 \mp i\,y\sin\phi_1]$$

$$\times\left[\frac{1}{\lambda - z_2}\,f_{(\pm)}{}^*(\lambda) + (-1)^I\,\frac{1}{\lambda + z_2}\,f_{(\mp)}{}^*(\lambda)\right]\left[\frac{1}{\mu - z_1}\,f_{(\pm)}(\mu) + (-1)^I\,\frac{1}{\mu + z_1}\,f_{(\mp)}(\mu)\right]$$

$$\times\frac{1}{y_2}\,[y\,z_1 - z\,y_1\cos\phi_1 + i\,y_1\sin\phi_1]\,[\cos\phi_1 + i\sin\phi_1]$$

$$= \frac{1}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \int d\Omega\left[\frac{1}{\lambda - z_2}\,f_{(\pm)}{}^*(\lambda) + (-1)^I\,f_{(\mp)}{}^*(\lambda)\right]\left[\frac{1}{\mu - z_1}f_{(\pm)}(\mu) + (-1)^I\right.$$

$$\left.\times\,\frac{1}{\mu + z_1}\,f_{(\mp)}(\mu)\right]$$

$$\times\left\{\frac{1}{1+z}\,[z_1{}^2 + z_2{}^2 - 2\,z\,z_1\,z_2] + [z + z_1 z_2] \pm [z_1 + z_2]\right\}, \quad \text{(C.20)}$$

and using (D.22), this is equal to

$$(1+z)\,\frac{1}{\pi^2}\int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\,\frac{1}{\pi}\int\frac{4\pi^2 dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}}$$

$$\times\,\frac{1}{z'-z}\left[\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1+z')^2} + \frac{\lambda\mu + z'}{1+z'}\right]\left[f_{(\pm)}{}^*(\lambda)\,f_{(\pm)}(\mu) + f_{(\mp)}{}^*(\lambda)\,f_{(\mp)}(\mu)\right]$$

$$\pm\,\frac{1}{z'-z}\cdot\frac{\lambda + \mu}{1+z'}\left[f_{(\pm)}{}^*(\lambda)\,f_{(\pm)}(\mu) - f_{(\mp)}{}^*(\lambda)\,f_{(\mp)}(\mu)\right]$$

(C.21 cont.)

$$+ (-1)^I \frac{1}{z'+z} \left[\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1-z')^2} - \frac{\lambda\mu + z'}{(1-z')^2} \right] \left[f_{(\pm)}^*(\lambda)\, f_{(\mp)}(\mu) + f_{(\mp)}^*(\lambda) f_{(\pm)}(\mu) \right]$$

$$\pm (-1)^I \frac{1}{z'+z} \cdot \frac{\lambda-\mu}{1-z'} \left[f_{(\pm)}^*(\lambda)\, f_{(\mp)}(\mu) - f_{(\mp)}^*(\lambda)\, f_{(\pm)}(\mu) \right] .$$

Thus we get

$$\int d\Omega \left[(1+z_1)\,(1+z_2)\, e^{i\phi_3} \chi_3^*(z_2)\, \chi_3(z_1) + (1-z_1)\,(1-z_2)\, e^{-i\phi_3} \chi_4^*(z_2) \chi_4(z_1) \right]$$

$$\times \exp i\,(\phi_2 + \phi_1) = (1+z)\, \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu\, \frac{1}{\pi}\, \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \frac{1}{z'-z} \left[\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1-z')^2} + \frac{\lambda\mu + z'}{1+z'} \right] \left[f_{(+)}^*(\lambda)\, f_{(+)}(\mu) + f_{(-)}^*(\lambda)\, f_{(-)}(\mu) \right]$$

$$+ \frac{1}{z'-z} \cdot \frac{\lambda+\mu}{1+z'} \left[f_{(+)}^*(\lambda)\, f_{(+)}(\mu) - f_{(-)}^*(\lambda)\, f_{(-)}(\mu) \right]$$

$$+ (-1)^I \frac{1}{z'+z} \left[\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1-z')^2} - \frac{\lambda\mu + z'}{1-z'} \right] \left[f_{(+)}^*(\lambda)\, f_{(-)}(\mu) + f_{(-)}^*(\lambda)\, f_{(+)}(\mu) \right]$$

$$+ (-1)^I \frac{1}{z'+z} \cdot \frac{\lambda-\mu}{1-z'} \left[f_{(+)}^*(\lambda)\, f_{(-)}(\mu) - f_{(-)}^*(\lambda)\, f_{(+)}(\mu) \right] \Bigg\} . \qquad \text{(C.21)}$$

Also we have

$$\int d\Omega\, y_1 y_2\, \chi_5^*(z_2)\, \chi_5(z_1) \exp i\,(\phi_2 + \phi_1)$$

$$= \int d\Omega\, y_1 y_2\, \frac{1}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \left[\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2} \right] \left[\frac{1}{\mu - z_1} + (-1)^I \frac{1}{\mu + z_1} \right]$$

(C.22 cont.)

$$\times f_5^*(\lambda) f_5(\mu) \frac{1}{y^2} \left[y z_1 - z y_1 \cos \phi_1 + i y_1 \sin \phi_1 \right] e^{i\phi_1}$$

$$= \frac{1}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \, f_5^*(\lambda) f_5(\mu) \int d\Omega \left[\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2} \right]$$

$$\times \left[\frac{1}{\mu - z_1} + (-1)^I \frac{1}{\mu + z_1} \right]$$

$$\times \left[\frac{1}{1+z} (z_1^2 + z_2^2 - 2z z_1 z_2) - (1 - z_1 z_2) \right] \tag{C.22}$$

which gives us, using (D.22),

$$(1+z) \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \, f_5^*(\lambda) f_5(\mu) \frac{1}{\pi} \int \frac{4 \pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \left\{ \frac{1}{z' - z} \left[\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1+z')^2} + \frac{\lambda\mu - 1}{1+z'} \right] + (-1)^I \frac{1}{z' + z} \left[\frac{\lambda^2 + \mu^2 - \mu\lambda\mu z'}{(1-z')^2} - \frac{\lambda\mu + 1}{1-z'} \right] \right\} . \tag{C.23}$$

If we combine (C.21) and (C.23), we will obtain

$$\mathrm{Im}_{el} \, \phi_3(p, z) = \frac{p}{4\pi} (1+z) \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \, \frac{1}{\pi} \int \frac{4\pi^2 \, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \left\{ \frac{1}{z' - z} \left[\frac{1}{E^2 m^4} \left(f_{(+)}^*(\lambda) f_{(+)}(\mu) + f_{(-)}^*(\lambda) f_{(-)}(\mu) \right) \right. \right.$$

$$\times \left(\frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{(1+z')^2} + \frac{\lambda\mu + z'}{1+z'} \right)$$

$$+ \frac{1}{E^2 m^4} \left(f_{(+)}^*(\lambda) f_{(+)}(\mu) - f_{(-)}^*(\lambda) f_{(-)}(\mu) \right) \frac{\lambda + \mu}{1 + z'}$$

(C.24 cont.)

$$- \frac{2}{m^6} f_5^*(\lambda) f_5(\mu) \left(\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1+z')^2} + \frac{\lambda\mu-1}{1+z'} \right) \Bigg]$$

$$+ (-1)^I \frac{1}{z'+z} \left[\frac{1}{E^2 m^4} \left(f_{(+)}^*(\lambda) f_{(-)}(\mu) + f_{(-)}^*(\lambda) f_{(+)}(\mu) \right) \left(\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1-z')^2} - \frac{\lambda\mu+z'}{1-z'} \right) \right.$$

$$+ \frac{1}{E^2 m^4} \left(f_{(+)}^*(\lambda) f_{(-)}(\mu) - f_{(-)}^*(\lambda) f_{(+)}(\mu) \right) \frac{\lambda-\mu}{1-z'}$$

$$\left. - \frac{2}{m^6} f_5^*(\lambda) f_5(\mu) \left(\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1-z')^2} - \frac{\lambda\mu+1}{1-z'} \right) \right] .$$

Similarly, we can derive an expression for $\mathrm{Im}_{el}\, \phi_4$ that will use the angular integrations of (D.23), and which, together with (C.24), leads to (7.21c) and (7.21d).

If we use B_{5jk}, we obtain

$$\mathrm{Im}_{el}\, \phi_5 = \frac{p}{4\pi} \int d\Omega\, e^{i\phi_1} \left\{ (E\, p^2\, m^3)^{-1}\, y_1 [\chi_1^*(z_2) + \chi_2^*(z_2)] \chi_5(z_1) \right.$$

$$\left. + (E\, m^5)^{-1}\, \chi_5^*(z_2) y_2 \left[(1+z_1) e^{i\phi_3} \chi_3(z_1) - (1-z_1)\, e^{-i\phi_3} \chi_4(z_1) \right] \right\} \quad \text{(C.25)}$$

and

$$d\Omega\, e^{i\phi_1} y_1\, \chi_{1,2}^*(z_2) \chi_5(z_1)$$

$$= \frac{1}{y}\, d\Omega\, (z_2 - z\, z_1) \left[\frac{z_2}{\pi} \int_{\lambda_1}^{\infty} \frac{d\lambda}{\lambda}\, f_{1,2}^*(\lambda) \left(\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2} \right) + g_{1,2}^* \right]$$

$$\times \left[\frac{1}{\pi} \int_{\lambda_1}^{\infty} d\mu \left(\frac{1}{\mu - z_1} + (-1)^I \frac{1}{\mu + z_1} \right) f_5(\mu) \right] ,$$

which, using (D.7) and (D.19), gives

$$y \frac{2}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \, f_{1,2}^{*}(\lambda) \, f_5(\mu) \, \frac{1}{\pi} \int \frac{4\pi^2 \, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}$$

$$\times \left[\frac{1}{z'-z} + (-1)^I \frac{1}{z'+z} \right] \left[\frac{\lambda - \mu z'}{1-z'^2} \right] . \tag{C.26}$$

For the other term we note that,

$$\int d\Omega \, e^{i\phi_1} \chi_5^{*}(z_2) y_2 (1 \pm z_1) \, e^{\pm i\phi_3} \chi_{3,4}(z_1)$$

$$= \frac{1}{y} \, d\Omega \, [(z_1 - z\, z_2)(1 \pm z_1) \pm y^2 \mp (z_1^2 + z_2^2 - 2\, z\, z_1\, z_2)]$$

$$\times \frac{1}{\pi} \int_{\lambda_1}^{\infty} d\lambda \left[\frac{1}{\lambda - z_2} + (-1)^I \frac{1}{\lambda + z_2} \right] f_5^{*}(\lambda)$$

$$\times \frac{1}{\pi} \int_{\lambda_1}^{\infty} d\mu \left[\frac{1}{\mu - z_1} f_{(\pm)}(\mu) + (-1)^I \frac{1}{\mu + z_1} f_{(\mp)}(\mu) \right]$$

$$= y \frac{1}{\pi^2} \int_{\lambda_1}^{\infty} d\lambda \int_{\lambda_1}^{\infty} d\mu \, \frac{1}{\pi} \int \frac{4\pi^2 \, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left[\frac{1}{z'-z} + (-1)^I \frac{1}{z'+z} \right] \frac{1}{1-z'^2}$$

$$\times \Big\{ f_5^{*}(\lambda) \, f_{(\pm)}(\mu) [-(\mu - \lambda z')(1 \pm \mu) \pm (1-z'^2) \mp (\lambda^2 + \mu^2 - 2\lambda\mu z')]$$

$$+ f_5^{*}(\lambda) \, f_{(\mp)}(\mu) [-(\mu - \lambda z')(1 \mp \mu) \pm (1-z'^2) \mp (\lambda^2 + \mu^2 - 2\lambda\mu z')] \Big\} . \tag{C.27}$$

Equation (7.21e) follows from (C.26) and (C.27).

APPENDIX D. THE ANGULAR INTEGRATIONS

In this appendix we collect and summarize the angular integrations used in Appendix C. These are all basically of the form

$$\int d\Omega \; \frac{1}{\lambda \pm z_2} \; \frac{1}{\mu \pm z_1} \; f(z, z_1, z_2) \; , \tag{D.1}$$

where f is a simple polynomial and

$$z_2 = z\, z_1 + y\, y_1 \cos \phi_1 , \tag{D.2}$$

and

$$d\Omega = d\, z_1 \, d\, \phi_1 \; . \tag{D.3}$$

We start with some very simple integrals:

$$\int d\Omega \; 1 = 4\pi , \tag{D.4}$$

$$\int d\Omega \; \frac{1}{\lambda \mp z_1} = 2\pi \int_{-1}^{+1} d\, z_1 \; \frac{1}{\lambda \mp z_1}$$

$$= 2\pi \, \ell n \left(\frac{\lambda + 1}{\lambda - 1} \right) \equiv L(\lambda) , \tag{D.5}$$

$$\int d\Omega \; \frac{z_1}{\lambda \mp z_1} = \pm \; d\Omega \left[\frac{\lambda}{\lambda \mp z_1} - 1 \right] = \pm \left[\lambda L(\lambda) - 4\pi \right] \tag{D.6}$$

and

$$\int d\Omega \; \frac{z_2 - z\, z_1}{\lambda \mp z_1} = \int d\Omega \; \frac{y\, y_1 \cos \phi_1}{\lambda \mp z_1} = 0 \; . \tag{D.7}$$

We now turn to the basic Mandelstam integral:

$$\int d\Omega \; \frac{1}{\lambda - z_2} \; \frac{1}{\mu - z_1} \equiv J(\lambda, \mu, z) = \frac{1}{\pi} \int_{z_0}^{\infty} \frac{4\pi^2 \, dz}{[K(\lambda, \mu, z)]^{\frac{1}{2}}} \; \frac{1}{z' - z} \; , \tag{D.8}$$

where

$$K(\lambda, \mu, z) = \lambda^2 + \mu^2 + z^2 - 2\lambda\mu z - 1, \tag{D.9}$$

$$z_0 = \lambda\mu + [(\lambda^2 - 1)(\mu^2 - 1)]^{\frac{1}{2}} , \tag{D.10}$$

$$\int d\Omega \; \frac{1}{\lambda + z_2} \; \frac{1}{\mu + z_1} = J(\lambda, \mu, z) \; , \tag{D.11}$$

$$\int d\Omega \; \frac{1}{\lambda \mp z_2} \; \frac{1}{\mu \pm z_1} = J(\lambda, \mu, -z) \; ,$$

$$\therefore \int d\Omega \left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2} \right] \left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1} \right]$$

$$= 2 \, [J(\lambda, \mu, z) \mp J(\lambda, \mu, -z)] = \frac{2}{\pi} \int_{z_0}^{\infty} \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left[\frac{1}{z' - z} \mp \frac{1}{z' + z} \right] . \tag{D.12}$$

Other derived integrals needed are:

$$\int d\Omega \frac{z_1 z_2}{(\lambda - z_2)(\mu - z_1)}$$

$$= \int d\Omega \; \frac{1}{(\lambda - z_2)(\mu - z_1)} \; [(z_1 - \mu)(z_2 - \lambda) + \mu(z_2 - \lambda) + \lambda(z_1 - \mu) + \lambda\mu]$$

$$= 4\pi - \mu \, L(\mu) - \lambda \, L(\lambda) + \lambda\mu \, J(\lambda, \mu, z), \tag{D.13}$$

$$\int d\Omega \left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2} \right] \left[\frac{1}{\mu - z_2} \mp \frac{1}{\mu + z_2} \right] z_1 \, z_2 \qquad \text{(D.14 cont.)}$$

$$= 2(1\pm 1)\,[\,4\pi - \mu\, L(\mu) - \lambda L(\lambda)] + 2\lambda\mu\,[\,J(\lambda,\mu,z) \pm J(\lambda,\mu,-z)\,]\ , \qquad (D.14)$$

$$\int d\Omega \quad \frac{z_2^{\,2}}{(\lambda - z_2)(\mu - z_1)} = \int d\Omega \quad \frac{1}{(\lambda - z_2)\,(\mu - z_1)}\ [\,(z_2 - \lambda) z_2 + \lambda(z_2 - \lambda) + \lambda^2]$$

$$= \lambda^2\, J(\lambda,\mu,z) - \lambda \int d\Omega \quad \frac{1}{\mu - z_1} \ - z \int d\Omega \quad \frac{z_1}{\mu - z_1}$$

$$= \lambda^2\, J(\lambda,\mu,z) - \lambda\, L(\mu) - z\,[\,\mu L(\mu) - 4\pi]\,;$$

$$\therefore \int d\Omega\, \frac{z_2(z_2 - z\, z_1)}{(\lambda - z_2)(\mu - z_1)} = \lambda(\lambda - \mu z)\, J(\lambda,\mu,z) - \lambda\, L(\mu) + \lambda\, z\, L(\lambda)\ . \qquad (D.15)$$

Now we have

$$\lambda(\lambda - \mu z)\, J(\lambda,\mu,z) = y^2 \left[\frac{\lambda(\lambda - \mu z) J(\lambda,\mu,z)}{(1+z)(1-z)} \right]$$

$$= y^2\ \frac{1}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}} \left[\frac{1}{z' - z} \quad \frac{\lambda(\lambda - \mu z')}{1 - z'^{\,2}} \right.$$

$$\left. + \ \frac{1}{2} \quad \frac{1}{1-z} \quad \frac{\lambda(\lambda - \mu)}{z' - 1} + \frac{1}{2} \quad \frac{1}{1+z} \quad \frac{\lambda(\lambda + \mu)}{z' + 1} \right]$$

$$= y^2\ \frac{1}{\pi} \int \frac{4\,\pi^2\, dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}} \quad \frac{1}{z' - z} \quad \frac{\lambda(\lambda - \mu z')}{1 - z'^{\,2}}$$

$$+ \ \frac{1}{2}\ (1+z)\lambda(\lambda - \mu)\ J(\lambda,\mu,1) + \frac{1}{2}\ (1-z)\ \lambda(\lambda + \mu)\ J(\lambda,\mu-1)\ . \qquad (D.16)$$

If we note that

$$J(\lambda,\mu,\pm 1) = \frac{1}{\lambda \mp \mu}\ [\,L(\mu) \mp L(\lambda)]\ , \qquad (D.17)$$

and substitute (D.16) in (D.15), we derive

$$\int d\Omega\, \frac{z_2(z_2 - z\, z_1)}{(\lambda - z_2)\ (\mu - z_1)} = y^2\ \frac{1}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda,\mu,z')]^{\frac{1}{2}}} \quad \frac{1}{z' - z} \quad \frac{\lambda(\lambda - \mu z')}{1 - z'^{\,2}}\ . \qquad (D.18)$$

Similarly we derive

$$\int d\Omega \; z_2(z_2 - z\, z_1)\left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1}\right]$$

$$= y^2 \; \frac{2}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left[\frac{1}{z' - z} \mp \frac{1}{z' + z}\right] \frac{\lambda(\lambda - \mu z')}{1 - z'^2}\,, \qquad \text{(D.19)}$$

and

$$\int d\Omega \; (z_1^{\,2} + z_2^{\,2} - 2z\, z_1\, z_2)\left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1}\right]$$

$$= y^2 \; \frac{2}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left[\frac{1}{z' - z} \mp \frac{1}{z' + z}\right] \frac{\lambda^2 + \mu^2 - 2\lambda\mu z'}{1 - z'^2}\,. \qquad \text{(D.20)}$$

This result enables us to deduce

$$\int d\Omega \left[\frac{1}{1+z} \quad (z_1^{\,2} + z_2^{\,2} - 2z\, z_1\, z_2) - (1 - z_1\, z_2)\right]\left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1}\right]$$

$$= \frac{2}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left\{\left[\frac{1}{z' - z} \mp \frac{1}{z' + z}\right]\left[\frac{(1-z)\,(\lambda^2 + \mu^2 - 2\lambda\mu z')}{1 - z'^2} - 1\right]\right.$$

$$\left. + \left[\frac{1}{z' - z} \pm \frac{1}{z' + z}\right] \lambda\mu\right\} + 2(1 \pm 1)\left[4\pi - \lambda\, L(\lambda) - \mu\, L(\mu)\right]. \qquad \text{(D.21)}$$

Now we have

$$\frac{1}{\pi} \int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \; \frac{1}{z' \mp z} \left[\frac{(1-z)\,(\lambda^2 + \mu^2 - 2\lambda\mu z')}{1 - z'^2} - 1 \pm \lambda\mu\right]$$

$$= (1+z) \; \frac{1}{\pi} \int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \; \frac{1}{z' \mp z} \; \frac{1}{1 \pm z'} \left[\frac{(1 \mp z')(\lambda^2 + \mu^2 - 2\lambda\mu z')}{1 - z'^2} - 1 \pm \lambda\mu\right]$$

$$+ \frac{1}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \frac{1}{z' \pm 1} \left[\frac{2(\lambda^2+\mu^2-2\lambda\mu z')}{1-z'^2} - 1 \pm \lambda\mu\right]$$

$$= (1 + z) \frac{1}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \frac{1}{z' \mp z} \frac{1}{1 \pm z'} \left[\frac{(1 \mp z')(\lambda^2+\mu^2-2\lambda\mu z')}{1-z'^2} - 1 \pm \lambda\mu\right]$$

$$\pm [\lambda L(\lambda) + \mu L(\mu) - 4\pi] .$$

Therefore, substituting in (D.21),

$$\int d\Omega \left[\frac{1}{1+z}(z_1^2 + z_2^2 - 2z z_1 z_2) - (1 - z_1 z_2)\right]\left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1}\right]$$

$$= (1+z) \frac{2}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left\{\frac{1}{z'-z}\left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1+z')^2} - \frac{1-\lambda\mu}{1+z'}\right]\right.$$

$$\left. \mp \frac{1}{z'+z}\left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1-z')^2} - \frac{1+\lambda\mu}{1-z'}\right]\right\} . \qquad \text{(D.22)}$$

Similarly, we have

$$d\Omega \left[\frac{1}{1-z}(z_1^2 + z_2^2 - 2z z_1 z_2) - (1 + z_1 z_2)\right]\left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1}\right]$$

$$= (1-z) \frac{2}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}} \left\{\frac{1}{z'-z}\left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1-z')^2} - \frac{1+\lambda\mu}{1-z'}\right]\right.$$

$$\left. \mp \frac{1}{z'+z}\left[\frac{\lambda^2+\mu^2-2\lambda\mu z'}{(1+z')^2} - \frac{1-\lambda\mu}{1+z'}\right]\right\} . \qquad \text{(D.23)}$$

Finally, we have

$$\int d\Omega\, z_1 \frac{1}{\lambda - z_2} \frac{1}{\mu - z_1} = \int d\Omega\, [(z_1 - \mu) + \mu] \frac{1}{\lambda - z_2} \frac{1}{\mu - z_1}$$

$$= \mu J(\lambda, \mu, z) - L(\lambda) ,$$

and

and

$$\int d\Omega\,(z_1 \pm z_2)\ \frac{1}{\lambda - z_2}\ \frac{1}{\mu - z_1} = (\mu \pm \lambda)\ J(\lambda, \mu, z) - L(\lambda) \mp L(\mu)\ . \qquad \text{(D.24)}$$

Now we have

$$J(\lambda, \mu, z) = (1 \pm z)\ \frac{1}{\pi}\int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}\ \frac{1}{1 \pm z'}\left[\frac{1}{z' - z} \pm \frac{1}{1 \pm z}\right]$$

$$= (1 \pm z)\ \frac{1}{\pi}\ \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}\ \frac{1}{1 \pm z'}\ \frac{1}{z' - z} + J(\lambda, \mu, \mp 1)\ .$$

We substitute in (D.24), and making use of (D.17), obtain

$$\int d\Omega\,(z_1 \pm z_2)\ \frac{1}{\lambda - z_2}\ \frac{1}{\mu - z_1} = (1 \pm z)\ (\mu + \lambda)\ \frac{1}{\pi}\int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}\ \frac{1}{1 \pm z'}\ \frac{1}{z' - z}\ ,$$

$$\int d\Omega\,(z_1 + z_2)\left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1}\right]$$

$$= (1 + z)\ \frac{2}{\pi}\int \frac{4\pi^2 dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}\left[\frac{1}{z' - z}\ \frac{\mu + \lambda}{1 + z'} \mp \frac{1}{z' + z}\ \frac{\mu - \lambda}{1 - z'}\right], \qquad \text{(D.25)}$$

and

$$\int d\Omega (z_1 - z_2)\left[\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right]\left[\frac{1}{\mu - z_1} \mp \frac{1}{\mu + z_1}\right]$$

$$= (1 - z)\ \frac{2}{\pi}\int \frac{4\pi^2\, dz'}{[K(\lambda, \mu, z')]^{\frac{1}{2}}}\left[\frac{1}{z' - z}\ \frac{\mu - \lambda}{1 - z'} \mp \frac{1}{z' + z}\ \frac{\mu + \lambda}{1 + z'}\right]. \qquad \text{(D.26)}$$

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FOOTNOTES

[1] The paper cited in Ref. (4) contains many ideas germane to an understanding of the present work. We make frequent reference to it throughout as GGMW.

[2] This approach is being explored by L. Balazs (Lawrence Radiation Laboratory Berkeley), private communication.

[3] C. f. the parallel argument in Ref. (3) for pion-nucleon scattering.

[4] By "elastic scattering" we mean to exclude pion production; all possible helicity flips and charge exchange processes will, of course, be considered.

[5] Our dot product is $A.B. = A_0 B_0 - \underset{\sim}{A} \cdot \underset{\sim}{B} = - \underset{\sim}{A}_4 B_4 - \underset{\sim}{A} \cdot \underset{\sim}{B}$. Our Dirac matrices $\underset{\sim}{\gamma} = \rho_2 \underset{\sim}{\sigma}$, $\gamma_4 = \rho_3$, $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = - \rho_1$, satisfy $\gamma_\mu^\dagger = \gamma_\mu$, $\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu}$. Our spinors are so normalised that $\bar{u} u = 1$. The $u_\lambda (p)$ for the two helicities λ are as in GGMW: the "large components" are positive numbers. An isotopic-spin spinor of definite 3- component of isotopic spin is an implicit factor of every spinor, except that occasionally a linear superposition of two such to produce definite total isotopic spin will be understood when a product of two spinors is written. Our G-parity "charge-conjugate" spinors are $v_\lambda(p) = - i C \tau_2 \bar{u}_\lambda(p)^T$, where $C = i \gamma_2 \gamma_4$; $\bar{v} v = - 1$.

[6] See also Amati, Leader and Vitale, Ref. (12).

[7] We use the rather cumbersome phrase "simplest box diagram" to avoid possible confusion. All the diagrams of Figs. 4 and 5 are sometimes called "box diagrams."

[8]The reduction of the general form of the S-matrix to such a form, involving only five amplitudes in each isotopic spin channel, is discussed in GGMW and in Refs. (5), and (12).

[9]We shall frequently use a notation as in (2.13a). If two sets of amplitudes A and B are linearly related, we will write A = (AB) B, defining the matrix (AB): thus $(AB)^{-1} = (BA)$. All our transformations will be a direct product of a (2 by 2) matrix acting on the label for total isospin and a (5 by 5) matrix acting on the index related to the ordinary spins. When the isotopic spin factor is the identity matrix, it will be ignored in the notation, as in (2.13), and we will say that the transformation "does not involve" the isotopic spin.

[10]That the number of amplitudes prior to consideration of the invariance properties is only 16 is trivial here, and is equally so if external lines are represented by two-component spinors, but involves the Dirac equations appropriate to the four external lines in the treatment of the form (2.8) given by GGMW, and in Refs. (5) and (16).

[11]The fast way of computing $(\overline{F}\ F)$ and a demonstration that it is in fact independent of s, t, u on the basis of our definition is given at the end of Appendix A.

[12]We omit a detailed field-theoretical discussion of the interaction term in the Lagrangian and of the propagator, which would be in order for i = 2, 3, 4. Extra contributions of the expressions given, if present, should be obtained by the requirement of maximal analyticity and unitarity within the context of S-matrix theory: c. f. Ref. (16), and for an example see Wong and Shaw (University of California, San Diego, at La Jolla) private communication.

[13] It is amusing to note that an immediate consequence of (5.3) is that the X_i of Eq. A.3 are merely the G amplitudes in the order 5, 2, 3, 4, 1.

[14] In Eqs. (6.21) and subsequently, the upper sign refers to I = 0, the lower to I = 1.

[15] The functions $L(\lambda)$ and $J(\lambda, \mu, 0)$ are defined by Eqs. (D.5) and (C.9) respectively.

[16] C.f. Equation (5.4).

[17] See also Refs. (5) and (12).

Table I. $\chi_i^I(s, z)$ as $s \to \infty$

i	z fixed $-1 < z < 1$	t fixed < 0 ("z = 1")	u fixed < 0 ("z = - 1")
1	$o(s^{7/4})$	$O(s^2)$	$O(s^2)$
2	$o(s^{7/4})$	$o(s^2)$	$o(s^2)$
3	$o(s^{3/4})$	$O(s)$	$o(s^2)$
4	$o(s^{3/4})$	$o(s^2)$	$O(s)$
5	$o(s^{3/4})$	$o(s)$	$o(s)$

Table II. $\bar{\chi}_i^I(u, \bar{z})$ as $u \to \infty$

i	$\bar{z}$ fixed $-1 < \bar{z} < 1$	t fixed < 0 ("$\bar{z}$ = 1")	s fixed < 0 ("$\bar{z}$ = - 1")
1	$o(u^{7/4})$	$O(u^2)$	$o(u^2)$
2	$o(u^{7/4})$	$o(u^2)$	$o(u^2)$
3	$o(u^{3/4})$	$O(u)$	$o(u^2)$
4	$o(u^{3/4})$	$o(u^2)$	$o(u)$
5	$o(u^{3/4})$	$o(u)$	$o(u)$

Table III. $\chi_i^I(u, \bar{z})$ as $u \to \infty$

i	$\bar{z}$ fixed $-1 < \bar{z} < 1$	t fixed < 0 ("$\bar{z} = 1$")	s fixed < 0 ("$\bar{z} = -1$")
1	$o(u^{7/4})$	$O(u^2)$	$o(u)$
2	$o(u^{7/4})$	$o(u^2)$	$o(u)$
3	$o(u^{3/4})$	$O(u)$	$o(1)$
4	$o(u^{3/4})$	$o(u^2)$	$o(1)$
5	$o(u^{3/4})$	$o(u)$	$o(1)$

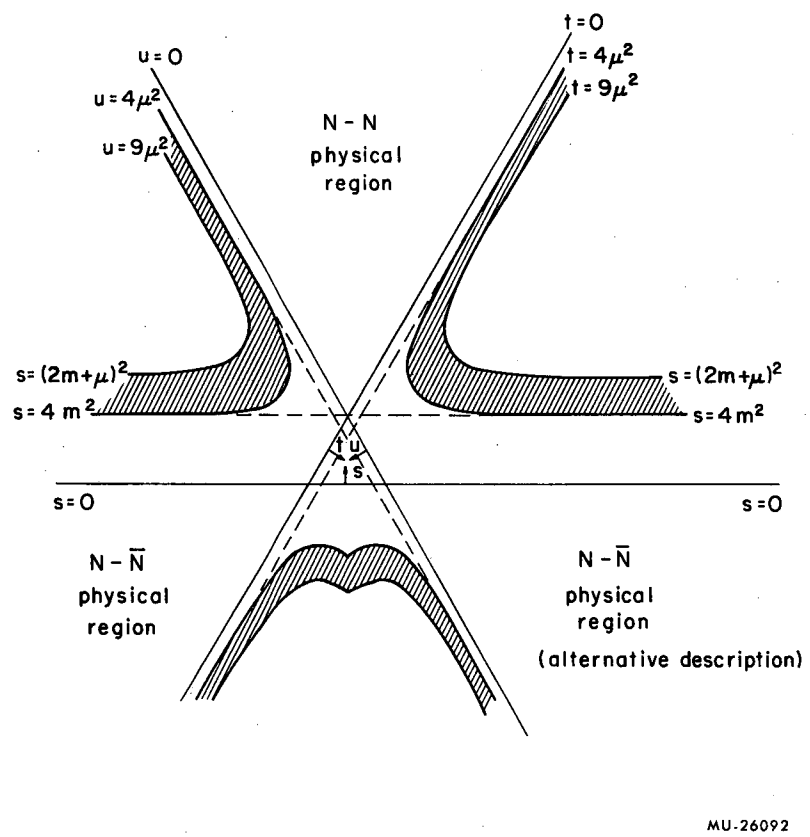


Fig. 1. The physical regions, and the strip regions of the double spectral functions. The latter are indicated by cross-hatching. This figure is not drawn to scale.

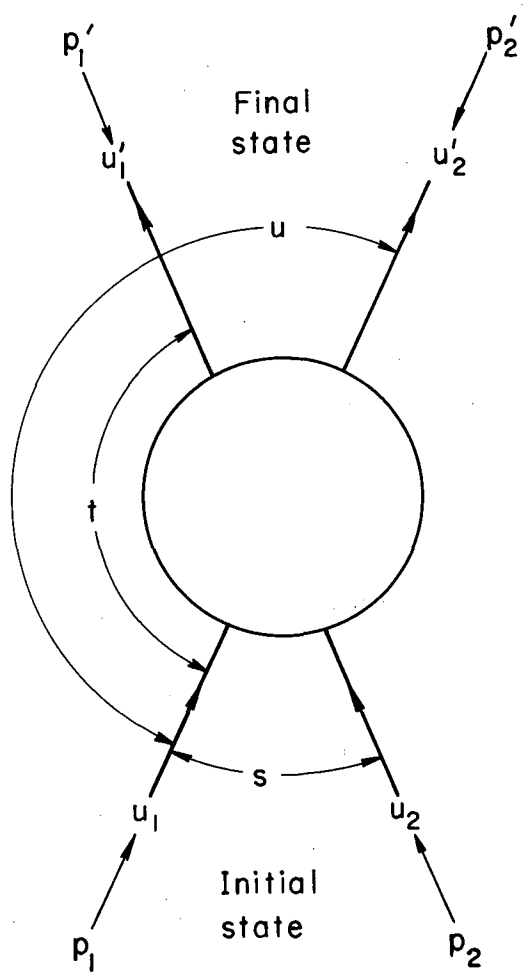
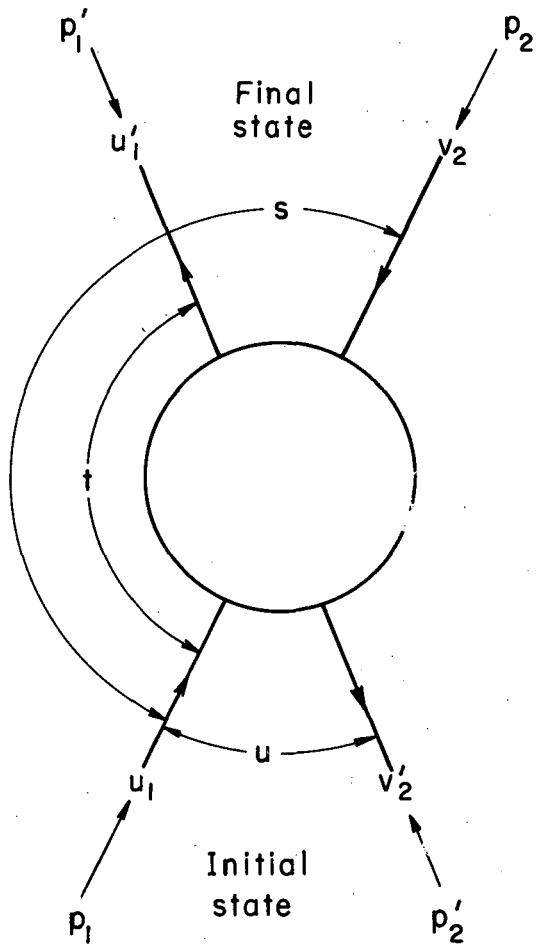


Fig. 2. Nucleon-nucleon elastic scattering. The arrows on the lines attached to the central "reaction zone" are in accordance with Feynman rules.



MU-26094

Fig. 3. Nucleon-antinucleon elastic scattering.

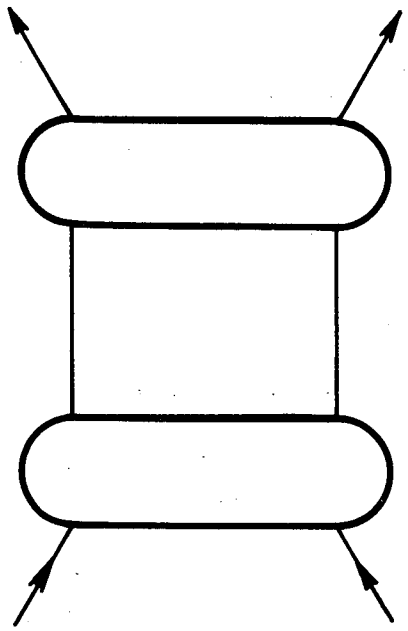


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Fig. 4. Landau-Cutkowsky diagram with two-particle intermediate state in the nucleon-nucleon channel.

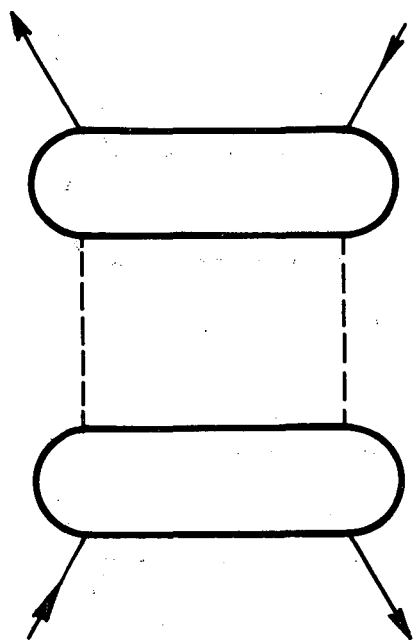

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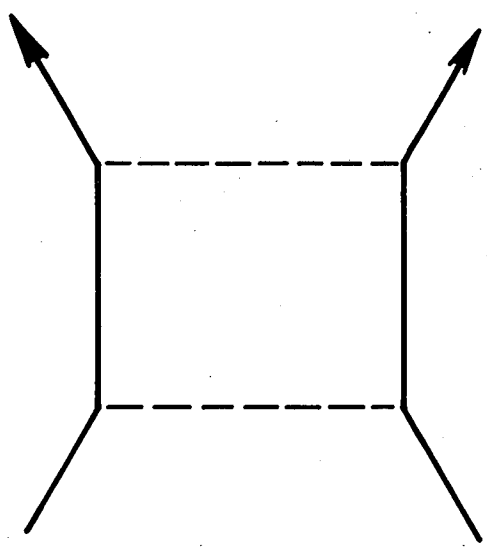
Fig. 5. Landau-Cutkowsky diagram with two-particle intermediate state in the nucleon-antinucleon channel.



MU-26097

Fig. 6. The simplest box diagram.

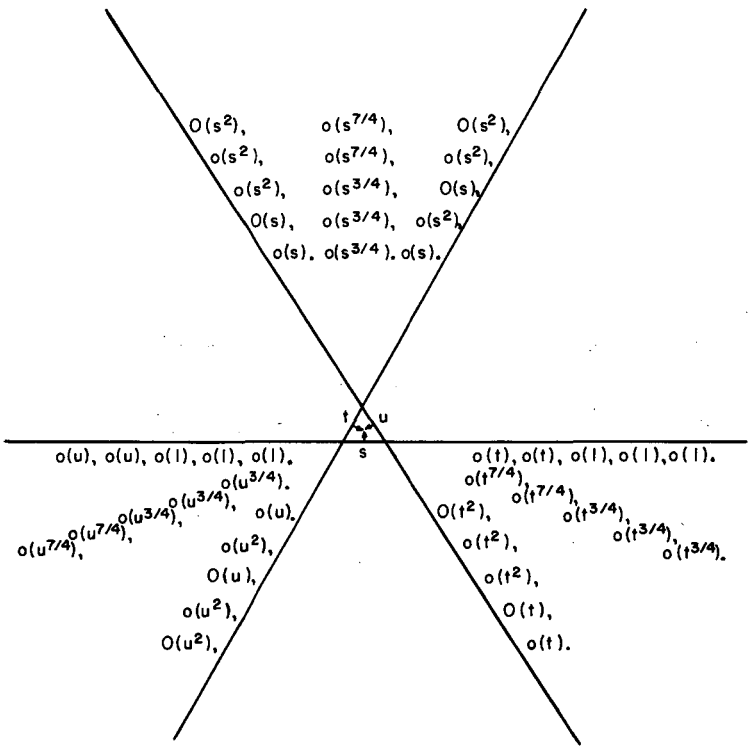


Fig. 9. Asymptotic properties of the χ amplitudes in physical regions. In each case the behavior of the χ_i are listed in order of i, with a period at the end of each list.

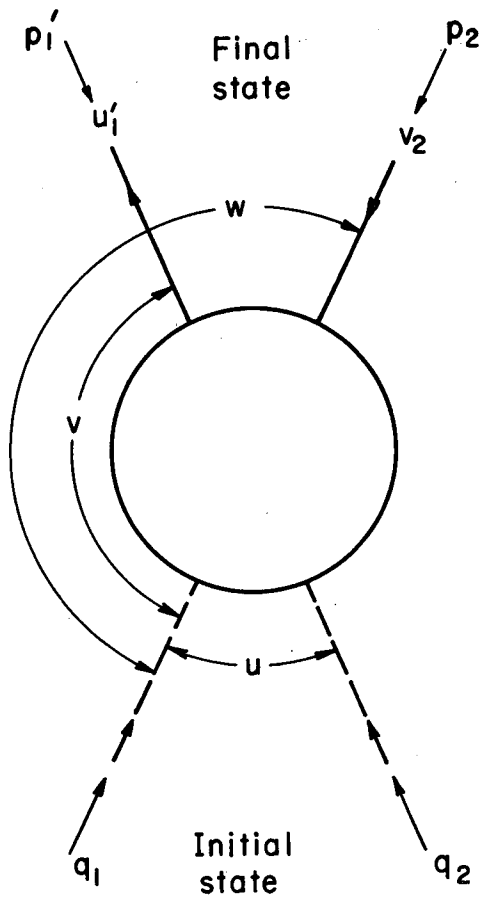


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Fig. 10. Diagram for the reaction $\pi\pi \rightarrow N\,\overline{N}$.

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